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ON THE DISTRIBUTION OF THE HILBERT TRANSFORM OF THE LOCAL TIME OF A SYMMETRIC LÉVY PROCESS[1]

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We derive simple explicit formulas for the Fourier–Laplace transforms of the Hilbert transform and related functionals of the local time of a symmetric Lévy process. These formulas generalize results of Biane and Yor for Brownian motion. The method of proof provides an explanation (of sorts) for the presence of the hyperbolic functions in such formulas.

1. Introduction. The paper of Biane and Yor [3] contains a wealth of information on the distribution of certain functionals of Brownian motion and related processes. Some of the most striking of their results concern the Hilbert transform of the Brownian local time. To recall one example from [3], let $(B_t)_{t \geq 0}$ be standard one-dimensional Brownian motion with $B_0 = 0$, and consider the fluctuating additive functional

$$H_t = \int_0^\infty (L_t^x - L_t^{-x}) \frac{dx}{\pi x} = \lim_{\varepsilon \downarrow 0} \int_0^t \frac{ds}{\pi B_s} 1_{\{|B_s| \geq \varepsilon\}}, \tag{1.1}$$

where $\{L_t^x : t \geq 0, x \in \mathbb{R}\}$ is Brownian local time. If T is an exponential random variable independent of (B_t) with mean q^{-1}, then ([3], page 67)

$$E(\exp(i\lambda H_T)) = \operatorname{sech}\left(\lambda (2q)^{-1/2}\right), \qquad \lambda \in \mathbb{R}. \tag{1.2}$$

This formula is derived in [3] through a combination of excursion theory, Brownian scaling and special properties of Bessel processes. In fact, (1.2) is just one of a group of explicit formulas in [3] for Fourier transforms of random variables related to (H_t).

Our aim in this paper is to show that (1.2) and various companion identities hold for a wide class of symmetric one-dimensional Lévy processes. One only has to recognize the expression $(2q)^{-1/2}$ for what it is:

$$(2q)^{-1/2} = \int_0^\infty e^{-qt} p(t, 0)\, dt,$$

where $p(t, x)$ is the Brownian transition density. Then the form of (1.2) appropriate to a suitable Lévy process is obtained by relacing $(2q)^{-1/2}$ by the analogous expression for the Lévy process.

Our proofs are more direct than those found in [3] and are based on moment calculations which yield a combinatorial explanation for the ubiquity of the

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hyperbolic functions in (1.2) and its companions. Of course, we would never have arrived at our results without the hint provided by the Brownian case.

The process (H_t) arises naturally in certain limit theorems for occupation times. See Yamada [10], [11] for the Brownian case and [5] for similar results in the context of stable processes.

We now proceed to set down our hypotheses precisely, after which we shall formulate our main results.

Throughout the paper, $X = (\Omega, \mathscr{F}, \mathscr{F}_t, \theta_t, X_t, P^x)$ will denote the canonical realization of a real-valued Lévy process. Thus X is a Hunt process with stationary independent increments and Lévy exponent ψ determined by

$$(1.3) \qquad P^0(\exp(i\lambda X_t)) = \exp(-t\psi(\lambda)), \qquad t \ge 0, \lambda \in \mathbb{R}.$$

We assume that X is *symmetric*, so that ψ is even (hence real). This means that ψ can be represented as

$$(1.4) \qquad \psi(\lambda) = \tfrac{1}{2}\sigma^2\lambda^2 + \int_0^\infty (1 - \cos \lambda x)\nu(dx),$$

where $\sigma \ge 0$ and ν is a measure on $]0, \infty[$ such that $\int_0^\infty (x^2 \wedge 1)\nu(dx) < \infty$. We assume further that

$$(1.5) \qquad \int_0^\infty [q + \psi(\lambda)]^{-1} d\lambda < \infty \quad \text{for some (and then all) } q > 0.$$

Because of (1.5), there are continuous transition densities $p(t, x)$ and resolvent densities $u^q(x)$ such that for $q > 0$ and $x \in \mathbb{R}$,

$$\int_0^\infty e^{-qt} p(t, x)\, dt = u^q(x),$$

$$(1.6) \qquad u^q(x) = \frac{1}{2\pi}\int_{-\infty}^\infty e^{-i\lambda x} \frac{d\lambda}{q + \psi(\lambda)} = \frac{1}{\pi}\int_0^\infty \frac{\cos \lambda x}{q + \psi(\lambda)}\, d\lambda$$

and

$$(1.7) \qquad P^x \int_0^\infty e^{-qt} f(X_t)\, dt = \int u^q(y - x) f(y)\, dy$$

for any bounded or positive Borel function f.

It is well known ([4], VI(4.11)) that (1.5) (in combination with the symmetry of X) guarantees the existence of local time $(L_t^x \colon t \ge 0, x \in \mathbb{R})$ for X. For each $x \in \mathbb{R}$, $(L_t^x)_{t \ge 0}$ is a continuous additive functional (CAF) of X such that the support of the measure $d_t L_t^x$ coincides almost surely with $\{t\colon X_t = x\}$. Moreover (L_t^x) can be normalized so that

$$(1.8) \qquad P^x \int_0^\infty e^{-qt}\, dL_t^y = u^q(y - x)$$

and then

$$\int_0^t f(X_s)\,ds = \int_{\mathbb{R}} f(x) L_t^x\,dx \tag{1.9}$$

for all $t \geq 0$ and all bounded Borel functions f almost surely.

To ensure that a jointly continuous version of local time can be chosen, we need to impose one more condition. Define

$$\delta(u) = \sup_{0<x\leq u} \frac{1}{\pi}\int_0^\infty \frac{1-\cos\lambda x}{1+\psi(\lambda)}\,d\lambda,$$

$$\rho(x) = \int_0^x \left[\log(1+u^{-2})\right]^{1/2} d\sqrt{\delta(u)}\,.$$

Our final hypothesis is

$$\int_0^1 x^{-1}\rho(x)\,dx < \infty. \tag{1.10}$$

Implicit in (1.10) is the finiteness of $\rho(1/2)$, which is equivalent to Barlow's condition

$$\sum_{n=1}^{\infty} \left(\delta(2^{-n})/n\right)^{1/2} < \infty \tag{1.11}$$

(see [1], especially page 29), and under which a version of (L_t^x) can be chosen so as to be jointly continuous in t and x. Moreover, by [1], (1.11) implies that there are constants $C(\omega, t_0)$ with $C(\omega, t_0) < \infty$ such that almost surely,

$$\sup_{0\leq t\leq t_0} \left|L_t^x(\omega) - L_t^y(\omega)\right| \leq C(\omega, t_0)\rho(|x-y|) \tag{1.12}$$

for all $x, y \in \mathbb{R}$ and $t_0 > 0$.

It is worth noting that both (1.5) and (1.10) hold provided

$$\sup\{\beta \geq 0\colon \lambda^{-\beta}\psi(\lambda) \to +\infty \text{ as } \lambda \to +\infty\} > 1.$$

In particular, any symmetric stable process of index $\alpha \in]1,2]$ satisfies our hypotheses.

As noted in (2.7) of [5], there is a set $\Lambda \in \mathscr{F}^*$ (the universal completion of $\mathscr{F}^0$) with $\theta_t\Lambda \subset \Lambda$ for all $t > 0$ and $P^x(\Lambda) = 1$ for all $x \in \mathbb{R}$ and a version of L_t^x such that (1.9), (1.12), the joint continuity of $(t, x) \mapsto L_t^x(\omega)$ and

$$L_{t+s}^x(\omega) = L_t^x(\omega) + L_s^x(\theta_t\omega) \tag{1.13}$$

hold identically for $\omega \in \Lambda$. Moreover, one can take $\omega \mapsto L_t^x(\omega)$ to be $\mathscr{F}_t \cap \mathscr{F}^*$ measurable for each t and x and one can assume that for $\omega \in \Lambda$,

$$L_t^x(\omega) = 0 \quad \text{whenever } |x| > \sup\{|X_s(\omega)|\colon 0 \leq s \leq t\}. \tag{1.14}$$

These properties of (L_t^x) will be used in the sequel without special mention.

In view of (1.10), (1.12) and (1.14), the integral

$$H_t := \frac{1}{\pi}\int_0^\infty y^{-1}(L_t^y - L_t^{-y})\,dy \tag{1.15}$$

is absolutely convergent almost surely, and defines a (fluctuating) CAF of X. Of course, H_t is the (negative of the) Hilbert transform of $x \mapsto L_t^x$, evaluated at $x = 0$. Evidently,

$$H_t = \lim_{\varepsilon \downarrow 0} \frac{1}{\pi}\int_0^t X_s^{-1} 1_{\{|X_s| \geq \varepsilon\}}\,ds = \text{p.v.}\frac{1}{\pi}\int_0^t X_s^{-1}\,ds.$$

It can be shown that (H_t) is a CAF of zero energy (in the sense of Fukushima [7]) and that $t \mapsto H_t$ is of unbounded variation over any interval on which $t \mapsto L_t^0$ is not constant.

NOTATION. In the sequel we shall write P for P^0 and L_t for L_t^0. The function

$$\kappa(q) = u^q(0) = \frac{1}{\pi}\int_0^\infty \frac{d\lambda}{q + \psi(\lambda)}, \qquad q > 0, \tag{1.16}$$

will play an important role in what follows. Note, for example, that if X is a stable process with $\psi(\lambda) = c|\lambda|^\alpha$, $\alpha > 1$, $c > 0$, then

$$\kappa(q) = \frac{q^{1/\alpha - 1}}{\alpha c^{1/\alpha}\sin(\pi/\alpha)}.$$

Our main results concern the P-distribution of H_S for various random times S. Among these are certain exit and entrance times associated with the zero set $\{t\colon X_t = 0\}$. We define for $t > 0$,

$$g(t) = \sup\{s \leq t\colon X_s = 0\} \qquad (\sup\phi = 0),$$
$$d(t) = \inf\{s > t\colon X_s = 0\} \qquad (\inf\phi = +\infty).$$

(1.17) THEOREM. *Let $T = T(q)$ be an exponentially distributed random variable independent of X with parameter $q > 0$. Then for $\lambda, \mu \in \mathbb{R}$,*

$$P(\exp(i\lambda H_T)) = \operatorname{sech}(\lambda\kappa(q)), \tag{1.18}$$

$$P\big(\exp(i\lambda H_{g(T)})\big) = \frac{\tanh(\lambda\kappa(q))}{\lambda\kappa(q)}, \tag{1.19}$$

$$P\big(\exp\big(i\lambda H_{g(T)} + i\mu(H_T - H_{g(T)})\big)\big) = \frac{\tanh(\lambda\kappa(q))}{\lambda}\cdot\frac{\mu}{\sinh(\mu\kappa(q))}. \tag{1.20}$$

In particular, $H_{g(T)}$ and $H_T - H_{g(T)}$ are independent.

Our Lévy process X is (point) recurrent if and only if the zero set $\{t: X_t = 0\}$ is unbounded a.s. P. As is well known, this is equivalent to the condition $P(L_\infty = \infty) = 1$, which in turn is equivalent to $\kappa(0) := \kappa(0+) = \infty$; see (3.16) below or [6], Section 5. These observations, together with the recurrence criterion ([2], 13.23), show that

$$(1.21) \qquad \kappa(0) = \infty \quad \text{if and only if} \quad \int_0^1 \psi(\lambda)^{-1}\, d\lambda = \infty.$$

Of course, $P(d(T) = \infty) > 0$ if X is transient (i.e., nonrecurrent).

(1.22) THEOREM. *Let T be an exponential random variable as in* (1.17). *Then for $\lambda, \mu \in \mathbb{R}$,*

$$(1.23) \qquad P\big(\exp(i\lambda H_{d(T)}); d(T) < \infty\big) = 1 - \frac{\tanh(\lambda\kappa(q))}{\tanh(\lambda\kappa(0))},$$

where $\tanh(\pm\infty) = \pm 1$ *and*

$$(1.24) \qquad \begin{aligned} &P\big(\exp\big(i\lambda H_{g(T)} + i\mu(H_{d(T)} - H_{g(T)})\big); d(T) < \infty\big) \\ &\quad = \frac{\tanh(\lambda\kappa(q))}{\lambda}\left[\frac{\mu}{\tanh(\mu\kappa(q))} - \frac{\mu}{\tanh(\mu\kappa(0))}\right]. \end{aligned}$$

In particular, $H_{g(T)}$ and $H_{d(T)} - H_{g(T)}$ are independent on $\{d(T) < \infty\}$.

REMARK. The independence assertions made in (1.17) and (1.22) are well-known consequences of excursion theory and are valid quite generally.

For our final result, let $\tau(t) = \inf\{s: L_s > t\}$ denote inverse local time at 0. It is a standard consequence of the strong Markov property that the process $(\tau(t), H_{\tau(t)})_{t \ge 0}$ is a Lévy process with values in $\mathbb{R}^+ \times \mathbb{R}$. (Note that this process has finite lifetime L_∞ if X is transient.) The distribution of $(\tau(t), H_{\tau(t)})_{t \ge 0}$ is determined in the following:

(1.25) THEOREM. *For $q > 0$ and $\lambda \in \mathbb{R}$,*

$$(1.26) \qquad P\big(\exp(-q\tau(t) + i\lambda H_{\tau(t)}); \tau(t) < \infty\big) = \exp(-t\lambda\coth(\lambda\kappa(q))).$$

In particular, if X is recurrent, then

$$(1.27) \qquad P\big(\exp(i\lambda H_{\tau(t)})\big) = \exp(-t|\lambda|),$$

so that $(H_{\tau(t)})_{t \ge 0}$ is a standard symmetric Cauchy process.

If X is a symmetric stable process with index $\alpha > 1$, then a scaling argument shows that $(H_{\tau(t)})$ must be a Cauchy process. But this scaling argument fails to show that the scale constant in (1.27) is 1. We have no sensible explanation for the remarkable formula (1.27) in the general case.

The rest of the paper is laid out as follows. In Section 2 we prove a combinatorial lemma which serves as the key ingredient in Section 3, where the moments of (H_t) are computed. The proofs of (1.17), (1.22) and (1.25) are in Section 4.

2. A combinatorial lemma. In this section we prepare two identities that will aid in the evaluation of certain multiple integrals arising from the moment calculations of Section 3.

We shall say that a real sequence $(x_1, \dots, x_k)$ is *alternating* (respectively, *reverse alternating*) provided $x_1 > x_2 < x_3 > \cdots$ (respectively, $x_1 < x_2 > x_3 < \cdots$). Let S_k denote the set of permutations of $\{1, 2, \dots, k\}$. We shall identify each $\sigma \in S_k$ with a sequence $(\sigma_1, \dots, \sigma_k)$ in the obvious way. We write E_k for the number of alternating permutations in S_k, setting $E_0 = 1$ for convenience. There is an obvious bijection between alternating and reverse alternating elements of S_k, so E_k also counts reverse alternating permutations. The exponential generating function of $\{E_k: k \geq 0\}$ can be expressed as follows:

$$\sum_{n=0}^{\infty} (-1)^n E_{2n} \frac{x^{2n}}{(2n)!} = \operatorname{sech} x, \qquad |x| < \frac{\pi}{2}, \tag{2.1a}$$

$$\sum_{n=0}^{\infty} (-1)^n E_{2n+1} \frac{x^{2n+1}}{(2n+1)!} = \tanh x, \qquad |x| < \frac{\pi}{2}. \tag{2.1b}$$

See, for example, formula (58) on page 149 of [9].

Let $T_k = \{-1, 1\}^k$, $k \geq 2$. Regard $\tau = (\tau_1, \dots, \tau_k) \in T_k$ as a choice of signs to be attached to a permutation $\sigma \in S_k$. Define, for $k \geq 2$, $\sigma \in S_k$, $\tau \in T_k$,

$$e_k(\tau, \sigma) = \prod_{j=1}^{k} \operatorname{sgn}(\tau_j \sigma_j - \tau_{j-1}\sigma_{j-1}), \quad \text{if } k \text{ is even;} \tag{2.2a}$$

$$e_k(\tau, \sigma) = \prod_{j=1}^{k-1} \operatorname{sgn}(\tau_{j+1} \sigma_{j+1} - \tau_j \sigma_j), \quad \text{if } k \text{ is odd.} \tag{2.2b}$$

By convention $\tau_0 \sigma_0 = 0$ and sgn is the usual sign function. Note that $e_k(\tau, \sigma)$ is unchanged if $(\sigma_1, \sigma_2, \dots, \sigma_k)$ is replaced by $(x_{\sigma(1)}, \dots, x_{\sigma(k)})$, where $0 < x_1 < \cdots < x_k$ is any sequence of real numbers.

(2.3) LEMMA. *Given $k \geq 2$, let $n := k/2$ or $(k-1)/2$ according as k is even or odd. Then*

$$2^{-k} \sum_{\tau \in T_k, \sigma \in S_k} e_k(\tau, \sigma) = (-1)^n E_k.$$

PROOF. The lemma is an immediate consequence of the following observations: (a) If $k \geq 2$ is even, then for $\sigma \in S_k$,

$$e_k(\tau, \sigma) = (-1)^n, \qquad \forall \tau \in T_k, \quad \text{if } \sigma \text{ is alternating,}$$

while

$$\sum_{\tau \in T_k} e_k(\tau, \sigma) = 0, \quad \text{if } \sigma \text{ is not alternating.}$$

(b) If $k \geq 3$ is odd, then for $\sigma \in S_k$,

$$e_k(\tau, \sigma) = (-1)^n, \qquad \forall \tau \in T_k, \quad \text{if } \sigma \text{ is reverse alternating,}$$

while

$$\sum_{\tau \in T_k} e_k(\tau, \sigma) = 0, \quad \text{if } \sigma \text{ is not reverse alternating.}$$

We shall prove only (a); the proof of (b) is quite similar.

We prove the alternating case of (a) by induction on $n = k/2$. One can check that $e_2(\tau, \sigma) = -1$ if $\sigma \in S_2$ is alternating by listing the four possibilities for τ. Suppose now that $e_{2n}(\tau, \sigma) = (-1)^n$ for all alternating $\sigma \in S_{2n}$ and all $\tau \in T_{2n}$. Let $\sigma \in S_{2n+2}$ be alternating. Then $\sigma_{2n} < \sigma_{2n+1} > \sigma_{2n+2}$, so by the remark following (2.2), we have

$$e_{2n+2}(\tau, \sigma) = (-1)^n \operatorname{sgn}(\Delta_{2n+1}) \operatorname{sgn}(\Delta_{2n+2}),$$

where $\Delta_j = \tau_j \sigma_j - \tau_{j-1}\sigma_{j-1}$. It is easy to check that since $\sigma_{2n} < \sigma_{2n+1} > \sigma_{2n+2}$, one has $\operatorname{sgn}(\Delta_{2n+1})\operatorname{sgn}(\Delta_{2n+2}) = -1$ for each of the eight possible choices of $(\tau_{2n}, \tau_{2n+1}, \tau_{2n+2})$. Thus $e_{2n+2}(\tau, \sigma) = (-1)^{n+1}$ for all $\tau \in T_{2n+2}$ and all alternating $\sigma \in S_{2n+2}$, and the induction is complete.

Now suppose $\sigma \in S_k$ is not alternating. Then there is a unique integer j, $1 \leq j < k$, such that $(\sigma_1, \sigma_2, \dots, \sigma_j)$ is alternating but $(\sigma_1, \dots, \sigma_{j+1})$ is not. If $j = 1$, then $\sigma_1 < \sigma_2$. If $1 < j < k$, then $\sigma_{j-1} > \sigma_j > \sigma_{j+1}$ or $\sigma_{j-1} < \sigma_j < \sigma_{j+1}$ according as j is even or odd. Given $\tau \in T_k$, define $\tilde{\tau} \in T_k$ by changing the sign of τ_j, leaving the other elements of τ unchanged. Since the effect on $e_k(\tau, \sigma)$ of the passage from τ to $\tilde{\tau}$ is local [only sgn $(\tau_1\sigma_1)$ is altered if $j = 1$, only $\operatorname{sgn}(\Delta_j)\operatorname{sgn}(\Delta_{j+1})$ is altered if $1 < j < k$], it is easy to check that $e_k(\tilde{\tau}, \sigma) = -e_k(\tau, \sigma)$. Since $\tau \mapsto \tilde{\tau}$ is a bijection of T_k, it follows that $\sum_{\tau \in T_k} e_k(\tau, \sigma) = 0$.

3. Moment calculations. In this section we establish several identities concerning the moments of (H_t). We begin by recording a general identity from which the others will be deduced.

Given a finite (positive) measure μ on $\mathbb{R}$, we can define a CAF of X by setting

$$A_t^\mu = \int L_t^x \mu(dx), \qquad t \geq 0.$$

In view of (1.8) and (1.9),

$$(3.1) \qquad P^x \int_0^\infty e^{-qt} g(X_t)\, dA_t^\mu = \int u^q(y - x) g(y) \mu(dy).$$

We write

$$\hat{\mu}(\xi) = \int e^{i\xi x}\mu(dx), \qquad \xi \in \mathbb{R},$$

for the Fourier transform of μ. The Fourier transform $\hat{f}$ of a function $f \in L^1(\mathbb{R})$ is defined analogously.

(3.2) LEMMA. *For $q > 0$, $k \in \mathbb{N}$,*

$$(3.3)\qquad P\int_0^\infty e^{-qt}H_t^k\,dA_t^\mu = \frac{k!i^k}{(2\pi)^{k+1}}\int_{\mathbb{R}^{k+1}}\left[\prod_{j=1}^k \frac{\operatorname{sgn}(\xi_{j+1}-\xi_j)}{q+\psi(\xi_j)}\right]\frac{\hat{\mu}(-\xi_{k+1})}{q+\psi(\xi_{k+1})}\,d\xi_1\cdots d\xi_{k+1},$$

the absolute convergence of both integrals being part of the assertion.

PROOF. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a bounded Borel function with compact support, so that $\hat{f}$ is bounded. By a routine calculation we have, for $k \in \mathbb{N}$,

$$\begin{aligned} I_k &:= P\int_0^\infty e^{-qt}\left[\int_0^t f(X_s)\,ds\right]^k dA_t^\mu \\ (3.4)\qquad &= k!P\int\cdots\int_{0<t_1<\cdots<t_k<t<\infty} e^{-qt}\left[\prod_{j=1}^k f(X_{t_j})\right]dt_1\cdots dt_k\,dA_t^\mu \\ &= k!\int_{\mathbb{R}^{k+1}}\left[\prod_{j=1}^k u^q(x_j-x_{j-1})f(x_j)\right] \\ &\qquad \times u^q(x_{k+1}-x_k)\,dx_1\cdots dx_k\,\mu(dx_{k+1}), \end{aligned}$$

where $x_0 = 0$. Using (1.6) and the symmetry of ψ, repeated application of Fubini's theorem (or the Parseval relation) yields

$$\begin{aligned} (3.5)\qquad I_k &= \frac{k!}{(2\pi)^{k+1}}\int_{\mathbb{R}^{k+1}}\left[\prod_{j=1}^k \frac{\hat{f}(\xi_{j+1}-\xi_j)}{q+\psi(\xi_j)}\right] \\ &\qquad \times \frac{\hat{\mu}(-\xi_{k+1})}{q+\psi(\xi_{k+1})}\,d\xi_1\cdots d\xi_{k+1}. \end{aligned}$$

[The manipulations involved in passing from (3.4) to (3.5) are easily justified since $\hat{\mu}$ is bounded and both f and $[q+\psi]^{-1}$ lie in $L^1(\mathbb{R})$.]

We now take $f(x) = f_\varepsilon(x) = (\pi x)^{-1}1_{\{\varepsilon\le|x|\le\varepsilon^{-1}\}}$ in (3.4). Observe that

$$\hat{f}_\varepsilon(\xi) = \left(\frac{2i}{\pi}\right)\int_{\xi\varepsilon}^{\xi/\varepsilon}\frac{\sin x}{x}\,dx, \qquad \xi \in \mathbb{R},$$

which tends boundedly and pointwise to $i\,\mathrm{sgn}(\xi)$ as $\varepsilon \downarrow 0$. Thus (3.5) implies that for each $k \in \mathbb{N}$, $q > 0$,

$$\lim_{\varepsilon \downarrow 0} P\int_0^\infty e^{-qt}\left[\int_0^t f_\varepsilon(X_s)\,ds\right]^k dA_t^\mu$$

$$(3.6)\qquad = \frac{k!\,i^k}{(2\pi)^{k+1}}\int_{\mathbb{R}^{k+1}}\left[\prod_{j=1}^{k}\frac{\mathrm{sgn}(\xi_{j+1}-\xi_j)}{q+\psi(\xi_j)}\right]\frac{\hat{\mu}(-\xi_{k+1})}{q+\psi(\xi_{k+1})}\,d\xi_1\cdots d\xi_{k+1}.$$

On the other hand, by (1.9) and (1.15),

$$(3.7)\qquad \lim_{\varepsilon \downarrow 0}\int_0^t f_\varepsilon(X_s)\,ds = \lim_{\varepsilon \downarrow 0}\int_\varepsilon^{\varepsilon^{-1}}\frac{1}{\pi y}\left[L_t^y - L_t^{-y}\right]dy = H_t$$

for all $t > 0$ almost surely. Now (3.6) for even $k \in \mathbb{N}$ shows that for each $r > 0$, the family of functions $(t, \omega) \mapsto |\int_0^t f_\varepsilon(X_s(\omega))\,ds|^r$, $0 < \varepsilon < 1$, is uniformly integrable relative to the finite measure $e^{-qt}\,dA_t^\mu(\omega)P(d\omega)$. Therefore, we can combine (3.6) and (3.7) to obtain the conclusion of the lemma. □

In two cases we can evaluate the multiple integral in (3.3) by exploiting symmetry. We note at this point that the P-distribution of $((H_s, L_t)$: $s \geq 0$, $t \geq 0)$ is unchanged if H_s is replaced by $-H_s$. This follows from (1.9) and (1.15) because of the symmetry of X. Thus

$$(3.8)\qquad P(H_t^k) = P\left(\int_0^t H_s^k\,dL_s\right) = 0, \qquad \forall\, t > 0,$$

whenever $k \in \mathbb{N}$ is odd.

(3.9) Proposition. *If $k \in \mathbb{N}$ is even, then for $q > 0$,*

$$(3.10)\qquad P\int_0^\infty qe^{-qt}H_t^k\,dt = [\kappa(q)]^k E_k$$

and

$$(3.11)\qquad P\int_0^\infty e^{-qt}H_t^k\,dL_t = [\kappa(q)]^{k+1}E_{k+1}/(k+1),$$

where E_k is as in Section 2.

Proof. We begin with the proof of (3.10). Fix an even $k \in \mathbb{N}$. For $\varepsilon > 0$, let $\mu_\varepsilon(dx) = \exp(-\varepsilon x^2/2)\,dx$. Then $\hat{\mu}(\xi)/2\pi = (2\pi\varepsilon)^{-1/2}\exp(-\xi^2/2\varepsilon)$ is an approximate identity as $\varepsilon \downarrow 0$. Since $\psi(0) = 0$ and sgn is continuous except at 0, if we substitute μ_ε for μ in (3.3), then we can pass to the limit as $\varepsilon \downarrow 0$ to obtain

$$(3.12)\quad \begin{aligned} &P\int_0^\infty e^{-qt}H_t^k\,dt \\ &= \frac{k!i^k}{(2\pi)^k}\int_{\mathbb{R}^k}\frac{\operatorname{sgn}(\xi_2-\xi_1)\cdots\operatorname{sgn}(\xi_k-\xi_{k-1})\operatorname{sgn}(-\xi_k)}{(q+\psi(\xi_1))\cdots(q+\psi(\xi_k))q}\,d\xi_1\cdots d\xi_k \\ &= \frac{k!i^k}{q(2\pi)^k}\int_{\mathbb{R}^k}\left[\prod_{j=1}^k\frac{\operatorname{sgn}(x_j-x_{j-1})}{q+\psi(x_j)}\right]dx_1\cdots dx_k,\qquad (x_0=0),\end{aligned}$$

where we have made the change of variables $(\xi_1,\dots,\xi_k)\to(x_k,\dots,x_1)$. To evaluate the final expression in (3.12), we partition each of the 2^k orthants of $\mathbb{R}^k$ into $k!$ sectors according to the relative magnitudes of $|x_j|$, $1\le j\le k$. (The hyperplanes forming the boundaries of these sectors can be ignored since they contribute nothing to the integral. Thus the sectors can be taken to be open.) Since ψ is even, each sector contributes

$$\begin{aligned}&\pm i^k(2\pi)^{-k}2^{-k}q^{-1}\int_{\mathbb{R}^k}[q+\psi(x_1)]^{-1}\cdots[q+\psi(x_k)]^{-1}\,dx_1\cdots dx_k\\ &\quad = \pm i^k2^{-k}q^{-1}\left[(2\pi)^{-1}\int[q+\psi(x)]^{-1}\,dx\right]^k = \pm i^k2^{-k}q^{-1}[\kappa(q)]^k\end{aligned}$$

to the final expression in (3.12), the $\pm$ sign being determined by the (constant) value of

$$(3.13)\qquad \operatorname{sgn}(x_1)\operatorname{sgn}(x_2-x_1)\cdots\operatorname{sgn}(x_k-x_{k-1})$$

over the sector. It follows that

$$(3.14)\qquad P\int_0^\infty qe^{-qt}H_t^k\,dt = i^k[\kappa(q)]^k2^{-k}N_k,$$

where N_k is the number of sectors for which (3.13) is $+1$ minus the number of sectors for which it is -1. But recalling the notation of Section 2, we see that there is an obvious bijection between our $2^kk!$ sectors and the cartesian product $T_k\times S_k$. Thus, since k is even, Lemma (2.3) yields

$$(3.15)\qquad i^k2^{-k}N_k = E_k.$$

Combining (3.14) and (3.15) we obtain (3.10). A similar argument leads from (3.3) with $\mu=\varepsilon_0$ to (3.11). □

REMARK. It is straightforward to verify that

$$(3.16)\qquad P\int_0^\infty qe^{-qt}(L_t)^k\,dt = k![\kappa(q)]^k,\qquad k\in\mathbb{N}.$$

Comparing this with (3.10), we see that

$$(3.17)\qquad P(H_t^k) = P(L_t^k)E_k/k!,\qquad k \text{ even}.$$

Similarly, one can deduce from (3.11) that

$$P\left(H_{g(t)}^k\right) = P\left(L_t^k\right)E_{k+1}/(k+1)!, \qquad k \text{ even}. \tag{3.18}$$

[See (4.5) below for the way to pass from (3.11) to (3.18).]

4. Proofs of (1.17), (1.22) and (1.25). Throughout this section $T = T(q)$ is an exponentially distributed random variable independent of X with parameter $q > 0$.

PROOF OF (1.17). Formula (1.18) is implicit in (3.10). Indeed $P(H_t^k) = 0$ if k is odd; so, for *complex* λ,

$$\begin{aligned} P(\exp(i\lambda H_T)) &= P\int_0^\infty qe^{-qt}\exp(i\lambda H_t)\,dt \\ &= \sum_{n=0}^\infty \frac{(-1)^n\lambda^{2n}}{(2n)!}P\int_0^\infty qe^{-qt}H_t^{2n}\,dt \\ &= \sum_{n=0}^\infty \frac{(-1)^n E_{2n}}{(2n)!}[\lambda\kappa(q)]^{2n}, \quad \text{by (3.10)}, \\ &= \operatorname{sech}(\lambda\kappa(q)), \quad \text{by (2.1a)}. \end{aligned} \tag{4.1}$$

Actually, the interchange of summation and integration above can be justified only if $|\lambda\kappa(q)| < \pi/2$. However the absolute convergence of the integral

$$\int e^{i\lambda x}P(H_T \in dx) \tag{4.2}$$

in the disk $|\lambda| < \pi/2\kappa(q)$ implies its absolute convergence throughout the strip $|\mathrm{Im}(\lambda)| < \pi/2\kappa(q)$. The integral in (4.2) is thus analytic in the same strip by Morera's theorem. Since $\operatorname{sech}(\lambda\kappa(q))$ is also analytic in $|\mathrm{Im}(\lambda)| < \pi/2\kappa(q)$, the extremes in (4.1) coincide in the strip, and in particular on the real line $\mathrm{Im}(\lambda) = 0$. Thus (1.18) is proved.

Before continuing we need to recall some basic facts about the excursions of X from 0. The reader can refer to [8], Section 7 for full details. Let G denote the set of left endpoints of the intervals contiguous to the zero set $\{t \ge 0: X_t = 0\}$ and let $R = \inf\{t > 0: X_t = 0\}$ denote the hitting time of 0. Then there is a σ-finite measure $\hat{P}$ on Ω such that if $Z \ge 0$ is a predictable process and $K = K_t(\omega)$ is a $\mathscr{B}(\mathbb{R}^+) \otimes \mathscr{F}^*$ measurable process, then

$$P\Big(\sum_{s\in G} Z_s K_s(\theta_s)\Big) = P\int_0^\infty Z_s \hat{P}(K_s)\,dL_s. \tag{4.3}$$

The measure $\hat{P}$ satisfies $\hat{P}(R=0) = 0$, and by (4.3) with $Z_s(\omega) = e^{-qs}$ and $K_s(\omega) = 1 - e^{-qR(\omega)}$,

$$\hat{P}(1-e^{-qR}) = \kappa(q)^{-1}, \qquad q > 0. \tag{4.4}$$

Proceeding to the proof of (1.19), note that if $t > 0$, then $g(t)$ is the unique point $s \in G$ such that $s \le t$ and $R \circ \theta_s > t - s$. Thus, using (4.3) and (4.4),

$$\begin{aligned} P\int_0^\infty qe^{-qt}\exp(i\lambda H_{g(t)})\,dt &= \int_0^\infty qe^{-qt}\,dt\,P\Bigg(\sum_{s\in G,\, s\le t} e^{i\lambda H_s}1_{\{R\circ\theta_s > t-s\}}\Bigg) \\ &= \int_0^\infty qe^{-qt}\,dt\,P\bigg(\int_0^t e^{i\lambda H_s}\hat{P}(R > t-s)\,dL_s\bigg) \qquad (4.5)\\ &= P\bigg(\int_0^\infty e^{-qs}e^{i\lambda H_s}\,dL_s\bigg)\hat{P}(1-e^{-qR}) \\ &= P\bigg(\int_0^\infty e^{-qs}e^{i\lambda H_s}\,dL_s\bigg)\Big/\kappa(q). \end{aligned}$$

But in the light of (2.1b), (3.8) and (3.11), we have

$$\begin{aligned} P\int_0^\infty e^{-qs}e^{i\lambda H_s}\,dL_s &= \sum_{n=0}^\infty \frac{(i\lambda)^{2n}}{(2n)!}P\int_0^\infty e^{-qs}H_s^{2n}\,dL_s \\ &= \sum_{n=0}^\infty \frac{(-1)^n\lambda^{2n}}{(2n)!}\frac{E_{2n+1}}{(2n+1)}[\kappa(q)]^{2n+1} \qquad (4.6)\\ &= \lambda^{-1}\tanh(\lambda\kappa(q)). \end{aligned}$$

The formal manipulations above can be justified as in the proof of (1.18). Combining (4.5) with (4.6) we obtain (1.19).

Following the pattern established in the above argument one can show that

$$(4.7)\quad P\Big(\exp\big(i\lambda H_{g(T)} + i\mu(H_T - H_{g(T)})\big)\Big) = \lambda^{-1}\tanh(\lambda\kappa(q))\,B(q,\mu),$$

where

$$B(q,\mu) = \int_0^\infty qe^{-qt}\hat{P}(e^{i\mu H_t}; t < R)\,dt.$$

But setting $\lambda = \mu$ in (4.7) and using (1.18), we find that $B(q,\mu) = \mu/\sinh(\mu\kappa(q))$. Substituting this into (4.7), we obtain (1.20). The proof of Theorem (1.17) is complete. □

Proof of (1.25). As noted already in Section 1, $Y_t := (\tau(t), H_{\tau(t)})$ is a Lévy process with values in $\mathbb{R}^+\times\mathbb{R}$ up to the lifetime L_∞. This lifetime is finite if and only if X is transient, in which case L_∞ has the exponential distribution under P, with mean $\kappa(0) < \infty$. [See (3.16).] In any event, there is a Lévy exponent $\varphi\colon \mathbb{R}^+\times\mathbb{R} \to \mathbb{R}^+$ such that for $q > 0$, $\lambda \in \mathbb{R}$,

$$(4.8)\quad P\Big(\exp\big(-q\tau(t) + i\lambda H_{\tau(t)}\big); \tau(t) < \infty\Big) = \exp(-t\varphi(q,\lambda)), \qquad t \ge 0.$$

Note that $\varphi(0,0) = \kappa(0)^{-1}$ (= 0 if X is recurrent). Using (4.6) and the change of variable $s = \tau(t)$, we obtain

$$P\int_0^\infty \exp\left(-q\tau(t) + i\lambda H_{\tau(t)}\right)1_{\{\tau(t)<\infty\}}\,dt$$
$$= P\int_0^\infty e^{-qs}e^{i\lambda H_s}\,dL_s = \lambda^{-1}\tanh(\lambda\kappa(q)).$$

Combining this with (4.8) we arrive at

$$\varphi(q,\lambda) = \lambda\coth(\lambda\kappa(q)), \tag{4.9}$$

which is (1.25). □

PROOF OF (1.22). The symmetry property of (H_t) noted above (3.8) implies that $Y_t = (\tau(t), H_{\tau(t)})_{t\ge 0}$ is equal in distribution (under P) to $(\tau(t), -H_{\tau(t)})_{t\ge 0}$. This means that the Lévy exponent φ of Y can be expressed as

$$\varphi(q,\lambda) = \varphi(0,0) + \iint (1 - e^{-qr}\cos\lambda x)\gamma(dr,dx), \quad q > 0,\ \lambda\in\mathbb{R}, \tag{4.10}$$

where γ is a measure on $]0,\infty[\times(\mathbb{R}\setminus\{0\})$ invariant under the mapping $(r,x)\mapsto(r,-x)$ and such that $\iint[(r+x^2)\wedge 1]\gamma(dr,dx)<\infty$. The absence of drift terms in (4.10) is explained by the symmetry of Y noted above and by the computation

$$\lim_{q\to\infty}[q\kappa(q)]^{-1} = \lim_{q\to\infty}\left[\frac{1}{\pi}\int_0^\infty \frac{q\,d\xi}{q+\psi(\xi)}\right]^{-1} = 0. \tag{4.11}$$

[Indeed letting $\lambda\to 0$ in (4.9) we see that the subordinator $(\tau(t))$ has Lévy exponent $\kappa(q)^{-1}$, so the limit in (4.11) represents the drift of $(\tau(t))$; cf. [8], (7.6), (7.16).]

It is well known that γ is the $\hat{P}$-distribution of (R, H_R); that is,

$$\gamma(dr,dx) = \hat{P}(R\in dr, H_R\in dx;\ R<\infty). \tag{4.12}$$

In fact, the jumps $J = \{(t, Y_t - Y_{t-}): 0 < |Y_t - Y_{t-}| < \infty\}$ of Y form a Poisson point process on $\mathbb{R}^+\times\mathbb{R}^+\times(\mathbb{R}\setminus\{0\})$ (killed at L_∞) with intensity $dt\,\gamma(dr,dx)$, and since H is continuous,

$$Y_t - Y_{t-} = \left(\tau(t)-\tau(t-), H_{\tau(t)} - H_{\tau(t-)}\right) = (R, H_R)\circ\theta_{\tau(t-)}.$$

But $\tau(t)-\tau(t-)>0$ if and only if $s=\tau(t-)\in G$, so (4.12) follows from (4.3).

Comparing (4.10) with (4.9) and using (4.12), we now find that

$$\hat{P}(1-\exp(-qR)\cos(\lambda H_R);\ R<\infty) = \lambda\coth(\lambda\kappa(q)) - \kappa(0)^{-1}. \tag{4.13}$$

Thus

$$\begin{aligned}\hat{P}\left((1-e^{-qR})e^{i\lambda HR};\ R<\infty\right) &= \hat{P}\left((1-e^{-qR})\cos(\lambda H_R);\ R<\infty\right)\\ &= \lambda\coth(\lambda\kappa(q)) - \lambda\coth(\lambda\kappa(0)).\end{aligned} \tag{4.14}$$

Consequently, by (4.3),

$$\begin{aligned}
&P\Big(\exp\big(i\lambda H_{g(T)} + i\mu(H_{d(T)} - H_{g(T)})\big); d(T) < \infty\Big)\\
&= \int_0^\infty qe^{-qt}\,dt\,P\bigg(\sum_{s\in G,\, s\le t} e^{i\lambda H_s}e^{i\mu H_R\circ\theta_s}1_{\{t-s<R\circ\theta_s<\infty\}}\bigg)\\
&= \int_0^\infty qe^{-qt}\,dt\,P\int_0^t e^{i\lambda H_s}\hat{P}(e^{i\mu H_R}; t-s<R<\infty)\,dL_s\\
&= P\bigg(\int_0^\infty e^{-qs}e^{i\lambda H_s}\,dL_s\bigg)\hat{P}\big((1-e^{-qR})e^{i\mu H_R}; R<\infty\big)\\
&= \lambda^{-1}\tanh(\lambda\kappa(q))\big[\mu\coth(\mu\kappa(q)) - \mu\coth(\mu\kappa(0))\big],
\end{aligned}$$

where the final equality follows from (4.6) and (4.14). This yields (1.24), and also (1.23) upon setting $\mu = \lambda$. □

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