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SEMI-MIN-STABLE PROCESSES

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We define a semi-min-stable (SMS) process $Y(t)$ in $[0, \infty)$ to be one which is stable under the simultaneous operations of taking the minima of n independent copies of $Y(t)$ (pointwise over time t) and rescaling space and time. We show that the only possible rescaling of time is by a fixed power of n and that SMS processes are essentially the only possible weak limits for large m of a process obtained by taking the minimum, pointwise over t, of m independent copies of a given process and then rescaling space and time. We describe the representation of a SMS process as the minimum of a Poisson process on a function space. We obtain a partial characterization of sample continuous SMS processes, similar to that of de Haan in the case of max-stable processes.

1. Introduction. A number of authors recently, notably de Haan (1984), de Haan and Pickands (1986) and Giné, Hahn and Vatan (1990), have considered the class of min-stable stochastic processes (or, equivalently, the class of max-stable processes). The main motivation for studying such processes is that they are the possible weak limits for large n of a process obtained by taking the rescaled minimum, at each time-point t, of n independent copies of some given process $X(t)$. This is a natural generalization of the study of multivariate sample extremes, which is by now well-established [see Resnick (1987) and references therein].

Sometimes, however, it is necessary to rescale time as well as space to get an interesting limiting process. This device is used by Brown and Resnick (1977), Eddy and Gale [(1981), Section 4] and Penrose (1991); for further motivation, see the discussion at the start of Hüsler and Reiss (1989). The limit process need not then be min-stable. Here we consider a new class of limit processes, suitable for this setting.

DEFINITION. For any $\alpha \in R$, define a stochastic process $(Y(t), t \geq 0)$ taking values in $[0, \infty)$ to be (simple) *semi-min-stable of order* α (which we shall sometimes abbreviate to α-SMS) if, for each positive integer n,

$$\left(n \min_{1 \leq i \leq n} Y_i(t), t \geq 0\right) =_d (Y(n^\alpha t), t \geq 0). \tag{1.1}$$

Here $Y_i(\cdot)$ are independent copies of the process $Y(\cdot)$, and $=_d$ refers to equality of finite-dimensional distributions. The distribution of $Y(0)$, if not concentrated at one point, is exponential, a standard extreme value distribu-

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tion. The case $\alpha = 0$ of a semi-min-stable process is a simple min-stable process.

The terminology "semi-min-stable" is motivated by the terminology "semi-stable" (or "self-similar") for a process whose distribution is invariant under rescaling of space and time [see Lamperti (1962)]. See Vervaat (1985) and references therein for more recent results on semistable processes.

Notation and conventions. The distribution function of any random variable X is written as $F_X(x)$. We write $\overline{F}_X(x)$ for $1 - F_X(x)$. We write Z_+ for the set of positive integers, R_+ for the interval $[0, \infty)$, and $\overline{R}_+$ for the interval $[0, \infty]$. The space of functions from $[0, \infty)$ to R_+ (respectively, $\overline{R}_+$) is written as $R_+^{R_+}$ (respectively, $\overline{R}_+^{R_+}$) and is equipped with the σ-algebra generated by the one-dimensional projections. We reserve the letter J for an arbitrary element of Z_+.

For any function $(f(t),\ t \geq 0)$ and any vector $\mathbf{t} = (t_1, \dots, t_J) \in R_+^J$, we abuse notation and write $f(\mathbf{t})$ for the vector $(f(t_1), \dots, f(t_J))$. For vectors $\mathbf{x} = (x_1, \dots, x_J)$ and $\mathbf{y} = (y_1, \dots, y_J)$, we write $\mathbf{x} \geq \mathbf{y}$ if $x_j \geq y_j$, $1 \leq j \leq J$.

Unless stated otherwise, all stochastic processes discussed here are defined on $[0, \infty)$ and take values in $[0, \infty)$. They are written as $(X(t),\ t \geq 0)$ or as $(X(t))_{t \geq 0}$ or $X(\cdot)$ for short. As already mentioned, we identify the distributions of any two processes with the same finite-dimensional distributions. If $Y(t) = Y(0)$ almost surely (a.s.) for all $t > 0$, we shall say the process $Y(\cdot)$ is *constant*.

Recall that a sequence of positive real numbers $(b_n,\ n \in Z_+)$ is said to be *regularly varying* if, for all $a > 0$, $b_{[na]}/b_n$ converges to a strictly positive limit as $n \to \infty$. Such a sequence always has a unique *index*, that is, a number α such that $b_{[na]}/b_n \to a^\alpha$ as $n \to \infty$, for all $a > 0$. See Theorem 1.9.5 of Bingham, Goldie and Teugels (1987).

Weak convergence. We introduce the following notion of weak convergence of stochastic processes. For vectors $\mathbf{t} \in R_+^J$, $\mathbf{t}_n \in R_+^J$, $n \in Z_+$, we write $\mathbf{t}_n \downarrow \mathbf{t}$ if each component of $\mathbf{t}_n$ converges to the corresponding component of $\mathbf{t}$ from above. Let A be a dense subset of R_+ with $0 \in A$. For processes $(X_n(\cdot))$, $n = 0, 1, 2, \dots$, we write $X_n(\cdot) \Rightarrow_A X_0(\cdot)$ if $X_n(\mathbf{t}_n)$ converges weakly to $X_0(\mathbf{t})$ in R_+^J whenever $J \in Z_+$, $\mathbf{t}_n \in R_+^J$, $\mathbf{t} \in A^J$ and $\mathbf{t}_n \downarrow \mathbf{t}$ as $n \to \infty$.

To motivate this notion of weak convergence, which we use in Theorem 2, we consider three examples. First, if $X_n(\cdot) =_d X_0(\cdot)$, and $X_0(\cdot)$ is right-continuous in probability, then $X_n(\cdot) \Rightarrow_A X_0(\cdot)$, with $A = R_+$.

We shall sometimes consider a *continuous* process, one with a version with a.s. continuous sample paths. For such a process, we assume without comment that we are considering this version. If $X(\cdot)$ and $X_n(\cdot)$, $n \in Z_+$, are continuous processes and the sequence $(X_n(\cdot))$ converges narrowly ("weakly") in $C[0, \infty)$ (with the locally uniform topology) to $X(\cdot)$ in the usual sense, then $X_n(\cdot) \Rightarrow_A X(\cdot)$, with $A = R_+$ [see Billingsley (1968), Theorem 5.5].

Let $D[0,\infty)$ be the space of Skorohod functions on $[0,\infty)$, with Skorohod's J_1 topology, as considered by Billingsley (1968) and amended to noncompact time intervals in Whitt (1980). Suppose $X(\cdot)$ and $X_n(\cdot)$, $n \in Z_+$, have versions in $D[0,\infty)$, such that the sequence $(X_n(\cdot))$ converges weakly to $X(\cdot)$ in $D[0,\infty)$. Set $A = \{t \ge 0: P[J_t] = 0\}$, where J_t is the event that $X(\cdot)$ has a discontinuity at t. Then $0 \in A$ and A is dense in R_+ [in fact, its complement is at most countable; see Billingsley (1968), page 124]. Also, $X_n(\cdot) \Rightarrow_A X(\cdot)$, by Theorem 5.5 of Billingsley (1968) and the fact that if $t \ge 0$ is a continuity point of a Skorohod function $x(\cdot)$ and $t_n \to t$, then, for any sequence of Skorohod functions $x_n(\cdot)$ which converge in the Skorohod J_1 topology to $x(\cdot)$, $x_n(t_n) \to x(t)$ [see Billingsley (1968), page 112].

Thus, weak convergence of $X_n(\cdot)$ to $X(\cdot)$ in $C[0,\infty)$ or $D[0,\infty)$ implies $X_n(\cdot) \Rightarrow_A X(\cdot)$ for an appropriate choice of dense A with $0 \in A$. There is no converse: If $X_n(\cdot)$ and $X(\cdot)$ are deterministic processes given by setting $X_n(\cdot)$ to be the characteristic function of $[1 - 2/n, 1 - 1/n)$ and $X(\cdot)$ to be zero everywhere, then $X_n(\cdot) \Rightarrow_A X(\cdot)$ with $A = R_+$, but $X_n(\cdot)$ does not converge to $X(\cdot)$ in $D[0,\infty)$. There are similar counterexamples in $C[0,\infty)$.

2. Motivating results.

The next two results are analogous to Theorems 1 and 2 of Lamperti (1962).

THEOREM 1. *Suppose $(Y_i(t))_{t\ge 0}$, $i \in Z_+$, are independent copies of a nonconstant process $(Y(t))_{t \ge 0}$ which is right-continuous in probability. Suppose $Y(0)$ has a nondegenerate distribution and, for constants $a_n, b_n > 0$,*

$$\left(\min_{1\le i\le n} Y_i(t),\, t \ge 0\right) =_d \left(a_n Y(b_n t),\, t \ge 0\right), \qquad n \in Z_+. \tag{2.1}$$

Then there exist a unique $\gamma \in 0$ and a unique $\alpha \in R$ such that $(Y(t)^\gamma)_{t\ge 0}$ is semi-min-stable of order α; that is, for all n, $a_n = n^{-1/\gamma}$ and $b_n = n^\alpha$.

THEOREM 2. *Suppose that $(X_i(t))_{t\ge 0}$, $i \in Z_+$, are independent copies of a process $(X(t))_{t\ge 0}$. Suppose $Y(\cdot)$ is a nonconstant process which is right-continuous in probability and $Y(0)$ has a nondegenerate distribution. Suppose there exists a dense subset A of R_+, and two sequences of constants $a_n > 0$, $b_n > 0$, such that*

$$\left(a_n \min_{1\le i\le n} X_i(b_n t),\, t \ge 0\right) \Rightarrow_A \left(Y(t),\, t \ge 0\right) \quad \text{as } n \to \infty. \tag{2.2}$$

Then, for some $\gamma > 0$ and $\alpha \in R$, $(Y(t)^\gamma)_{t \ge 0}$ is semi-min-stable of order α. Also, (a_n) is regularly varying with index γ^{-1} and (b_n) is regularly varying with index $(-\alpha)$ as $n \to \infty$.

REMARKS. A converse to Theorem 2 is trivial. If $Y(\cdot)$ is α-SMS and right-continuous in probability, there exist $X(\cdot)$ and $a_n > 0$ and $b_n > 0$ such that (2.2) holds. Just take $X(\cdot) = Y(\cdot)$.

A similar (but easier) argument to the proof of Theorem 2 shows that if in (2.2) we are given $b_n = n^{-\alpha}$, and if we assume only convergence (not right-continuous convergence) of all finite-dimensional distributions and make no continuity assumptions on $Y(\cdot)$, we may still conclude that $Y(\cdot)^\gamma$ is α-SMS for some $\gamma > 0$.

3. Spectral and other representations of semi-min-stable processes. The next two theorems provide representations for an α-SMS process in terms of a Poisson process on a space of functions. Later (Theorem 5) we shall derive a more intuitive representation for members of a large subclass of the continuous α-SMS processes.

The following representation is based on the spectral representation of max-infinitely divisible processes by Balkema, de Haan and Karandikar (1991). For background information on Poisson process, see Resnick (1987). Recall that for vectors $\mathbf{x}$ and $\mathbf{t}$ in R_+^J, the notation $f(\mathbf{t}) \geq \mathbf{x}$ means $f(t_j) \geq x_j$, $1 \leq j \leq J$.

THEOREM 3. (i) *Suppose μ is σ-finite measure on $\overline{R}_+^{R_+}$. Suppose that for all $a > 0$, $J \in Z_+$, $\mathbf{x}$ and $\mathbf{t}$ in R_+^J we have*

$$a\mu\big(\{f\colon f(\mathbf{t}) \geq \mathbf{x}\}^c\big) = \mu\big(\{f\colon f(a^\alpha \mathbf{t}) \geq a\mathbf{x}\}^c\big) \tag{3.1}$$

(so μ has infinite total mass). Then the $\overline{R}_+$-valued process $Y'(\cdot)$ given by

$$Y'(t) = \inf_{i \geq 1}\big(f_i(t)\big), \qquad t \geq 0, \tag{3.2}$$

where $\{f_i,\ i \in Z_+\}$ is a Poisson process on $\overline{R}_+^{R_+}$ with intensity μ, is α-SMS.

(ii) *Suppose that $Y(\cdot)$ is an α-SMS process which is right-continuous in probability. Then there exists a σ-finite measure μ on $\overline{R}_+^{R_+}$, satisfying* (3.1) *for all $a > 0$, $\mathbf{x}$ and $\mathbf{t}$ in R_+^J, such that $Y(\cdot) =_d Y'(\cdot)$, where the process $Y'(\cdot)$ is given by* (3.2).

REMARKS. We shall refer to the measure μ of the Theorem 3(ii) as a *spectral measure* of the SMS process $Y(\cdot)$ [Giné, Hahn and Vatan (1990) prefer the term *max-Lévy measure*]. In the case that the α-SMS process $Y(\cdot)$ is continuous, we might expect that it has a spectral measure which concentrates on continuous functions. Using results of Giné, Hahn and Vatan (1990), we obtain the following theorem.

THEOREM 4. *If $Y(\cdot)$ is a continuous α-SMS process and $Y(0)$ has a nondegenerate distribution, then $Y(\cdot)$ has a spectral measure μ which concentrates on $C_h(R_+)$, where we define $h\colon R_+ \to \overline{R}_+$, and $C_h(R_+)$ by*

$$h(t) = \inf\{x\colon P[Y(t) \leq x] = 1\}, \qquad t \geq 0,$$

and

$$C_h(R_+) = \big\{f \in C\big(R_+, \overline{R}_+\big)\colon f \leq h \text{ pointwise},\ f \not\equiv h\big\},$$

equipped with the σ-algebra generated by the one-dimensional projections. Moreover, this spectral measure is unique.

Theorem 3 implies the following alternative characterization of SMS processes.

COROLLARY 1. *A process $Y(\cdot)$ which is right-continuous in probability is α-SMS if and only if, for all $a > 0$ and $b > 0$,*

$$(3.3) \qquad \left(\min\{Y_{(a)}(t), Y'_{(b)}(t)\}\right)_{t \ge 0} =_d \left(Y_{(a+b)}(t)\right)_{t \ge 0},$$

where we define $Y_{(a)}(t) = a^{-1}Y(ta^{\alpha})$, and $Y'(\cdot)$ is an independent copy of $Y(\cdot)$.

4. Examples.

EXAMPLE 1. Let $\alpha \in R$, and let $(Z(t), t \ge 0)$ be an arbitrary measurable [see Doob (1953)] stochastic process taking values in $\overline{R}_+$. Suppose $c > 0$ and $\mathscr{P} = \{X_1, X_2, \dots\}$ is a homogeneous rate-c Poisson process on R_+. Suppose $Z_i(\cdot)$, $i \in Z_+$, are independent copies of $Z(\cdot)$, which are also independent of $\mathscr{P}$. Then an α-SMS, $\overline{R}_+$-valued process $Y(\cdot)$ can be obtained by setting

$$(4.1) \qquad Y(t) = \inf_{i \ge 1} X_i Z_i(X_i^{-\alpha} t), \qquad t \ge 0.$$

PROOF. By Propositions 3.8 and 3.7 of Resnick (1987), the random set of functions $\{X_i Z_i(X_i^{-\alpha}\,\cdot\,), i \ge 1\}$ is a realization of a Poisson process on $\overline{R}_+^{R_+}$, with intensity μ, say, where for $\mathbf{t}$ and $\mathbf{x}$ in R_+^J,

$$(4.2) \qquad \mu\left[\{f: f(\mathbf{t}) \ge \mathbf{x}\}^c\right] = c\int_0^{\infty} (1 - P[uZ(u^{-\alpha}\mathbf{t}) \ge \mathbf{x}])\,du.$$

For any $a > 0$, the change of variable $\xi = au$ in (4.2) gives us (3.1), so that $Y(\cdot)$ is α-SMS by Theorem 3(i). □

The next three examples are special cases of Example 1. In Theorem 5 we characterize a large class of continuous semi-min-stable processes as special cases of Example 1.

EXAMPLE 2 [Penrose (1991)]. Let $d > 0$, $c > 0$. Let $\mathscr{P} = \{P_1, P_2, P_2, \dots\}$ be a Poisson process on R_+ with intensity μ given by $\mu([0, x]) = cx^{d/2}$, $x \ge 0$. Given a realization of $\mathscr{P}$, let $(X_i(t), t \ge 0)$, $i \in Z_+$, be independent squared Bessel processes of dimension d [BESQ(d) processes], with initial positions $X_i(0) = P_i$, $i \in Z_+$. Recall that a BESQ(d) process is a diffusion on $[0, \infty)$ with generator L, say, where $Lf(x) = 2xf''(x) + df'(x)$, for $f \in C_K^2(0, \infty)$. See Revuz and Yor (1991). Set

$$Y(t) = \inf\{X_i(t): i \in Z_+\}, \qquad t \ge 0.$$

Then $(Y(t)^{d/2}, t \ge 0)$ is a stationary, continuous SMS process of order $2/d$; in fact, it is a special case of Example 1, as we now show.

Let $(R^2(t), t \ge 0)$ be a BESQ(d) process starting at 1 [i.e., $R(0) = 1$ a.s.]. By the scaling property of the BESQ(d) process [see, for example, Revuz and Yor (1991), page 413], the processes $R_i^2(\cdot)$ given by $R_i^2(t) = P_i^{-1}X_i(P_i t)$ are independent copies of $R^2(\cdot)$, which are also independent of $\mathscr{P}$. Setting $Z_i(t) = (R_i^2(t))^{d/2}$, we have

$$Y(t)^{d/2} = \inf_{i \geq 1} P_i^{d/2} Z_i(P_i^{-1} t). \tag{4.3}$$

The point process with points at $\{P_1^{d/2}, P_2^{d/2}, \dots\}$ is a homogeneous Poisson process on R_+, so (4.3) shows $Y(\cdot)^{d/2}$ is a special case of Example 1, with $\alpha = 2/d$.

To see stationarity, observe that the measure μ is invariant for the BESQ(d) transition function [see Liggett (1985), Proposition I.2.13; the restriction of L to $C_K^\infty[0, \infty)$ is a core for L; see Ethier and Kurtz (1986), page 371]. Hence, if the random point measure $\eta(t)$ on R_+ is defined to have atoms at the points $\{X_i(t),\ i \in Z_+\}$, then the point-measure-valued process $(\eta(t),\ t \geq 0)$ is stationary (and Markov), so $(Y(t))^{d/2}$ [which is determined by $\eta(t)$] is a stationary process.

EXAMPLE 3. In Brown and Resnick (1977), the limit process, denoted $M(\cdot)$, of a sequence of rescaled maxima of Brownian motions is given by

$$M(t) = \sup_{i \geq 1} (T_i + W_i^{**}(t)), \qquad t \geq 0,$$

where $\{T_i,\ i \geq 1\}$ is an enumeration of the points of a Poisson process on R with intensity $e^{-x}\,dx$, and $W_i^{**}(\cdot)$, $i \geq 1$, are independent Wiener processes, independent of $\{T_i\}$, with drift $-\frac{1}{2}$, and $W_i^{**}(0) = 0$. So

$$\exp(-M(t)) = \inf_{i \geq 1} \{X_i Z_i(t)\},$$

where $\{X_i\} = \{\exp(-T_i)\}$ is a Poisson process on R_+ with Lebesgue measure as intensity, and $Z_i(t) = \exp(-W_i^{**}(t))$, $i \geq 1$. This representation shows that $\exp(-M(\cdot))$ is a special case of Example 1, with $\alpha = 0$; thus, $\exp(-M(\cdot))$ is simple min-stable.

This may at first surprise the reader, since in the limiting procedure of Brown and Resnick (1977), time is rescaled by multiplication by a sequence of constants approaching 0, whereas the limit process has $\alpha = 0$. However, Brown and Resnick obtained $M(\cdot)$ as the weak limit of a process $M_n(\cdot)$ of the form

$$M_n(t) = \max_{i \leq n} a_n(X_i(b_n t)) + c_n,$$

where $\{X_i(\cdot),\ i \geq 1\}$ are independent Brownian motions with initial positions having a normal distribution, and a_n, b_n and c_n are constants. By comparison with (2.2), we see Brown and Resnick allowed themselves the addition of an extra constant c_n in their renormalization procedure.

EXAMPLE 4. If $Y(\cdot)$ is α-SMS, it is immediate that the law of $Y(0)$, if nondegenerate, is exponential. If $\alpha = 0$, the same is true of $Y(t)$ for all t [see de Haan (1984)]. But if $\alpha \neq 0$, if we make no continuity assumption on $Y(\cdot)$, then for $t > 0$, $Y(t)$ may have any distribution on R_+, as we shall now show.

Let $\alpha \neq 0$, $t > 0$, and let F be an arbitrary distribution function on R_+. Let G be the left-continuous inverse of the function $\log\{1/(1 - F(\cdot))\}$ [see Section 0.2 of Resnick (1987) for details]. Let $(Z(u),\ u \geq 0)$ be the deterministic process given by

$$Z(u) = (u/t)^{1/\alpha} G\big((u/t)^{-1/\alpha}\big),$$

and let $Z_i(u) = Z(u)$, $i \in Z_+$. Let $Y(\cdot)$ be given by (4.1). Then $Y(\cdot)$ is a special case of Example 1, with $Z(\cdot)$ deterministic. In particular, $Y(\cdot)$ is α-SMS. The distribution of $Y(t)$ is given as follows, where $I[\ ]$ denotes the indicator function:

$$\begin{aligned} P[Y(t) \geq y] &= \exp\left\{ -\int_0^\infty I[xZ(tx^{-\alpha}) < y]\, dx \right\} \\ &= \exp\left\{ -\int_0^\infty I[G(x) < y]\, dx \right\} \\ &= 1 - F(y - 0), \end{aligned}$$

where by definition $F(y - 0) = \sup\{F(x): x < y\}$ [the last equality is from Resnick (1987), Exercise 0.2.2]. It follows that $Y(t)$ has the prescribed distribution function F.

Note that, in Example 4, $Y(\cdot)$ may not be continuous in probability at 0. Such a continuity restriction on Y may restrict the possible finite-dimensional distributions for $Y(\cdot)$.

EXAMPLE 5 (Essentially due to S. T. Rachev). Suppose $\mathscr{P} = \{(S_i, U_i): i = 1, 2, 3, \dots\}$ is a Poisson process on $R_+ \times R_+$ with Lebesgue measure as intensity. Suppose $Z_i(\cdot)$ are independent copies of an arbitrary measurable stochastic process $Z(\cdot)$, which are also independent of $\mathscr{P}$. Then, for any α and γ with $\alpha\gamma > -1$, an α-SMS process $Y(\cdot)$ can be obtained by setting

$$Y(t) = \inf_{i \geq 1} U_i^{1/(1+\alpha\gamma)} Z_i(t^\gamma S_i), \qquad t \geq 0.$$

PROOF. The random set of functions $\{f_i,\ i \geq 1\}$, defined by $f_i(t) = U_i^{1/(1+\alpha\gamma)} Z_i(t^\gamma S_i)$, $t \geq 0$, is a realization of a Poisson process on $\overline{R}_+^{R_+}$, with intensity μ, say, where, for $\mathbf{t}$ and $\mathbf{x}$ in R_+^J,

$$\begin{aligned} \mu\{f: f(\mathbf{t}) \geq \mathbf{x}\}^c &= \int_0^\infty \int_0^\infty P\Big[\big(u^{1/(1+\alpha\gamma)} Z(st_j^\gamma)\big) < x_j, \text{ some } j \leq J\Big]\, du\, ds \\ &= E\int_0^\infty \int_0^\infty I\Big\{ u^{1/(1+\alpha\gamma)} \min_{1 \leq j \leq J} Z(st_j^\gamma) < x_j \Big\}\, du\, ds \qquad (4.4) \\ &= E\int_0^\infty \max_{1 \leq j \leq J} \big(x_j / Z(st_j^\gamma)\big)^{1+\alpha\gamma}\, ds, \end{aligned}$$

where $I\{\ \}$ denotes indicator function. Hence, for $a > 0$,

$$\begin{aligned}\mu\{f\colon f(a^{\alpha}\mathbf{t}) \geq a\mathbf{x}\}^c &= E\int_0^{\infty} \max_{j\leq J}\left(ax_j/Z\left(sa^{\alpha\gamma}t_j^{\gamma}\right)\right)^{1+\alpha\gamma} ds \\ &= E\int_0^{\infty} \max_{j\leq J}\left(ax_j/Z\left(\sigma t_j^{\gamma}\right)\right)^{1+\alpha\gamma} d\sigma\, a^{-\alpha\gamma}\end{aligned} \tag{4.5}$$

(where we changed variable to $\sigma = sa^{\alpha\gamma}$), and comparison of (4.4) and (4.5) shows that (3.1) holds. Thus $Y(\cdot)$ is α-SMS by Theorem 3(i). □

5. Spectral decomposition. By analogy with de Haan [(1984), Theorem 3] [see also de Haan and Pickands (1986), Theorem 2.1], one might ask if *all* semi-min-stable processes are given by Example 1 (or by Example 5). That is, we wish to decompose the spectral measure of a SMS process into a product. For continuous processes with an extra condition on the spectral measure, this is possible.

THEOREM 5. *Suppose $Y(\cdot)$ is a continuous α-SMS process, such that $Y(0)$ has a nondegenerate distribution and the spectral measure μ on $C_h(R_+)$, given by Theorem 4, satisfies*

$$\mu\{f\colon f(0) = \infty\} = 0. \tag{5.1}$$

Then there exists a continuous process $Z(\cdot)$ with $Z(0) = 1$ a.s. for which the construction of Example 1 gives us a process with the same finite-dimensional distributions as those of $Y(\cdot)$.

6. Proof of theorems. Before proving our theorems, we need the following simple lemmas.

LEMMA 1. *Suppose the process $Y(\cdot)$ is right-continuous in probability. If for some $b > 0$, $b \neq 1$, we have*

$$(Y(bt), t \geq 0) =_d (Y(t), t \geq 0), \tag{6.1}$$

then $Y(\cdot)$ is constant.

PROOF. For $\varepsilon > 0$, $t \geq 0$, (6.1) implies $P[|Y(t) - Y(0)| > \varepsilon] = P[|Y(b^n t) - Y(0)| > \varepsilon]$, and by letting $n \to \mp\infty$ according to whether $b \gtrless 1$ we have $Y(t) = Y(0)$ a.s. □

LEMMA 2. *Suppose $Y_n(\cdot)$, $n \in Z_+$, $X(\cdot)$ and $X'(\cdot)$ are stochastic processes, with $X(\cdot)$ and $X'(\cdot)$ right-continuous in probability, and are not constant. Suppose that for some dense $A \subset R_+$, with $0 \in A$, and for some sequence (β_n) of strictly positive numbers,*

$$(Y_n(t), t \geq 0) \Rightarrow_A (X(t), t \geq 0) \tag{6.2}$$

and

$$(Y_n(\beta_n t), t \geq 0) \Rightarrow_A (X'(t), t \geq 0). \tag{6.3}$$

Then, for some $\beta \in (0, \infty)$, $\beta_n \to \beta$ *and* $(X'(t), t \geq 0) =_d (X(\beta t), t \geq 0)$.

PROOF. If $\beta_n \to \infty$ along some subsequence, then, by (6.3),

$$(Y_n(t), t \geq 0) = Y_n(\beta_n(t/\beta_n), t \geq 0) \Rightarrow_A (X'(0), t \geq 0)$$

along that subsequence, where the limit is a constant process. By comparison with (6.2), $X(\cdot)$ is a constant process.

Therefore, since $X(\cdot)$ is assumed nonconstant, the sequence (β_n) is bounded away from ∞. So there exists $\beta \in [0, \infty)$ such that $\beta_n \to \beta$ along a subsequence. Take $\varepsilon_n \downarrow 0$ so that $\beta_n(1 + \varepsilon_n) \downarrow \beta$ along that subsequence. By (6.2),

$$Y_n(\beta_n(1 + \varepsilon_n)t, t \geq 0) \Rightarrow_A (X(\beta t), t \geq 0)$$

along the subsequence. By comparison with (6.3), $X'(\cdot) =_d X(\beta\, \cdot\,)$. Also, by Lemma 1 the sequential limit β is unique. Finally, $\beta > 0$ since $X'(\cdot)$ is nonconstant. □

PROOF OF THEOREM 1. By (2.1) and the special case $X(\cdot) =_d Y(\cdot)$ of Theorem 2, there exist $\gamma > 0$ and $\alpha \in R$ such that the process $(Y(\cdot))^\gamma$ is α-SMS. For each $n \in Z_+$, by consideration of $Y(0)$ in (2.1), γ is unique and there is no other choice of a_n except $n^{-1/\gamma}$. Also, by Lemma 1, α is unique and there is no other choice of (b_n) except $b_n = n^{-\alpha}$. □

PROOF OF THEOREM 2. By (2.2), $Y(0)$ is the nondegenerate weak limit of the random variables $a_n \min_{i \leq n} X_i(0)$. Hence by one-dimensional extreme value theory [for example, Proposition 0.3 of Resnick (1987)], there exist positive c and γ such that

$$\overline{F}_{Y(0)}(x) = \exp(-(cx)^\gamma), \qquad x > 0.$$

Set $X = X(0)$. By Propositions 1.13 and 0.2 of Resnick (1987), the function $G(x) := 1/F_X(x^{-1})$ is regularly varying of order γ at infinity, and $a_n^{-1} \sim c \sup\{x\colon 1/F_X(x) \geq n\}$, $n \to \infty$. The last expression implies $a_n \sim c^{-1} \inf\{x\colon G(x) \geq n\}$, and, by Proposition (0.8)(v) of Resnick (1987), (a_n) is regularly varying of order $1/\gamma$.

Now use the fact that $(-Y(t), t \geq 0)$ is max-infinitely divisible [for a definition, see Balkema, de Haan and Karandikar (1991)]. By the proof of Theorem 2.4 of that paper (using some dense subset of A where Balkema, de Haan and Karandikar use the set of rationals), there is a function $f(t, u)$ from $R_+ \times R$ to $\overline{R}_+$, measurable in u, such that if $\{U_i, i \geq 1\}$ is a homogeneous rate-1 Poisson process on R, then $Y(\cdot) =_d Y'(\cdot)$, where we define

$$Y'(t) = \inf_{i \geq 1} f(t, U_i), \qquad t \geq 0. \tag{6.4}$$

Let $k \in (0, \infty)$. Let $\{V_i, i \geq 1\}$ be a homogeneous rate-k Poisson process on R and define

$$Y^{(k)}(t) = \inf_{i \geq 1} f(t, V_i), \qquad t \geq 0. \tag{6.5}$$

Then, for any $\mathbf{t} \in R_+^J$ and $\mathbf{x} \in R_+^J$,

$$P\left[Y^{(k)}(\mathbf{t}) \geq \mathbf{x}\right] = P\left[Y(\mathbf{t}) \geq \mathbf{x}\right]^k. \tag{6.6}$$

In particular,

$$\left(Y^{(k)}(t), t \geq 0\right) =_d \left(\min_{1 \leq i \leq k} (Y_i(t)), t \geq 0\right), \qquad k \in Z_+. \tag{6.7}$$

For any $k > 0$, any $\mathbf{t} \in A^J$, and any sequence $\mathbf{t}_n \downarrow \mathbf{t}$ and any continuity point $\mathbf{x} \in R_+^J$ of the distribution function of $Y(\mathbf{t})$, by (2.2)

$$\begin{aligned} P&\left\{a_n \min_{i \leq n} X_i(b_{[nk]}\mathbf{t}_n) \geq \mathbf{x}\right\} \\ &= \left(P\left\{a_{[nk]} X_1(b_{[nk]}\mathbf{t}_n) \geq (a_{[nk]}/a_n)\mathbf{x}\right\}^{[nk]}\right)^{n/[nk]} \\ &\to \left(P\{Y(\mathbf{t}) \geq k^{1/\gamma}\mathbf{x}\}\right)^{1/k} \quad \text{as } n \to \infty. \end{aligned} \tag{6.8}$$

Thus,

$$\left(a_n \min_{1 \leq i \leq n} X_i(b_{[nk]}t), t \geq 0\right) \Rightarrow_A \left(k^{-1/\gamma} Y^{(1/k)}(t), t \geq 0\right). \tag{6.9}$$

By comparison with (2.2) and by use of Lemma 2, (b_n) is regularly varying; moreover, if $(-\alpha)$ is the index of regular variation, then

$$\left(k^{-1/\gamma} Y^{(1/k)}(t), t \geq 0\right) =_d \left(Y(k^{-\alpha}t), t \geq 0\right), \tag{6.10}$$

so that taking $k = 1/n$, with n an integer, by (6.7) $(Y(\cdot))^\gamma$ is α-SMS as desired. □

Proof of Theorem 3. (i) The process $Y'(\cdot)$ given by (3.2) satisfies

$$P[Y'(\mathbf{t}) \geq \mathbf{x}] = \exp\left(-\mu\{f\colon f(\mathbf{t}) \geq \mathbf{x}\}^c\right), \qquad \mathbf{t}, \mathbf{x} \in R_+^J.$$

By applying (3.1) to $a \in Z_+$, we immediately find that $Y'(\cdot)$ is α-SMS.

(ii) As in the proof of Theorem 2, the min-infinite divisibility of $Y(\cdot)$ implies that there is a function $f(t, u)$ from $R_+ \times R$ to $\overline{R}_+$, measurable in u, such that $Y(\cdot) =_d Y'(\cdot)$, with $Y'(\cdot)$ given by (6.4); that is, $Y'(t) = \inf_i\{f(t, U_i)\}$, where $\{U_i\}$ is a homogeneous rate-1 Poisson process on R.

Define the measure μ on $\overline{R}_+^{R_+}$ to be the image of Lebesgue measure on R under the mapping $u \mapsto f(\cdot, u)$, which is measurable. Set $f_i(t) = f(t, U_i)$. By Proposition 3.7 of Resnick (1987) (in which E_2 does not have to have a countable base), $\{f_i\} = \{f(\cdot, U_i)\}$ is a Poisson process in $\overline{R}_+^{R_+}$ with intensity μ. Hence, (3.2) holds; that is, $Y'(\cdot)$ is the minimum (pointwise over time) of a Poisson process in $\overline{R}_+^{R_+}$ with intensity μ. We have, for any $\mathbf{t}$ and $\mathbf{x}$ in R_+^J,

$$p[Y(\mathbf{t}) \geq \mathbf{x}] = \exp\left(-\mu\left(\{f\colon f(\mathbf{t}) \geq \mathbf{x}\}^c\right)\right).$$

So for $a > 0$, by (6.10) (setting $k = 1/a$) and (6.6),

$$\begin{aligned} \mu\left(\{f\colon f(a^\alpha \mathbf{t}) \geq a\mathbf{x}\}^c\right) &= -\log P[Y(a^\alpha \mathbf{t}) \geq a\mathbf{x}] \\ &= -\log P\left[aY^{(a)}(\mathbf{t}) \geq a\mathbf{x}\right] \\ &= -a \log P[Y(\mathbf{t}) \geq \mathbf{x}], \end{aligned}$$

and (3.1) follows. □

PROOF OF THEOREM 4. By definition, $h(t) = \inf\{x\colon P[Y(t) \le x] = 1\}$, that is,

$$h(t) = \inf\{x\colon \mu\{f\colon f(t) \le x\} = \infty\},$$

where μ is a spectral measure of $Y(\cdot)$, as given by Theorem 3(ii). Note that $h(0) = \infty$ by the nondegeneracy of $Y(0)$. By (3.1), $h(a^{\alpha}t) = ah(t)$ for all $a > 0$, $t > 0$, so that either $h \equiv \infty$ or $\alpha \ne 0$ and, for some $c \in (0, \infty)$,

$$h(t) = ct^{1/\alpha}, \qquad t > 0. \tag{6.11}$$

If $\alpha > 0$, (6.11) and continuity of $Y(\cdot)$ would imply $Y(0) = 0$ almost surely, contradicting the assumed nondegeneracy of $Y(0)$. So (6.11) is possible only when $\alpha < 0$, and in all cases h is continuous.

Let $K > 0$. Let $C_h[0, K]$ denote the set of continuous functions $f\colon [0, K] \to \overline{R}_+$ with $f(t) \le h(t)$, $0 \le t \le K$, and $f(t) < h(t)$, for some $t \le K$. Define $Y_0(t) = g(Y(t))$, where $g\colon [0, \infty] \to [0, 1]$ is a decreasing homeomorphism. The process $(Y_0(t), 0 \le t \le K)$ is sample continuous and max-infinitely divisible on a compact parameter space. By continuity of h and Theorem 2.4 of Giné, Hahn and Vatan (1990), there exists a unique infinite sigma-finite measure μ_K on $C_h[0, K]$ for which $(Y(t), 0 \le t \le K) =_d (Y'(t), 0 \le t \le K)$, with Y' given by (3.2), where $\{f_i, i \ge 1\}$ is a rate-μ_K Poisson process on $C_h[0, K]$.

The measures μ_K, $K > 0$, are consistent in the sense that for all $\mathbf{t} = (t_1, \dots, t_J) \in R_+^J$ and Borel $A \subset (\prod_{j=1}^{J}[0, h(t_j)]) \setminus \{h(\mathbf{t})\}$, $\mu_K\{f\colon f(\mathbf{t}) \in A\}$ is the same for all $K \ge \max_{j \le J}(t_j)$. By a sigma-finite version of Kolmogorov's consistency theorem, there is a measure ν on $\overline{R}_+^{Q_+}$, where Q_+ denotes the nonnegative rationals, which concentrates on functions $(f(q), q \in Q_+)$ for which $f \le h$ pointwise and $f(q) < h(q)$, for some $q \in Q_+$, such that ν is consistent with all the μ_K. Moreover, ν is unique by a monotone class argument.

Under the mapping

$$(f(q), q \in Q_+) \mapsto \Big(\limsup_{r \to t,\, r \in Q_+} f(r), t \ge 0\Big),$$

from $\overline{R}_+^{Q_+}$ to $\overline{R}_+^{R_+}$, the measure ν induces a measure μ on $\overline{R}_+^{R_+}$ which concentrates on continuous functions (and is consistent with ν_K for each K). This is the desired spectral measure for $Y(\cdot)$. □

PROOF OF COROLLARY 1. If (3.3) holds, then (1.1) holds by induction on n.

Conversely, if $Y(\cdot)$ is α-SMS, then, by Theorem 3, $Y(\cdot)$ has a spectral measure μ satisfying (3.1), which we may rewrite as

$$a \log P[Y(\mathbf{t}) \ge \mathbf{x}] = \log P\big[Y_{(a)}(\mathbf{t}) \ge \mathbf{x}\big], \qquad \mathbf{x} \in R_+^J, \mathbf{t} \in R_+^J, a > 0.$$

Hence, for $a > 0$, $b > 0$ we have

$$\begin{aligned}\log P\Big[\min\big(Y_{(a)}(t_j), Y'_{(b)}(t_j)\big) \ge x_j, 1 \le j \le J\Big] &= (a + b) \log P[Y(\mathbf{t}) \ge \mathbf{x}]\\ &= \log P\big[Y_{(a+b)}(\mathbf{t}) \ge \mathbf{x}\big]. \qquad \square\end{aligned}$$

PROOF OF THEOREM 5. For $f \in C_h(R_+)$, define $f_{(a)}$ as in Corollary 1, that is, $f_{(a)}(t) = a^{-1}f(a^\alpha t)$, $t \geq 0$. Then $f_{(a)} \in C_h(R_+)$, by (6.11). Define a measurable transformation $T\colon C_h(R_+) \to C_h(R_+)$ by $T(f) \equiv 0$ if $f(0) = 0$ or $f(0) = \infty$,

$$(Tf)(t) = f(0)^{-1} f\big(f(0)^\alpha t\big), \qquad t \geq 0,$$

otherwise. It is easy to check that for $a > 0$ and $f \in C_h(R_+)$,

$$T(f_{(a)}) \equiv Tf. \tag{6.12}$$

Let μ be the spectral measure on $C_h(R_+)$ given by Theorem 4. By (3.1) and a monotone class argument, we have for all measurable $A \subset C_h(R_+)$ and $a > 0$,

$$a\mu(A) = \mu\{f\colon f_{(a)} \in A\}. \tag{6.13}$$

By (3.1) and the assumption (5.1) that μ concentrates on functions f with $f(0) < \infty$, we have, for some constant $c > 0$,

$$\mu\{f\colon f(0) \in \cdot\} = c\,\text{Lebesgue}(\cdot). \tag{6.14}$$

For measurable $A \subset C_h(R_+)$ and $a > 0$, we have, by (6.12) and (6.13), that

$$\begin{aligned} \mu\{f\colon f(0) \leq a, Tf \in A\} &= \mu\{f\colon f_{(ac)}(0) \leq 1/c, T(f_{(ac)}) \in A\} \\ &= ac\mu\{f\colon f(0) \leq 1/c, Tf \in A\} \\ &= acP[Z \in A], \end{aligned} \tag{6.15}$$

where we define the law of the stochastic process $Z(\cdot)$ to be the measure μT^{-1}, restricted to functions f with $f(0) \leq 1/c$ (a probability measure). That is, $P[Z \in \cdot] = \mu\{f\colon Tf \in \cdot,\ f(0) \leq 1/c\}$. Note that $Z(0) = 1$ and $Z(\cdot)$ is continuous almost surely.

For $x > 0$ write $\lceil x \rceil_n$ for $[nx + 1]/n$ and $\lceil x \rceil_n^\alpha$ for $(\lceil x \rceil_n)^\alpha$. Since μ concentrates on continuous functions, for $\mathbf{t} \in (0, \infty)^J$ we have the following convergence in R_+^{J+1} for μ-almost every f:

$$\begin{aligned} (f(0), f(\mathbf{t})) &= \lim_{n\to\infty} \big(\lceil f(0) \rceil_n, \lceil f(0) \rceil_n f(0)^{-1} f(\lceil f(0) \rceil_n^{-\alpha} f(0)^\alpha \mathbf{t})\big) \\ &= \lim_{n\to\infty} \big(\lceil f(0) \rceil_n, \lceil f(0) \rceil_n Tf(\lceil f(0) \rceil_n^{-\alpha} \mathbf{t})\big) \quad \text{a.e. } (d\mu). \end{aligned} \tag{6.16}$$

Let $b > 0$. By (6.14), $\mu\{f\colon f(0) \leq b + 1\} < \infty$ and $\mu\{f\colon f(0) = b\} = 0$. So by (6.16), for almost every $\mathbf{x} \in R_+^J$,

$$\begin{aligned} &\mu\big(\{f\colon f(0) \leq b\} \cap \{f\colon f(\mathbf{t}) \geq \mathbf{x}\}^c\big) \\ &\quad = \lim_{n\to\infty} \mu\Big(\{f\colon \lceil f(0) \rceil_n \leq b\} \cap \big\{f\colon \lceil f(0) \rceil_n Tf(\lceil f(0) \rceil_n^{-\alpha} \mathbf{t}) \geq \mathbf{x}\big\}^c\Big) \\ &\quad = \lim_{n\to\infty} \sum_{i=1}^{[nb]} \mu\Big(\{f\colon \lceil f(0) \rceil_n = (i/n)\} \cap \big\{f\colon Tf((i/n)^{-\alpha} \mathbf{t}) \geq (n/i)\mathbf{x}\big\}^c\Big); \end{aligned}$$

by (6.15) this equals

$$(6.17)\qquad \begin{aligned} c \lim_{n\to\infty} \sum_{i=1}^{[nb]} n^{-1} P\big[Z((i/n)^{-\alpha}\mathbf{t}) \geq (n/i)\mathbf{x}\big]^c \\ = c\int_0^b P[uZ(u^{-\alpha}\mathbf{t}) \geq \mathbf{x}]^c\, du, \end{aligned}$$

at least if the integrand on the right-hand side of (6.17) is Riemann-integrable. Assuming for the moment that (6.17) is valid for almost every $\mathbf{x} \in R_+^J$, we have on taking $b \to \infty$ [and using the assumption (5.1)] that for each $\mathbf{t} \in R_+^J$, (4.2) holds for almost every $\mathbf{x} \in R_+^J$ and hence for all $\mathbf{x} \in R_+^J$. Thus, μ can be obtained by the construction of Example 1.

It remains to prove (6.17). The left-hand side of (6.17) equals

$$(6.18)\qquad c\int_0^{\lceil b\rceil_n - n^{-1}} \Big(1 - P\big[Z(\lceil u\rceil_n^{-\alpha}\mathbf{t}) \geq \lceil u\rceil_n^{-1}\mathbf{x}\big]\Big)\, du.$$

By continuity of $Z(\cdot)$, for each $u > 0$, for almost every $\mathbf{x}$ we have

$$(6.19)\qquad P\big[Z(\lceil u\rceil_n^{-\alpha}\mathbf{t}) \geq \lceil u\rceil_n^{-1}\mathbf{x}\big] \to P\big[Z(u^{-\alpha}\mathbf{t}) \geq u^{-1}\mathbf{x}\big], \qquad n \to \infty.$$

Since $Z(\cdot)$ is continuous, it is measurable [Doob (1953), Theorem 2.5]. Hence, $P[Z(\mathbf{t}) \geq \mathbf{x}]$ is a (jointly) Borel measurable function of $(\mathbf{t}, \mathbf{x})$. See Halmos [(1974), Section 35, Theorem A].

Hence, for fixed $\mathbf{t}$, the left-hand side of (6.19) is jointly Borel measurable in $\mathbf{x}$ and u; also, since the mapping $(u, \mathbf{x}) \mapsto (u^{-\alpha}\mathbf{t}, u^{-1}\mathbf{x})$ is continuous, the right-hand side of (6.19) is jointly Borel measurable in $\mathbf{x}$ and u, and by a standard argument the set of $(\mathbf{x}, u)$ for which (6.19) holds is jointly Borel measurable in $\mathbf{x}$ and u.

By Fubini's theorem, for almost every $\mathbf{x}$, for almost all u (6.19) holds. Finally, the integrand in the expression (6.18) for the left-hand side of (6.17) is uniformly bounded by 1 and is zero outside $\{0 \leq u \leq b\}$, so that (6.19) and dominated convergence give us (6.17) for almost every $\mathbf{x}$.

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REFERENCES

Balkema, A. A., de Haan, L. and Karandikar, R. L. (1991). The maximum of n independent stochastic processes. Preprint, Univ. Amsterdam.

Billingsley, P. (1968). *Convergence of Probability Measures*. Wiley, New York.

Bingham, N. H., Goldie, C. M. and Teugels, J. L. (1987). *Regular Variation. Encyclopedia of Math.* **27**. Cambridge Univ. Press.

Brown, B. M. and Resnick, S. I. (1977). Extreme values of independent stochastic processes. *J. Appl. Probab.* **14** 732–739.

de Haan, L. (1984). A spectral representation for max-stable processes. *Ann. Probab.* **12** 1194–1204.

DE HAAN, L. and PICKANDS, J., III. (1986). Stationary min-stable processes. *Probab. Theory Related Fields* **72** 477–492.

DOOB, J. L. (1953). *Stochastic Processes*. Wiley, New York.

EDDY, W. F. and GALE, J. D. (1981). The convex hull of a spherically symmetric sample. *Adv. in Appl. Probab.* **13** 751–763.

ETHIER, S. N. and KURTZ, T. G. (1986). *Markov Processes: Characterization and Convergence*. Wiley, New York.

GINÉ, E., HAHN, M. G. and VATAN, P. (1990). Max-infinitely divisible and max-stable sample continuous processes. *Probab. Theory Related Fields* **87** 139–165.

HALMOS, P. (1974). *Measure Theory*. Springer, Berlin. (First published 1950.)

HÜSLER, J. and REISS, R. D. (1989). Maxima of normal random vectors: Between independence and complete dependence. *Statist. Probab. Lett.* **7** 283–286.

LAMPERTI, J. (1962). Semi-stable stochastic processes. *Trans. Amer. Math. Soc.* **104** 62–78.

LIGGETT, T. M. (1985). *Interacting Particle Systems*. Springer, Berlin.

PENROSE, M. D. (1991). Minima of independent Bessel processes and distances between Brownian particles. *J. London Math. Soc. (2)* **43** 355–366.

RESNICK, S. I. (1987). *Extreme Values, Regular Variation and Point Processes*. Springer, Berlin.

REVUZ, D. and YOR, M. (1991). *Continuous Martingales and Brownian Motion*. Springer, Berlin.

VERVAAT, W. (1985). Sample path properties of self-similar processes with stationary increments. *Ann. Probab.* **13** 1–27.

WHITT, W. (1980). Some useful functions for functional limit theorems. *Math. Oper. Res.* **5** 67–85.

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