The Annals of Probability
1992, Vol. 20, No. 3, 1369–1384

ASYMPTOTIC SERIES AND EXIT TIME PROBABILITIES

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This paper is concerned with accurate asymptotic estimates for exit time probabilities associated with nearly deterministic Markov diffusions. The exit time probabilities are expressed as asymptotic series of WKB type in a small parameter, which measures the strength of the random Brownian motion inputs. This series is valid in certain regions in which the minimum action function $u(x, s)$ is a smooth function of state x and time s. The function u is a solution to the corresponding Hamilton–Jacobi PDE of first order.

1. Introduction. Let $D \subset \Re^n$ be a bounded domain with smooth boundary ∂D and consider the nondegenerate stochastic differential equation

$$
\begin{aligned}
dx_t^\varepsilon &= b^\varepsilon(x_t^\varepsilon, t)\,dt + \sqrt{\varepsilon}\,\sigma(x_t^\varepsilon)\,dw_t, \qquad t > s, \\
x_s^\varepsilon &= x \in D,
\end{aligned} \tag{1.1}
$$

where $\varepsilon > 0$ is a small parameter. Let $\tau^\varepsilon = \tau_{x,s}^\varepsilon$ denote the exit time from D of the process x^ε. For any fixed $T > 0$, define the *exit time probability function* by

$$
q^\varepsilon(x, s) = P_{x,s}(\tau^\varepsilon \le T). \tag{1.2}
$$

We wish to expand q^ε in powers of ε in the form of an asymptotic series

$$
\begin{aligned}
q^\varepsilon = \exp\left(-u/\varepsilon - v/\sqrt{\varepsilon} - w\right)\Big[1 + \sqrt{\varepsilon}\,\phi_1 + \varepsilon\phi_2 + \cdots \\
+\varepsilon^{m/2}\phi_m + o(e^{m/2})\Big], \quad \text{as } \varepsilon \downarrow 0,
\end{aligned} \tag{1.3}
$$

for *any* $m \ge 0$, valid in certain regions $N \Subset \overline{D} \times [0, T)$ on which the function u is smooth. The function u is the solution to a Hamilton–Jacobi equation, and the remaining terms in the expansion are solutions to transport equations which can be solved explicitly using the method of characteristics. If the vector field b^ε is independent of ε, then $v = 0$ and the series (1.3) involves only integer powers of ε.

Freidlin and Wentzell (1984) proved that

$$
\lim_{\varepsilon \to 0} \varepsilon \log q^\varepsilon = -u. \tag{1.4}
$$

This result has also been proved using stochastic control methods [Fleming (1978)] and vanishing viscosity methods [Fleming and Souganidis (1986a)]. To

Received May 1990; revised April 1991.
[1]Partially supported by NSF Grant MCS-81-2940, AFOSR Grant AFOSR-89-0015 and ARO Grant DAAL 03-86-K-0171.
[2]Partially supported by ARO Grant DAAL 03-86-K-0171 to Brown University.
AMS 1980 *subject classifications*. 60F10, 35C20.
Key words and phrases. Large deviations, exit time probabilities, WKB asymptotic series.

prove (1.4), Freildin and Wentzell apply a general large deviation asymptotic principle,

$$P_{x,s}(x^{\varepsilon}_{sT} \in A) \asymp \exp\left(-\frac{1}{\varepsilon} \inf_{\theta \in A} I(\theta)\right) \quad \text{as } \varepsilon \downarrow 0, \tag{1.5}$$

where $A \subset C([s,T]), \mathfrak{R}^n)$ and $I(\cdot)$ is the action function for the process x^{ε} (see Section 2), to the set $A = A_{x,s}$, where

$$A_{x,s} = \{\theta \in C([s,T], \mathfrak{R}^n) \colon \theta_s = x,\ \theta_t \notin D \text{ for some } t \in [s,T]\}.$$

This yields $u(x,s) = \inf_{\theta \in A_{x,s}} I(\theta)$, so that u is the value function for a deterministic calculus of variations problem.

As far as asymptotic series are concerned, Fleming (1971) obtained an expansion for the value function for a class of stochastic control problems with small noise. Also, Fleming and Souganidis (1986a) use an analytical approach associated with the vanishing viscosity method to obtain an asymptotic series for the solutions to certain quasilinear elliptic PDE. In both cases, the series obtained are valid in certain regions where the limit function is smooth.

In principle, either of the methods could be applied to our problem after first making the logarithmic transformation $u^{\varepsilon} = -\varepsilon \log q^{\varepsilon}$. Instead, in this paper we make a sequence of factorizations [cf. Sheu (1986)] and employ the Feynman–Kac formula to evaluate certain limits. The method yields a rather direct proof and characterization of (1.3). A convergence result is presented in Section 2 and applied in Section 3 to obtain a general asymptotic series expansion of the type we need. The detailed result (Theorem 4.1) concerning (1.3) is given in Section 4. As another application of our method, we obtain in Section 5 a semiclassical asymptotic series expansion for the solution of an imaginary time version of Schrödinger's equation. In Theorem 4.1 two assumptions, (A1) and (A2), are made. Assumption (A1) implies that the flow of the unperturbed ($\varepsilon = 0$) version of (1.1) is into D, while (A2) states that the unique minimizing $\theta \in A_{x,s}$ exits from D before time T and has no conjugate points. We do not know the appropriate asymptotic series for q^{ε} when the minimizing θ exits at time T.

We also mention related work by Azencott (1985). He obtains expansions of the general form

$$P(x^{\varepsilon}_{sT} \in A) = \exp(-\Lambda_0/\varepsilon - \Lambda_1/\sqrt{\varepsilon} - \Lambda_2)\left[a_0 + \sqrt{\varepsilon}\, a_1 + \cdots + \varepsilon^{m/2} a_m + o(\varepsilon^{m/2})\right]$$

as $\varepsilon \downarrow 0$ for *some* $m \geq 0$ (depending on the data), where $\Lambda_0 = \inf_{\theta \in A} I(\theta)$. This result depends on the smoothness of the boundary ∂A of A near the minimizing extremal $\theta^* \in \partial A$. The proof uses an expansion of a process related to x^{ε}_{sT}, from which the terms in the series are computed. We do not know whether or not the set $A_{x,s}$ defined above satisfies Azencott's condition.

2. A convergence result. In this section we prove a convergence result which will be used in the next section. Let us fix $T > T' > 0$. Let $\beta \in C(\mathfrak{R}^n \times$

$[0, T], \mathfrak{R}^n)$ be such that $\beta(\cdot, s)$ is also a Lipschitz function:

$$|\beta(x,s) - \beta(y,s)| \le L|x-y|, \qquad x, y \in \mathfrak{R}^n,\ s \in [0,T].$$

Consider the differential equation

$$\begin{aligned} &\dot{\xi}_t = \beta(\xi_t, t), \qquad s < t < T, \\ &\xi_s = x. \end{aligned} \tag{2.1}$$

Let N be an open subset of $\mathfrak{R}^n \times [0, T']$. For $(x, s) \in N$, define

$$\sigma = \sigma_{x,s} = \inf\{t > s : (\xi_t, t) \notin N\},$$

$$y = y_{x,s} = \xi_\sigma, \qquad z = z_{x,s} = (y, \sigma),$$

$$\gamma(x,s) = \{(\xi_t, t) : s \le t \le \sigma\},$$

where ξ satisfies (2.1).

Definition 2.1. We say that N is a *region of strong regularity* (RSR) provided $\partial N = \Gamma_1 \cup \Gamma_2$, where

$$\Gamma_1 = \{z_{x,s} : (x,s) \in N\}$$

is a C^∞ manifold, Γ_1 is a relatively open subset of ∂N and $\gamma(x, s)$ crosses Γ_1 nontangentially.

Note that if $(x, s) \in N$, then $\gamma(x, s) \subset N \cup \Gamma_1$.

Let v denote the solution of

$$\begin{aligned} &\frac{\partial v}{\partial s} + \beta \cdot Dv = -g - hv \quad \text{in } N, \\ &v = v_0 \quad \text{on } \Gamma_1, \end{aligned} \tag{2.2}$$

where $g, h, v_0 \in C_b(\mathfrak{R}^n \times [0, T])$, the space of continuous real valued functions defined on $\mathfrak{R}^n \times [0, T]$ equipped with the supremum norm. We note that (2.1) are the characteristic differential equations for (2.2). When N is a RSR the method of characteristics gives the representation

$$\begin{aligned} v(x,s) &= v_0(z)\exp\left(\int_s^\sigma h(\xi_t, t)\,dt\right) \\ &\quad + \int_s^\sigma \exp\left(\int_s^t h(\xi_r, r)\,dr\right) g(\xi_t, t)\,dt \end{aligned} \tag{2.3}$$

for $(x, s) \in N$.

Let $\beta^\varepsilon \in C(\mathfrak{R}^n \times [0, T], \mathfrak{R}^n)$ satisfy

$$\beta^\varepsilon \to \beta \quad \text{uniformly as} \quad \varepsilon \to 0,$$

and

$$|\beta^\varepsilon(x,s) - \beta^\varepsilon(y,s)| \le K|x-y|,\ x, y \in \mathfrak{R}^n,\ s \in [0,T],\ \varepsilon > 0. \tag{2.4}$$

Also, let $v_0^\varepsilon, g^\varepsilon, h^\varepsilon \in C_b(\mathfrak{R}^n \times [0,T])$ satisfy

$$\begin{aligned} v_0^\varepsilon &\to v_0, \\ g^\varepsilon &\to g, \\ h^\varepsilon &\to h, \end{aligned} \tag{2.5}$$

as $\varepsilon \to 0$ uniformly in $\overline{N}$ and a uniform Lipschitz condition of the type (2.4).

Consider the stochastic differential equation

$$\begin{aligned} d\xi_t^\varepsilon &= \beta^\varepsilon(\xi_t^\varepsilon, t)\,dt + \sqrt{\varepsilon}\,\sigma(\xi_t^\varepsilon)\,dw_t, \qquad s < t < T, \\ \xi_s^\varepsilon &= x, \end{aligned} \tag{2.6}$$

where $\sigma \in C_b(\mathfrak{R}^n) \cap C^\infty(\mathfrak{R}^n)$ satisfies the nondegeneracy condition

$$a(x) \equiv \sigma(x)\sigma(x)' \geq \alpha_0 I \tag{2.7}$$

for some $\alpha_0 > 0$. The stochastic differential equation (2.6) is viewed as a random perturbation of the ordinary differential equation (2.1), and the associated theory is presented in Freidlin and Wentzell (1984).

Let $P_{x,s}^\varepsilon$ denote the distribution of $\xi_{sT}^\varepsilon = \{\xi_t^\varepsilon;\ s \leq t \leq T\}$ on $C([s,T], \mathfrak{R}^n)$. Using Freidlin and Wentzell (1984), Theorem 5.3.1, and Varadhan (1984), Theorem 2.4, we deduce:

LEMMA 2.1. *$\{P_{x,s}^\varepsilon\}$ obeys the large deviation principle (LDP) uniformly in $(x,s) \in \mathfrak{R}^n \times [0,T]$ with action function*

$$I_{x,s}(\theta) = \begin{cases} \frac{1}{2}\int_s^T \left[\dot{\theta}_t - \beta(\theta_t, t)\right]' a^{-1}(\theta_t)\left[\dot{\theta}_t - \beta(\theta_t,t)\right] dt, \\ \qquad \text{if } \theta_s = x, \text{ and } \theta \text{ is absolutely continuous}, \\ +\infty, \quad \text{otherwise}. \end{cases}$$

The action function I governs the asymptotic behavior of the measures P^ε according to (1.5).

For $(x,s) \in N$ define

$$\sigma_{x,s}^\varepsilon = \inf\{t > s \colon (\xi_t^\varepsilon, t) \notin N\},$$
$$y^\varepsilon = \xi_{\sigma^\varepsilon}^\varepsilon, \qquad z^\varepsilon = (y^\varepsilon, \sigma^\varepsilon).$$

We consider functions $v^\varepsilon \in C(\overline{N}) \cap C^{2,1}(N)$ satisfying the PDE

$$\begin{aligned} \frac{\partial v^\varepsilon}{\partial s} + \frac{\varepsilon}{2}\operatorname{tr}\left(a(x)D^2 v^\varepsilon\right) + \beta^\varepsilon \cdot Dv^\varepsilon &= -g^\varepsilon - h^\varepsilon v^\varepsilon \quad \text{in } N, \\ v^\varepsilon &= v_0^\varepsilon \quad \text{on } \Gamma_1, \end{aligned} \tag{2.8}$$

and the weak estimate

$$\left\{ \begin{array}{l} \text{for all } \gamma > 0, \text{ there exist } C > 0,\ \varepsilon_0 > 0 \text{ (depending perhaps on } \gamma) \\ \text{such that } |v^\varepsilon| \leq Ce^{\gamma/\varepsilon} \text{ in } N \text{ for all } 0 < \varepsilon \leq \varepsilon_0. \end{array} \right\} \tag{2.9}$$

The function v^ε has the Feynman–Kac representation

$$(2.10)\qquad \begin{aligned} v^\varepsilon(x,s) &= E_{x,s}\left[v^\varepsilon(z^\varepsilon)\exp\left(\int_s^{\sigma^\varepsilon} h^\varepsilon(\xi_t^\varepsilon,t)\,dt\right)\right] \\ &\quad + E_{x,s}\left[\int_s^{\sigma^\varepsilon} \exp\left(\int_s^t h^\varepsilon(\xi_r^\varepsilon,r)\,dr\right)g^\varepsilon(\xi_t^\varepsilon,t)\,dt\right]. \end{aligned}$$

We wish to prove that $v^\varepsilon \to v$ as $\varepsilon \to 0$. An immediate consequence of Lemma 2.1 is that

$$\xi_{sT}^\varepsilon \to_P \xi_{sT} \quad \text{as } \varepsilon \to 0,$$

where ξ_{sT} is the solution of (2.1) on $[s,T]$. In fact, for each $\delta > 0$ there exists $C > 0$, $\varepsilon_0 > 0$ such that if $0 < \varepsilon \le \varepsilon_0$, then

$$(2.11)\qquad P_{x,s}(\|\xi^\varepsilon - \xi\|_{sT} > \delta) \le e^{-C/\varepsilon} \quad \text{for all } (x,s) \in \Re^n \times [0,T].$$

More importantly:

LEMMA 2.2. *Assume that N is a RSR. For each $\delta > 0$, there exist $C > 0$, $\varepsilon_0 > 0$ such that $0 \le \varepsilon \le \varepsilon_0$ implies*

$$(2.12)\qquad P_{x,s}(|z^\varepsilon - z| > \delta) \le e^{-C/\varepsilon}$$

uniformly on compact subsets of $N \cup \Gamma_1$.

PROOF. Let $K \subset N \cup \Gamma_1$ be compact and let $(x,s) \in K$. Then the fact that $\gamma(x,s)$ crosses Γ_1 nontangentially implies that for some $\alpha > 0$,

$$(\xi_t,t) \in N \quad \text{for } s \le t < \sigma,$$

$$(\xi_t,t) \notin \overline{N} \quad \text{for } \sigma < t \le \sigma + \alpha.$$

Then if $|z^\varepsilon - z| > \delta$, we must have

$$\|\xi^\varepsilon - \xi\|_{sT} > \delta'$$

for some $\delta' > 0$. But then

$$P(|z^\varepsilon - z| > \delta) \le P(\|\xi^\varepsilon - \xi\|_{sT} > \delta')$$

and the result follows from (2.11). □

THEOREM 2.1. *Assume that N is a RSR, and that* (2.9) *holds. Then*

$$\lim_{\varepsilon\to 0} v^\varepsilon = v$$

uniformly on compact subsets of $N \cup \Gamma_1$.

PROOF. Assume for notational simplicity that $g^\varepsilon \equiv g \equiv 0$; the general case is similar. In what follows, the letter C will be used to denote any constant. Let $K \subset N \cup \Gamma_1$ be compact and $(x,s) \in K$. Define

$$B = B_{x,s} = \{\|\xi_{sT}^\varepsilon - \xi_{sT}\| < \delta\} \cap \{|z^\varepsilon - z| < \delta\},$$

where $\delta > 0$ is chosen small enough to ensure $z^\varepsilon \in \Gamma_1$ on the set $B_{x,s}$, for $(x,s) \in K$.

Choose $C > 0$, $\varepsilon_1 > 0$ such that if $0 \le \varepsilon \le \varepsilon_1$, then (2.11) and (2.12) hold. Choose $\gamma < C/2$ and let $\varepsilon_0 \le \varepsilon_1$ be such that (2.9) holds. Now

$$\begin{aligned} v^\varepsilon(x,s) &= E_{x,s}\left[v^\varepsilon(z^\varepsilon)\exp\left(\int_s^{\sigma^\varepsilon} h^\varepsilon(\xi_t^\varepsilon, t)\,dt\right); B\right] \\ &\quad + E_{x,s}\left[v^\varepsilon(z^\varepsilon)\exp\left(\int_s^{\sigma^\varepsilon} h(\xi_t^\varepsilon, t)\,dt\right); B^c\right] \\ &= (A) + (B). \end{aligned}$$

Then if $0 < \varepsilon \le \varepsilon_0$,

$$\begin{aligned} |(B)| &\le Ce^{\gamma/\varepsilon}P(B^c) \\ &\le Ce^{\gamma/\varepsilon}e^{-C/\varepsilon} \\ &\le Ce^{-C/2\varepsilon}. \end{aligned}$$

Next

$$\begin{aligned} &\left|\int_s^{\sigma^\varepsilon} h^\varepsilon(\xi_t^\varepsilon, t)\,dt - \int_s^{\sigma} h(\xi_t, t)\,dt\right| \\ &\quad \le \int_s^{\sigma^\varepsilon}|h^\varepsilon(\xi_t^\varepsilon, t) - h^\varepsilon(\xi_t, t)|\,dt + \int_s^{\sigma^\varepsilon}|h^\varepsilon(\xi_t, t) - h(\xi_t, t)|\,dt \\ &\qquad + \left|\int_\sigma^{\sigma^\varepsilon} h(\xi_t, t)\,dt\right| \\ &\quad \le C\|\xi^\varepsilon - \xi\|_{sT} + C\|h^\varepsilon - h\| + C|\sigma - \sigma^\varepsilon|. \end{aligned}$$

Now using (2.3),

$$\begin{aligned} &|(A) - v(x,s)| \\ &\quad \le E_{x,s}\left[v_0(z)\int_0^1 \exp\left(\lambda\int_s^{\sigma^\varepsilon} h^\varepsilon(\xi_t^\varepsilon, t)\,dt + (1-\lambda)\int_s^{\sigma} h(\xi_t, t)\,dt\right)d\lambda\right. \\ &\qquad \left.\times\left|\int_s^{\sigma^\varepsilon} h^\varepsilon(\xi_t^\varepsilon, t)\,dt - \int_s^{\sigma} h(\xi_t, t)\,dt\right| + C|v_0^\varepsilon(z^\varepsilon) - v_0(z)|; B\right] \\ &\quad \le CE_{x,s}[\|\xi^\varepsilon - \xi\|_{sT} + \|h^\varepsilon - h\| + |\sigma - \sigma^\varepsilon| + |z^\varepsilon - z| + \|v_0^\varepsilon - v_0\|; B] \\ &\quad \le C\delta + C\|h^\varepsilon - h\| + C\delta + C\delta + \|v_0^\varepsilon - v_0\|. \end{aligned}$$

Thus

$$\limsup_{\varepsilon \to 0}|v^\varepsilon(x,s) - v(x,s)| \le C\delta.$$

Since δ can be chosen arbitrarily small, the theorem is proved. □

3. Asymptotic series. In this section we obtain an asymptotic series expansion for the solution Z^ε of

$$(3.1)\qquad \begin{aligned} &\frac{\partial Z^\varepsilon}{\partial s} + \frac{\varepsilon}{2}\operatorname{tr}(a(x)D^2Z^\varepsilon) + \beta^\varepsilon\cdot D^\varepsilon = -h^\varepsilon Z^\varepsilon \quad \text{in } N \\ &Z^\varepsilon = e^{-\Psi_2\Phi^\varepsilon} \quad \text{on } \Gamma_1, \end{aligned}$$

valid in a RSR N, for use in later sections. Here, Z^ε satisfies an estimate of the form (2.9) and we assume for each $m = 0, 1, \dots,$

$$(3.2)\quad \begin{aligned} &\beta^\varepsilon = \beta + \sqrt{\varepsilon}\,\beta_1 + \varepsilon\beta_2 + \cdots + \varepsilon^{m/2}\beta_m + o(\varepsilon^{m/2}) \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{uniformly in } \Re^n \times [0,T], \\ &\left.\begin{aligned} h^\varepsilon &= h + \sqrt{\varepsilon}\,h_1 + \varepsilon h_2 + \cdots + \varepsilon^{m/2}h_m + o(\varepsilon^{m/2}) \\ \Phi^\varepsilon &= 1 + \sqrt{\varepsilon}\,\Phi_1 + \varepsilon\Phi_2 + \cdots + \varepsilon^{m/2}\Phi_m + o(\varepsilon^{m/2}) \end{aligned}\right\} \text{uniformly in } \overline{N}, \end{aligned}$$

as $\varepsilon \downarrow 0$, where Ψ_2 and the terms in the series for h^ε and Φ^ε belong to $C_b(\Re^n \times [0,T]) \cap C_b^\infty(N \cup \Gamma_1)$. The terms for β^ε belong to $C(\Re^n \times [0,T], \Re^n) \cap C^\infty(N \cup \Gamma_1, \Re^n)$, and we assume that β^ε, h^ε and Φ^ε satisfy a uniform Lipschitz condition of the type (2.4).

THEOREM 3.1. *Let N be a RSR and assume Z^ε satisfies* (3.1), (3.2) *and* (2.9). *Then for each $m = 0, 1, 2, \dots$ we have*

$$(3.3)\quad \begin{aligned} Z^\varepsilon(x,s) = e^{-w(x,s)}\big[&1 + \sqrt{\varepsilon}\,\phi_1(x,s) + \varepsilon\phi_2(x,s) + \cdots \\ &+ \varepsilon^{m/2}\phi_m(x,s) + o(\varepsilon^{m/2})\big] \end{aligned}$$

as $\varepsilon \downarrow 0$ uniformly on compact subsets of $N \cup \Gamma_1$. Here, $w, \phi_m \in C^\infty(N \cup \Gamma_1)$ and satisfy ($\phi_0 \equiv 1$, $\phi_m \equiv 0$ if $m < 0$)

$$(3.4)\quad \begin{aligned} \frac{\partial w}{\partial s} + \beta \cdot Dw &= h \quad \text{in } N, \\ w &= \Psi_2 \quad \text{on } \Gamma_1, \end{aligned}$$

$$(3.5)\quad \begin{aligned} \frac{\partial \phi_m}{\partial s} + \beta \cdot D\phi_m = -\Bigg[&\sum_{i=1}^m \beta_i \cdot D\phi_{m-i} - Dw\,a \cdot D\phi'_{m-2} + \frac{1}{2}\operatorname{tr}(aD^2\phi_{m-2}) \\ &+ \frac{1}{2}\big(Dw\,a\,Dw' - \operatorname{tr}(aD^2w)\big)\phi_{m-2} \\ &+ \sum_{i=1}^m (-\beta_i \cdot Dw + h_i)\phi_{m-i}\Bigg] \quad \text{in } N, \\ \phi_m = \Phi_m \quad &\text{on } \Gamma_1. \end{aligned}$$

REMARK. The terms in the expansion (3.3) can be computed using the method of characteristics.

PROOF. (a) Applying Theorem 2.1, we see that

$$\lim_{\varepsilon \to 0} Z^\varepsilon = Z \quad \text{uniformly on compact subsets of } N \cup \Gamma_1,$$

where Z satisfies

$$\begin{aligned} \frac{\partial Z}{\partial s} + \beta \cdot DZ &= -hZ \quad \text{in } N, \\ Z &= e^{-\Psi_2} \quad \text{on } \Gamma_1. \end{aligned}$$

By the method of characteristics,

$$Z(x,s) = \exp\left(-\Psi_2(y,\sigma) + \int_s^{\sigma} h(\xi_t,t)\,dt\right).$$

Now set

$$w = -\log Z.$$

Thus $w \in C^\infty(N \cup \Gamma_1)$ solves (3.4).

(b) Let $\phi_0 \equiv 1$, $\phi_m \equiv 0$ if $m < 0$ and recursively set

$$\phi_m^\varepsilon = \frac{\phi_{m-1}^\varepsilon - \phi_{m-1}}{\sqrt{\varepsilon}},$$

with $\phi_0^\varepsilon = Z^\varepsilon/Z$. Define similarly β_m^ε, and so forth. Let $K \subset N \cup \Gamma_1$ be a compact subset such that the interior K° is also a RSR. Let $\tilde{\beta}^\varepsilon \in C(\Re^n \times [0,T], \Re^n)$ satisfy

$$\tilde{\beta}^\varepsilon = \beta^\varepsilon - \varepsilon\, Dw\, a \quad \text{in } K,$$

$$\left|\tilde{\beta}^\varepsilon(x,s) - \tilde{\beta}^\varepsilon(y,s)\right| \le L|x-y| \quad \text{in } \Re^n \times [0,T]$$

and

$$\tilde{\beta}^\varepsilon \to \beta \quad \text{uniformly in } \Re^n \times [0,T] \text{ as } \varepsilon \to 0.$$

Select functions $g^\varepsilon, h^\varepsilon \in C_b(\Re^n \times [0,T])$ satisfying

$$\begin{aligned} g^\varepsilon = {} & \sum_{i=1}^m \beta_i^\varepsilon \cdot D\phi_{m-i} - Dw\, aD\left(\phi_{m-2} + \sqrt{\varepsilon}\,\phi_{m-1}\right)' \\ & + \tfrac{1}{2}\operatorname{tr}\left(aD^2\left(\phi_{m-2} + \sqrt{\varepsilon}\,\phi_{m-1}\right)\right) \\ & + \tfrac{1}{2}\left(Dw\, a\, Dw' - \operatorname{tr}(a D^2 w)\right)\left(\phi_{m-2} + \sqrt{\varepsilon}\,\phi_{m-1}\right) \\ & + \sum_{i=1}^m \left(-\beta_i^\varepsilon \cdot Dw + h_i^\varepsilon\right)\phi_{m-i} \quad \text{in } K, \end{aligned}$$

$$g^\varepsilon \to g \quad \text{uniformly in } K \text{ as } \varepsilon \to 0,$$

where $g \in C(\Re^n \times [0,T])$ is a function which equals minus the term appearing in the right-hand side of (3.5) in K and

$$h^\varepsilon = \sqrt{\varepsilon}\left[-\beta_1^\varepsilon \cdot Dw + h_1^\varepsilon + \frac{\sqrt{\varepsilon}}{2}\left(Dw\, a\, Dw' - \operatorname{tr}(a D^2 w)\right)\right] \quad \text{in } K.$$

Then ϕ_m^ε satisfies $(m \ge 1)$

$$\frac{\partial \phi_m^\varepsilon}{\partial s} + \frac{\varepsilon}{2}\operatorname{tr}\left(a D^2 \phi_m^\varepsilon\right) + \tilde{\beta}^\varepsilon \cdot D\phi_m^\varepsilon = -g^\varepsilon - h^\varepsilon \phi_m^\varepsilon, \quad \text{in } K^\circ,$$

$$\phi_m^\varepsilon = \Phi_m^\varepsilon \quad \text{on } K \cap \Gamma_1.$$

Now for any $\gamma > 0$,

$$|\phi_m^\varepsilon| \le \frac{C}{\varepsilon^{m/2}} e^{\gamma/2\varepsilon} \le C e^{\gamma/\varepsilon} \quad \text{in } K,$$

for all sufficiently small $\varepsilon > 0$. Now apply Theorem 2.1 to see that

$$\lim_{\varepsilon \to 0} \phi_m^\varepsilon = \phi_m \quad \text{uniformly on } K,$$

where ϕ_m satisfies (3.5). □

4. Exit time probability function. Let $D \subset \mathfrak{R}^n$ be smooth and bounded, with outward unit normal vector ν defined on ∂D. Fix $T > 0$, and on some probability space $(\Omega, \mathscr{F}, P)$ consider the Markov process $(x^\varepsilon, P_{x,s})$ where

$$dx_t^\varepsilon = b^\varepsilon(x_t^\varepsilon, t)\, dt + \sqrt{\varepsilon}\, \sigma(x_t^\varepsilon)\, dw_t, \qquad s < t < T, \tag{4.1}$$
$$x_s^\varepsilon = x \in D,$$

and w is a given standard $\mathfrak{R}^n$-valued Wiener process. The diffusion coefficient σ satisfies the conditions of Section 2, and the vector field b^ε belongs to $C(\mathfrak{R}^n \times [0, T], \mathfrak{R}^n)$ and obeys the Lipschitz condition

$$|b^\varepsilon(x, t) - b^\varepsilon(y, t)| \le L|x - y| \quad \text{for all } x, y \in \mathfrak{R}^n,\ 0 < t < T,\ \varepsilon > 0.$$

We assume

$$b^\varepsilon = b + \sqrt{\varepsilon}\, b_1 + \varepsilon b_2 + \cdots + \varepsilon^{m/2} b_m + o(\varepsilon^{m/2}) \tag{4.2}$$

uniformly as $\varepsilon \downarrow 0$, for any $m \ge 0$, and the terms in the series belong to $C^\infty(\mathfrak{R}^n \times [0, T], \mathfrak{R}^n)$. We also assume

$$b \cdot \nu < 0 \quad \text{on } \partial D. \tag{A1}$$

Among other things, (A1) ensures that the expansion (1.3) is nontrivial, with $u(x, s) > 0$, where u is as in (4.5). The ordinary differential equation corresponding to (4.1) is

$$\dot{x}_t = b(x_t, t), \quad s < t < T, \tag{4.3}$$
$$x_s = x,$$

and the action function is given in Lemma 2.1 (with b replacing β).

The *exit time* $\tau^\varepsilon = \tau_{x,s}^\varepsilon$ is defined by

$$\tau_{x,s}^\varepsilon = \inf\{t > s : x_t^\varepsilon \notin D\},$$

where $x_s^\varepsilon = x$. The *exit time probability function* $q^\varepsilon(x, s)$ defined by (1.2) belongs to $C^{2,1}(A)$ for all compact $A \subset \overline{D} \times [0, T)$ and is the solution of

$$\frac{\partial q^\varepsilon}{\partial s} + \frac{\varepsilon}{2} \operatorname{tr}(a(x) D^2 q^\varepsilon) + b^\varepsilon \cdot D q^\varepsilon = 0 \quad \text{in } D \times (0, T), \tag{4.4}$$
$$q^\varepsilon(x, s) = 1 \quad \text{on } \partial D \times [0, T),$$
$$q^\varepsilon(x, T) = 0 \quad \text{if } x \in D.$$

Define

$$u(x,s) = \inf_{\theta \in A_{x,s}} I(\theta)$$

$$(4.5)\qquad = \inf_{\theta \in C([s,T],\Re^n),\ \theta_s = x} \left\{ \frac{1}{2}\int_s^{T\wedge\tau} \left[\dot\theta_t - b(\theta_t,t)\right]' a^{-1}(\theta_t)\left[\dot\theta_t - b(\theta_t,t)\right] dt + \chi(\theta_{T\wedge\tau}) \right\},$$

where $\tau = \tau_{x,s}(\theta) = \inf\{t > s: \theta_t \notin D,\ \theta_s = x\}$ and

$$\chi(x) = \begin{cases} +\infty, & \text{if } x \in D, \\ 0, & \text{if } x \in \partial D. \end{cases}$$

The principal term in the series expansion of q^ε arises from the following large deviation result.

THEOREM 4.1 [Freidlin and Wentzell (1984)]. *We have*

$$\lim_{\varepsilon \to 0} \varepsilon \log q^\varepsilon = -u$$

uniformly on compact subsets of $\overline{D} \times [0,T)$.

REMARK. One can give a PDE proof of Theorem 4.1 by showing that $u^\varepsilon = -\varepsilon \log q^\varepsilon$ converges uniformly on compact subsets to the unique viscosity solution of the Hamilton–Jacobi equation [Fleming and Souganidis (1986b)]

$$(4.6)\qquad \begin{aligned} &\frac{\partial u}{\partial s} + b \cdot Du - \frac{1}{2} Du\, a\, Du' = 0 \quad \text{in } D \times (0,T), \\ &u(x,s) = 0 \quad \text{on } \partial D \times [0,T], \\ &u(x,s) \to +\infty \quad \text{as } s \uparrow T \text{ if } x \in D. \end{aligned}$$

The limit function $u \in C(\overline{D} \times [0,T))$ satisfies the equation in (4.6) in the classical sense at each point (x,s) where u is differentiable [Fleming (1969), Theorem 1] and has the calculus of variations representation (4.5) valid in $\overline{D} \times [0,T)$.

We next present the remainder of the series, employing the results of Sections 2 and 3. Consider an open set $N \subset D \times [0,T']$, where $T' < T$, such that $u \in C^\infty(\overline{N})$. Using the notation of Section 2, we take $\beta \in C(\Re^n \times [0,T'],\Re^n)$ such that

$$\beta(x,s) = b(x,s) - Du(x,s)a(x), \qquad (x,s) \in \overline{N},$$

$$\beta(x,s) - \beta(y,s)| \le L|x-y|, \qquad x,y \in \Re^n,\ s \in [0,T'].$$

[The extension of β to $(\Re^n \times [0,T']) \setminus \overline{N}$ is otherwise arbitrary.] We assume

(A2) $\quad N \subset D \times [0,T']$, where $T' < T$, is a RSR with respect to β.

For $(x,s) \in N$, there exists a unique $\xi \in A_{x,s}$, which gives the minimum of

(4.5) [Fleming (1969), page 520]. Moreover, classical arguments in calculus of variations [Fleming (1969), Section 2] imply that

$$\begin{aligned} &\dot{\xi}_t = \beta(\xi_t, t), \qquad s < t < \sigma, \\ &\xi_s = x, \end{aligned} \tag{4.7}$$

where (ξ_t, t) exits from N at time $\sigma < T'$ and $(\xi_\sigma, \sigma) \in \Gamma_1$ with $\Gamma_1 \subset \partial D \times (0, T')$. In particular, under assumption (A1) there exists $\delta > 0$ such that $N = D_\delta \times [0, T']$ is a RSR satisfying (A2) with $u > 0$ and $u \in C^\infty(\overline{N})$, where $D_\delta = \{x \in D\colon \operatorname{dist}(x, \partial D) < \delta\}$.

THEOREM 4.2. *Let N satisfy* (A2) *and $u \in C^\infty(\overline{N})$. Then for each $m = 0, 1, 2, \dots$ we have*

$$\begin{aligned} P_{x,s}(\tau^\varepsilon \le T) &= \exp\bigl(-u(x,s)/\varepsilon - v(x,s)/\sqrt{\varepsilon} - w(x,s)\bigr) \\ &\quad \times \bigl[1 + \sqrt{\varepsilon}\,\phi_1(x,s) + \varepsilon\phi_2(x,s) + \cdots \\ &\qquad + \varepsilon^{m/2}\phi_m(x,s) + o(\varepsilon^{m/2})\bigr] \end{aligned} \tag{4.8}$$

as $\varepsilon \downarrow 0$ uniformly on compact subsets of N. Here, $v, w, \phi_m \in C^\infty(N)$ and satisfy ($\phi_0 \equiv 1$, $\phi_m \equiv 0$ if $m < 0$)

$$\begin{aligned} &\frac{\partial v}{\partial s} + (b - Du\,a)\cdot Dv = b_1 \cdot Du \quad \text{in } N, \\ &v(x,s) = 0 \quad \text{on } \partial D \times [0,T) \cap \overline{N}, \end{aligned} \tag{4.9}$$

$$\begin{aligned} &\frac{\partial w}{\partial s} + (b - Du\,a)\cdot Dw \\ &\quad = -\left[\frac{1}{2}\operatorname{tr}(aD^2u) - \frac{1}{2} Dv\,a\,Dv' + b_1\cdot Dv + b_2 \cdot Du\right] \quad \text{in } N, \\ &w = 0 \quad \text{on } \partial D \times [0,T) \cap \overline{N}, \end{aligned} \tag{4.10}$$

$$\begin{aligned} &\frac{\partial \phi_m}{\partial s} + (b - Du\,a)\cdot D\phi_m \\ &\quad = -\Biggl[\sum_{i=1}^m b_i \cdot D\phi_{m-i} - Dw\,a \cdot D\phi'_{m-2} + \frac{1}{2}\operatorname{tr}\bigl(aD^2\phi_{m-2}\bigr) \\ &\qquad + \frac{1}{2}\bigl(Dw\,a\,Dw' - \operatorname{tr}(aD^2w)\bigr)\phi_{m-2} - Dv\,a\,D\phi_{m-1} \\ &\qquad - \frac{1}{2}\operatorname{tr}(aD^2v)\phi_{m-1} + Dv\,a\,Dw \\ &\qquad - \sum_{i=1}^m (b_i\cdot Dw + b_{i+1}\cdot Dv + b_{i+2}\cdot Du)\phi_{m-i}\Biggr] \quad \text{in } N, \\ &\phi_m = 0 \quad \text{on } \partial D \times [0,T) \cap \overline{N}. \end{aligned} \tag{4.11}$$

REMARK. The characteristics of all these first order PDE satisfy (4.7).

PROOF. Define

$$Z^{\varepsilon}(x,s) = q^{\varepsilon}(x,s)\exp\big(u(x,s)/\varepsilon + v(x,s)/\sqrt{\varepsilon}\big).$$

This type of factorization appears in Sheu (1986). Let $K \subset N \cup \Gamma_1$ be compact with K° also a RSR for β. Let $\beta^{\varepsilon} \in C(\mathfrak{R}^n \times [0,T'], \mathfrak{R}^n)$ satisfy

$$\beta^{\varepsilon} = b^{\varepsilon} - \big(Du + \sqrt{\varepsilon}\, Dv\big)a \quad \text{in } K,$$

$$\beta^{\varepsilon} \to \beta \quad \text{uniformly in } \mathfrak{R}^n \times [0,T'] \text{ as } \varepsilon \to 0,$$

$$|\beta^{\varepsilon}(x,s) - \beta^{\varepsilon}(y,s)| \le L|x-y|, \qquad x,y \in \mathfrak{R}^n,\ s \in [0,T'],\ \varepsilon > 0,$$

and let $h^{\varepsilon} \in C_b(\mathfrak{R}^n \times [0,T'])$ satisfy

$$h^{\varepsilon} = -\big[\tfrac{1}{2}\operatorname{tr}(aD^2u) + \sqrt{\varepsilon}\operatorname{tr}\big((aD^2v) - Dv\,a\,Dv'\big) + b_1^{\varepsilon}\cdot Dv + b_2^{\varepsilon}\cdot Du\big] \quad \text{in } K.$$

Then

$$\frac{\partial Z^{\varepsilon}}{\partial s} + \frac{\varepsilon}{2}\operatorname{tr}\big(a(x)\,D^2Z^{\varepsilon}\big) + \beta^{\varepsilon}\cdot DZ^{\varepsilon} = -h^{\varepsilon}Z^{\varepsilon} \quad \text{in } K^{\circ}, \tag{4.12}$$

$$Z^{\varepsilon} = 1 \quad \text{on } \partial D \times [0,T) \cap K.$$

Now for each $\gamma > 0$,

$$0 \le Z^{\varepsilon} \le C\exp\big((|u^{\varepsilon} - u| + \sqrt{\varepsilon}\,C)/\varepsilon\big)$$
$$\le Ce^{\gamma/\varepsilon}$$

for all sufficiently small $\varepsilon > 0$. Applying Theorem 2.1, we see that

$$\lim_{\varepsilon\to 0} Z^{\varepsilon} = Z \quad \text{uniformly on } K.$$

To complete the proof, apply the result of Theorem 3.1 to the solution Z^{ε} of (4.12). □

REMARKS. (i) Theorem 4.2 holds in any RSR satisfying (A2), whether or not (A1) was used in the construction of N. For example, if

$$b\cdot\nu > 0 \quad \text{on } \partial D,$$

then $u(x,s) \equiv 0$ in $\overline{D}_{\delta} \times [0,T'] = N$, for some $\delta > 0$ and $T' < T$, and so N is a RSR. In this case,

$$P_{x,s}(\tau^{\varepsilon} \le T) = 1 + o(\varepsilon^m)$$

as $\varepsilon \downarrow 0$ for any $m \ge 0$. Here $v \equiv w \equiv \phi_m \equiv 0$ in N.

(ii) If $b^{\varepsilon} \equiv b$ is independent of $\varepsilon > 0$, then the series involves only integer powers of ε. In this case, with σ equal to the identity matrix, we have for each $m = 0,1,2,\dots,$

$$P_{x,s}(\tau^{\varepsilon} \le T) = \exp[-u(x,s)/\varepsilon - w(x,s)]$$
$$\times\big[1 + \varepsilon\psi_1(x,s) + \cdots + \varepsilon^m\psi_m(x,s) + o(\varepsilon^m)\big]$$

as $\varepsilon \downarrow 0$ uniformly on compact subsets of N. Here, $w, \psi_m \in C^\infty(N)$ and satisfy ($\psi_0 \equiv 1$)

$$\frac{\partial w}{\partial s} + (b - Du)\cdot Dw = \left(-\frac{1}{2}\Delta u\right) \quad \text{in } N,$$

$$w = 0 \quad \text{on } \delta D \times [0,T) \cap \overline{N},$$

$$\frac{\partial \psi_m}{\partial s} + (b - Du)\cdot D\psi_m = -\left[\frac{1}{2}(|Dw|^2 - \Delta w)\psi_{m-1} - Dw \cdot D\psi_{m-1} + \frac{1}{2}\Delta\psi_{m-1}\right] \quad \text{in } N,$$

$$\psi_m = 0 \quad \text{on } \partial D \times [0,T) \cap \overline{N}.$$

EXAMPLE. Let $n = 1$, $D = (-1,1)$, $b^\varepsilon(x,s) = -x$, $\sigma(x) = 1$. The minimum problem (4.5) can be solved explicitly, by elementary calculus of variations. For $|x| < 1$ and $x \neq 0$, the minimizing $\xi \in A_{x,s}$ is unique, and $u \in C^1((D \setminus \{0\}) \times [0,T))$. If $\exp(s - T) < x < 1$, then ξ_t exits D at time $\sigma < T$ and $\xi_\sigma = 1$. Thus $N_1 = \{(x,s)\colon \exp(s - T) < x < 1\}$ is a RSR. Similarly, $N_2 = \{(x,s)\colon (-x,s) \in N_1\}$ is a RSR. Theorem 4.1 applies in N_1 and N_2. If $(x,s) \in N_1 \cup N_2$, then up to order 1 we have as $\varepsilon \to 0$,

$$P_{x,s}(\tau^\varepsilon \le T) = \exp\left(-(1 - x^2)/\varepsilon - \log|x|\right)\left[1 + \varepsilon(1/x^2 - 1)/2 + o(\varepsilon)\right].$$

If on the other hand $0 < x < \exp(s - T)$, then $\sigma = T$. We do not know the appropriate form for an asymptotic series expansion for the exit probability in that case. It would be helpful to have accurate asymptotic estimates for $q^\varepsilon(x,s)$ when $T - s$ is small and x near $\partial D = \{-1, 1\}$.

5. The Cauchy problem. We wish to obtain an asymptotic series expansion for the solution q^ε of the Cauchy problem

$$(5.1) \qquad \begin{aligned} &\frac{\partial q^\varepsilon}{\partial s} + \frac{\varepsilon}{2}\operatorname{tr}(a(x)\,D^2 q^\varepsilon) + b^\varepsilon \cdot Dq^\varepsilon + \frac{1}{\varepsilon}V^\varepsilon q^\varepsilon = 0 \quad \text{in } \Re^n \times (0,T),\\ &q^\varepsilon(x,T) = C_\varepsilon e^{-\Psi^\varepsilon/\varepsilon} \quad \text{if } x \in \Re^n. \end{aligned}$$

When $b^\varepsilon = 0$ and a is the identity matrix, this is an "imaginary time" analogue of the problem of semiclassical limit for Schrödinger's equation.

We take b^ε as in Section 4 and $C_\varepsilon > 0$ is a constant such that $\lim_{\varepsilon \to 0} \varepsilon \log C_\varepsilon = 0$, and we assume for each $m = 0, 1, \dots,$

$$(5.2) \qquad \begin{aligned} V^\varepsilon &= V + \sqrt{\varepsilon}\,V_1 + \varepsilon V_2 + \cdots + \varepsilon^{m/2} V_m + o(\varepsilon^{m/2}),\\ \Psi^\varepsilon &= \Psi + \sqrt{\varepsilon}\,\Psi_1 + \varepsilon \Psi_2 + \cdots + \varepsilon^{m/2}\Psi_m + o(\varepsilon^{m/2}), \end{aligned}$$

uniformly as $\varepsilon \downarrow 0$, where the terms in the series for V^ε belong to $C_b(\Re^n \times [0,T]) \cap C^\infty(\Re^n \times [0,T])$, while the terms for Ψ^ε belong to $C_b(\Re^n) \cap C^\infty(\Re^n)$.

The functions V^ε and Ψ^ε are assumed to satisfy a uniform Lipschitz condition of the type (2.4). Let

$$\Phi^\varepsilon = \exp\left(-\left(\Psi^\varepsilon - \Psi - \sqrt{\varepsilon}\,\Psi_1 - \varepsilon\Psi_2\right)/\varepsilon\right);$$

then $\Phi^\varepsilon \sim \exp(-\sqrt{\varepsilon}\,\Psi_3 - \varepsilon\Psi_4 - \cdots)$ satisfies (3.2).

Define

$$u^\varepsilon(x,s) = -\varepsilon \log q^\varepsilon(x,s) \quad \text{in } \mathfrak{R}^n \times [0,T].$$

Then u^ε is the solution of the quasilinear PDE

$$\begin{aligned} &\frac{\partial u^\varepsilon}{\partial s} + \frac{\varepsilon}{2}\operatorname{tr}(a(x)D^2u^\varepsilon) + b^\varepsilon \cdot Du^\varepsilon \\ &\qquad - \frac{1}{2}Du^\varepsilon\, a\, Du^{\varepsilon\prime} - V^\varepsilon = 0 \quad \text{in } \mathfrak{R}^n \times (0,T), \\ &u^\varepsilon(x,T) = -\varepsilon \log C_\varepsilon + \Psi^\varepsilon(x) \quad \text{for } x \in \mathfrak{R}^n. \end{aligned} \tag{5.3}$$

By assumption, V^ε and $u^\varepsilon(\cdot,T)$ are bounded independent of ε for $0 < \varepsilon \le 1$. Therefore, for suitable constants a_i, b_i, $i = 1,2$, $a_1 + b_1 s$ is a subsolution of (5.3) and $a_2 + b_2$ is a supersolution of (5.3). This implies, by the maximum principle for parabolic PDE, that $|u^\varepsilon(x,s)| \le C$ for all $(x,s) \in \mathfrak{R}^n \times [0,T]$ and $0 < \varepsilon \le 1$. An easy adaptation of the method of Barles and Perthame (1988) shows that

$$\lim_{\varepsilon \to 0} u^\varepsilon = u$$

uniformly on compact subsets of $\mathfrak{R}^n \times [0,T]$, where $u \in C_b(\mathfrak{R}^n \times [0,T])$ is the unique viscosity solution of the Hamilton–Jacobi equation

$$\begin{aligned} &\frac{\partial u}{\partial s} + b \cdot Du - \frac{1}{2}Du\, a\, Du' - V = 0 \quad \text{in } \mathfrak{R}^n \times (0,T), \\ &u(x,T) = \Psi(x) \quad \text{for } x \in \mathfrak{R}^n. \end{aligned} \tag{5.4}$$

The function u has a calculus of variations representation similar to (4.5):

$$u(x,s) = \inf\{J(\theta) : \theta \in C([s,t],\mathfrak{R}^n),\ \theta \text{ absolutely continuous},\ \theta_s = x\},$$

where

$$J(\theta) = \int_s^T \left(\tfrac{1}{2}\left[\dot\theta_t - b(\theta_t,t)\right]' a^{-1}(\theta_t)\left[\dot\theta_t - b(\theta_t,t)\right] - V(\theta_t,t)\right) dt + \Psi(\theta_T).$$

We consider an open set $N \subset \mathfrak{R}^n \times [0,T]$ for which the previously defined function u belongs to $C^\infty(\overline{N})$. We define β as was done in Section 4 and assume that N is a RSR with respect to β. We write $\Gamma_1 = \partial N \cap (\mathfrak{R}^n \times \{T\})$. By Theorem 2 of Fleming (1969), there exists a closed set $E \subset \mathfrak{R}^n \times [0,T]$ of Hausdorff dimension less than or equal to n such that each $(x,s) \in (\mathfrak{R}^n \times [0,T] \setminus E$ belongs to a RSR.

THEOREM 5.1. *Let $N \subset \mathfrak{R}^n \times [0,T]$ be a RSR, with $u \in C^\infty(\overline{N})$. Then for each $m = 0,1,2,\dots,$ we have*

$$
\begin{aligned}
q^\varepsilon(x,s) &= C_\varepsilon \exp\left(-u(x,s)/\varepsilon - v(x,s)/\sqrt{\varepsilon} - w(x,s)\right) \\
&\quad \times\left[1 + \sqrt{\varepsilon}\,\phi_1(x,s) + \varepsilon\phi_2(x,s) + \cdots \right. \\
&\quad \left. + \varepsilon^{m/2}\phi_m(x,s) + o(\varepsilon^{m/2})\right]
\end{aligned}
\tag{5.5}
$$

as $\varepsilon \downarrow 0$ uniformly on compact subsets of $N \cup \Gamma_1$. Here, $v, w, \phi_m \in C^\infty(N \cup \Gamma_1)$ and satisfy ($\phi_0 \equiv 1$, $\phi_m \equiv 0$ if $m < 0$)

$$
\begin{aligned}
&\frac{\partial v}{\partial s} + (b - Du\,a)\cdot Dv = b_1 \cdot Du + V_1 \quad \text{in } N,\\
&v(x,T) = \Psi_1(x) \quad \text{for } x \text{ with } (x,T) \in \Gamma_1,
\end{aligned}
\tag{5.6}
$$

$$
\begin{aligned}
&\frac{\partial w}{\partial s} + (b - Du\,a)\cdot Dw \\
&\quad = -\left[\frac{1}{2}\operatorname{tr}(aD^2u) - \frac{1}{2}Dv\,a\,Dv' + b_1\cdot Dv + b_2 \cdot Du + V_2\right] \quad \text{in } N,\\
&w(x,T) = \Psi_2(x) \quad \text{for } x \text{ with } (x,T) \in \Gamma_1,
\end{aligned}
\tag{5.7}
$$

$$
\begin{aligned}
&\frac{\partial \phi_m}{\partial s} + (b - Du\,a)\cdot D\phi_m \\
&\quad = -\left[\sum_{i=1}^m b_i \cdot D\phi_{m-i} - Dw\,a\cdot D\phi'_{m-2} + \frac{1}{2}\operatorname{tr}(aD^2\phi_{m-2})\right. \\
&\qquad + \frac{1}{2}\left(Dw\,a\,Dw' - \operatorname{tr}(aD^2w)\right)\phi_{m-2} - Dv\,a\,D\phi_{m-1} \\
&\qquad - \frac{1}{2}\operatorname{tr}(aD^2v)\phi_{m-1} + Dv\,a\,Dw \\
&\qquad \left. - \sum_{i=1}^m (b_i\cdot Dw + b_{i+1}\cdot Dv + b_{i+2}\cdot Du + V_{i+2})\phi_{m-i}\right] \quad \text{in } N,\\
&\phi_m(x,T) = \Phi_m(x) \quad \text{for } x \text{ with } (x,T)\in\Gamma_1.
\end{aligned}
\tag{5.8}
$$

REMARK. The series (5.5) is an imaginary time analogue of Maslov's expansion in quantum mechanics; see Fleming (1983).

PROOF. Define

$$Z^\varepsilon(x,s) = C_\varepsilon^{-1}q^\varepsilon(x,s)\exp\left(u(x,s)/\varepsilon + v(x,s)/\sqrt{\varepsilon}\right)$$

and proceed as in the proof of Theorem 4.2. □

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