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SOME LARGE-DEVIATION THEOREMS FOR BRANCHING DIFFUSIONS[1]

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A branching diffusion process is studied when its diffusivity decreases to 0 at the rate of $\varepsilon \ll 1$ and its branching/transmutation intensity increases at the rate of ε^{-1}. We derive the action functionals which describe some large deviations of the processes as ε tends to 0. The branching diffusion processes are closely related to systems of semilinear parabolic differential equations.

Introduction. We investigate large deviation probabilities for a family of branching diffusions with transmutation. Our motivation can be described using the following family of binary branching Brownian motions (BM):

(H1) A particle starts from the position $x \in \mathbb{R}^d$ and executes a small BM, $\varepsilon^{1/2}$(BM), $0 < \varepsilon \ll 1$.
(H2) Its lifetime exceeds t with probability $\exp(-ct/\varepsilon)$, where c is a positive constant.
(H3) At death it is replaced by two descendants.
(H4) Each descendant, starting from where its ancestor dies, repeats the process (H1)–(H3) and so on. All diffusions and lifespans are assumed independent of one another.

Denote by Q_x^ε the probability distribution of this branching BM. Note that the probability space is a set of sample trees $Z \equiv (Z_t, 0 < t)$ rooted at x, where Z_t denotes the collection of positions of particles existent at time t. For a particle existent at time T, we call its ancestral path a branch; for two particles existent at time T, we call their ancestral paths a 2-branch. For convenience, T is a fixed number throughout this paper. Our study is motivated by the following problems.

PROBLEM 1. What is the asymptotic probability as $\varepsilon \to 0$ that the sample tree Z contains a branch close to φ: $[0, T] \to \mathbb{R}^d$?

PROBLEM 2. Conditioned that the sample tree Z contains a branch φ, how many branches are there in a tiny neighborhood of φ?

PROBLEM 3. What is the asymptotic probability in Problem 1 if we replace a branch by a 2-branch?

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For the neighboring model of random walks, Problems 1 and 2 were mentioned and conjectured in [14]–[16], and partial solutions were given in [14] and [15]. There is no overlap with our method. We are grateful to the editor for pointing out these references.

If there is no branching mechanism in the process Q_x^ε, then Problems 2 and 3 are trivial and the solution to Problem 1 is well known (cf. [9] and [18]). Roughly speaking, the probability that the $\varepsilon^{1/2}$(BM) contains the branch (i.e., follows the path) φ: $[0, T] \to \mathbb{R}^d$ is asymptotically

$$\exp \frac{-1}{\varepsilon} \int_0^T \frac{1}{2} |\dot{\varphi}(s)|^2 \, ds.$$

Here $\dot{\varphi}(s)$ is $d\varphi(s)/ds$. Such a function as

$$\varphi \to \frac{1}{\varepsilon} \int_0^T \frac{1}{2} |\dot{\varphi}(s)|^2 \, ds$$

is called an action functional, rate function or entropy.

For Q_x^ε defined in (H1)–(H4), the same event occurs with a greater probability because there are typically $\exp(ct/\varepsilon)$ many particles at time t. Some computations then lead to the following conjectures.

CONJECTURE 1. The Q_x^ε probability that the sample tree contains the branch φ: $[0, T] \to \mathbb{R}^d$ is asymptotically like

$$\exp\left\{ \frac{1}{\varepsilon} \min_{0 \le t \le T} \left[ct - \int_0^t \frac{1}{2} |\dot{\varphi}(s)|^2 \, ds \right] \right\},$$

that is,

$$\exp\left\{ \frac{-1}{\varepsilon} \max_{0 \le t \le T} \int_0^t \left(\frac{1}{2} |\dot{\varphi}(s)|^2 - c \right) ds \right\} \quad \text{as } \varepsilon \to 0.$$

CONJECTURE 2. Conditioned that the sample tree contains the branch φ, there are typically

$$\exp\left\{ \frac{1}{\varepsilon} \max_{0 \le t \le T} \left[c(T - t) - \int_t^T \frac{1}{2} |\dot{\varphi}(s)|^2 \, ds \right] \right\},$$

that is,

$$\exp\left\{ \frac{1}{\varepsilon} \max_{0 \le t \le T} \int_t^T \left(c - \frac{1}{2} |\dot{\varphi}(s)|^2 \right) ds \right\}$$

many branches close to φ as $\varepsilon \to 0$.

In Section 1 a rigorous version of Conjecture 1 is proved (Theorem 1). In Section 2, Conjecture 2 is precisely formulated and is partially proved (Theorem 2). In Section 3 we give a meaning and a proof to Problem 3 (Theorem 3). The unsolved part of Conjecture 2 and some other large deviations for the number of particles are discussed elsewhere.

Theorems 1, 2 and 3 are, in fact, proved for a broad class of branching diffusion processes. Briefly speaking, we allow a branching diffusion particle to transmute (change type) and allow the diffusivity, lifetime and offspring distribution to depend on both the position and the type of the particle. The offspring distribution can be supercritical (mean $m > 1$), critical ($m = 1$) or subcritical ($m < 1$).

In Section 4 we point out some applications and further generalizations.

1. Large deviations for the branches. We consider the following generalizations of (H1)–(H4):

(H1′) A type-k, $k = 1, 2$, particle starts from $x \in \mathbb{R}^d$ and executes a small diffusion generated by $\varepsilon^{1/2} D_k \Delta/2$, where D_k: $\mathbb{R}^d \to (0, \infty)$ is Lipschitz continuous.

(H2′) Its lifetime exceeds t with probability $\exp[(-1/\varepsilon)\int_0^t \alpha_k(X_s)\,ds]$ given its path X_s, $0 \le s \le t$, where α_k: $\mathbb{R}^d \to (0, \infty)$ is continuous.

(H3′) At death it is replaced by n_1 type-1 descendants and n_2 type-2 descendants with probability $p_k(n_1, n_2, y)$, where y is the ancestor's position at death and $p_k(n_1, n_2, \cdot)$ is continuous for each $k = 1, 2$, $n_1, n_2 \ge 0$, $\sum_{n_1, n_2 \ge 0} p_k(n_1, n_2, y) \equiv 1$.

(H4′) Each descendant, starting from where its ancestor dies, repeats the process (H1′)–(H3′) and so on. All diffusions, lifespans and offspring distributions are assumed independent of one another.

Let $P^\varepsilon_{x,k}$ denote the probability distribution, $E^\varepsilon_{x,k}$ the expectation and Z the sample tree of this process. Q^ε_x can be regarded as the special case where $D_1(x) = D_2(x) \equiv 1$, $\alpha_1(x) = \alpha_2(x) \equiv c$,

$$\sum_{n_1+n_2=2} p_1(n_1, n_2, x) = 1$$

and there is therefore no need to distinguish particle types.

Listed below are some important functions associated with the process $P^\varepsilon_{x,k}$.

Birth rates (*Malthusian parameter*).

$$c_{1,1}(x) := \alpha_1(x)\left\{\left[\sum_{n_1, n_2} n_1 p_1(n_1, n_2, x)\right] - 1\right\},$$

$$c_{1,2}(x) := \alpha_1(x) \sum_{n_1, n_2} n_2 p_1(n_1, n_2, x),$$

$$c_{2,1}(x) := \alpha_2(x) \sum_{n_1, n_2} n_1 p_2(n_1, n_2, x),$$

$$c_{2,2}(x) := \alpha_2(x)\left\{\left[\sum_{n_1, n_2} n_2 p_2(n_1, n_2, x)\right] - 1\right\}, \qquad x \in \mathbb{R}^d.$$

Hamiltonian.

$$H(x,p) := \text{the largest eigenvalue of}$$
$$\begin{bmatrix} \frac{1}{2}D_1(x)|p|^2 + c_{1,1}(x) & c_{1,2}(x) \\ c_{2,1}(x) & \frac{1}{2}D_2(x)|p|^2 + c_{2,2}(x) \end{bmatrix},$$

equals $|p|^2/2 + c$ in the case of Q_x^ε as in (H1)–(H4), $x, p \in \mathbb{R}^d$.

Lagrangian. $L(x,q) := \sup_{p \in \mathbb{R}^d}[q \cdot p - H(x,p)]$, equals $|q|^2/2 - c$ in the case of Q_x^ε, $x, q \in \mathbb{R}^d$.

Action functional for the branches. $S\colon C(\mathbb{R}^d) \to [0,\infty]$ is defined as

$$S(\varphi) := \begin{cases} \max_{0 \le a \le T} \int_0^a L(\varphi(s), \dot\varphi(s))\, ds, & \text{if } \varphi \text{ is absolutely continuous (a.c.)}, \\ +\infty, & \text{otherwise}, \end{cases}$$

where $C(\mathbb{R}^d)$ denotes the set of all continuous functions: $[0,T] \to \mathbb{R}^d$. We impose on $C(\mathbb{R}^d)$ the uniform topology induced by the metric ρ:

$$\rho(\varphi,\psi) := \max_{0 \le s \le T} |\varphi(s) - \psi(s)|, \qquad \varphi, \psi \in C(\mathbb{R}^d).$$

We make the following assumptions on our branching diffusions $P_{x,k}^\varepsilon$:

(A1) There exists $K \in N$ such that if $n_1 > K$ or $n_2 > K$, then $p_k(n_1, n_2, x) \equiv 0$ for $k = 1,2$; $\inf\{c_{k,m}(x)\colon (k,m) = (1,2) \text{ or } (2,1) \text{ and } x \in \mathbb{R}^d\} > 0$.

(A2) $D_k(x), \alpha_k(x)$, $k = 1,2$, are bounded functions.

The essential part of these two assumptions is that $c_{k,m}(x)$, $k, m = 1,2$, are finite. The remainder is mainly for simplicity of presentation.

THEOREM 1. *Assume conditions* (A1) *and* (A2). *Then the functional* $\varepsilon^{-1}S$ *is the action functional for the branches of the family of branching diffusions* $P_{x,k}^\varepsilon$, $\varepsilon \to 0$, $k = 1,2$. *By this statement we mean the following in the uniform topology* ρ:

(1.1) *S is lower semicontinuous and the level set* $\{\varphi\colon S(\varphi) \le A$ *and* $\varphi(0) = x\}$ *is compact for each* $0 \le A < \infty$ *and* $x \in \mathbb{R}^d$.

(1.2) *If C is a closed subset of* $C(\mathbb{R}^d) \cap \{\varphi\colon \varphi(0) = x\}$, *then* $\limsup_{\varepsilon \to 0} \varepsilon \ln P_{x,k}^\varepsilon\{$*the sample tree Z contains a branch in* $C\} \le -\inf_{\varphi \in C} S(\varphi)$ *for* $k = 1,2$.

(1.3) *If G is an open subset of* $C(\mathbb{R}^d) \cap \{\varphi\colon \varphi(0) = x\}$, *then* $\liminf_{\varepsilon \to 0} \varepsilon \ln P_{x,k}^\varepsilon\{$*the sample tree Z contains a branch in* $G\} \ge -\inf_{\varphi \in G} S(\varphi)$ *for* $k = 1,2$.

We establish a series of Lemmas 1–5 to prove Theorem 1.

LEMMA 1.1. *The functional S is lower semicontinuous in the uniform topology ρ and so are the functionals*

$$\varphi \to \int_0^T L(\varphi(s), \dot{\varphi}(s)\, ds \quad \text{and} \quad \varphi \to \min_{0 \le a \le T} \int_a^T L(\varphi(s), \dot{\varphi}(s))\, ds.$$

PROOF. We only prove this for the functional S. For the other two functionals the modification needed is simple. For $\varphi \in C(\mathbb{R}^d)$, a partition $\pi = (t_0, t_1, \dots, t_N)$, $0 = t_0 < t_1 < \cdots < t_N = T$ and $1 \le j \le N$, define a Hamiltonian $H_{\varphi, \pi, j}$, $H_{\varphi, \pi, j}(p) :=$ the largest eigenvalue of the matrix

$$\begin{bmatrix} \left[\max\limits_{t_{j-1} \le t \le t_j} D_1(\varphi(t))\right] \dfrac{|p|^2}{2} + \max\limits_{t_{j-1} \le t \le t_j} C_{1,1}(\varphi(t)), & \max\limits_{t_{j-1} \le t \le t_j} C_{1,2}(\varphi(t)) \\ \max\limits_{t_{j-1} \le t \le t_j} C_{2,1}(\varphi(t)), & \left[\max\limits_{t_{j-1} \le t \le t_j} D_2(\varphi(t))\right] \dfrac{|p|^2}{2} + \max\limits_{t_{j-1} \le t \le t_j} C_{2,2}(\varphi(t)) \end{bmatrix}, \qquad p \in \mathbb{R}^d.$$

Denote by $L_{\varphi, \pi, j}(q)$ the corresponding Lagrangian:

$$L_{\varphi, \pi, j}(q) := \sup_{p \in \mathbb{R}^d} \left[q \cdot p - H_{\varphi, \pi, j}(p)\right], \qquad q \in \mathbb{R}^d.$$

Then we have

$$\begin{aligned} \int_{t_{j-1}}^{t_j} L(\varphi(s), \dot{\varphi}(s))\, ds &\ge \int_{t_{j-1}}^{t_j} L_{\varphi, \pi, j}(\dot{\varphi}(s))\, ds \\ &\ge (t_j - t_{j-1}) L_{\varphi, \pi, j}\left(\frac{\varphi(t_j) - \varphi(t_{j-1})}{t_j - t_{j-1}}\right), \end{aligned}$$

where the last step uses Jensen's inequality.

Let Π denote the set of all partitions of $[0, T]$. From the definition of the functional S, the following representation is valid:

$$S(\varphi) = \sup_{\pi = (t_0, t_1, \dots, t_N) \in \Pi} \max_{m = 0, \dots, N} \sum_{j=1}^m (t_j - t_{j-1}) L_{\varphi, \pi, j}\left(\frac{\varphi(t_j) - \varphi(t_{j-1})}{t_j - t_{j-1}}\right).$$

Since

$$\varphi \to \max_{m = 0, \dots, N} \sum_{j=1}^m (t_j - t_{j-1}) L_{\varphi, \pi, j}\left(\frac{\varphi(t_j) - \varphi(t_{j-1})}{t_j - t_{j-1}}\right)$$

is a continuous function from $C(\mathbb{R}^d)$ to $[0, \infty)$ in the uniform topology ρ for each partition $\pi \in \Pi$, this representation of S proves the lower semicontinuity. □

LEMMA 1.2. *Let $0 < \theta < 1/2$ and $\|\varphi\|_\theta$ be the Hölder norm of order θ of $\varphi \in C(\mathbb{R}^d)$, that is,*

$$\|\varphi\|_\theta := \sup_{0 \le s < t \le T} \frac{|\varphi(t) - \varphi(s)|}{|t - s|^\theta}.$$

Then

$$\limsup_{A\to\infty}\limsup_{\varepsilon\to 0}\varepsilon \ln P^{\varepsilon}_{x,k}\{\text{the sample tree } Z \text{ contains a branch } Y \text{ with } \|Y\|_\theta > A\} = -\infty \text{ for each } k = 1, 2 \text{ and } x \in \mathbb{R}^d.$$

PROOF. For clarity of argument we first give a proof for the special branching diffusion Q^{ε}_x as defined by (H1)–(H4) and then point out the modification needed for the general case.

Denote by N the total number of particles existent at time T. First, note that

$$(1.4)\quad \begin{aligned} I &:= Q^{\varepsilon}_x\{Z \text{ contains a branch } Y \text{ with } \|Y\|_\theta > A\} \\ &\le Q^{\varepsilon}_x\left\{N \ge \exp\left(\frac{B}{\varepsilon}\right)\right\} \\ &\quad + Q^{\varepsilon}_x\left\{N < \exp\left(\frac{B}{\varepsilon}\right) \text{ and at least one branch } Y \text{ with } \|Y\|_\theta > A\right\} \end{aligned}$$

for any positive B.

By Chebyshev's inequality and by the fact that each branch is, in distribution, equivalent to an $\varepsilon^{1/2}$(BM), we see that

$$(1.5)\quad I \le E^{\varepsilon}_x\{N\}\exp\left(\frac{-B}{\varepsilon}\right) + \text{Prob}\{\|\varepsilon^{1/2}(\text{BM})\|_\theta > A\}\exp\left(\frac{B}{\varepsilon}\right).$$

It suffices to show the following:

$$(1.6)\quad \limsup_{\varepsilon\to 0}\varepsilon \ln E^{\varepsilon}_x\{N\} = Tc,$$

$$(1.7)\quad \limsup_{A\to\infty}\limsup_{\varepsilon\to 0}\varepsilon \ln \text{Prob}\{\|(\text{BM})\|_\theta > \varepsilon^{-1/2}A\} = -\infty,$$

because these, together with (1.5), imply

$$\limsup_{A\to\infty}\limsup_{\varepsilon\to 0} I \le \max[Tc - B, -\infty] = Tc - B$$

and B can be arbitrarily large. (1.6) is easy to compute. (1.7) is known (see, e.g., [2]). For Q^{ε}_x, Lemma 1.2 is proved.

If $P^{\varepsilon}_{x,k}$ is a general branching diffusion as defined by (H1′)–(H4′), (A1) and (A2), we define a time change $\tau(t)$, depending on the branch $(\nu, x) := (\nu_t, X_t; 0 \le t \le T) = (\nu^{\varepsilon}_t, X^{\varepsilon}_t; 0 \le t \le T)$, namely,

$$\tau(t) := \int_0^t D_{\nu_s}(X_s)\,ds \quad \text{for } 0 \le t \le T.$$

Regard $t = t(\tau)$, also depending on (ν, X), as the inverse of $\tau(t)$ and denote by $Z_{t(\tau)}$ the random tree process using time parameter τ. It is then sufficient to imitate the proof for Q^{ε}_x. □

For a given $\psi \in C(\mathbb{R}^d)$ with $\psi(0) = x$, $S(\psi) < \infty$ and a positive number δ consider $\Omega(\psi, \delta)$, the δ-neighborhood of ψ in reversed time:

$$\Omega(\psi, \delta) := \{(t, y) \colon 0 < t \le T \text{ and } |y - \psi(T - t)| < \delta\}.$$

Let $(U_1^\varepsilon(t, y), U_2^\varepsilon(t, y))$ be the solution of the following system of linear differential equations:

$$\left(\frac{\partial}{\partial t} - \varepsilon D_1(y)\Delta/2\right) U_1^\varepsilon(t, y) = \varepsilon^{-1}[c_{1,1}(y) U_1^\varepsilon + c_{1,2}(y) U_2^\varepsilon],$$

$$\left(\frac{\partial}{\partial t} - \varepsilon D_2(y)\Delta/2\right) U_2^\varepsilon = \varepsilon^{-1}[c_{2,1}(y) U_1^\varepsilon + c_{2,2}(y) U_2^\varepsilon] \tag{1.8}$$

in $\Omega(\psi, \delta)$, boundary value $\equiv 0$ and initial value $\equiv 1$.

We derive from [8, Theorem 1] the following Proposition 1.3. It is proved by large-deviation estimates and the Laplace asymptotic method.

PROPOSITION 1.3. *The solutions $(U_1^\varepsilon, U_2^\varepsilon)$ to the problem (1.8) satisfy the following*:

$$\lim_{\varepsilon \to 0} \varepsilon \ln U_k^\varepsilon(T, x) = \sup\left\{\int_0^T - L(\varphi(s), \dot\varphi(s))\, ds \colon (T - s, \varphi(s)) \in \Omega(\psi, \delta) \text{ for } 0 \le s < T \text{ and } \varphi(0) = x\right\}, \quad k = 1, 2. \tag{1.9}$$

There exists a function $f(\varepsilon)$, independent of (t, y) and with $\lim_{\varepsilon \downarrow 0} f(\varepsilon) = 0$ such that

$$\varepsilon \ln U_k^\varepsilon(t, y) \le f(\varepsilon) + \sup\left\{\int_0^t - L(\varphi(s), \dot\varphi(s))\, ds \colon (t - s, \varphi(s)) \in \Omega(\psi, \delta) \text{ for } 0 \le s < t \text{ and } \varphi(0) = y\right\} \tag{1.10}$$

$$\text{for } k = 1, 2,\ (t, y) \in \Omega(\psi, \delta).$$

LEMMA 1.4. *If $S(\psi) < \infty$, $\psi(0) = x$ and $N(\psi, \delta, T)$ is defined as the number of branches X with $\rho(X, \psi) < \delta$, that is, $(T - s, X_s) \in \Omega(\psi, \delta)$ for $0 \le s < T$, then*

$$\limsup_{\delta \to 0} \limsup_{\varepsilon \to 0} \varepsilon \ln P_{x,k}^\varepsilon\{N(\psi, \delta, T) \ge 1\} \le -S(\psi) \quad \text{for } k = 1, 2.$$

PROOF. First, we extend the definition of $N(\psi, \delta, T)$:

$$(1.11)\quad \begin{aligned} &N(G_{T-t}\psi, \delta, t) := \text{the number of branches } X \text{ with} \\ &\max_{0\le s<t} |X_s - (G_{T-t}\psi)(s)| < \delta, \text{ where } (G_{T-t}\psi)(s) := \psi(T-t+s). \end{aligned}$$

It is known (cf. [3], [10]–[12] and [17]) that

$$(1.12)\qquad E^{\varepsilon}_{y,k}\{N(G_{T-t}\psi, \delta, t)\} = U^{\varepsilon}_k(t,y), \quad \text{the solution of (1.8)}.$$

Since $P^{\varepsilon}_{x,k}\{N(\psi,\delta,t) \ge 1\} \le E^{\varepsilon}_{x,k}\{N(\psi,\delta,t)\} = U^{\varepsilon}_k(T,x)$, by Proposition 1.3 we get

$$\begin{aligned} &\limsup_{\delta\to 0}\limsup_{\varepsilon\to 0} \varepsilon \ln P^{\varepsilon}_{x,k}\{N(\psi,\delta,T) \ge 1\} \\ &\quad \le \limsup_{\delta\to 0} \sup\left\{\int_0^T -L(\varphi(s),\dot\varphi(s))\,ds : (T-s,\varphi(s)) \in \Omega(\psi,\delta) \text{ for} \right. \\ &\qquad\qquad \left. 0 \le s < T \text{ and } \varphi(0) = x\right\} \quad \text{by Lemma 1.1} \\ &\quad = \int_0^T -L(\psi(s),\dot\psi(s))\,ds. \end{aligned}$$

Had we replaced T by a and repeated the same argument, we would have obtained

$$\limsup_{\delta\to 0}\limsup_{\varepsilon\to 0} \varepsilon \ln P^{\varepsilon}_{x,k}\{N(\psi,\delta,a) \ge 1\} \le \int_0^a -L(\psi(s),\dot\psi(s))\,ds.$$

In view of the fact that $P^{\varepsilon}_{x,k}\{N(\psi,\delta,a) \ge 1\}$ is decreasing in a, the proof is complete. □

LEMMA 1.5. *If $S(\psi) < \infty$, $\psi(0) = x$ and $N(\psi,\delta,T)$ is as in Lemma 1.4, then*

$$\liminf_{\delta\to 0}\liminf_{\varepsilon\to 0} \varepsilon \ln P^{\varepsilon}_{x,k}\{N(\psi,\delta,T) \ge 1\} \ge -S(\psi), \qquad k = 1,2.$$

PROOF. First, we investigate the moment generating function of $N(G_{T-t}\psi, \delta, t)$ which is defined in (1.11). For $0 \le \theta < 1$, define

$$u^{\varepsilon}_k(t,y,\theta) := E^{\varepsilon}_{y,k}\{1 - (1-\theta)^{N(G_{T-t}\psi,\delta,t)}\}, \qquad k = 1,2,\ (t,y) \in \Omega(\psi,\delta).$$

In view of the fact that $1 - (1-\theta)^n \le n\theta$ for all $n \ge 0$, (1.12) and (1.10) in Proposition 1.3 ensure that $u^{\varepsilon}_k(t,y,\theta)$ converges to 0 uniformly in $(t,y) \in \Omega(\psi,\delta)$ as $\varepsilon \to 0$ if θ is chosen to be sufficiently small (depending on ε and δ). More precisely, if

$$M_{\psi,\delta} := \sup_{\varphi:\,\rho(\varphi,\psi)<\delta}\ \max_{0\le t\le T} \int_{T-t}^T -L(\varphi(s),\dot\varphi(s))\,ds$$

and

$$\theta_{\varepsilon,\delta,\delta_1} := \exp\frac{-(M_{\psi,\delta}+\delta_1)}{\varepsilon}, \quad \text{where } \delta_1 \text{ is a positive number},$$

then we have

$$\beta_{\varepsilon,\delta,\delta_1} := \sup_{k=1,2,\,(t,y)\in\Omega(\psi,\delta)} u_k^\varepsilon(t,y,\theta_{\varepsilon,\delta,\delta_1}) \to 0 \quad \text{as } \varepsilon \to 0.$$

Our assumptions (A1) and (A2) ensure that $B > 0$ exists such that

$$\begin{aligned}\alpha_k(y)\Bigg[1 - u_k^\varepsilon(t,y,\theta_{\varepsilon,\delta,\delta_1}) \\ - \sum_{n_1,n_2=0}^{K} p_k(n_1,n_2,y)(1-u_1^\varepsilon)^{n_1}(1-u_2^\varepsilon)^{n_2}\Bigg] \\ \geq \left[c_{k,1}(y) - B\beta_{\varepsilon,\delta,\delta_1}\right]u_1^\varepsilon(t,y,\theta_{\varepsilon,\delta,\delta_1}) + \left[c_{k,2}(y) - B\beta_{\varepsilon,\delta,\delta_1}\right]u_2^\varepsilon,\end{aligned} \tag{1.13}$$

for all $(t,y) \in \Omega(\psi,\delta)$ for any fixed $\delta, \delta_1 > 0$ and sufficiently small ε.

By looking into the time when the original particle dies, it can be derived that $(u_1^\varepsilon(t,y,\theta), u_2^\varepsilon(t,y,\theta))$ satisfies the following differential equation (cf. [3], [10]–[12] and [17]):

$$\begin{aligned}\left[\frac{\partial}{\partial t} - \varepsilon D_k(y)\Delta/2\right]u_k^\varepsilon(t,y,\theta) \\ = \varepsilon^{-1}\alpha_k(y)\Bigg[(1-u_k^\varepsilon) \\ - \sum_{n_1,n_2} p_k(n_1,n_2,y)(1-u_1^\varepsilon)^{n_1}(1-u_2^\varepsilon)^{n_2}\Bigg], \\ k = 1,2 \text{ for } (t,y) \text{ in } \Omega(\psi,\delta),\end{aligned} \tag{1.14}$$

boundary value $\equiv 0$ and initial value $\equiv \theta$.

From (1.13), (1.14) and a comparison principle for parabolic equations, it follows that

$$u_k^\varepsilon(t,y,\theta_{\varepsilon,\delta,\delta_1}) \geq V_k^\varepsilon(t,y) \quad \text{for } k = 1,2, \tag{1.15}$$

where $(V_1^\varepsilon(t,y), V_2^\varepsilon(t,y))$ satisfies

$$\begin{aligned}\left[\frac{\partial}{\partial t} - \varepsilon D_k(y)\Delta/2\right]V_k^\varepsilon(t,y) \\ = \varepsilon^{-1}\left\{\left[c_{k,1}(y) - B\beta_{\varepsilon,\delta,\delta_1}\right]V_1^\varepsilon + \left[c_{k,2}(y) - B\beta_{\varepsilon,\delta,\delta_1}\right]V_2^\varepsilon\right\}, \\ k = 1,2 \text{ for } (t,y) \text{ in } \Omega(\psi,\delta),\end{aligned}$$

boundary value $\equiv 0$ and initial value $\equiv \theta_{\varepsilon,\delta,\delta_1}$.

In view of assumption (A1), $c_{1,2}(y) - B\beta_{\varepsilon,\delta,\delta_1}$ and $c_{2,1}(y) - B\beta_{\varepsilon,\delta,\delta_1}$ are uniformly positive as $\varepsilon \to 0$ and Proposition 1.3 is thus applicable to $V_k^\varepsilon(t,y)$. A simple computation shows

$$\begin{aligned}\lim_{\varepsilon\to 0}\varepsilon \ln V_k^\varepsilon(T,x) &= -(M_{\varphi,\delta}+\delta_1)\\ &\quad + \sup_{\varphi(0)=x,\ \rho(\varphi,\psi)<\delta}\int_0^T -L(\varphi(s),\dot\varphi(s))\,ds\\ &\geq -\delta_1 - M_{\varphi,\delta} + \int_0^T -L(\psi(s),\dot\psi(s))\,ds\\ &= -\delta_1 + \inf_{\varphi:\ \rho(\varphi,\psi)<\delta}\left[\min_{0\leq a\leq T}\int_a^T L(\varphi(s),\dot\varphi(s))\,ds\right]\\ &\quad + \int_0^T -L(\psi(s),\dot\psi(s))\,ds.\end{aligned}$$

Lemma 1.1 then implies

$$\begin{aligned}&\liminf_{\delta\to 0}\liminf_{\varepsilon\to 0}\varepsilon\ln V_k^\varepsilon(t,x)\\ &\quad\geq -\delta_1 + \min_{0\leq a\leq T}\int_a^T L(\psi(s),\dot\psi(s))\,ds\\ &\qquad + \int_0^T -L(\psi(s),\dot\psi(s))\,ds \qquad (1.16)\\ &\quad= -\delta_1 - \max_{0\leq a\leq T}\int_0^a L(\psi(s),\dot\psi(s))\,ds\\ &\quad= -\delta_1 - S(\psi)\quad \text{for } k=1,2.\end{aligned}$$

Since $P_{x,k}^\varepsilon\{N(\psi,\delta,T)\geq 1\}\geq E_{x,k}^\varepsilon\{1-(1-\theta)^{N(\psi,\delta,T)}\} := u_k^\varepsilon(T,x,\theta)$ for $0\leq\theta<1$, (1.15) and (1.16) yield the desired result upon letting δ_1 tend to 0. □

PROOF OF THEOREM 1. The lower semicontinuity of S is proved in Lemma 1.1. To prove the compactness of the level sets, we only need to show that $\{\varphi: S(\varphi)\leq A$ and $\varphi(0)=x\}$ is precompact in the uniform topology for each $A<\infty$. To prove this, recall $\overline{D} := \sup_{x\in\mathbb{R}^d,\ k=1,2} D_k(x)$ and define

$$\overline{C}_{i,j} := \sup_{x\in\mathbb{R}^d} c_{i,j}(x),\qquad i,j=1,2\ [\text{finite by assumptions (A1) and (A2)}],$$

$$\lambda := \text{the largest eigenvalue of the matrix } \overline{C} = \begin{bmatrix}\overline{C}_{1,1} & \overline{C}_{1,2}\\ \overline{C}_{2,1} & \overline{C}_{2,2}\end{bmatrix},$$

$$\begin{aligned}H^*(p) &:= \text{the largest eigenvalue of the matrix } \left(\overline{D}\frac{|p|^2}{2}\text{ identity} + \overline{C}\right)\\ &= \overline{D}\frac{|p|^2}{2}+\lambda,\qquad p\in\mathbb{R}^d,\end{aligned}$$

$$L^*(q) := \sup_{p\in\mathbb{R}^d}[q\cdot p - H^*(p)] = \frac{|q|^2}{2\overline{D}} - \lambda,\qquad q\in\mathbb{R}^d.$$

It follows easily that if φ is a.c., then

$$S(\varphi) = \int_0^T L(\varphi(s), \dot{\varphi}(s))\, ds \geq \int_0^T L^*(\dot{\varphi}(s))\, ds = \frac{1}{2\overline{D}} \int_0^T |\dot{\varphi}(s)|^2\, ds - T\lambda.$$

Thus $\{\varphi\colon\ S(\varphi) \leq A$ and $\varphi(0) = x\}$ is contained in $\{\varphi\colon\ (1/2\overline{D})\int_0^T |\dot{\varphi}(s)|^2\, ds \leq A + T\lambda$ and $\varphi(0) = x\}$, which is precompact in the uniform topology. The desired compactness is established and (1.1) is proved.

To prove the upper bound (1.2), note that

$$\begin{aligned}
&P^{\varepsilon}_{x,k}\{Z \text{ contains a branch } X \in C\}\\
&\quad \leq P^{\varepsilon}_{x,k}\{Z \text{ contains a branch } Y \text{ with } \|Y\|_{1/4} > A\}\\
&\qquad + P^{\varepsilon}_{x,k}\{Z \text{ contains a branch } X \in C \text{ with } \|X\|_{1/4} \leq A\}\\
&\quad := I^{\varepsilon}_1(A) + I^{\varepsilon}_2(A).
\end{aligned}$$

Thus

$$\begin{aligned}
&\limsup_{\varepsilon \to 0} \varepsilon \ln P^{\varepsilon}_{x,k}\{Z \text{ contains a branch } X \in C\}\\
&\quad \leq \max\left[\limsup_{\varepsilon \to 0} \varepsilon \ln I^{\varepsilon}_1(A),\ \limsup_{\varepsilon \to 0} \varepsilon \ln I^{\varepsilon}_2(A)\right].
\end{aligned} \tag{1.17}$$

By Lemma 1.2 and the fact that if C is closed, then $C \cap \{\varphi\colon\ \|\varphi\|_{1/4} \leq A\}$ is compact, (1.17) guarantees that to prove (1.2) it suffices to verify $\limsup_{\varepsilon \to 0} \varepsilon \ln P^{\varepsilon}_{x,k}\{Z$ contains a branch X in a compact subset $K\} \leq -\inf_{\varphi \in K} S(\varphi)$. This limiting inequality is a rather simple consequence of Lemma 1.4. The lower bound (1.3) follows easily from Lemma 1.5. □

2. Large deviations for the number of branches. To be precise, Conjecture 2 consists of the following two parts [again assume $S(\psi) < \infty$ and $\psi(0) = x$]:

If $A > \max_{0 \leq a \leq T} \int_a^T -L(\psi(s), \dot{\psi}(s))\, ds$, then

$$\limsup_{\delta \to 0} \limsup_{\varepsilon \to 0} \varepsilon \ln P^{\varepsilon}_{x,k}\left\{N(\psi, \delta, T) \geq \exp \frac{A}{\varepsilon} \,\middle|\, N(\psi, \delta, T) \geq 1\right\} < 0.$$

If $A < \max_{0 \leq a \leq T} \int_a^T -L(\psi(s), \dot{\psi}(s))\, ds$, then

$$\limsup_{\delta \to 0} \limsup_{\varepsilon \to 0} \varepsilon \ln P^{\varepsilon}_{x,k}\left\{N(\psi, \delta, T) \leq \exp \frac{A}{\varepsilon} \,\middle|\, N(\psi, \delta, T) \geq 1\right\} < 0.$$

In Theorem 2 we shall prove that the first limit is in fact $-\infty$. The second limit will be calculated elsewhere.

THEOREM 2. *For $S(\psi) < \infty$ and $\psi(0) = x$, we have the following*:

$$\limsup_{\delta\to 0}\limsup_{\varepsilon\to 0} \varepsilon \ln E^{\varepsilon}_{x,k}\{N(\psi,\delta,T)^n\} \le -S(\psi) + n \max_{0\le a\le T}\int_a^T - L(\psi(s),\dot\psi(s))\,ds \quad \textit{for each } n \ge 1. \tag{2.1}$$

If $A > \max_{0\le a\le T}\int_a^T - L(\psi(s),\dot\psi(s))\,ds$, *then*

$$\limsup_{\delta\to 0}\limsup_{\varepsilon\to 0} \varepsilon \ln P^{\varepsilon}_{x,k}\left\{N(\psi,\delta,T) \ge \exp\frac{A}{\varepsilon}\,\middle|\, N(\psi,\delta,T) \ge 1\right\} = -\infty. \tag{2.2}$$

PROOF. For notational simplicity we first prove this lemma for the process Q^{ε}_x defined in the Introduction. Modifications needed for the general case will be pointed out in the end.

Let $n!w^{\varepsilon}_n(t,y)$ denote the nth moment of $N(G_{T-t}\psi,\delta,t)$, that is,

$$w^{\varepsilon}_n(t,y) := E^{\varepsilon}_y\{N(G_{T-t}\psi,\delta,t)^n\}, \qquad n \in N,$$

where E^{ε}_y is the expectation associated with Q^{ε}_y. Denote by $v^{\varepsilon}(t,y,\lambda)$ the moment generating function

$$v^{\varepsilon}(t,y,\lambda) := E^{\varepsilon}_y\{\exp[-\lambda N(G_{T-t}\psi,\delta,t)]\}, \qquad \lambda \ge 0.$$

We have the following equation for $v^{\varepsilon}(t,y,\lambda)$ (cf. [3], [10]–[12] and [17]):

$$\left(\frac{\partial}{\partial t} - \frac{\varepsilon\Delta}{2}\right)v^{\varepsilon} = \frac{c}{\varepsilon}\left[(v^{\varepsilon})^2 - v^{\varepsilon}\right] \quad \text{in } \Omega(\psi,\delta),$$

$$\text{boundary value} \equiv 1, \qquad \text{initial value} \equiv \exp(-\lambda).$$

By this differential equation and the formal expansion

$$v^{\varepsilon}(t,y,\lambda) := 1 + \sum_{n=1}^{\infty}(-1)^n w^{\varepsilon}_n(t,y)\lambda^n,$$

one arrives at the following sequence of formal differential equations (cf. [13]):

$$\left(\frac{\partial}{\partial t} - \frac{\varepsilon\Delta}{2}\right)w^{\varepsilon}_1 = \frac{c}{\varepsilon}w^{\varepsilon}_1 \quad \text{in } \Omega(\psi,\delta), \tag{2.3}$$

$$\text{boundary value} \equiv 0, \qquad \text{initial value} \equiv 1.$$

For $n \ge 2$,

$$\left(\frac{\partial}{\partial t} - \frac{\varepsilon\Delta}{2}\right)w^{\varepsilon}_n = \frac{c}{\varepsilon}w^{\varepsilon}_n + \frac{c}{\varepsilon}\sum_{i=1}^{n-1} w^{\varepsilon}_i w^{\varepsilon}_{n-i} \quad \text{in } \Omega(\psi,\delta), \tag{2.4}$$

$$\text{boundary value} \equiv 0, \qquad \text{initial value} \equiv 1.$$

For $n \ge 1$ particles existent at time t, we shall call their ancestral paths $(\varphi_1,\varphi_2,\dots,\varphi_n) := \varphi^n$ an n-branch during the time interval $[0,t]$. Denote by $C_{t,n}$ the set of n-branches during $[0,t]$. Define the splitting time Γ: $C_{t,2} \to [0,t]$

and the function $F_{t,n}\colon C_{t,n} \to (-\infty, \infty]$ as

$$\Gamma(\varphi_1, \varphi_2) := \max\{b\colon 0 \le b \le t \text{ and } \varphi_1(s) = \varphi_2(s) \text{ for } 0 \le s \le b\},$$

$$\begin{aligned} F_{t,n}(\varphi^n) := \int_0^t - L(\varphi_1(s), \dot{\varphi}(s))\, ds + \int_{\Gamma(\varphi_2, \varphi_1)}^t - L(\varphi_2(s), \dot{\varphi}_2(s))\, ds \\ + \int_{\max[\Gamma(\varphi_3, \varphi_1), \Gamma(\varphi_3, \varphi_2)]}^t - L(\varphi_3(s), \dot{\varphi}_3(s))\, ds + \cdots \\ + \int_{\max[\Gamma(\varphi_n, \varphi_1), \dots, \Gamma(\varphi_n, \varphi_{n-1})]}^t - L(\varphi_n(s), \dot{\varphi}_n(s))\, ds. \end{aligned}$$

It is easy to check that $F_{t,n}$ is symmetric with respect to $\varphi_1, \dots, \varphi_n$. We claim that:

(2.5) There exist $f_1(\varepsilon), f_2(\varepsilon), \dots,$ independent of $(t, y) \in \Omega(\psi, \delta)$ and $\lim_{\varepsilon \to 0} f_n(\varepsilon) = 0$ for each $n \ge 1$, such that $\varepsilon \ln w_n^\varepsilon(t, y) \le f_n(\varepsilon) + \sup\{F_{t,n}(\varphi^n)\colon (t - s, \varphi_j(s)) \in \Omega(\psi, \delta)$ for $0 \le s < t$ and $\varphi_j(0) = y$ for $1 \le j \le n\}$.

Recall that $-L(x, q) = c - \frac{1}{2}|q|^2$ in the case of Q_x^ε. (2.1) follows from the assertion (2.5) by the following computation:

$$\begin{aligned} &\limsup_{\delta \to 0} \limsup_{\varepsilon \to 0} \varepsilon \ln w_n^\varepsilon(T, x) \\ &\le \limsup_{\delta \to 0} \sup_{\varphi^n \in C_{T,n}} \big\{F_{T,n}(\varphi^n)\colon \big(T - s, \phi_j(s)\big) \in \Omega(\psi, \delta) \text{ for } 0 \le s < T \\ &\qquad\qquad \text{and } \varphi_j(0) = x \text{ for } 1 \le j \le n\big\} \\ &\le \limsup_{\delta \to 0} \Bigg[\sup_{\varphi_1} \bigg\{ \int_0^T - L\big(\varphi_1(s), \dot{\varphi}_1(s)\big)\, ds\colon \\ &\qquad\qquad (T - s, \varphi_1(s)) \in \Omega(\psi, \delta) \text{ for } 0 \le s < T \bigg\} \\ &\qquad + \sum_{j=2}^n \sup_{\varphi_j, a} \bigg\{ \int_a^T - L\big(\varphi_j(s), \dot{\varphi}_j(s)\big)\, ds\colon \\ &\qquad\qquad \big(T - s, \varphi_j(s)\big) \in \Omega(\psi, \delta) \text{ for } 0 \le s < T \bigg\} \Bigg], \end{aligned}$$

which by Lemma 1.1 equals

$$\begin{aligned} &\int_0^T - L\big(\psi(s), \dot{\psi}(s)\big)\, ds + (n - 1) \max_{0 \le a \le T} \int_a^T - L\big(\psi(s), \dot{\psi}(s)\big)\, ds \\ &= \min_{0 \le a \le T} \int_0^a - L\big(\psi(s), \dot{\psi}(s)\big)\, ds + n \max_{0 \le a \le T} \int_a^T - L\big(\psi(s), \dot{\psi}(s)\big)\, ds \\ &= -S(\psi) + n \max_{0 \le a \le T} \int_a^T - L\big(\psi(s), \dot{\psi}(s)\big)\, ds. \end{aligned}$$

Next, we prove the assertion (2.5) by induction on n. For the problem (2.3) the following Feynman–Kac formula holds:

$$w_1^\varepsilon(t,y) = M_y^\varepsilon\left\{\exp\left[\frac{1}{\varepsilon}\int_0^t c\,ds\right]; (t-s,\zeta_s) \in \Omega(\psi,\delta) \text{ for } 0 \le s < t\right\},$$

where M_y^ε denotes the expectation with reference to $\varepsilon^{1/2}$(BM) starting from y and ζ_s denotes the sample path. A Laplace asymptotic method then shows

$$\begin{aligned}\limsup_{\varepsilon\to 0} \varepsilon \ln w_1^\varepsilon(t,y) &\le \sup\left\{\int_0^t \left(c - \tfrac{1}{2}|\dot\varphi(s)|^2\right) ds : (t-s,\dot\varphi(s)) \in \Omega(\psi,\delta)\right.\\ &\qquad\qquad \left.\text{for } 0 \le s < t \text{ and } \varphi(0) = y\right\}\\ &= \sup\left\{\int_0^t - L(\varphi(s),\dot\varphi(s))\,ds : (t-s,\varphi(s)) \in \Omega(\psi,\delta)\right.\\ &\qquad\qquad \left.\text{for } 0 \le s < t \text{ and } \varphi(0) = y\right\}.\end{aligned}$$

By the local uniformity property of the large deviations for small BM, it is readily checked that (2.5) holds for $n = 1$.

The solution $w_n^\varepsilon(t,y)$ of the problem (2.4) admits the following Feynman–Kac representation:

$$\begin{aligned}w_n^\varepsilon(t,y) &= w_1^\varepsilon(t,y)\\ &\quad + \sum_{i=1}^{n-1} M_y^\varepsilon\left\{\int_0^{\tau_t[\zeta]} w_i^\varepsilon(t-s,\zeta_s) w_{n-i}^\varepsilon(t-s,\zeta_s) \exp\left[\frac{1}{\varepsilon}\int_0^s c\,db\right] ds\right\},\end{aligned}$$

where $\tau_t[\zeta] := \min\{s: (t-s,\zeta_s) \notin \Omega(\psi,\delta)\}$.

Applying the smaller-n cases of (2.5), the Laplace asymptotic formula gives

$$\begin{aligned}\limsup_{\varepsilon\to 0} \varepsilon \ln w_n^\varepsilon(t,y) \le \sup\bigl\{&F_{t,n}(\varphi^n) : \bigl(t-s,\varphi_j(s)\bigr) \in \Omega(\psi,\delta)\\ &\text{for } 0 \le s < t,\ \varphi_j(0) = y \text{ for } 1 \le j \le n\bigr\}.\end{aligned}$$

Again, the local uniformity property implies (2.5) and ends the induction. (2.1) is completely proved in the case of Q_x^ε.

For the general branching diffusions $P_{x,k}^\varepsilon$, we consider a related Markov diffusion–transmutation process $(\zeta_s^\varepsilon, \nu_s^\varepsilon)$ uniquely characterized by

> $d\zeta_t^\varepsilon = \varepsilon^{1/2} D_{\nu_t^\varepsilon}(\zeta_t^\varepsilon)^{1/2}\,dB_t$, where B_t is a d-dimensional BM, ν_s^ε is $\{1,2\}$-valued with $\operatorname{Prob}\{\nu_{t+b}^\varepsilon = l \mid \zeta_t^\varepsilon = y$ and $\nu_t^\varepsilon = m\} = \dfrac{c_{m,l}(y)}{\varepsilon} b + o(b)$ as $b \to 0$ for $(m,l) = (1,2)$ or $(2,1)$, $\zeta_0^\varepsilon = x$ and $\nu_0^\varepsilon = k$.

Denote the corresponding expectation by $M_{x,k}^\varepsilon$. The modification needed consists in replacing $\varepsilon^{1/2}$(BM) by the diffusion–transmutation process $(\zeta_t^\varepsilon, \nu_t^\varepsilon)$.

The large-deviation action functional for $(\zeta_t^\varepsilon, \nu_t^\varepsilon)$ as $\varepsilon \to 0$ is derived in [8]. Correspondingly, we define

$$w_{k,n}^\varepsilon(t,y) := \frac{1}{n!} E_{y,k}^\varepsilon\{N(G_{T-t}\psi, \delta, t)^n\}, \qquad n \in \mathbb{N}.$$

The differential equations for $w_{k,n}^\varepsilon$ are

$$\left(\frac{\partial}{\partial t} - \varepsilon D_k(y)\frac{\Delta}{2}\right) w_{k,1}^\varepsilon(t,y)$$
$$= \varepsilon^{-1}[c_{k,1}(y) w_{k,1}^\varepsilon + c_{k,2}(y) w_{k,2}^\varepsilon] \quad \text{for } k = 1,2 \text{ in } \Omega(\psi,\delta),$$
$$\text{boundary values} \equiv 0, \qquad \text{initial values} \equiv 1,$$

and for $n \geq 2$,

$$\left(\frac{\partial}{\partial t} - \varepsilon D_k(y)\Delta/2\right) w_{k,n}^\varepsilon(t,y)$$
$$= \varepsilon^{-1}[c_{k,1}(y) w_{k,1}^\varepsilon + c_{k,2}(y) w_{k,2}^\varepsilon]$$
$$+ G_n(y, w_{1,1}^\varepsilon, \dots, w_{1,n-1}^\varepsilon, w_{2,1}^\varepsilon, \dots, w_{2,n-1}^\varepsilon) \quad \text{for } k=1,2 \text{ in } \Omega(\psi,\delta),$$
$$\text{boundary values} \equiv 0, \qquad \text{initial values} \equiv 1,$$

where G_n is such that $G_n(y, w_{1,1}^\varepsilon\lambda, \dots, w_{1,n-1}^\varepsilon\lambda^{n-1}, w_{2,1}^\varepsilon\lambda, \dots, w_{2,n-1}^\varepsilon\lambda^{n-1})$ is a homogeneous polynomial in λ of order n for each $y \in \mathbb{R}^d$. For $w_{k,1}^\varepsilon(t,y)$ the following Feynman–Kac formula is valid:

$$w_{k,1}^\varepsilon(t,y) = M_{y,k}^\varepsilon\left\{\exp\left[\frac{1}{\varepsilon}\int_0^t \left(c_{\nu_s,1}(\zeta_s) + c_{\nu_s,2}(\zeta_s)\right) ds\right]; \right.$$
$$\left. (t-s, \zeta_s) \in \Omega(\psi,\delta) \text{ for } 0 \leq s < t\right\}.$$

A Laplace asymptotic method shows that

$$\lim_{\varepsilon \to 0} \varepsilon \ln w_{k,1}^\varepsilon(T,x)$$
$$\leq \sup\left\{\int_0^T - L(\varphi(s), \dot\varphi(s))\, ds \colon (T-s, \varphi(s)) \in \Omega(\psi,\delta)\right.$$
$$\left. \text{for } 0 \leq s < t \text{ and } \varphi(0) = x\right\}.$$

This result, together with the needed local uniformity property for large deviations, has been stated in Proposition 1.3. Since the same kind of arguments as for M_y^ε works nicely for $M_{y,k}^\varepsilon$, we omit the details.

By (2.1), the Chebyshev inequality and Lemma 1.5, we get

$$\limsup_{\delta \to 0} \limsup_{\varepsilon \to 0} \varepsilon \ln P_{x,k}^\varepsilon\left\{N(\psi,\delta,T) \geq \exp\frac{A}{\varepsilon} \,\middle|\, N(\psi,\delta,T) \geq 1\right\}$$
$$\leq n\left[\max_{0 \leq a \leq T}\int_a^T - L(\psi(s), \dot\psi(s))\, ds - A\right].$$

Letting n tend to ∞ yields (2.2) and completes the proof. □

3. Large deviations for the 2-branches. Recall that a 2-branch is an element of $C(\mathbb{R}^d)^2$. Under the process $P^{\varepsilon}_{x,k}$, a 2-branch $X^2 = (X_1, X_2)$ satisfies $X_1(0) = x = X_2(0)$ a.s. We introduce two functionals Γ and Θ: $C(\mathbb{R}^d)^2 \to [0, T]$ and a functional S_2: $C(\mathbb{R}^d)^2 \to [0, \infty]$:

$$\Gamma\varphi^2 := \max\{a : 0 \le a \le T,\ \varphi_1(t) = \varphi_2(t) \text{ for } 0 \le t \le a\},$$

where $\varphi^2 := (\varphi_1, \varphi_2)$. ΓX^2 means the splitting time of X_1 and X_2, that is, the death time of the most recent ancestor of the branches X_1 and X_2:

$$\Theta\varphi^2 := \max\left\{b : 0 \le b \le \Gamma\varphi^2,\ \int_0^b L(\varphi_1(s), \dot\varphi_1(s))\,ds = \max_{0 \le a \le \Gamma\varphi^2} \int_0^a L(\varphi_1(s), \dot\varphi_1(s))\,ds\right\}. \tag{3.1}$$

Action functional for the 2-branches:

$$S_2(\varphi^2) := \begin{cases} \displaystyle\int_0^{\Theta\varphi^2} L(\varphi_1(s), \dot\varphi_1(s))\,ds \\ \displaystyle\quad + \sum_{i=1}^{2} \max\left[0, \max_{\Gamma\varphi^2 \le a \le T} \int_{\Theta\varphi^2}^{a} L(\varphi_i(s), \dot\varphi_i(s))\,ds\right], & \text{if } \varphi^2 := (\varphi_1, \varphi_2) \text{ is a.c.,} \\ +\infty, & \text{otherwise.} \end{cases}$$

Let $\theta := \Theta\varphi^2$. The following formula is easy to check:

$$S_2(\varphi^2) = S(\varphi_1, \Gamma\varphi^2) + \sum_{i=1}^{2} S(G_\theta\varphi_i, T - \theta) = S(\varphi_1, \theta) + \sum_{i=1}^{2} S(G_\theta\varphi_i, T - \theta), \tag{3.2}$$

where $S(\varphi, b)$ is defined as $S(\varphi)$ in Section 1 but with T replaced by b and $(G_\theta\varphi)(s) := \varphi(\theta + s)$.

We impose on $C(\mathbb{R}^d)^2$ the topology induced by the metric ρ_2:

$$\rho_2(\varphi^2, \psi^2) := \min\{\max[\rho(\varphi_1, \psi_1), \rho(\varphi_2, \psi_2)], \max[\rho(\varphi_1, \psi_2), \rho(\varphi_2, \psi_1)]\}.$$

ρ_2 gives a uniform topology on the symmetrized space $C(\mathbb{R}^d)^2/\{(\varphi_1, \varphi_2) = (\varphi_2, \varphi_1)\}$.

If the particle does not multiply under the branching mechanism, then the probability that the sample tree Z contains a 2-branch is 0. To anticipate a nontrivial large-deviation phenomenon for the 2-branches, it is therefore natural to assume the following multiplying condition:

(3.3) For each $k = 1, 2$, $x \in \mathbb{R}^d$, there exist n_1, n_2 such that $n_1 + n_2 \ge 2$ and $p_k(n_1, n_2, x) > 0$.

THEOREM 3. *If condition* (3.3) *is satisfied in addition to assumptions* (A1) *and* (A2), *then* $\varepsilon^{-1}S_2$ *is the action functional for the 2-branches of the family of branching diffusions* $P^{\varepsilon}_{x,k}$, $\varepsilon \to 0$, $k = 1, 2$. *This statement means the following in the* ρ_2 *topology*:

(3.4) S_2 *is lower semicontinuous and the level set* $\{\varphi^2\colon S_2(\varphi^2) \le A$ *and* $\varphi^2(0) = (x, x)\}$ *is compact for each* $0 \le A < \infty$ *and* $x \in \mathbb{R}^d$.

(3.5) *If* C *is a closed subset of* $C(\mathbb{R}^d)^2 \cap \{\varphi^2\colon \varphi^2(0) = (x,x)\}$, *then* $\limsup_{\varepsilon\to 0} \varepsilon \ln P^{\varepsilon}_{x,k}\{$*the sample tree* Z *contains a* 2-*branch in* $C\} \le -\inf_{\varphi^2 \in C} S_2(\varphi^2)$ *for* $k = 1, 2$.

(3.6) *If* G *is an open subset of* $C(\mathbb{R}^d)^2 \cap \{\varphi^2\colon \varphi^2(0) = (x,x)\}$, *then* $\liminf_{\varepsilon\to 0} \varepsilon \ln P^{\varepsilon}_{x,k}\{$*the sample tree* Z *contains a* 2-*branch in* $G\} \ge -\inf_{\varphi^2 \in G} S_2(\varphi^2)$ *for* $k = 1, 2$.

PROOF. From (3.2) and Lemma 1.1, we see that S_2 is the sum of three lower semicontinuous (in the ρ topology) functions, which is readily checked to be lower semicontinuous in the ρ_2 topology. To verify the property of compact levels, it suffices to prove that, for each A, $\Lambda = \Lambda(A, x) := \{\varphi^2\colon S_2(\varphi^2) \le A$ and $\varphi^2(0) = (x, x)\}$ is precompact. Recall that $\overline{D} := \sup_{k=1,2,\, x\in\mathbb{R}^d} D_k(x)$. From the definition of S_2, we have

$$S_2(\varphi^2) \ge \frac{1}{2\overline{D}}\left[\int_0^{\Theta\varphi^2} |\dot{\varphi}_1(s)|^2\, ds + \sum_{i=1}^{2} \int_{\Theta\varphi^2}^{T} |\dot{\varphi}_i(s)|^2\, ds\right] - 2T\lambda$$

$$\ge \frac{1}{2\overline{D}} \int_0^T |\dot{\varphi}_i(s)|^2\, ds - 2T\lambda \quad \text{for } i = 1, 2, \text{ where } (\varphi_1, \varphi_2) = \varphi^2.$$

So,

$$\Lambda \subset \left\{\varphi^2\colon \varphi_1(0) = x = \varphi_2(0),\ \int_0^T |\dot{\varphi}_i(s)|^2\, ds \le 2\overline{D}(2T\lambda + A),\ i = 1, 2\right\},$$

which clearly is precompact. (3.4) is proved.

To prove the upper bound estimate (3.5), we use Lemma 1.2 and the same kind of argument as used in the proof of Theorem 1. It turns out that we only need to show

$$\limsup_{\delta\to 0} \limsup_{\varepsilon\to 0} \varepsilon \ln P^{\varepsilon}_{x,k}\{Z \text{ contains a 2-branch } X^2 \text{ with } \rho_2(X^2, \psi^2) < \delta\} \le -S_2(\psi^2) \quad \text{for } S_2(\psi^2) < \infty. \tag{3.7}$$

From now on we fix a ψ^2 with $S_2(\psi^2) < \infty$ and $\psi^2(0) = (x, x)$. Let $\theta := \Theta\psi^2$.

For each $A > 0$ we have

$$
\begin{aligned}
&P^{\varepsilon}_{x,k}\{Z \text{ contains } X^2 \text{ with } \rho_2(X^2,\psi^2) < \delta\} \\
&\quad \le P^{\varepsilon}_{x,k}\left\{N(\psi_1,\delta,\theta) \ge \exp\frac{A}{\varepsilon}\right\} \\
&\qquad + P^{\varepsilon}_{x,k}\Big\{1 \le N(\psi_1,\delta,\theta) < \exp\frac{A}{\varepsilon}, \text{ at least one of these branches} \\
&\qquad\qquad \text{stays in the } \delta\text{-neighborhood of } \psi_1 \text{ and at least one in the} \\
&\qquad\qquad \delta\text{-neighborhood of } \psi_2 \text{ during the time interval } [\theta, T]\Big\} \\
&\quad := B_1(\varepsilon,\delta,A) + B_2(\varepsilon,\delta,A).
\end{aligned}
\tag{3.8}
$$

From Theorem 2 (T replaced by θ) and the fact that, by (3.1),

$$
\begin{aligned}
&\max_{0\le a\le\theta}\int_a^\theta -L(\psi_1(s),\dot\psi_1(s))\,ds \\
&\quad = \left[\max_{0\le a\le\theta}\int_0^a L(\psi_1(s),\dot\psi_1(s))\,ds\right] - \int_0^\theta L(\psi_1(s),\dot\psi_1(s))\,ds \\
&\quad = \max_{0\le a\le\Gamma\psi^2}\int_0^a L(\psi_1(s),\dot\psi_1(s))\,ds \\
&\qquad - \max_{0\le a\le\Gamma\psi^2}\int_0^a L(\psi_1(s),\dot\psi_1(s))\,ds = 0,
\end{aligned}
$$

we get

$$
\limsup_{\delta\to 0}\limsup_{\varepsilon\to 0}\varepsilon\ln B_1(\varepsilon,\delta,A) = -\infty \quad \text{for each } A > 0. \tag{3.9}
$$

Next, we estimate $B_2(\varepsilon,\delta,A)$:

$$
\begin{aligned}
B_2(\varepsilon,\delta,A) &\le P^{\varepsilon}_{x,k}\{N(\psi_1,\delta,\theta)\ge 1\} \\
&\quad \times \prod_{i=1}^{2}\left(\exp\frac{A}{\varepsilon}\right)\sup_{|y-\psi_1(\theta)|<\delta,\ m=1,2} P^{\varepsilon}_{y,m}\{N(G_\theta\psi_i,\delta,T-\theta)\ge 1\}.
\end{aligned}
$$

From its proof Lemma 1.4 is readily checked to hold with the following uniformity property:

$$
\begin{aligned}
&\limsup_{\delta\to 0}\limsup_{\varepsilon\to 0}\varepsilon\ln\sup_{|y-\psi_1(\theta)|<\delta,\ m=1,2} P^{\varepsilon}_{y,m}\{N(G_\theta\psi_i,\delta,T-\theta)\ge 1\} \\
&\quad \le -S(G_\theta\psi_i, T-\theta).
\end{aligned}
$$

This implies

$$
\begin{aligned}
&\limsup_{\delta\to 0}\limsup_{\varepsilon\to 0}\varepsilon\ln B_2(\varepsilon,\delta,A) \\
&\quad \le -S(\psi_1,\theta) + 2A - \sum_{i=1}^{2} S(G_\theta\psi_i, T-\theta),
\end{aligned}
\tag{3.10}
$$

which by (3.2) equals $2A - S_2(\psi^2)$. (3.7) follows from (3.8)–(3.10) and from letting A tend to 0. The upper bound (3.5) is established.

The lower estimate (3.6) is equivalent to

$$\begin{aligned}
&\liminf_{\delta\to 0}\liminf_{\varepsilon\to 0} \varepsilon \ln P^{\varepsilon}_{x,k}\{Z \text{ contains a 2-branch } X^2 \\
&\qquad\qquad\qquad\qquad \text{with } \rho_2(X^2,\psi^2) < \delta\} \\
&\quad \geq -S_2(\psi^2) \text{ for } \psi^2 \text{ such that } S_2(\psi^2) < \infty \text{ and } x \in \mathbb{R}^d.
\end{aligned} \tag{3.11}$$

This limiting inequality is easy to derive for $\theta = 0$ or T. We omit the details. To prove this for $0 < \theta < T$, first observe that

$$\begin{aligned}
&P^{\varepsilon}_{x,k}\{Z \text{ contains } X^2 \text{ with } \rho_2(X^2,\psi^2) < \delta\} \\
&\quad \geq P^{\varepsilon}_{x,k}\{N(\psi_1,\gamma,\theta) \geq 2\} \\
&\qquad \times \prod_{i=1}^{2} \inf_{|y-\psi_1(\theta)|<\gamma,\ m=1,2} P^{\varepsilon}_{y,m}\{N(G_\theta\psi_i,\delta,T-\theta) \geq 1\}, \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{for any } 0 < \gamma < \delta \\
&\quad := I(\varepsilon,\gamma)\prod_{i=1}^{2} J_i(\varepsilon,\gamma).
\end{aligned} \tag{3.12}$$

By (1.3) in Theorem 1 we have

$$\begin{aligned}
&\liminf_{\varepsilon\to 0} \varepsilon \ln P^{\varepsilon}_{y,m}\{N(G_\theta\psi_i,\delta,T-\theta) \geq 1\} \\
&\quad \geq -S(y - \psi_1(\theta) + G_\theta\psi_1, T-\theta) \quad \text{for } |y - \psi_1(\theta)| < \delta.
\end{aligned}$$

From the proof of Theorem 1 the following local uniformity is readily checked. There exists $0 < \gamma(\delta) < \delta$ such that

$$\begin{aligned}
&\liminf_{\varepsilon\to 0} \varepsilon \ln J_i(\varepsilon,\gamma(\delta)) \\
&\quad \geq -\sup_{|y-\psi_1(\theta)|\leq\gamma(\delta)} S(y - \psi_1(\theta) + G_\theta\psi_i, T-\theta) \quad \text{for } i = 1,2.
\end{aligned} \tag{3.13}$$

By (1.3) in Theorem 1 and the multiplying condition (3.3), it is easy to see that

$$\liminf_{\varepsilon\to 0} I(\varepsilon,\gamma(\delta)) \geq -S(\psi_1,\theta). \tag{3.14}$$

The limiting inequality (3.11) now follows from (3.12)–(3.14) and the continuity property of $D_k(x)$ and $c_{k,m}(x)$, $k, m = 1,2$. This ends the lower estimate and Theorem 3. □

4. Some remarks. Our results allow us to evaluate many asymptotic probabilities.

EXAMPLE 1.

$$\lim_{\varepsilon\to 0}\varepsilon\ln P^{\varepsilon}_{0,k}\Big\{\text{the sample tree } Z \text{ contains a branch } X \text{ with } \int_0^T |X_s|^p\,ds > 1\Big\} = -\inf\Big\{S(\varphi)\colon \int_0^T|\psi(s)|^p\,ds > 1 \text{ and } \varphi(0)=0\Big\}.$$

EXAMPLE 2.

$$\lim_{\varepsilon\to 0}\varepsilon\ln P^{\varepsilon}_{x,k}\Big\{Z \text{ contains a 2-branch } X^2 := (X_1, X_2) \text{ with diameter } \max_{k,m=1,2,\,0\le s,t\le T}|X_k(s)-X_m(t)|>1\Big\}$$
$$= -\inf\Big\{S_2(\varphi^2)\colon \varphi^2 := (\varphi_1,\varphi_2),\ \max_{k,m=1,2,\,0\le s,t\le T}|\varphi_k(s)-\varphi_m(t)|>1 \text{ and } \varphi^2(0)=(x,x)\Big\}.$$

EXAMPLE 3.

$$\lim_{\varepsilon\to 0}\varepsilon\ln P^{\varepsilon}_{x,k}\{\text{there are some particles in the domain } A\subset\mathbb{R}^d \text{ at time } T\}$$
$$= -\inf\{S(\varphi)\colon \varphi(0)=x \text{ and } \varphi(T)\in A\}$$
$$= -\inf_{\varphi\colon \varphi(0)=x,\,\varphi(T)\in A}\ \max_{0\le a\le T}\int_0^a L(\varphi(s),\dot\varphi(s))\,ds.$$

EXAMPLE 4.

$$\lim_{\varepsilon\to 0}\varepsilon\ln P^{\varepsilon}_{x,k}\{\text{there are some particles in } A \text{ and some in } B \text{ at time } T\}$$
$$= -\inf\{S_2(\varphi^2)\colon \varphi^2(0)=(x,x),\ \varphi_1(T)\in A \text{ and } \varphi_2(T)\in B\},$$

where A and B are two disjoint domains.

By the fact that $u^{\varepsilon}_k(t,x,A) := P^{\varepsilon}_{x,k}\{\text{there are some particles in } A \text{ at time } t\}$ is the the solution of the system of reaction–diffusion equations

$$\Big(\frac{\partial}{\partial t}-\varepsilon D_k(y)\Delta/2\Big)u^{\varepsilon}_k(t,x,A) = \varepsilon^{-1}\alpha_k(x)\Big[(1-u^{\varepsilon}_k)-\sum_{n_1,n_2}p_k(n_1,n_2,x)(1-u^{\varepsilon}_1)^{n_1}(1-u^{\varepsilon}_2)^{n_2}\Big] \quad (4.1)$$

for $k=1,2$ in $(0,\infty)\times\mathbb{R}^d$,

with

$$u^{\varepsilon}_k(0,x,A)=\chi_A(x),\quad \text{the index function of } A, \text{ for } k=1,2,\ x\in\mathbb{R}^d, \quad (4.2)$$

we have the following PDE interpretation of Examples 3 and 4:

3′. $$\lim_{\varepsilon\to 0} \varepsilon \ln u_k^\varepsilon(t,x) = -\inf_{\varphi:\,\varphi(0)=x,\,\varphi(T)\in A}\ \max_{0\le a\le t}\int_0^a L(\varphi(s),\dot\varphi(s))\,ds$$

for $k = 1, 2$, $t > 0$, $x \in \mathbb{R}^d$.

4′. $$\lim_{\varepsilon\to 0}\varepsilon \ln[u_k^\varepsilon(t,x,A) + u_k^\varepsilon(t,x,B) - u_k^\varepsilon(t,x,A\cup B)] = -\inf_{\varphi^2:\,\varphi^2(0)=(x,x),\,\varphi_1(t)\in A,\,\varphi_2(t)\in B} S_2(\varphi^2).$$

Statement 3′ is closely connected with the phenomenon of wave front propagation in the reaction–diffusion system (4.1) as $\varepsilon \to 0$, studied by many authors (cf. [1], [5]). Statement 3′ distinguishes itself from [1] and [5] in that it considers space-nonhomogeneous systems and gives a new formula for the wave fronts. For wave front propagation in the case of one reaction–diffusion equation, see, for example, [4], [6] and [7]; in the case of a weakly coupled reaction–diffusion system, see [7].

With minor modifications, one can generalize our results to incorporate small diffusions with drift (corresponding to the elliptic operators $\varepsilon\sum_{i,j=1}^d a_k^{i,j}(x)\partial^2/\partial x_i\partial x_j + \sum_{i=1}^d b_k^i(x)\partial/\partial x_i$, $k = 1, 2$), more than two particle types and n-branches, $n \ge 3$. It is also possible to adopt the Hölder norm of any order $\theta < \frac{1}{2}$ rather than the uniform topology.

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