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A NOTE ON THE CONVERGENCE OF SUMS OF INDEPENDENT RANDOM VARIABLES[1]

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Let X_n, $n \geq 1$, be a sequence of independent random variables, and let F_N be the distribution function of the partial sums $\sum_{n=1}^{N} X_n$. Motivated by a conjecture of Erdös in probabilistic number theory, we investigate conditions under which the convergence of $F_N(x)$ at two points $x = x_1, x_2$ with different limit values already implies the weak convergence of the distributions F_N. We show that this is the case if $\sum_{n=1}^{\infty} \rho(X_n, c_n) = \infty$ whenever $\sum_{n=1}^{\infty} c_n$ diverges, where $\rho(X, c)$ denotes the Levy distance between X and the constant random variable c. In particular, this condition is satisfied if $\liminf_{n \to \infty} P(X_n = 0) > 0$.

1. Introduction. A function $f\colon \mathbf{N} \to \mathbf{R}$ is called additive if $f(nm) = f(n) + f(m)$ for any coprime integers n and m. Given an additive function f, one can define, for each $N \in \mathbf{N}$, a distribution function

$$F_N(x) = \frac{1}{N} \#\{n \leq N\colon f(n) \leq x\} \tag{1.1}$$

and investigate the behavior of F_N as $N \to \infty$. An old conjecture of Erdös, stated as Problem 1 in Elliott [3] (page 330), asserts that in order for the sequence F_N to be (weakly) convergent, it is sufficient that there exist two numbers $x_1 < x_2$ such that

$$\lim_{N \to \infty} \left(F_N(x_2) - F_N(x_1) \right) \quad \text{exists and is positive.} \tag{1.2}$$

As noted in Elliott [3] (page 331), standard techniques from probabilistic number theory show that this conjecture is equivalent to the following purely probabilistic statement.

Let a_n, $n \geq 1$, be a sequence of real numbers and let X_n be a sequence of independent random variables assuming the values a_n and 0 with probabilities $1/p_n$ and $1 - 1/p_n$, respectively, where p_n is the nth prime number. In order for the distributions

$$F_N(x) = P\left(\sum_{n=1}^{N} X_n \leq x \right) \tag{1.3}$$

to be weakly convergent, it is sufficient that (1.2) holds for some $x_1 < x_2$.

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The conjecture remains open to date. Some partial results have been obtained by Paul [5] and Babu [1] (Chapter 4). In particular, Babu showed that the conclusion of Erdös' conjecture holds if (1.2) is replaced by the stronger condition that the limit $\lim_{N\to\infty}(F_N(z) - F_N(x_1))$ exists for $x_1 \le z \le x_2$ and is not a linear function of z.

It is natural to expect that the probabilistic version of Erdös' conjecture, if true, holds for much more general sequences of independent random variables X_n than those arising in connection with additive functions. In this direction, Paul [5] suggested that the condition

$$\lim_{n\to\infty} P(X_n = 0) = 1 \tag{1.4}$$

might already be sufficient in order for the conclusion of Erdös' conjecture to be valid. On the other hand, without any a priori condition on a sequence of independent random variables X_n, the conclusion is not true in general. A trivial example is obtained by taking X_n to be equal to some constant c_n with probability 1. If the series $\sum_{n=1}^{\infty} c_n$ diverges but has bounded partial sums, then the distributions F_N are not convergent, but (1.2) holds for any sufficiently large x_2 and sufficiently small x_1. More generally, if

$$\sum_{n=1}^{\infty} \rho(X_n, c_n) < \infty,$$

where

$$\rho(X, c) = \inf\{\varepsilon > 0: P(X < c - \varepsilon) \le \varepsilon,\ P(X > c + \varepsilon) \le \varepsilon\} \tag{1.5}$$

denotes the Lévy distance between X and the constant random variable c, then the distributions F_N converge if and only if the series $\sum_{n=1}^{\infty} c_n$ converges, but (1.2) may be satisfied in either case. We shall therefore assume that

$$\sum_{n=1}^{\infty} \rho(X_n, c_n) = \infty \quad \text{if} \quad \sum_{n=1}^{\infty} c_n \text{ diverges.} \tag{1.6}$$

Our principal result shows that under this condition the conclusion of Erdös' conjecture is valid, provided (1.2) is strengthened to

$$L_i = \lim_{N\to\infty} F_N(x_i) \quad \text{exists for } i = 1, 2 \text{ and } L_1 \ne L_2. \tag{1.7}$$

THEOREM 1. *Let X_n, $n \ge 1$, be a sequence of independent random variables satisfying* (1.6). *In order for the distributions* (1.3) *to converge, it is sufficient that* (1.7) *holds for some $x_1 < x_2$.*

It is easy to see that $\rho(X_n, c_n) \ge \min(|c_n|, P(X_n = 0))$. In particular, if $\sum_{n=1}^{\infty} \rho(X_n, c_n)$ converges [so that $\rho(X_n, c_n) \to 0$ as $n \to \infty$] and

$$\liminf_{n\to\infty} P(X_n = 0) > 0, \tag{1.8}$$

then $\rho(X_n, c_n) \ge |c_n|$ holds for all sufficiently large n. Thus we see that condition (1.8), which is slightly weaker than (1.4), implies the hypothesis (1.6) of the theorem, and we obtain the following corollary.

COROLLARY 1. *Let X_n, $n \geq 1$, be a sequence of independent random variables satisfying* (1.8). *In order for the distributions* (1.3) *to converge, it is sufficient that* (1.7) *holds for some $x_1 < x_2$.*

By specializing this result to the random variables arising from Erdös' conjecture and translating it back into a statement about additive arithmetic functions, we obtain Erdös' original conjecture in the slightly weaker form where (1.2) is replaced by (1.7).

COROLLARY 2. *Let f: $\mathbf{N} \to \mathbf{R}$ be an additive function. In order for the distributions* (1.1) *to converge, it is sufficient that* (1.7) *holds for some $x_1 < x_2$.*

While we cannot decide whether Erdös' condition (1.2) is already sufficient in Corollaries 1 and 2, we show in our second theorem that under the more general hypotheses of Theorem 1, (1.7) cannot be replaced by (1.2).

THEOREM 2. *There exists a sequence X_n, $n \geq 1$, of independent random variables satisfying* (1.6) *such that* (1.2) *holds for some $x_1 < x_2$, but the distributions* (1.3) *do not converge.*

2. Preliminaries. As is well known, the weak convergence of the distributions of the partial sums of a series $\sum_{n=1}^{\infty} X_n$ of independent random variables is equivalent to the almost sure (a.s.) convergence of that series. Moreover, by Kolmogorov's three series theorem, $\sum_{n=1}^{\infty} X_n$ converges almost surely if and only if the series

$$\sum_{n=1}^{\infty} \operatorname{Var}(X_n^{\varepsilon}), \qquad \sum_{n=1}^{\infty} E(X_n^{\varepsilon}), \qquad \sum_{n=1}^{\infty} P(|X_n| > \varepsilon),$$

converge, where ε is any fixed positive number and X_n^{ε} denotes the random variable X_n truncated at $\pm\varepsilon$. We shall need the following related result, which can be found in Doob [2] (Theorem III.2.9).

LEMMA 1. *Let X_n, $n \geq 1$, be a sequence of independent random variables satisfying, for some constant K,*

$$\limsup_{N \to \infty} P\left(\left| \sum_{n=1}^{N} X_n \right| \leq K \right) > 0. \tag{2.1}$$

Then there exist constants a_n, $n \geq 1$, such that the series $\sum_{n=1}^{\infty}(X_n - a_n)$ converges a.s.

We define the range of a random variable X as the set

$$R(X) = \{x \in \mathbf{R}: P(|X - x| \leq \varepsilon) > 0 \text{ for every } \varepsilon > 0\}, \tag{2.2}$$

that is, the support of the probability measure on $\mathbf{R}$ induced by X. The next lemma describes the form of this set when X is given as an a.s. convergent series of independent random variables.

LEMMA 2. *Let $\sum_{n=1}^{\infty} X_n$ be an a.s. convergent series of independent random variables and let X denote its sum. Suppose that for every $\varepsilon > 0$ and $n \geq n_0 = n_0(\varepsilon)$, there exist numbers $c_n = c_n(\varepsilon) \in R(X_n)$ with $|c_n| \leq \varepsilon$ such that*

$$\limsup_{N \to \infty} \left| \sum_{n=n_0}^{N} c_n \right| = \infty. \tag{2.3}$$

Then $R(X)$ is equal to $\mathbf{R}$ or an interval of the form $(-\infty, a]$ or $[a, \infty)$ for some $a \in \mathbf{R}$.

PROOF. To show that $R(X)$ has the required form, it obviously suffices to show that if $a_0 \in R(X)$, then $[a_0, \infty) \subset R(X)$ or $(-\infty, a_0] \subset R(X)$.

We fix $a_0 \in R(X)$ and a positive number ε and we let c_n, $n \geq n_0 = n_0(\varepsilon)$, be given as in the lemma. By (2.3), the partial sums $\sum_{n=n_0}^{N} c_n$ are then either unbounded from above or unbounded from below. We shall show that in the first case

$$R(X) \cap [a - 5\varepsilon, a + 5\varepsilon] \neq \varnothing \tag{2.4}$$

holds for every $a > a_0$. A similar argument will give (2.4) in the second case for every $a < a_0$. Since at least one of the two cases holds for arbitrarily small values of ε and $R(X)$ is a closed set, this will imply that $R(X)$ contains at least one of the intervals $[a_0, \infty)$ or $(-\infty, a_0]$. It therefore remains to show that if

$$\limsup_{N \to \infty} \sum_{n=n_0}^{N} c_n = \infty, \tag{2.5}$$

then (2.4) holds for every $a > a_0$.

By the assumption $a_0 \in R(X)$, we have

$$P(|X - a_0| \leq \varepsilon) > 0.$$

Since X is the limit in distribution of the partial sums $S_N = \sum_{n=1}^{N} X_n$, it follows that

$$P(|S_N - a_0| \leq 2\varepsilon) > 0, \qquad N \geq N_0, \tag{2.6}$$

and

$$P(|X - S_N| \leq \varepsilon) > 0, \qquad N \geq N_0, \tag{2.7}$$

with a suitable $N_0 = N_0(\varepsilon) \geq n_0(\varepsilon)$.

Now let $a > a_0$ be given and choose $N_1 \geq N_0 + 1$ such that

$$\left| \sum_{n=N_0+1}^{N_1} c_n - (a - a_0) \right| \leq \varepsilon.$$

This is possible by (2.5) and the conditions $N_0 \geq n_0(\varepsilon)$ and $|c_n| \leq \varepsilon$. Since $c_n \in R(X_n)$, we have $P(|X_n - c_n| \leq \varepsilon/N_1) > 0$ for each n and thus

$$P\left(\left|S_{N_1} - S_{N_0} - (a - a_0)\right| \leq 2\varepsilon\right) \geq P\left(\left|\sum_{n=N_0+1}^{N_1} (X_n - c_n)\right| \leq \varepsilon\right)$$
$$\geq \prod_{n=N_0+1}^{N_1} P\left(|X_n - c_n| \leq \frac{\varepsilon}{N_1}\right) > 0.$$

Combining this with (2.6) and (2.7), we deduce

$$\begin{aligned} &P(|X - a| \leq 5\varepsilon) \\ &\quad \geq P\left(|S_{N_0} - a_0| \leq 2\varepsilon\right) P\left(\left|S_{N_1} - S_{N_0} - (a - a_0)\right| \leq 2\varepsilon\right) P\left(|X - S_{N_1}| \leq \varepsilon\right) \\ &\quad > 0, \end{aligned}$$

which implies (2.4). □

3. Proof of Theorem 1. Let X_n, $n \geq 1$, be a sequence of independent random variables satisfying (1.6) and (1.7) for some $x_1 < x_2$. By (1.7), the hypothesis (2.1) of Lemma 1 is satisfied for any $K > \max(|x_1|, |x_2|)$, and it follows that, with suitable constants a_n, the series $\sum_{n=1}^{\infty}(X_n - a_n)$ is a.s. convergent. Let

$$Y_n = X_n - a_n, \qquad Y = \sum_{n=1}^{\infty} Y_n,$$

and set

$$G_N(x) = P\left(\sum_{n=1}^{N} Y_n \leq x\right) = F_N(x + A_N), \qquad G(x) = P(Y \leq x),$$

where F_N is defined by (1.3) and $A_N = \sum_{n=1}^{N} a_n$. Since the series $\sum_{n=1}^{\infty} Y_n$ is a.s. convergent, it converges also in distribution, and we have

$$G_N(x) \to G(x), \qquad N \to \infty, \tag{3.1}$$

at every continuity point of G. On the other hand, from (1.7) we have

$$G_N(x_i - A_N) = F_N(x_i) \to L_i, \qquad N \to \infty,\ i = 1,2. \tag{3.2}$$

If the limit $A = \lim_{N\to\infty} A_N$ exists, then (3.1) implies that the distributions $F_N(x) = G_N(x - A_N)$ converge to the distribution $F(x) = G(x - A)$ and we are done. Therefore, it remains to prove the convergence of the sequence A_N.

We first note that the numbers A_N must be bounded, for if $A_{N'} \to \infty$ on some subsequence $\{N'\}$, say, then we have

$$G_{N'}(x_i - A_{N'}) \to 0, \qquad N' \to \infty,\ i = 1,2,$$

and hence, by (3.2), $L_1 = L_2$, contradicting our assumption in (1.7). We may therefore assume that the lower and upper limits

$$\underline{A} = \liminf_{N\to\infty} A_N, \qquad \bar{A} = \limsup_{N\to\infty} A_N$$

are finite. We suppose that A_N is not convergent, so that $\underline{A} < \bar{A}$, and we shall show that this leads to a contradiction to our assumption (1.6).

We first show that, for $i = 1, 2$,

$$G(x) = L_i, \qquad x_i - \bar{A} < x < x_i - \underline{A}. \tag{3.3}$$

To this end we fix two increasing sequences $\{N'\}$ and $\{N''\}$ of positive integers such that

$$\underline{A} = \lim_{N'\to\infty} A_{N'}, \qquad \bar{A} = \lim_{N''\to\infty} A_{N''}.$$

For any $\delta > 0$ such that the points $x_i - \underline{A} - \delta$ and $x_i - \bar{A} + \delta$, $i = 1, 2$, are continuity points of $G(x)$ we then have, by (3.1), (3.2) and the monotonicity of G and G_N,

$$L_i = \lim_{N'\to\infty} G_{N'}(x_i - A_{N'}) \geq \lim_{N'\to\infty} G_{N'}(x_i - \underline{A} - \delta) = G(x_i - \underline{A} - \delta)$$

and similarly

$$L_i \leq G\big(x_i - \bar{A} + \delta\big),$$

so that

$$G\big(x_i - \bar{A} + \delta\big) \geq L_i \geq G(x_i - \underline{A} - \delta).$$

By the monotonicity of G, this forces $G(x)$ to be equal to L_i on the interval $(x_i - \bar{A} + \delta, x - \underline{A} - \delta)$, and since δ may be taken arbitrarily small, (3.3) follows.

Next, note that the range $R(Y)$ of Y, as defined in (2.2), is equal to the set of points of increase of the function $G(x)$. By (3.3) and the hypotheses $x_1 < x_2$ and $L_1 \neq L_2$ in (1.7), this set does not contain any points from the intervals $(x_i - \bar{A}, x_i - \underline{A})$, $i = 1, 2$, but it must contain at least one point from the intermediate interval $[x_1 - \underline{A}, x_2 - \bar{A}]$. Thus it cannot be of the form guaranteed in Lemma 2. To obtain the desired contradiction, it suffices therefore to show that the series $\sum_{n=1}^{\infty} Y_n$ satisfies the hypotheses of that lemma.

By construction, $\sum_{n=1}^{\infty} Y_n$ is an a.s. convergent series of independent random variables. Thus it remains to show that for every $\varepsilon > 0$ there exist numbers $c_n \in R(Y_n)$ with $|c_n| \leq \varepsilon$ for which (2.3) holds. Fix $\varepsilon > 0$ and let

$$c_n^- = \inf\{y \in R(Y_n) \colon |y| \leq \varepsilon\}, \qquad c_n^+ = \sup\{y \in R(Y_n) \colon |y| \leq \varepsilon\}.$$

The three series theorem implies that

$$\sum_{n=1}^{\infty} P\big(|Y_n| > \varepsilon\big) < \infty, \tag{3.4}$$

so that in particular $P(|Y_n| \leq \varepsilon) > 0$ for all sufficiently large n, say $n \geq n_0 = n_0(\varepsilon)$. Hence $R(Y_n) \cap [-\varepsilon, \varepsilon]$ is nonempty and $c_n^{\pm}$ is well-defined for $n \geq n_0$.

Note that $c_n^\pm \in R(Y_n)$, since $R(Y_n)$ is a closed set. If one of the two series $\sum_{n=n_0}^\infty c_n^+$ and $\sum_{n=n_0}^\infty c_n^-$ has unbounded partial sums, then (2.3) holds with $c_n = c_n^+$ or $c_n = c_n^-$. Therefore we may assume that

$$\limsup_{N\to\infty} \sum_{n=n_0}^{N} c_n^+ < \infty, \qquad \liminf_{N\to\infty} \sum_{n=n_0}^{N} c_n^- > -\infty. \tag{3.5}$$

Let

$$m_n = \frac{c_n^+ + c_n^-}{2}, \qquad \Delta_n = \frac{c_n^+ - c_n^-}{2},$$

so that $c_n^\pm = m_n \pm \Delta_n$. Then $\Delta_n \geq 0$, and (3.5) implies that

$$\sum_{n=n_0}^{\infty} \Delta_n < \infty. \tag{3.6}$$

Moreover, by the definition of $c_n^\pm$ we have for $n \geq n_0$,

$$P(|Y_n - m_n| > \Delta_n) = P(Y_n \notin [c_n^-, c_n^+]) \leq P(|Y_n| > \varepsilon). \tag{3.7}$$

Now let

$$\rho_n = \rho(X_n, a_n + m_n) = \rho(Y_n, m_n).$$

By (1.5) we have for $n \geq n_0$,

$$\tfrac{1}{2}\rho_n < P(|Y_n - m_n| \geq \tfrac{1}{2}\rho_n) \leq P(|Y_n - m_n| > \Delta_n)$$

provided $\Delta_n < \rho_n/2$, and thus in any case

$$\rho_n \leq 2(P(|Y_n - m_n| > \Delta_n) + \Delta_n).$$

By (3.7), (3.4) and (3.6) it follows that

$$\begin{aligned}\sum_{n=n_0}^{\infty} \rho_n &\leq 2 \sum_{n=n_0}^{\infty} (P(|Y_n - m_n| > \Delta_n) + \Delta_n)\\ &\leq 2 \sum_{n=n_0}^{\infty} P(|Y_n| > \varepsilon) + 2 \sum_{n=n_0}^{\infty} \Delta_n < \infty.\end{aligned}$$

Hence the series in (1.6) converges with $c_n = a_n + m_n$ for $n \geq n_0$. On the other hand, since by (3.7),

$$\begin{aligned}|E(Y_n^\varepsilon) - m_n| &\leq E(|Y_n^\varepsilon - m_n|)\\ &\leq \Delta_n + (\varepsilon + |m_n|)P(|Y_n - m_n| > \Delta_n) \leq \Delta_n + 2\varepsilon P(|Y_n| > \varepsilon)\end{aligned}$$

and, by (3.6) and the three series theorem, the series

$$\sum_{n=n_0}^{\infty} \Delta_n, \qquad \sum_{n=n_0}^{\infty} E(Y_n^\varepsilon), \qquad \sum_{n=n_0}^{\infty} P(|Y_n| > \varepsilon)$$

are convergent, $\sum_{n=n_0}^\infty m_n$ converges. Since we assumed that $\sum_{n=1}^\infty a_n$ diverges, it follows that $\sum_{n=n_0}^\infty c_n = \sum_{n=n_0}^\infty (a_n + m_n)$ diverges also, and we have reached a contradiction to (1.6). This completes the proof of Theorem 1. □

4. Proof of Theorem 2. Let Y_n, $n \geq 1$, be a sequence of independent random variables satisfying

$$\sum_{n=1}^{\infty} \rho(Y_n, d_n) = \infty \tag{4.1}$$

for every sequence d_n of real numbers and such that the series $\Sigma_{n=1}^{\infty} Y_n$ is a.s. convergent and has a continuous limit distribution. We shall show that the random variables $X_n = Y_n + (-1)^n$ then satisfy (1.2) and (1.6), but the distributions F_N do not converge.

We have

$$F_N(x) = P\left(\sum_{n=1}^{N} \left(Y_n + (-1)^n\right) \leq x\right) = \begin{cases} G_N(x), & \text{if } N \text{ is even,} \\ G_N(x+1), & \text{if } N \text{ is odd,} \end{cases}$$

where $G_N(x) = P(\Sigma_{n=1}^{N} Y_n \leq x)$. Since, by the assumptions on Y_n, $G_N(x)$ converges to a continuous distribution $G(x)$ as $N \to \infty$, it follows that

$$\lim_{N\to\infty} F_{2N}(x) = G(x), \qquad \lim_{N\to\infty} F_{2N+1}(x) = G(x+1) \tag{4.2}$$

for every $x \in \mathbf{R}$. This shows that the distributions F_N are not convergent.

Next, let

$$D(x) = G(x+1) - G(x).$$

By the properties of $G(x)$ as a continuous distribution function, the function $D(x)$ is nonnegative, continuous, not identically zero and tends to 0 as $|x| \to \infty$. It follows that there exists an $x_0 \in \mathbf{R}$ such that $D(x_0) = \max_{x \in \mathbf{R}} D(x) > 0$. Moreover, given any $x_1 < x_0$ with $0 < D(x_1) < D(x_0)$, there exists a number $x_2 > x_0$ such that $D(x_2) = D(x_1)$, or equivalently,

$$G(x_2 + 1) - G(x_1 + 1) = G(x_2) - G(x_1).$$

From this and (4.2) it follows that the limit $\lim_{N\to\infty}(F_N(x_2) - F_N(x_1))$ exists and is equal to $G(x_2) - G(x_1)$ for any such pair $x_1 < x_2$. Moreover, this limit must be positive, for otherwise $G(x)$ would be constant and $D(x) = G(x+1) - G(x)$ nondecreasing on the interval (x_1, x_2), contradicting our assumption that $x_0 < x_2$ and $D(x_0) > D(x_2)$. Hence (1.2) is satisfied.

Finally we note that condition (1.6) follows from (4.1) and the relation $\rho(X_n, c_n) = \rho(Y_n + (-1)^n, c_n) = \rho(Y_n, d_n)$ with $d_n = (-1)^{n+1} + c_n$. This completes the proof.

We remark that a sequence of random variables Y_n satisfying the above conditions can easily be constructed. For example, we may take Y_n to be $1/n$ and $-1/n$ with probabilities $\frac{1}{2}$ each. The a.s. convergence of the series $\Sigma_{n=1}^{\infty} Y_n$ follows from the three series theorem. The continuity of the limit distribution is a consequence of a theorem of Lévy [4] which states that an a.s. convergent series $\Sigma_{n=1}^{\infty} Y_n$ of independent random variables has a continuous limit distribution if and only if, for any sequence c_n, $n \geq 1$, $\sum_{n=1}^{\infty} P(Y_n \neq c_n)$ diverges. Since $\rho(Y_n, c) \geq 1/2n$ for any constant c, the condition (4.1) is also satisfied. □

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