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BOOK REVIEW

H. KUNITA, *Stochastic Flows and Stochastic Differential Equations*. Cambridge University Press, Cambridge, 1990, 346 pages, \$69.50.

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A stochastic flow is a family of random mappings $\phi_{s,t}$, $0 \le s \le t \le T$, of R^d (or other space) into itself, satisfying the composition relation $\phi_{t,u} \circ \phi_{s,t} = \phi_{s,u}$ if $s \le t \le u$. There are interesting cases with coalescence, but most treatments have dealt with homeomorphic or diffeomorphic flows. Then $\overline{\phi}$ defined by $\overline{\phi}_{s,t} = \phi_{s,t}$ for $s \le t$ and $\overline{\phi}_{s,t} = (\phi_{t,s})^{-1}$ for $s \ge t$, is a flow satisfying the composition relation for $0 \le s, t \le T$. If ϕ_t, $t \ge 0$ is a homeomorphism-valued process, then $\phi_{s,t} := \phi_t \circ (\phi_s)^{-1}$, $0 \le s$, $t \le T$ is a flow. If ϕ is a flow defined for $0 \le s, t \le T$, the restriction to $s \le t$ is called a *forward* flow, and is usually considered for fixed s. The restriction to $s \ge t$ is a *backward* flow. We speak mostly of forward flows, but there are parallel backward results, and the interplay between the two is important. ϕ_t denotes $\phi_{0,t}$.

In an important case $\phi_{s,t}(x)$ is continuous in s, t and x, and $s_1 \le t_1 \le s_2 \le t_2 \le \cdots$ implies that $\phi_{s_1,t_1}, \phi_{s_2,t_2}, \ldots$ are independent mappings; such flows are called *Brownian*. For Brownian flows with some regularity conditions, the paths of a set of k points are a dk-dimensional diffusion. In the time-homogeneous case, the law of $\phi_{s,t}$ depends on $t - s$.

It is known that Brownian stochastic flows arise as solutions of Itô systems in R^d:

$$d\phi^i_{s,t}(x) = \sum_{j=1}^{m} \sigma^i_j(\phi_{s,t}(x), t)\, dW_{jt} + b^i(\phi_{s,t}(x), t)\, dt, \qquad t \ge s; \tag{1}$$

$$\phi^i_{s,s}(x) = x^i, \qquad 1 \le i \le d.$$

Here the W's are independent Wiener processes in R^1. The σ's and b's satisfy familiar conditions. If we change σ without changing $\sigma\sigma^T$, the one-point motions are unchanged but in general the flow will be different. However, the flow is determined by the two-point motions, or alternatively by the *infinitesimal mean* (drift) $b(x,t)$ and the *infinitesimal covariance matrix*:

$$\begin{aligned} a^{ij}(x,y,t) &= \sum_k \sigma^i_k(x,t)\sigma^j_k(y,t) \\ &= \lim_{u \downarrow t} E\{(\phi^i_{tu}(x) - x^i)(\phi^j_{tu}(y) - y^j)\}/(u-t), \\ b^i(x,t) &= \lim_{u \downarrow t} E\{\phi^i_{tu}(x) - x^i\}/(u-t). \end{aligned} \tag{2}$$

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a^{ij} is nonnegative definite, considered as a function of the two variables (i, x) and (j, y). $a^{ij}(x, x, t)$ is the diffusion matrix of the one-point motion. It is through a and b that our intuition best relates to the flow.

In the case of temporally homogeneous Brownian flows we can use the treatment of Baxendale (1984), based on reproducing kernel theory, to go from given a and b to stochastic equations. Here

$$a^{ij}(x,y) = \sum_{1 \le \alpha < \infty} V_\alpha^i(x) V_\alpha^j(y) \tag{3}$$

in the sense of reproducing kernel theory. The flow is then given by the (usually infinite) system

$$\phi_t^i(x) = x^i + \int_0^t \sum_\alpha V_\alpha^i(\phi_s(x))\, dW_\alpha(s) + \int_0^t b^i(\phi_s(x))\, ds. \tag{4}$$

Although in a sense the theory of Brownian flows is the theory of (possibly infinite) Itô systems, the flow theory puts much emphasis on the behavior of $\phi_{st}(x)$ as a function of x, and one is also concerned with the effect of the mappings ϕ_{st} on the shapes or measures of subsets transported by the flow. The reviewer will not try to cover the history of the subject, which owes much to Kunita and, to cite a few more researchers, Baxendale, Bismut, Carverhill, Elworthy, LeJan, Malliavin, Ruelle and Watanabe. The pioneering work on stochastic differential equations by Itô, Stroock and Varadhan, and others must be kept in mind. There is a section of historical remarks at the end of the book.

Before following Kunita into the general mathematical framework of flows, Brownian and non-Brownian, we observe that there are jump-type or piecewise smooth flows which have limiting Brownian flows. A few examples:

(a) Solutions of systems of ordinary differential equations with random coefficients. A few examples are the papers of Wong and Zakai (1965), Kesten and Papanicolaou (1979) and Kunita (1984). In a frequently treated case, certain of the input functions become white noise in the limit.

(b) "Stirrings" or other jump-type processes: randomly selected transformations applied more and more frequently to R^d at random times and places [Harris (1981), stimulated by the symmetric simple exclusion process of Spitzer (1970); Matsumoto and Shigekawa (1985)]. A randomized modification of the idealized nonrandom stirring of a fluid in Aref (1984) has a limiting Brownian stochastic flow in $R^2 \setminus \{0\}$ [Jakel (1989)].

Although Brownian flows do not have velocities, the infinitesimal covariance matrix for a Brownian flow resembles mathematically the "correlation tensor" of velocities in the statistical theory of turbulence [Monin and Yaglom (1975)]. In each case, incompressibility and isotropy are related to properties of the matrix, and if isotropy and spatial homogeneity are present, a decomposition of a^{ij} into potential and solenoidal parts [Itô (1956)] is useful [Baxendale and Harris (1986) and LeJan (1985)]. Morris (1989) has given conditions under which a similar decomposition holds for spatially homogenous nonisotropic flows.

To appreciate the mathematical framework of the book, go back to (1) and put (fixing $s = 0$, say)

$$F^i(x,t) = \sum_{j=1}^{m} \int_0^t \sigma_j^i(x,r)\, dW_j(r) + \int_0^t b^i(x,r)\, dr, \qquad (5)$$

$$t \geq 0,\ i = 1, 2, \ldots, d.$$

$F(x, t)$ is a vector-valued semimartingale, the Itô terms on the right side of (5) giving a vector-valued local martingale. Write (1) in the form

$$\phi_t(x) := \phi_{0,t}(x) = x + \int_0^t F(\phi_r(x), dr). \qquad (6)$$

F is the (Itô forward) *random infinitesimal generator* (r.i.g.) of ϕ.

Now start with any R^d-valued semimartingale process $F(x, t)$ continuous in $(x, t) \in R^d \times [0, \infty)$ and satisfying additional regularity conditions. Following LeJan and Watanabe (1984) and Kunita, we define for R^d-valued predictable processes $g(r) \equiv g_r$ a vector integral of the Itô (resp., Stratonovich) type, denoted by

$$\int_0^t F(g_r, dr) \qquad \left(\text{resp., } \int_0^t F(g_r, o\, dr)\right), \qquad (7)$$

which is consistent with the usual definitions when F is as in (5). Thus the first (resp., second) integral in (7) is a limit of sums [put $g_i = g(r_i)$]

$$\sum [F(g_i, r_{i+1}) - F(g_i, r_i)], \qquad t_i < r_{i+1},$$

[resp., $(1/2)\sum[F(g_{i+1}, r_{i+1}) - F(g_{i+1}, r_i) + F(g_i, r_{i+1}) - F(g_i, r_i)]$.] Then there is a unique forward flow (not in general Brownian) satisfying (6), that is, for which F is the r.i.g. Conversely, given a flow, we can find a unique semimartingale F for which (6) holds. Using $\int_0^t F(\phi_r(x), o\, dr)$ as the integral in (6), F is the *Stratonovich* forward r.i.g. There is also a backward r.i.g. for each type of integral.

Suppose an R^d-valued semimartingale $F(x, t)$ with a filtration $\mathbf{F}_t$ has the form $F(x, t) = M(x, t) + B(x, t)$, where, for fixed x, $M(x, t)$ is a continuous local martingale and $B(x, t)$ is continuous and of bounded variation on finite t-intervals. Assume that the joint quadratic variation of $M_i(x, \cdot)$ and $M_j(y, \cdot)$ can be written in the form $A_{ij}(x, y, t) = \int_0^t a^{ij}(x, y, r)\, dr$, while $B_i(x, t) = \int_0^t b^i(x, r)\, dr$, where the a^{ij} and the b^i are predictable as functions of r. [More generally dr may have to be replaced by $A(dr)$, where A is a continuous increasing process.] Then $a(x, y, t)$ and $b(x, t)$ are by definition the *local characteristics* of F. Almost surely, for the corresponding flow ϕ,

$$b^i(x,t) = \lim_{u \downarrow t} E\{\phi_{t,u}^i(x) - x^i | \mathbf{F}_t\}/(u - t), \qquad (8)$$

$$a^{ij}(x,y,t) = \lim_{u \downarrow t} E\{(\phi_{t,u}^i(x) - x^i)(\phi_{t,u}^j(y) - y^j) | \mathbf{F}_t\}/(u - t). \qquad (9)$$

The Brownian case arises exactly when these limits are nonrandom. In this case (6) becomes an Itô system like (1), except that in general a countable infinity of Wiener processes is required. Non-Brownian flows arise naturally as backward Brownian flows, or by composition of Brownian flows [LeJan and Watanabe (1984)].

A number of technical problems have to be solved in the above treatment. In particular, a generalized Itô formula is required to express functions of the form $F(g_t, t)$, where g_t is a continuous semimartingale with values in the space of x. Bismut (1981) and other researchers have contributed here.

Kunita and others have shown that if the r.i.g. is smooth enough, the solution of (6) is jointly continuous in s, t and x, has continuous partial derivatives of several orders in x, and is homeomorphic or diffeomorphic. A thorough treatment of such results is given in the book. On the other hand, nonsmooth correlations may lead to flows with coalescence [Harris (1984) and Darling (1987)].

As is known, a diffusion may have an explosion time. For flows, the situation is rather complicated. For example, a flow can be constructed in $R^2 \setminus \{0\}$ such that for fixed x, the path starting at x is a.s. defined for all t, and on each finite t-interval stays outside some random open ball containing the origin; but for fixed $t > 0$ there are a.s. points $y \in R^2 \setminus \{0\}$ such that the path starting at y enters every open ball containing $\{0\}$ during $[0, t]$. Both global and local existence results are given in the book.

How does a Brownian flow $\{\phi_t\}$ behave as $t \to \infty$? One approach is to study the effect of $\{\phi_t\}$ on a nonrandom Borel measure Π in R^d. Define $(\phi_t^{-1}\Pi)(B) = \Pi(\phi_t(B))$. Under certain conditions $\phi_t^{-1}\Pi = \Pi$. (Π-shrinking and Π-expanding flows are also considered.) If Π is Lebesgue, the flow is called *incompressible*. A necessary and sufficient condition for incompressibility is that the divergence of the Stratonovich r.i.g. is 0; similar results are given for the preservation of other measures. Thus a Brownian flow in R^2 with the Stratonovich r.i.g.

$$F(x,t) = \sum_{\alpha=1}^{m} A_\alpha(x) W_t^\alpha \tag{10}$$

is incompressible if and only if $\operatorname{div} A_\alpha(x) = 0$. In R^2 this implies existence of a scalar random stream function $G(x, t)$ which, in a limiting sense, describes the transport across the segment $0x$ during the time-interval $[0, t]$. [See Pei, (1991).]

Suppose the one-point motion has an invariant probability Λ and a strictly positive definite covariance matrix $a^{ij}(x, x)$. Then almost surely $\phi_t^{-1}\Lambda$ converges weakly (as a measure) to a probability measure Λ_∞ such that $E\Lambda_\infty(B) = \Lambda(B)$. Unless a special divergence condition holds, the measure Λ_∞ is a.s. singular with respect to Λ. Other related results are given. LeJan [e.g., (1984, 1987)] has done much on this subject.

The asymptotic behavior of a flow for large t may be studied through the Lyapunov exponents, which are frequently defined in terms of the linearized flow. In the present book Lyapunov exponents are introduced in terms of such quantities as

$$\limsup_{t \to \infty} \log \Pi(\phi_t(B))/t$$

for measures Π and Borel sets B. One would like to have more on the relationship of this approach to the usual one.

Fundamental to the analysis of flows is a study of convergence of a family of flows $\{\phi_\varepsilon = \phi_\varepsilon(x, t),\ \varepsilon > 0\}$ or a sequence $\{\phi_n\}$ to a flow ϕ. The joint convergence of a family of flows ϕ_ε and their r.i.g.'s F_ε is considered. Denote the mappings $x \to \phi(x, t)$ and $x \to F(x, t)$ by $\phi(t)$ and $F(t)$, respectively; similarly for $\phi_\varepsilon(t)$ and $F_\varepsilon(t)$. We have weak convergence *as flows* of (ϕ_ε) to ϕ if the joint law of the process $(\phi_\varepsilon(t), F_\varepsilon(t),\ 0 \le t \le T)$ converges weakly to that of $(\phi(t), F(t))$. Various choices may be made for the spaces of mappings. For example $F_\varepsilon(t)$ may be in the space of C^k-mappings and $\phi_\varepsilon(t)$ in the space of C^k-diffeomorphisms. A less stringent convergence "as diffusions" occurs when all finite-point processes $t \to (\phi_\varepsilon(x_1, t), \dots, \phi_\varepsilon(x_k, t);\ F_\varepsilon(x_1, t), \dots, F_\varepsilon(x_k, t))$ converge weakly to the corresponding process for (ϕ, F).

The limit theorems are of great generality, and include some types studied earlier by various authors. An interesting case is the system of random ordinary differential equations

$$dx/dt = \sum_{l=1}^{r} F_l(x, t) v_l^\varepsilon(t) + F_0(x, t), \tag{11}$$

where the $F_l(x, t)$ are continuous R^d-valued nonrandom functions and the v_l^ε are processes with mean 0 whose time integrals tend to Wiener processes $B_l(t)$ as $\varepsilon \downarrow 0$. Under appropriate additional conditions, the solution tends to the stochastic flow determined by

$$d\phi_t = \sum_{l=1}^{r} F_l(\phi_t, t) \circ dB_l(t) + F_0(\phi_t)\, dt. \tag{12}$$

Under other conditions, there may be a more complicated limiting equation.

A section about flows on manifolds establishes the fundamentals, discusses the action of a flow on a tensor field and treats Kunita's earlier work on compositions and decompositions of flows.

The book concludes with a treatment of stochastic partial differential equations. A special case of the results in the book shows the intimate relationship between such equations and stochastic flows. Let $\phi_{s,t}$ be a stochastic flow in R^d generated by the semimartingale $-F(x, t)$:

$$d\phi_{s,t} = -F(\phi_{s,t}, \circ dt), \qquad t \ge s,\ s \text{ fixed}. \tag{13}$$

Let

$$\psi_{s,t} = (\phi_{s,t})^{-1}, \qquad t \ge s;\ \psi_t = \psi_{0,t}. \tag{14}$$

For s fixed, $\{\psi_{s,t},\ t \ge s\}$ is a backward flow. If f is a smooth function

$R^d \to R^d$, an extension of Itô's theorem to backward flows gives

$$f(\psi_t(x)) = f(x) + \sum_i \int_0^t F^i(x, \circ dr)(\partial/\partial x^i)(f \circ \psi_r)(x). \tag{15}$$

Putting $u(x,t) = f(\psi_t(x))$, (15) becomes

$$u(x,t) = f(x) + \sum_i \int_0^t F^i(x, \circ dr)(\partial/\partial x^i) u(x,r). \tag{16}$$

In other words, the solution of the integrated linear stochastic PDE (16) is exhibited in terms of the backward flow ψ_t. Equations of this type have been treated by Ogawa (1973) in connection with wave propagation in a randomly fluctuating medium. The book also treats quasilinear and nonlinear first-order equations, where more involved methods are required.

Besides their physical applications, first-order equations have applications to the solution of second-order equations. The second-order form considered is

$$\begin{aligned} u(x,t) &= f(x) + \int_0^t L_r u(x,r)\, dr + \sum_{i=1}^d \int_0^t F^i(x, \circ dr) \\ &\quad \cdot \partial u(x,r)/\partial x^i + \int_0^t F^{d+1}(x, \circ dr) u(x,r), \end{aligned} \tag{17}$$

where L_r is a second-order operator on x. Applications of this linear equation to nonlinear filtering are treated. To give a taste of Kunita's probabilistic method, consider a very special case in R^1, which however is too special for the filtering application.

Let $Lh(x) = (1/2)\sigma^2(x)\, d^2h(x)/dx^2$, $F(x,t) = W_t$, $F^{d+1} = 0$, so (17) becomes

$$u(x,t) = f(x) + \int_0^t \tfrac{1}{2}\sigma^2(x) u''(x,r)\, dr + \int_0^t u'(x,r) \circ dW(r). \tag{18}$$

We introduce a semimartingale $X(x,t) = \sigma(x)\overline{W}_t - (1/2)\sigma(x)\sigma'(x)t$, where $\overline{W}$ is independent of W. The local characteristics of X are $a(x,y) = \sigma(x)\sigma(y)$ and $b(x) = -(1/2)\sigma(x)\sigma'(x)$. These are picked so that $a(x,x) = \sigma^2(x)$, while $-(1/2)\sigma(x)\sigma'(x)$ is the drift of the diffusion associated with L (here 0) minus the "Stratonovich correction term" associated with $a(x,y)$.

Let ϕ be the flow having the Stratonovich r.i.g. $-W_t - X(x,t)$ and put $\psi_t = (\phi_{0t})^{-1}$. Using the previous result on first-order equations, it is shown that the solution of (18) is

$$u(x,t) = \overline{E} f(\psi_t(x)), \tag{19}$$

where $\overline{E}$ is conditional expectation with W given.

The reviewer has very few complaints about the book. The exposition is well organized and goes deep. The inclusion of additional easy special cases as examples would assist the reader. Since the conditions for some of the theorems are necessarily quite technical (e.g., a certain process is required to be a backward and forward semimartingale), it would help to state readily verifiable sufficient conditions for some of the theorems. Little is said about the geometrical aspects of flows associated with work on Lyapunov exponents. This has been treated thoroughly elsewhere, but a brief discussion here would help the reader appreciate this important aspect of flows.

Taken as a whole, the book is an admirable and impressive treatment of the foundations of stochastic flow theory, with a wide enough framework to handle many applications.

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