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ONE-DIMENSIONAL STRATONOVICH DIFFERENTIAL EQUATIONS[1]

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We consider one-dimensional stochastic differential equations of the Stratonovich type:

$$dX_t = \sum_i \sigma_i(t, w, X_t) \circ dZ_t^i + \sum_k h_k(t, w, X_t)\, dA_t^k,$$

where Z^i are continuous semimartingales, and A^k are continuous finite variation processes. We extend the definition of the Fisk–Stratonovich integral for a large class of coefficients σ_i, and under suitable conditions we prove existence and uniqueness for that equation.

1. Introduction. The aim of this work is to give sense and to prove existence and uniqueness for Stratonovich differential equations of the following type:

$$(1.1)\qquad \begin{aligned} dX_t &= \sum_i \sigma_i(t, w, X_t) \circ dZ_t^i + \sum_j h_j(t, w, X_t)\, dA_t^j, \\ X_0 &= X_0, \end{aligned}$$

where X, Z^i are continuous one-dimensional semimartingales and A^j are continuous finite variation processes. Usually, it is assumed that σ_i is smooth enough so $\sigma_i(t, x, X_t)$ is a semimartingale and the quadratic covariation $[\sigma_i(\cdot, \cdot, X), Z^i]$ is well defined. Actually, we only need the existence of the mentioned quadratic covariation plus the existence of the Itô integral $\int_0^t \sigma_i(s, w, X_s)\, dZ_s^i$. Meyer [17] has proved that if $\sigma: \mathbb{R} \to \mathbb{R}$ is the antiderivative of a cadlag function σ' (right continuous with left limits), then $[\sigma(X), Z]$ is the limit of sums of the form $\Sigma(\sigma(X_{t_{i+1}}) - \sigma(X_{t_i}))(Z_{t_{i+1}} - Z_{t_i})$, and this limit is equal to $\int_0^t \sigma'(X_s)\, d[X, Z]_s$, even though $\sigma(X)$ is not necessarily a semimartingale. This is the starting point of our work.

Now, I shall describe briefly how this paper is organized. In Section 2 we introduce the class of functions σ_i we shall consider (we call this class $\mathscr{UAD}$). In Section 3 we prove two important results: an extension of the time–occupation formula and a generalized Itô formula. At the end of Section 3 we extend the definition of the Fisk–Stratonovich integral: $\int_0^t \sigma(s, w, X_s) \circ dZ_s$, where $\sigma \in \mathscr{UAD}$. Moreover, we prove that $\int_0^t \sigma(X_s) \circ dZ_s$ can be consistently defined for σ just Lipschitz.

In Section 4 we prove the existence of a maximal and a minimal solution (Theorem 4.14) of (1.1). Finally, in Section 5 we prove uniqueness for (1.1)

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(Theorems 5.8 and 5.15). In the main theorems of Sections 4 and 5, we have in mind that $(Z^1, \dots, Z^n)$ is a standard n-dimensional Brownian motion, but we proved them in a more general situation, in which the structure of the quadratic covariation $[Z^i, Z^j]$ is special. We decided to restrict ourselves to this case because we are interested in considering equations like (1.1), where σ_i are as general as possible.

As a general reference for stochastic integration, as well as properties of semimartingales, we refer the reader to Protter [20].

A bit of notation. Capital letters will usually denote processes; in particular, U, X, Y, Z will be continuous semimartingales and A, C, F will be continuous finite variation processes. If a function f depends on (t, w, x), then $f(t, w, \cdot)$ will represent the function $x \to f(t, w, x)$, where t, w are assumed to be fixed; similar interpretations can be made for $f(t, \cdot, x)$, $f(\cdot, \cdot, x)$, and so on. $f(t, w, x-)$ denotes the left limit, that is, $f(t, w, x-) = \lim_{y \nearrow x} f(t, w, y)$. In a similar way we define $f(t, w, x+)$.

2. Preliminaries. We start with the definition of the class of functions we shall work with. At the beginning we shall consider nonrandom functions, and later on we extend the definition to random functions.

DEFINITION 2.1. We say $f \in \mathscr{AD}$ (antiderivative of the class $\mathscr{D}$) iff f is absolutely continuous and f' admits a version in $\mathscr{D} = \{h: \mathbb{R} \to \mathbb{R}/h$ is right continuous with left limits$\}$ (henceforth if $h \in \mathscr{D}$ we say h is cadlag), that is, $f' = h\,dx$-a.s., for some $h \in \mathscr{D}$.

For $f \in \mathscr{AD}$ we shall denote by f' the cadlag version of its derivative, unless another version is specified. Now we give a useful result.

LEMMA 2.2. *The following are equivalent*:

(a) $f \in \mathscr{AD}$.

(b) *f is absolutely continuous and f' has a version which is the uniform limit on compacts of functions of finite variation. That is, there exists a sequence of functions (h_n) of finite variation on compacts and a function h such that* $\forall \Lambda \subseteq \mathbb{R}$ *compact*, $\sup_{x \in \Lambda} |h_n(x) - h(x)| \to_{n \to \infty} 0$ *and* $f' = h\; dx$-*a.s.*

(c) *f is continuous, and*

$$f'^{+}(x) = \lim_{h \downarrow 0} \frac{f(x+h) - f(x)}{h}$$

exists at all points (*we call it the right derivative*) *and it is cadlag*.

PROOF. The equivalence between (a) and (b) is deduced from the facts: $h \in \mathscr{D}$ iff h is the uniform limit on compacts of cadlag functions of bounded variation on compacts, and every function of bounded variation is equal to a cadlag function except at most on a countable set.

(c) $\Rightarrow$ (b) By a well-known result [see Rooij and Schikhof [21], Theorem 15.3], since f'^{+} is Riemann integrable over bounded sets, we have

$$f(x) = f(0) + \int_0^x f'^{+}(u)\,du,$$

from which (b) holds.

(b) $\Rightarrow$ (c) If $f(x) = f(0) + \int_0^x h(u)\,du$ where h is cadlag, then

$$f'^{+}(x) = h(x) \text{ and (c) holds.}$$ □

An important property from the semimartingale theory point of view is the following corollary.

COROLLARY 2.3. *$\mathscr{AD}$ is closed under composition.*

PROOF. Let h and g be in $\mathscr{AD}$; then they are locally Lipschitz and so their composition is locally Lipschitz. In particular, $(h \circ g)'$ exists a.e. We shall prove that $(h \circ g)'$ has a cadlag version. Let $\tilde{h}'$ and $\tilde{g}'$ be the cadlag versions of h' and g' and let $l(x) = \tilde{h}'(g(x)) \cdot \tilde{g}'(x)$ which has left and right limits, so is equal to a cadlag function except on a countable set.

Let $\varphi \in C_0^\infty$ with support contained in $[0,1]$ such that $\varphi \geq 0$, and $\int_0^1 \varphi(u)\,du = 1$. Define $\varphi_n(u) = n\varphi(nu)$ and consider $f_n(x) = \int \tilde{h}'(y)\varphi_n(y - x)\,dx$. Then f_n is continuous and

$$|f_n(x) - \tilde{h}'(x)| \leq \int_x^{x+1/n} |\tilde{h}'(y) - \tilde{h}'(x)| \varphi_n(y - x)\,dy,$$

which tends to 0 because $\tilde{h}'$ is right continuous. Let

$$h_n(y) = \int_0^y f_n(z)\,dz;$$

then

$$\int_0^x f_n(g(u))\tilde{g}'(u)\,du = h_n(g(x)) - h_n(g(0)),$$

because h_n is C' and then $(h_n \circ g)'(u) = f_n(g(u)) \cdot \tilde{g}'(u)$ at every point u at which g is differentiable. Finally, by the dominated convergence theorem (henceforth DCT), we conclude that

$$h(g(x)) - h(g(0)) = \int_0^x \tilde{h}'(g(u))\tilde{g}'(u)\,du,$$

from which l is a version of $(h \circ g)'$. □

$\mathscr{AD}$ is a rich class of functions and has the basic properties we need to get general results in the study of Stratonovich differential equations. For example, Meyer [17] proved that if $f \in \mathscr{AD}$ and P_n is a sequence of refining partitions of $[0,t]$ with mesh (P_n) tending to 0, then

$$\sum_{P_n}\left(f(X_{s_{i+1}})-f(X_{s_i})\right)(Z_{s_{i+1}}-Z_{s_i}) \xrightarrow[n\to\infty]{} \int_0^t f'(X_s)\,d[X,Z]_s$$

for X and Z continuous semimartingales, where the convergence is UCP (uniform on compacts in probability).

In particular,

$$\sum_{P_n}\frac{f(X_{s_{i+1}})+f(X_{s_i})}{2}(Z_{s_{i+1}}-Z_{s_i})$$

$$\xrightarrow[n\to\infty]{} \int_0^t f(X_s)\,dZ_s+\frac{1}{2}\int_0^t f'(X_s)\,d[X,Z]_s.$$

Thus the Stratonovich integral can be defined for $f(X_t)$. Notice that, in general, $f(X_t)$ is not a semimartingale (even for $f\in C^1$). For example, if B is a one-dimensional Brownian motion, then $f(B_t)$ is a semimartingale iff f is the difference of two convex functions (see Çinlar, Jacod, Protter and Sharpe [6]).

We shall extend our class $\mathscr{AD}$ to a class which we shall call $\mathscr{UAD}$ (uniformly in $\mathscr{AD}$). In what follows we assume a fixed probability space $(\Omega,\mathscr{F},P)$ with a filtration $(\mathscr{F}_t)_{t\in\mathbb{R}}$, which satisfies the usual hypotheses.

DEFINITION 2.4. Let $f\colon \mathbb{R}_+\times\Omega\times\mathbb{R}\to\mathbb{R}$. We say $f\in\mathscr{UAD}$ if there exists an adapted increasing and continuous process A_t ($A_{0-}=0$), and a function $g\colon \mathbb{R}_+\times\Omega\times\mathbb{R}\to\mathbb{R}$ such that:

(2.1) $g(\cdot,\cdot,x)$ is adapted and jointly measurable for every x. $g(s,w,\cdot)\in\mathscr{AD}$ uniformly on (s,w), that is, there exists a sequence $h_n\colon \mathbb{R}_+\times\Omega\times\mathbb{R}\to\mathbb{R}$, such that $h_n(s,w,\cdot)$ is of finite variation on compacts, $h_n(\cdot,\cdot,x)$ is adapted and jointly measurable and $\forall\ t$ and $\Lambda(\subseteq\mathbb{R})$ compact we have

(2.1a) $\sup_n \sup_{s<t,\,x\in\Lambda}|h_n(s,w,x)|<\infty$;

(2.1b) $\int_0^t h_n(s,w,\cdot)\,dA_s$ is of finite variation on compacts;

(2.1c) $\sup_{s<t,\,x\in\Lambda}\left|h_n(s,w,x)-\frac{\partial g}{\partial x}(s,w,x)\right| \xrightarrow[n\to\infty]{} 0$, where $\frac{\partial g}{\partial x}$ is the cadlag version;

(2.1d) $\int_0^t|g(u,w,0)|\,dAu<\infty$.

(2.2) $f(t,w,x)=\int_0^t g(s,w,x)\,dA_s=\int_{0^+}^t g(s,w,x)\,dA_s+g(0,w,x)A_0$.

We shall use the notation $f=g\cdot A$.

REMARKS.

1. In the previous definition we can consider A to be increasing. In fact, if $|dA|$ represents the total variation, then
$$f(t,w,x) = \int_0^t g(s,w,x)\frac{dA_s}{|dA_s|}|dA_s|,$$
where $dA_s/|dA_s|$ is the Radon-Nikodym derivative, which is bounded by 1.
2. If f_1 and f_2 are in $\mathscr{U}\mathscr{A}\mathscr{D}$, then $f_1 = g_1 \cdot A_1$ and $f_2 = g_2 \cdot A_2$, where (A_i) are increasing. Let $A = A_1 + A_2$; then $f_i = (g_i\, dA_i/dA) \cdot dA$. That is, both f_1 and f_2 can be represented with respect to the same increasing process A.
3. If $A_s = 1_{s\geq 0}$ and $g(s,w,x) = h(x) \in \mathscr{A}\mathscr{D}$, then $f(s,w,x) = g \cdot A = h(x)$, from which $\mathscr{U}\mathscr{A}\mathscr{D} \supseteq \mathscr{A}\mathscr{D}$.

LEMMA 2.5. *Let $f \in \mathscr{U}\mathscr{A}\mathscr{D}$ where $f = g \cdot A$. Then*:

(a) $\forall$ s, w, $f(s,w,\cdot\,) \in \mathscr{A}\mathscr{D}$; *in particular*, $\partial f/\partial x(s,w,\cdot\,)$ *has a cadlag version*.

(b) *There is a sequence $f_n(s,w,x)$, a difference of two convex functions on x, such that*
$$\forall\, t, \Lambda(\subseteq \mathbb{R}) \text{ compact}, \qquad \sup_{s\leq t,\, x\in\Lambda}\left|\frac{\partial f}{\partial x}(s,w,x) - \frac{\partial f_n}{\partial x}(s,w,x)\right| \xrightarrow[n\to\infty]{} 0,$$
where $\partial f/\partial x$ and $\partial f_n/\partial x$ are the cadlag versions.

(c)
$$\frac{\partial f}{\partial x}(s,w,x) = \int_0^s \frac{\partial g}{\partial x}(u,w,x)\, dA_u$$
and
$$\begin{aligned} f(s,w,x) &= f(s,w,0) + \int_0^x \frac{\partial f}{\partial x}(s,w,y)\, dy \\ &= f(s,w,0) + \int_0^x\int_0^s \frac{\partial g}{\partial x}(u,w,y)\, dA_u\, dy \\ &= f(s,w,0) + \int_0^s\int_0^x \frac{\partial g}{\partial x}(u,w,y)\, dy\, dA_u. \end{aligned}$$

PROOF.

(a) Take
$$g_n(s,w,x) = g(s,w,0) + \int_0^x h_n(s,w,y)\, dy,$$
where h_n is given by (2.1). Then $g_n(s,w,\cdot\,)$ is the difference of two convex functions [because $h_n(s,w,\cdot\,)$ is of bounded variation]. If $\bar{h}_n(s,w,x) = h_n(s,w,x+)$, then $l(s,w,x) = \lim_n \bar{h}_n(s,w,x)$ exists and it is uniform in

(s, x) on compact sets. Moreover,

$$g(s,w,x) = g(s,w,0) + \int_0^x l(s,w,y)\,dy$$

and $l(s, w, \cdot)$ is cadlag.

By (2.1a), $l(\cdot, w, \cdot)$ is locally bounded and $l(\cdot, \cdot, \cdot)$ is measurable, so by Fubini's theorem we get

$$\begin{aligned} f(s,w,x) &= \int_0^s \int_0^x l(u,w,y)\,dy\,dA_u + \int_0^s g(u,w,0)\,dA_u \\ &= \int_0^x \int_0^s l(u,w,y)\,dA_u\,dy + \int_0^s g(u,w,0)\,dA_u \\ &= \int_0^x \int_0^s l(u,w,y)\,dA_u\,dy + f(s,w,0), \end{aligned}$$

from which $f(s, w, \cdot) \in \mathscr{A}\mathscr{D}$.

(b) It is enough to take $f_n(s, w, x) = f(s, w, 0) + \int_0^x \int_0^s h_n(u, w, y)\,dA_u\,dy$.

(c) This is an application of Fubini's theorem. □

3. Fisk–Stratonovich integral. In this section we shall prove two important results, which will give us the basis for the definition of the Fisk–Stratonovich integral in the following setting.

Given that X and Z are continuous semimartingales and $f \in \mathscr{U}\mathscr{A}\mathscr{D}$, how can we define $\int_0^t f(s, w, X_s) \circ dZ_s$? Our definition will extend the classical definition ($h \in C^2$):

$$\int_0^t h(X_s) \circ dZ_s = \int_0^t h(X_s)\,dZ_s + \frac{1}{2}[h(X), Z]_t^c.$$

For that reason, it is obvious that we need to give sense to the quadratic covariation $[f(\cdot, \cdot, X), Z]_t$. It turns out that we need a generalized version of the time–occupation formula and a generalized Itô formula for a "dense" subclass of functions in $\mathscr{U}\mathscr{A}\mathscr{D}$.

LEMMA 3.1. *Let X be a continuous semimartingale, $X_0 = 0$. Then there exists a set $\mathscr{A} \in \mathscr{F}$ of probability 0 such that $\forall\, w \in \mathscr{A}^c$, $\forall\, f\colon \mathbb{R}_+ \times \Omega \times \mathbb{R} \to \mathbb{R}$ measurable and bounded, $\forall\, t \geq 0$,*

$$\int_0^t f(s,w,X_s)\,d[X,X]_s = \int_{\mathbb{R}} \int_0^t f(s,w,a) L(a,ds)\,da, \tag{3.1}$$

where $L(a, t) = L_t^a$ is the local time of X.

PROOF. We know that except on a set $\mathscr{A}_0$ of measure 0 the following properties are satisfied:

(i) X is continuous.
(ii) $[X, X]_.$ is continuous and increasing.
(iii) L_t^a is jointly continuous in t and cadlag in a, and it is increasing in t.

For any $w \in \mathscr{A}_0^c$ and for any f, measurable and bounded, both sides of (3.1) are continuous in t.

By the usual time–occupation formula and the continuity of the processes involved, we conclude $\exists\ \mathscr{A} \in \mathscr{F}$, $\mathscr{A} \supset \mathscr{A}_0$ and $P(\mathscr{A}) = 0$ such that $\forall\ w \in \mathscr{A}^c$, $\forall\ t \in \mathbb{R}_+$, $\forall\ c, d \in \mathbb{Q}$ (rationals),

$$\int_0^t 1_{[c,d)}(X_s)\, d[X, X]_s = \int_{\mathbb{R}} 1_{[c,d)}(a) L_t^a\, da.$$

Thus the random measures

$$\mu(f) = \int_0^t f(X_s)\, d[X, X]_s \quad \text{and} \quad \nu(f) = \int_0^t f(a) L_t^a\, da$$

agree for $w \in \mathscr{A}^c$ over the intervals with rational endpoints. Hence $\mu = \nu$, proving (3.1). □

Now we shall obtain a generalized Itô formula for $f \in \mathscr{UAD}$, $f = g \cdot A$, where $g(s, w, \cdot)$ is the difference of two convex functions. This result for $g(s, w, \cdot)\ C^2$ was already obtained by Protter [20], Chapter 5, Theorem 18.

First, we introduce some notation. If g: $\mathbb{R} \to \mathbb{R}$ is a convex function, it is well known, that the second distributional derivative is a σ-finite measure μ, such that

$$\forall\ K < \infty, \qquad \mu[-K, K] < \infty$$

and

$$\forall\ |x| < K, \qquad g(x) - \frac{1}{2}\int_{-K}^{K} |x - y| \mu(dy) = ax + b,$$

where a and b depend on K.

THEOREM 3.2. *Let $f \in \mathscr{UAD}$, $f = g \cdot A$ with $g(s, w, \cdot)$ a difference of two convex functions, and let $\mu(s, w, \cdot)$ be the second derivative as a distribution (on compacts). Then, if X is a continuous semimartingale with local time L_t^a, we have*

$$f(t, w, X_t) = f(0, w, X_0) + \int_{0^+}^t g(s, w, X_s)\, dA_s + \int_{0^+}^t \frac{\partial f}{\partial x}(s, w, X_s -)\, dX_s$$
$$+ \frac{1}{2}\int_0^t \int_{\mathbb{R}} (L_t^y - L_s^y) \mu(s, w, dy)\, dA_s,$$

where

$$\frac{\partial f}{\partial x}(s, w, x -) = \lim_{y \uparrow x} \frac{\partial f}{\partial x}(s, w, y).$$

In order to prove this result, we need a couple of lemmas, which we shall prove first.

LEMMA 3.3. *Let* $0 \le s \le t$ *and* $f: \mathbb{R}^2 \times \Omega \to \mathbb{R}$ *be* $B(\mathbb{R}^2) \otimes F_s$ *measurable, X a continuous semimartingale and ν a signed measure on* $\mathbb{R}$, *with absolute variation* $|\nu|$.

Assume

$$(3.2)\qquad W_u = \left(\int_{\mathbb{R}} (f(X_u, y, w))^2 |\nu(dy)|\right)^{1/2} 1_{\{s \le u \le t\}} \in L(X),$$

that is, it is integrable with respect to X (see Protter [20], *Chapter* 4), *a condition satisfied, for example, if f is bounded. Then the following Fubini-type result holds*:

$$(3.3)\qquad \int_{\mathbb{R}}\int_s^t f(X_u, y, w)\, dX_u \nu(dy) = \int_s^t \int_{\mathbb{R}} f(X_u, y, w) \nu(dy)\, dX_u.$$

PROOF. This follows by the usual Fubini theorem for semimartingales (Protter [20]) on taking the Jordan decomposition of ν. □

LEMMA 3.4. *Assume $\mu(s, w, dy)$ is a random measure such that $\mu(\cdot, \cdot, dy)$ is measurable, adapted, and there is a compact set $K \subset \mathbb{R}$ such that* $\forall\, s$, $\forall\, \mathscr{A} \in \mathscr{B}(\mathbb{R})$, $\mathscr{A} \subset K^c \Rightarrow \mu(s, w, \mathscr{A}) = 0$.

If $(M_y)_{y \in \mathbb{R}}$ *is a cadlag process such that for z fixed,* $E(M_y/\mathscr{F}_z)$ *has a cadlag version, and for every compact* $\Lambda \subseteq \mathbb{R}$,

$$(3.4)\qquad \sup_{y \in \Lambda} |M_y| \in L^2(P),$$

and A is an adapted increasing and continuous process such that

$$\int_0^t \int_K |\mu|(s, w, dy)\, dA_s \in L^2(P),$$

then if $t \le z$,

$$(3.5)\qquad E\left(\int_0^t \int_{\mathbb{R}} M_y \mu(s, w, dy)\, dA_s \middle| \mathscr{F}_z\right) = \int_0^t \int_{\mathbb{R}} E(M_y | \mathscr{F}_z) \mu(s, w, dy)\, dA_s.$$

PROOF. Assume $K = [a, b]$; then $|\mu|(s, w[a, b]^c) = 0$. Consider $f_n(s, w, y) = 1_{[0,t]}(s) \sum_{i=1}^{m(n)} M_{y_{i+1}^n} 1_{\{y_i^n < y \le y_{i+1}^n\}}$, where

$$\mathscr{P}_n = \{y_0^n = a < y_1^n < \cdots < y_{m(n)+1}^n = b\}$$

and mesh$(\mathscr{P}_n) \to 0$. Then $f_n \to_{n \to \infty} 1_{[0,t]}(s) M_y$, $y \in [a, b]$. By the DCT,

$$g_n = \int_0^t \int_{\mathbb{R}} f_n(s, w, y) \mu(s, w, dy)\, dA_s \xrightarrow[n \to \infty]{} \int_0^t \int_{\mathbb{R}} M_y \mu(s, w, dy)\, dA_s.$$

Since

$$|g_n| \le \left(\sup_{y \in [a,b]} |M_y| \int_0^t \int_{\mathbb{R}} |\mu|(s, w, dy)\, dA_s\right) \in L^1(P),$$

we conclude by the DCT for conditional expectations that

$$E(g_n | \mathscr{F}_z) \xrightarrow[n\to\infty]{} E\left(\int_0^t \int_{\mathbb{R}} M_y \mu(s, w, dy)\, dA_s \middle| \mathscr{F}_z \right),$$

in L^1 and a.s.

Now

$$E(g_n | \mathscr{F}_z) = E\left(\int_0^t \sum_i \int_{y_i^n}^{y_{i+1}^n} M_{y_{i+1}^n} \mu(s, w, dy)\, dA_s \middle| \mathscr{F}_z \right)$$

$$= \sum_i E\left\{ M_{y_{i+1}^n} \int_0^t \int_{y_i^n}^{y_{i+1}^n} \mu(s, w, dy)\, dA_s \middle| \mathscr{F}_z \right\}.$$

Since $t \le z$, we have $\int_0^t \int_{y_i^n}^{y_{i+1}^n} \mu(s, w, dy)\, dA_s \in \mathscr{F}_z$. Thus

$$E(g_n | \mathscr{F}_z) = \sum_i E(M_{y_{i+1}^n} | \mathscr{F}_z) \int_0^t \int_{y_i^n}^{y_{i+1}^n} \mu(s, w, dy)\, dA_s,$$

which converges a.s. to $\int_0^t \int_{\mathbb{R}} E(M_y | \mathscr{F}_z) \mu(s, w, dy)\, dA_s$, because $E(M_. | \mathscr{F}_z)$ is cadlag.

Notice that both sides of (3.5) are continuous in t, so the null set involved in (3.5) can be taken independent of t. □

COROLLARY 3.5. *Assume μ, A satisfy the hypotheses of the previous lemma and (M_v^y) is a process, jointly cadlag in y and continuous in v, such that for y fixed, M_v^y is a $(\mathscr{F}_t)$-martingale and for v fixed, M_v^y satisfies* (3.4). *Then*

$$N_v = \int_0^t \int_{\mathbb{R}} M_v^y \mu(s, w, dy)\, dAs,$$

$v \ge t$, is a continuous martingale w.r.t. $(\mathscr{F}_v)_{v \ge t}$.

PROOF. Let $v \ge t$ and $v' \ge t$ and $K \subset \mathbb{R}$ such that $\forall\, s, w, |\mu|(s, w, K^c) = 0$. Then

$$|N_v - N_{v'}| \le \sup_{y \in K} |M_v^y - M_{v'}^y| \int_0^t \int_{\mathbb{R}} |\mu|(s, w, dy)\, dA_s,$$

from which we conclude N has continuous paths. The martingale property is an application of Lemma 3.4 and the fact

$$M_z^y = E(M_v^y | \mathscr{F}_z), \quad \text{which is cadlag in } y. \qquad \square$$

PROOF OF THEOREM 3.2. By linearity we can assume that $g(s, w, \cdot\,)$ is a convex function. Moreover, we shall assume that there is a compact set Λ such that

$$\forall\, t, \forall\, w, \qquad \mu(t, w, \Lambda^c) = 0.$$

At the end of the proof we shall relax this hypothesis. We have

$$g(s,w,x) = a(s,w)x + b(s,w) + \frac{1}{2}\int |x-y|\mu(s,w,dy),$$

where

$$b(s,w) = g(s,w,0) - \frac{1}{2}\int |y|\mu(s,w,dy)$$

and

$$a(s,w) = g(s,w,1) - b(s,w) - \frac{1}{2}\int |1-y|\mu(s,w,dy).$$

Since $\tilde{f} = (a \cdot A)x + b \cdot A$ obviously satisfies the result, we can assume without loss of generality that

$$g(s,w,x) = \frac{1}{2}\int |x-y|\mu(s,w,dy).$$

In this way,

$$\frac{\partial g}{\partial x}(s,w,x) = \frac{1}{2}\int \operatorname{sign}(x-y)\mu(s,w,dy),$$

where

$$\operatorname{sign}(u) = \begin{cases} 1, & u > 0, \\ -1, & u \le 0. \end{cases}$$

Then $\partial g/\partial x(s,w,\cdot)$ is the left-continuous version.

Let $\varphi\colon \mathbb{R} \to \mathbb{R}_+$ be C^∞ with support contained in $[-1,0]$, such that

$$\int_{-\infty}^{\infty} \varphi(u)\,du = 1.$$

Define

$$g_n(s,w,x) = n\int_{-\infty}^{\infty} g(s,w,x+y)\varphi(ny)\,dy.$$

Then $g_n(s,w,\cdot)$ is C^∞ and if $x < x'$,

$$\begin{aligned} \frac{\partial g_n}{\partial x}(s,w,x) &= n\int_{-\infty}^{\infty} \frac{\partial g}{\partial x}(s,w,x+y)\varphi(ny)\,dy \\ &\le n\int_{-\infty}^{\infty} \frac{\partial g}{\partial x}(s,w,x'+y)\varphi(ny)\,dy, \end{aligned}$$

because $\partial g/\partial x(s,w,\cdot)$ is nondecreasing, so

$$\frac{\partial g_n}{\partial x}(s,w,x) \le \frac{\partial g_n}{\partial x}(s,w,x').$$

By the left continuity of $\partial g/\partial x(s, w, \cdot)$, we get

$$\forall\, s, w, x, \qquad \frac{\partial g_n}{\partial x}(s, w, x) \to \frac{\partial g}{\partial x}(s, w, x) \quad \text{when } n \to \infty.$$

Also, by continuity of $g(s, w, \cdot)$, we obtain $g_n(s, w, x) \to g(s, w, x)$. Consider $f_n = g_n \cdot A$. We have

$$\frac{\partial^3 g_n}{\partial x^3} = n^3 \int_{-\infty}^{\infty} \frac{\partial g}{\partial x}(s, w, v)\varphi''(n(v - x))\, dv,$$

from which and (2.1a) and (2.1b) we conclude

$$\forall\, t \geq 0, \forall\, K(\subseteq \mathbb{R}) \text{ compact}, \qquad \sup_{s \leq t,\, x \in K} \left| \frac{\partial^3 g_n}{\partial x^3}(s, w, x) \right| < \infty.$$

Thus we can apply Theorem 18 in Chapter 5 of Protter [20] to $f_n = g_n \cdot A$:

$$\begin{aligned} f_n(t, w, X_t) &= f_n(0, w, X_0) + \int_0^t g_n(s, w, X_s)\, dA_s \\ &\quad + \int_0^t \frac{\partial f_n}{\partial x}(s, w, X_s)\, dX_s + C_t^n, \end{aligned}$$

where

$$\begin{aligned} C_t^n &= \frac{1}{2}\int_0^t \frac{\partial^2 f_n}{\partial x^2}(s, w, X_s)\, d[X, X]_s \\ &= \frac{1}{2}\int_0^t \int_0^s \frac{\partial^2 g_n}{\partial x^2}(u, w, X_s)\, dA_u\, d[X, X]_s. \end{aligned}$$

Since $\partial^2 g_n/\partial x^2 \geq 0$, A_t and $[X, X]_t$ are increasing, we conclude that C_t^n is increasing in t, for each n.

Let $X = M + F$ be the canonical decomposition of X into a continuous local martingale M and a continuous finite variation process F. By stopping we can assume that X, $[X, X]$, M, F, A and

$$\begin{aligned} \int_0^t \mu(s, w, \mathbb{R})\, dA_s &\leq \int_0^t \int_{\mathbb{R}} (|y| + |1 - y|)\mu(s, w, dy)\, dA_s \\ &= 2\left(\int_0^t \{g(s, w, 0) + g(s, w, 1)\}\, dA_s\right) \end{aligned}$$

are bounded by $K < \infty$. So

$$\forall\, w, \qquad \sup_t |X_t| \leq K, \qquad \sup_t [X, X]_t \leq K, \qquad A_\infty \leq K.$$

We have that

$$\frac{\partial f_n}{\partial x}(t, w, X_t) \to \frac{\partial f}{\partial x}(t, w, X_t -) \quad \text{for all } t$$

and

$$\begin{aligned}\left|\frac{\partial f_n}{\partial x}(s, w, X_s)\right| &= \left|\int_0^s \frac{\partial g_n}{\partial x}(u, w, X_s)\, dA_u\right| \\ &\le \int_0^s \left|\frac{\partial g_n}{\partial x}(u, w, X_s)\right| dA_u \\ &\le n\int_0^s \int_{-\infty}^{\infty} \left|\frac{\partial g}{\partial x}(u, w, X_s + y)\right| \varphi(ny)\, dy\, dA_u \\ &\le \frac{n}{2}\int_{-\infty}^{\infty}\int_0^s \mu(u, w, \mathbb{R})\, dA_u\, \varphi(ny)\, dy \le \frac{K}{2}.\end{aligned}$$

Then by the DCT for semimartingales (Protter [20], Chapter 4, Theorem 32), we have

$$\int_0^s \frac{\partial f_n}{\partial x}(u, w, X_u)\, dX_u \to \int_0^s \frac{\partial f}{\partial x}(u, w, X_u -)\, dX_u,$$

uniformly in probability for $s \le t$. By the usual DCT,

$$\int_0^t g_n(s, w, X_s)\, dA_s \to \int_0^t g(s, w, X_s)\, dA_s$$

$[|g_n(s, w, X_s)| \le \sup_{u \le t,\, |x| \le K+1} |\partial g/\partial x(u, w, x)| K + |g(s, w, 0)|$, which is dA_s-integrable]. Thus

$$f(t, w, X_t) = f(0, w, X_0) + \int_0^t g(s, w, X_s)\, dA_s + \int_0^t \frac{\partial f}{\partial x}(s, w, X_s -)\, dX_s + C_t,$$

where C_t is the limit of C_t^n, which exists because all the other terms converge. Then C is a continuous, increasing and adapted process. In particular, $f(t, w, X_t)$ is a continuous semimartingale.

Let $\mathscr{P}^n = \{t_0^n = 0 < t_1^n < \cdots < t_m^n = t\}$ be a sequence of refining partitions of $[0, t]$ such that mesh $\mathscr{P}^n \to 0$. We shall omit the dependence of t_i on n. Then $f(t, w, X_t) - f(0, w, X_0) = S_1(n) + S_2(n)$, where

$$S_1(n) = \sum_i f(t_{i+1}, w, X_{t_{i+1}}) - f(t_i, w, X_{t_{i+1}}),$$

$$S_2(n) = \sum_i f(t_i, w, X_{t_{i+1}}) - f(t_i, w, X_{t_i}).$$

Since

$$f(t_{i+1}, w, X_{t_{i+1}}) - f(t_i, w, X_{i+1}) = \int_{t_i}^{t_{i+1}} g(u, w, X_{t_{i+1}})\, dA_u,$$

we have

$$\left| S_1(n) - \int_{0^+}^{t} g(u, w, X_u)\, dA_u \right|$$

$$\leq \sum_i \int_{t_i}^{t_{i+1}} \left| g(u, w, X_u) - g(u, w, X_{t_{i+1}}) \right| dA_u$$

$$\leq \sup_{u \leq t,\, |x| \leq K+1} \left| \frac{\partial g}{\partial x}(u, w, x) \right| \left\{ \sup_i \sup_{u \in (t_i, t_{i+1})} |X_{t_{i+1}} - X_u| \right\} (A_t - A_0),$$

which tends to 0 as mesh $\mathscr{P}^n \to 0$.

Since $g(s, w, x) = \frac{1}{2} \int |x - y| \mu(s, w, dy)$, we have

$$S_2(n) = \frac{1}{2} \sum_i \int_0^{t_i} \int_{\mathbb{R}} \left(|X_{t_{i+1}} - y| - |X_{t_i} - y| - J_i(y, w) \right) \mu(s, w, dy)\, dA_s$$

$$+ \frac{1}{2} \sum_i \int_0^{t_i} \int_{\mathbb{R}} J_i(y, w) \mu(s, w, dy)\, dA_s$$

$$= S_{21}(n) + S_{22}(n),$$

where

$$J_i(y, w) = \int_{t_i^+}^{t_{i+1}} \operatorname{sign}(X_v - y)\, dF_v + L_{t_{i+1}}^y - L_{t_i}^y.$$

By Fubini's theorem,

$$S_{22}(n) = \frac{1}{2} \sum_i \int_{t_i^+}^{t_{i+1}} \int_0^{t_i} \int_{\mathbb{R}} \operatorname{sign}(X_v - y) \mu(s, w, dy)\, dA_s\, dF_v$$

$$+ \frac{1}{2} \int_0^t \int_{\mathbb{R}} \sum_i (L_{t_{i+1}}^y - L_{t_i}^y) 1_{s \leq t_i} \mu(s, w, dy)\, dA_s$$

$$= S_{221}(n) + S_{222}(n).$$

Using the DCT, we have

$$S_{221}(n) = \sum_i \int_{t_i^+}^{t_{i+1}} \frac{\partial f}{\partial x}(t_i, w, X_v-)\, dF_v \to \int_{0^+}^{t} \frac{\partial f}{\partial x}(v, w, X_v-)\, dF_v.$$

By continuity properties of the local time,

$$S_{222}(n) \to \frac{1}{2} \int_0^t \int_{\mathbb{R}} (L_t^y - L_s^y) \mu(s, w, dy)\, dA_s.$$

Thus we can conclude that $S_{21}(n)$ converges to a continuous process, which we call J_t. So far we have proved

$$f(t,w,X_t) = \int_{0^+}^t \frac{\partial f}{\partial x}(s,w,X_s-)\,dM_s + \text{continuous finite variation terms.}$$

On the other hand,

$$f(t,w,X_t) = f(0,w,X_0) + \int_{0^+}^t g(s,w,X_s)\,dA_s + \int_{0^+}^t \frac{\partial f}{\partial x}(s,w,X_s-)\,dF_s$$
$$+ \frac{1}{2}\int_0^t\int_{\mathbb{R}} (L_t^y - L_s^y)\mu(s,w,dy)\,dA_s + J_t.$$

Thus, to finish the proof, we need to show that

$$J_t = \int_{0^+}^t \frac{\partial f}{\partial x}(s,w,X_s-)\,dM_s.$$

By the uniqueness of the decomposition for continuous semimartingales, this is equivalent to proving that J_t is a local martingale.

Let

$$S_{21}(n)_u = \frac{1}{2}\sum_i \int_0^{t_i}\int_{\mathbb{R}} (H^y_{t_{i+1}\wedge u} - H^y_{t_i\wedge u})\mu(s,w,dy)\,dA_s,$$

where

$$H_t^y = |X_t - y| - \int_{0+}^t \operatorname{sign}(X_v - y)\,dF_v - L_t^y.$$

By Tanaka's formula for fixed y, $(H^y_{t_{i+1}\wedge u} - H^y_{t_{i+1}\wedge u})1_{\{u\ge t_i\}}$ is a martingale. It is also jointly cadlag in y and continuous in u. By Corollary 3.5, $(S_{21}(n)_u)_{u\le t}$ is a continuous martingale, and

$$
\begin{aligned}
&E\Big((S_{21}(n)_u)^2\Big)\\
&= \frac{1}{4}\sum_i E\left\{\int_0^{t_i}\int_{\mathbb{R}} (H^y_{t_{i+1}\wedge u} - H^y_{t_i\wedge u})\mu(s,w,dy)\,dA_s\right\}^2\\
&\le \frac{1}{4}\sum_i E\left\{\int_0^{t_i}\int_{\mathbb{R}} (H^y_{t_{i+1}\wedge u} - H^y_{t_i\wedge u})^2\mu(s,w,dy)\,dA_s\int_0^t\int_{\mathbb{R}}\mu(s,w,dy)\,dA_s\right\}\\
&\le \frac{K}{4}\sum_i E\int_0^{t_i}\int_{\mathbb{R}} ([H^y,H^y]_{t_{i+1}\wedge u} - [H^y,H^y]_{t_i\wedge u})\mu(s,w,dy)\,dA_s.
\end{aligned}
$$

The last line is justified by an application of Corollary 3.5 to the process

$$(H^y_{t_{i+1}\wedge u} - H^y_{t_i\wedge u})^2 - ([H^y,H^y]_{t_{i+1}\wedge u} - [H^y,H^y]_{t_i\wedge u}).$$

Using the definition of H_t^y, we have

$$[H^y,H^y]_t = \int_0^t \operatorname{sign}(X_v - y)^2\,d[X,X]_v = [X,X]_t.$$

Then

$$\sup_{n\in\mathbb{N},\, u\le t} E\left((S_{21}(n))_u^2\right) \le \frac{K}{4} E\left(\int_0^t \int_{\mathbb{R}} [X,X]_t \mu(s,w,dy)\, dA_s\right) < \infty.$$

The sequence $(S_{21}(n))$ of martingales is UI (uniformly integrable). Since (J_t) is the UCP limit of $S_{21}(n)$, then J_t is a continuous martingale.

Finally, we shall reduce the general case (μ not necessarily concentrated on a fixed compact set) to the one just proved. Again we shall assume that X is bounded by K (by stopping). Define

$$g_K(s,w,x) = \begin{cases} g(s,w,K+1) + \dfrac{\partial g}{\partial x}(s,w,K+1)\{x-(K+1)\}: \\ \hfill x \ge K+1, \\ g(s,w,x): |x| < K+1, \\ g(s,w,-(K+1)) + \dfrac{\partial g}{\partial x}(s,w,-(K+1))(x+(K+1)): \\ \hfill x \le -(K+1). \end{cases}$$

Then $\mu_K(s,w,dx)$ is concentrated on $[-(K+1), K+1]$ and $\mu(s,w,\mathscr{A}) = \mu_K(s,w,\mathscr{A})$ if $\mathscr{A} \subset (-(K+1),(K+1))$, from which the result follows. □

The definition of the Fisk–Stratonovich integral depends on the existence of a certain quadratic covariation. For that matter the following result is essential.

THEOREM 3.6. *Let $f, h \in \mathscr{U}\mathscr{A}\mathscr{D}$, $f = g\cdot A$, $h = l \cdot A$ and X, Z be continuous semimartingales. Consider a sequence $(\mathscr{P}_n)$ of random partitions tending to the identity, that is,*

$$\mathscr{P}_n = \{T_0^n = 0 \le T_1^n \le T_2^n \le \cdots \le T_{m(n)}^n < \infty \ a.s.\},$$

where T_i^n are stopping times, which satisfy

$$\operatorname{mesh}(\mathscr{P}_n) = \max_i (T_{i+1}^n - T_i^n) \xrightarrow[n\to\infty]{} 0 \quad a.s.$$

and

$$T_{m(n)}^n \nearrow \infty \quad a.s.$$

Define

$$\Delta_{n,i}^t f = f\left(T_{i+1}^n \wedge t, w, X_{T_{i+1}^n}^t\right) - f\left(T_i^n \wedge t, w, X_{T_i^n}^t\right),$$

$$\Delta_{n,i}^t h = h\left(T_{i+1}^n \wedge t, w, Z_{T_{i+1}^n}^t\right) - h\left(T_i^n \wedge t, w, Z_{T_i^n}^t\right).$$

Then

$$\sum_i \Delta^t_{n,i} f \Delta^t_{n,i} h \xrightarrow[n\to\infty]{} \int_{0^+}^t \frac{\partial f}{\partial x}(s, w, X_s -) \frac{\partial h}{\partial x}(s, w, Z_s -)\, d[X, Z]_s$$

in UCP (uniformly on compacts in probability), where $\partial f / \partial x$ *and* $\partial h / \partial x$ *are the cadlag versions.*

PROOF. We can assume X and Z are bounded by γ. Let (ν^k) be an approximation of $\partial g / \partial x$ such that:

(i) $\nu^k(s, w, \cdot\,)$ is of finite variation on compacts, and it is cadlag.
(ii) $\forall\ t$, $\forall\ \Lambda(\subseteq \mathbb{R})$ compact,

$$\sup_k \sup_{s \le t,\, x \in \Lambda} |\nu^k(s, w, x)| < \infty \quad \text{a.s.}$$

and

$$\lim_k \sup_{s \le t,\ x \in \Lambda} \left| \nu^k(s, w, x) - \frac{\partial g}{\partial x}(s, w, x) \right| = 0.$$

In a similar way, ξ^k is an approximation of $\partial l / \partial x$ satisfying (i) and (ii). Let

$$g^k(s, w, x) = f(s, w, 0) + \int_0^x \nu^k(s, w, y)\, dy$$

and

$$l^k(s, w, x) = l(s, w, 0) + \int_0^x \xi^k(s, w, y)\, dy.$$

Define $f^k = g^k \cdot A$, $h^k = l^k \cdot A$. Then by Theorem 3.2 we have

$$f^k(t, w, X_t) = f^k(0, w, X_0) + \int_0^t \frac{\partial f^k}{\partial x}(u, w, X_u -)\, dX_u + F^k_t,$$

$$h^k(t, w, Z_t) = h^k(0, w, Z_0) + \int_0^t \frac{\partial h^k}{\partial x}(u, w, Z_u -)\, dZ_u + H^k_t,$$

where F^k, H^k are continuous finite variation processes. Then

$$\begin{aligned}
&[f^k(\cdot, w, X_\cdot), h^k(\cdot, w, Z_\cdot)]_t \\
&\quad = \int_{0^+}^t \frac{\partial f^k}{\partial x}(u, w, X_u -) \frac{\partial h^k}{\partial x}(u, w, Z_u -)\, d[X, Z]_u \\
&\qquad + f^k(0, w, X_0) h^k(0, w, Z_0)
\end{aligned}$$

and so

$$\sum_i \Delta^t_{n,i} f^k \Delta^t_{n,i} h^k \xrightarrow[n\to\infty]{\text{UCP}} \int_{0^+}^t \frac{\partial f^k}{\partial x}(s,w,X_s-)\frac{\partial h^k}{\partial x}(s,w,Z_s-)\,d[X,Z]_s. \tag{3.6}$$

Now

$$\begin{aligned}
&\left|\sum_i \Delta^s_{n,i} f^k \Delta^s_{n,i} h^k - \sum_i \Delta^s_{n,i} f \Delta^s_{n,i} h\right| \\
&\quad\le \left|\sum_i \left(\Delta^s_{n,i} f - \Delta^s_{n,i} f^k\right)\Delta^s_{n,i} h\right| + \left|\sum_i \left(\Delta^s_{n,i} h - \Delta^s_{n,i} h^k\right)\Delta^s_{n,i} f^k\right| \\
&\quad\le \left(\sum_i (\Delta^s_{n,i} h)^2\right)^{1/2}\left(\sum_i \left(\Delta^s_{n,i} f - \Delta^s_{n,i} f^k\right)^2\right)^{1/2} \\
&\qquad + \left(\sum_i \left(\Delta^s_{n,i} f^k\right)^2\right)^{1/2}\left(\sum_i \left(\Delta^s_{n,i} h - \Delta^s_{n,i} h^k\right)^2\right)^{1/2}.
\end{aligned} \tag{3.7}$$

We also have

$$\begin{aligned}
\left(\sum_i (\Delta^s_{n,i} h)^2\right)^{1/2} &\le \left(\sum_i \left(\Delta^s_{n,i} h - \Delta^s_{n,i} h^k\right)^2\right)^{1/2} \\
&\quad + \left(\sum_i \left(\Delta^s_{n,i} h^k\right)^2\right)^{1/2}.
\end{aligned} \tag{3.8}$$

We have the following estimates:

$$\begin{aligned}
\Delta^s_{n,i} f - \Delta^s_{n,i} f^k &= f\left(T^n_{i+1}\wedge s, w, X^s_{T^n_{i+1}}\right) - f\left(T^n_i \wedge s, w, X^s_{T^n_i}\right) \\
&\quad - \left[f^k\left(T^n_{i+1}\wedge s, w, X^s_{T^n_{i+1}}\right) - f^k\left(T^n_i \wedge s, w, X^s_{T^n_i}\right)\right] \\
&= \int_0^{T^n_{i+1}\wedge s}\left\{g\left(u,w,X^s_{T^n_{i+1}}\right) - g^k\left(u,w,X^s_{T^n_{i+1}}\right)\right\} dA_u \\
&\quad - \int_0^{T^n_i\wedge s}\left\{g\left(u,w,X^s_{T^n_i}\right) - g^k\left(u,w,X^s_{T^n_i}\right)\right\} dA_u \\
&= \int_{T^n_i\wedge s}^{T^n_{i+1}\wedge s}\left\{g\left(u,w,X^s_{T^n_{i+1}}\right) - g^k\left(u,w,X^s_{T^n_{i+1}}\right)\right\} dA_u \\
&\quad + \int_0^{T^n_i\wedge s}\Big\{g\left(u,w,X^s_{T^n_{i+1}}\right) - g\left(u,w,X^s_{T^n_i}\right) \\
&\qquad - \left[g^k\left(u,w,X^s_{T^n_{i+1}}\right) - g^k\left(u,w,X^s_{T^n_i}\right)\right]\Big\} dA_u.
\end{aligned}$$

So

$$\left|\Delta^s_{n,i} f - \Delta^s_{n,i} f^k\right| \le C(s,k)\left(A_{T^n_{i+1}\wedge s} - A_{T^n_i \wedge s}\right) + \tilde{C}(s,k)\left|X^s_{T^n_{i+1}} - X^s_{T^n_i}\right| A_s,$$

where

$$C(s,k) = \sup_{u\le s,\, |x|\le\gamma} \left|g(u,w,x) - g^k(u,w,x)\right|,$$

$$\tilde{C}(s,k) = \sup_{u\le s,\, |x|\le\gamma} \left|\frac{\partial g}{\partial x}(u,w,x) - \nu^k(u,w,x)\right|.$$

Notice that $C(s,k) \le \tilde{C}(s,k)\cdot\gamma$. Then

$$\sup_{s\le t}\sum_i \left(\Delta^s_{n,i} f - \Delta^s_{n,i} f^k\right)^2 \le M(t,k) + \left(\tilde{C}(t,k)A_t\right)^2 \sum_i \left(X^t_{T^n_{i+1}} - X^t_{T^n_i}\right)^2, \tag{3.9}$$

where $M(t,k) \to_{k\to\infty} 0$ and $M(t,k)$ does not depend on n. In a similar way,

$$\sup_{s\le t}\sum_i \left(\Delta^s_{n,i} h - \Delta^s_{n,i} h^k\right)^2 \le N(t,k) + \left(\tilde{D}(t,k)A_t\right)^2 \sum_i \left(Z^t_{T^n_{i+1}} - Z^t_{T^n_i}\right)^2, \tag{3.10}$$

where $N(t,k) \to_{k\to\infty} 0$, $N(t,k)$ does not depend on n, and

$$\tilde{D}(t,k) = \sup_{u\le t,\, |x|\le\gamma} \left|\frac{\partial l}{\partial x}(u,w,x) - \xi^k(u,w,x)\right| \xrightarrow[k\to\infty]{} 0.$$

Using the same technique, we conclude

$$\sup_{s\le t}\sum_i \left(\Delta^s_{n,i} f^k\right)^2 \le F(t) + G(t)\sum_i \left(X^t_{T^n_{i+1}} - X^t_{T^n_i}\right)^2, \tag{3.11}$$

where F and G do not depend on k or n, and

$$\sup_{s\le t}\sum_i \left(\Delta^s_{n,i} h^k\right)^2 \le F(t) + G(t)\sum_i \left(Z^t_{T^n_{i+1}} - Z^t_{T^n_i}\right)^2. \tag{3.12}$$

Finally, let

$$L^n(f,h) = \sup_{s\le t}\left|\sum_i \Delta^s_{n,i} f\,\Delta^s_{n,i} h - \int_{0^+}^t \frac{\partial f}{\partial x}(u,w,X_u)\frac{\partial h}{\partial x}(u,w,Z_u)\,d[X,Z]_u\right|$$
$$\le L^{k,n}_1 + L^{k,n}_2 + L^{k,n}_3,$$

where

$$L^{k,n}_1 = \sup_{s\le t}\left|\sum_i \Delta^s_{n,i} f\,\Delta^s_{n,i} h - \sum_i \Delta^s_{n,i} f^k\,\Delta^s_{n,i} h^k\right|,$$

$$L^{k,n}_2 = L^n(f^k, h^k),$$

$$L^{k,n}_3 = L^k_3 = \sup_{s\le t}\left|\int_{0^+}^t \left(\frac{\partial f^k}{\partial x}\frac{\partial h^k}{\partial x} - \frac{\partial f}{\partial x}\frac{\partial h}{\partial x}\right) d[X,Z]_s\right|.$$

For $\varepsilon > 0$,

$$P[L^n(f,h) \geq \varepsilon] \leq \sum_{i=1}^{3} P\left[L_i^{k,n} \geq \frac{\varepsilon}{3}\right].$$

Thus, for any k,

$$\limsup_n P[L^n(f,h) \geq \varepsilon] \leq \sum_{i=1}^{3} \limsup_n P\left[L_i^{k,n} \geq \frac{\varepsilon}{3}\right].$$

By (3.6) for k fixed, $L_2^{k,n} \to_{n\to\infty} 0$ in probability. By (3.7) to (3.12) we get the following bound for $L_1^{k,n}$:

$$\left(L_1^{k,n}\right)^2 \leq \rho(t,k) + \phi(t,k)\left\{\left(\sum_i \left(X_{T_{i+1}^n}^t - X_{T_i^n}^t\right)^2\right)^{1/2} + \left(\sum_i \left(X_{T_{i+1}^n}^t - Z_{T_i^n}^t\right)^2\right)^{1/2}\right\}^2,$$

where ρ, ϕ are independent of n, and they tend to 0 a.s. as $k \to \infty$. So

$$\limsup_n P\left[L_1^{k,n} \geq \frac{\varepsilon}{3}\right] \leq P\left\{\rho(t,k) + \phi(t,k)\left([X,X]_t^{1/2} + [Z,Z]_t^{1/2}\right)^2 \geq \frac{\varepsilon}{3}\right\},$$

which tends to 0 as $k \to \infty$. Since L_3^k does not depend on n, we get

$$\limsup_n P[L^n(f,h) \geq \varepsilon] \leq \limsup_k P\left[L_3^k \geq \frac{\varepsilon}{3}\right],$$

which is 0 because L_3^k tends to 0 a.s. □

The previous result shows the existence of the quadratic covariation between $f(X)$ and $h(Z)$, which allows us to define the Fisk–Stratonovich integral of $f(X)$ with respect to dZ.

DEFINITION 3.7. If X and Z are continuous semimartingales and f, h are in $\mathscr{UAD}$, we define $[f(X), h(Z)]$ as

$$[f(X),h(Z)]_t = \int_{0^+}^{t} \frac{\partial f}{\partial x}(s,w,X_s-)\frac{\partial h}{\partial x}(s,w,Z_s-)\,d[X,Z]_s + f(0,w,X_0)h(0,w,Z_0) \tag{3.13}$$

and

$$[f(X),h(Z)]_t^c = \int_{0^+}^{t} \frac{\partial f}{\partial x}(s,w,X_s-)\frac{\partial h}{\partial x}(s,w,Z_s-)\,d[X,Z]_s. \tag{3.14}$$

That is, $[f(X), h(Z)]^c$ is the continuous part of $[f(X), h(Z)]$, and is 0 at $t = 0$.

Notice that even though $f(X)$ and $h(Z)$ are not necessarily semimartingales, the quadratic covariation $[f(X), h(Z)]$ is well defined, and moreover it is a limit of sums in the sense of Theorem 3.6.

The quadratic covariation between $f(X)$ and $h(Z)$ also satisfies the Kunita–Watanabe inequality

$$|[f(X),h(Z)]_t| \leq ([f(X),f(X)]_t)^{1/2}([h(Z),h(Z)]_t)^{1/2}. \tag{3.15}$$

Now, we shall prove that in (3.13) [or (3.14)] we can replace $\partial f/\partial x(s, w, x-)$ by $\partial f/\partial x(s, w, x)$. More generally, we have the following lemma.

LEMMA 3.8. *Let X, Z be continuous semimartingales and f, h be in $\mathscr{U}\mathscr{A}\mathscr{D}$. If*

$$\tilde{f}_x(s, w, x) = \int_0^s \tilde{g}_x(u, w, x)\, dA_u$$

and

$$\tilde{h}_x(s, w, x) = \int_0^s \tilde{l}_x(u, w, x)\, dA_u,$$

where

$$\forall\,(u, w), \qquad \lambda\left\{x : \tilde{g}_x(u, w, x) \neq \frac{\partial g}{\partial x}(u, w, x)\right\} = 0$$

(λ is the Lebesgue measure and $\partial g/\partial x$ is the cadlag version), and similarly for $\tilde{l}_x$. Then

$$[f(X), h(Z)]_t^c = \int_{0^+}^t \tilde{f}_x(s, w, X_s)\tilde{h}_x(s, w, Z_s)\, d[X, Z]_s.$$

PROOF. Let

$$I_t = \left|\int_{0^+}^t \left\{\tilde{f}_x(s, w, X_s) - \frac{\partial f}{\partial x}(s, w, X_s-)\right\}\frac{\partial h}{\partial x}(s, w, Z_s-)\, d[X, Z]_s\right|.$$

I_t is a continuous process with paths of bounded variation. We shall prove that $I_t = 0$ a.s. By the Kunita–Watanabe inequality, we have

$$I_t \le \left(\int_{0^+}^t \left(\tilde{f}_x - \frac{\partial f}{\partial x}\right)^2 d[X, X]_s\right)^{1/2} \left(\int_{0^+}^t \left\{\frac{\partial h}{\partial x}(s, w, Z_s-)\right\}^2 d[Z, Z]_s\right)^{1/2},$$

but

$$\begin{aligned}
&\int_{0^+}^t \left(\tilde{f}_x - \frac{\partial f}{\partial x}\right)^2 d[X, X]_s \\
&\quad= \int_{0^+}^t \left(\int_0^s \tilde{g}_x(u, w, X_s) - \frac{\partial g}{\partial x}(u, w, X_s-)\, dA_u\right)^2 d[X, X]_s \\
&\quad\le A_t \int_{0^+}^t \int_0^t \left(\tilde{g}_x(u, w, X_s) - \frac{\partial g}{\partial x}(u, w, X_s-)\right)^2 dA_u\, d[X, X]_s \\
&\quad= A_t \int_0^t \int_{0^+}^t \left(\tilde{g}_x(u, w, X_s) - \frac{\partial g}{\partial x}(u, w, X_s-)\right)^2 d[X, X]_s\, dA_u \\
&\quad= A_t \int_0^t \int_{\mathbb{R}} L_t^a(X)\left(\tilde{g}_x(u, w, a) - \frac{\partial g}{\partial x}(u, w, a-)\right)^2 da\, dA_u.
\end{aligned}$$

The last equality is due to the occupation–time formula. For each (u, w) fixed,

$$\lambda\left\{a: \tilde{g}_x(u, w, a) \neq \frac{\partial g}{\partial x}(u, w, a-)\right\} = 0$$

[$\partial g/\partial x(u, w, \cdot\,)$ and $\partial g/\partial x(u, w, \cdot\, -\,)$ differ at most on a countable set], from which $I_t = 0$. By a similar argument we have

$$J_t = \left| \int_{0^+}^t \tilde{f}_x(s, w, X_s)\left\{\frac{\partial h}{\partial x}(s, w, Z_s-) - \tilde{h}_x(s, w, Z_s)\right\} d[X, Z]_s \right| = 0,$$

from which the result follows. □

Since

$$\frac{\partial f}{\partial x}(s, w, x-) = \int_0^s \frac{\partial g}{\partial x}(u, w, x-)\, dA_u$$

and

$$\left\{x: \frac{\partial g}{\partial x}(u, w, x) \neq \frac{\partial g}{\partial x}(u, w, x-)\right\}$$

is countable, we have that in (3.13) [or (3.14)] we can replace $\partial f/\partial x(s, w, X_s-\,)$ by $\partial f/\partial x(s, w, X_s)$.

DEFINITION 3.9. Let X and Z be continuous semimartingales and $f \in \mathscr{U}\mathscr{A}\mathscr{D}$. Then we define the Fisk–Stratonovich integral as

$$\begin{aligned} &\int_0^t f(s, w, X_s) \circ dZ_s \\ (3.16) \qquad &= \int_0^t f(s, w, X_s)\, dZ_s + \frac{1}{2}[f(X), Z]_t^c \\ &= \int_0^t f(s, w, X_s)\, dZ_s + \frac{1}{2}\int_{0^+}^t \frac{\partial f}{\partial x}(s, w, X_s)\, d[X, Z]_s. \end{aligned}$$

One is tempted to define the Fisk–Stratonovich integral to be the last line of (3.16), for all functions f such that the right-hand side makes sense. There are two objections to this. First of all, in general, we would not have that the Fisk–Stratonovich integral is the limit of sums of the type

$$(3.17) \qquad \sum_{P_n} \frac{f(s_{i+1}, w, X_{s_{i+1}}) + f(s_i, w, X_{s_i})}{2}\{Z_{s_{i+1}} - Z_{s_i}\}.$$

Second, the last term of (3.16) could depend on the particular version of $\partial f/\partial x$ we have chosen. For example, let $f(t, w, x) = |x - B_t(w)|$, where B_t is a one-dimensional Brownian motion. Then the cadlag version of $\partial f/\partial x$ is sign($x - B_t$), where

$$\operatorname{sign}(u) = \begin{cases} 1, & u \geq 0, \\ -1, & u < 0, \end{cases}$$

but

$$\int_0^t \frac{\partial f}{\partial x}(s, w, B_s)\, d[B, B]_s = t \neq -t = \int_0^t \frac{\partial f}{\partial x}(s, w, B_s -)\, d[B, B]_s.$$

Notice that $f \notin \mathscr{U}\mathscr{A}\mathscr{D}$, because for x fixed, $f(\cdot, w, x)$ is not of bounded variation.

One can define the Fisk–Stratonovich integral for $f(X_t)$, when f is absolutely continuous (and does not depend on t, w) whose derivative $f' \in L^2_{\text{loc}}(\mathbb{R}, d\lambda)$, as

$$\int_0^t f(X_s) \circ dZ_s = \int_0^t f(X_s)\, dZ_s + \frac{1}{2}\int_{0^+}^t f'(X_s)[X, Z]_s, \tag{3.18}$$

where f' is any version of the derivative of f. There are two things to verify. First, the last integral in (3.18) exists for any version of f', and second, the integral does not depend on the version of f' we use.

Let us prove first that (3.18) makes sense for any particular version of f'. In fact,

$$\begin{aligned} \left| \int_{0^+}^t f'(X_s)\, d[X, Z]_s \right| &\leq \left(\int_0^t (f'(X_s))^2\, d[X, X]_s \right)^{1/2} \left(\int_0^t d[Z, Z]_s \right)^{1/2} \\ &\leq \left(\int_{\mathbb{R}} (f'(a))^2 L_t^a(X)\, da \right)^{1/2} ([Z, Z]_t)^{1/2}. \end{aligned}$$

We know that $\forall\ a > X_t^* = \sup_{s \leq t}|X_s|$, $L_t^a(X) = 0$, and $\sup_a L_t^a(X) < \infty$ a.s. (L_t is cadlag). Then

$$\begin{aligned} &\left| \int_{0^+}^t f'(X_s)\, d[X, Z]_s \right| \\ &\quad \leq \left(\sup_a L_t^a(X)[Z, Z]_t \right)^{1/2} \left\{ \int_{-X_t^*}^{X_t^*} (f'(a))^2\, da \right\}^{1/2} < \infty \quad \text{a.s.} \end{aligned}$$

The same sort of technique proves that (3.18) does not depend on the version of f' we use. The problem is that we do not know if $\int_0^t f(X_s) \circ dZ_s$ is the limit of sums of the type (3.17). For that to happen, we feel that some regularity of f' is needed. It seems that something like "f' must be Riemann integrable over compact sets" is enough.

From now on we shall assume that $\partial f/\partial x(s, w, \cdot)$ is the cadlag version unless we specify what version we are using.

4. Existence. In this section we shall prove that under certain conditions the Stratonovich differential equation (4.1) has a strong solution.

$$(4.1)\qquad \begin{aligned} dX_t &= \sum_{i=1}^{n} \sigma_i(t,w,X_t)\circ dZ_t^i + \sum_{k=1}^{m} h_k(t,w,X_t)\,dA_t^k, \\ X_0 &= X_0 \in \mathscr{F}_0, \end{aligned}$$

where Z^i are continuous semimartingales, and A^k are continuous finite variation processes (henceforth CFV).

That is, there is a semimartingale X_t with continuous paths which satisfies (4.1). We shall say that X_t is strict if and only if X_t is adapted to the augmented right-continuous filtration (G_t) associated with

$$\mathscr{A}_t = \sigma\{X_0, (Z_s^i), (A_s^k), (\sigma_i(s,\cdot,x)), (h_k(s,\cdot,x)), s \le t, x \in \mathbb{R}\}.$$

If $\mathscr{A}_\infty = \vee_t \mathscr{A}_t$ and $N = \{A \in \mathscr{A}_\infty\colon P(A)(1-P(A)) = 0\}$, then $G_t = (\bigcap_{u>t}\mathscr{A}_u) \vee N$.

Since we are working in one dimension, we also expect to find a maximal and a minimal solution. In the Itô case, Barlow and Perkins [3] have a nice result in this direction. Moreover, in this paper there are two ideas very important for us. First is the LT condition (which I shall explain in a moment) and a way to approximate the coefficients σ_i, h_k to obtain a sequence of solutions, of these approximated equations, which will tend to the minimal solution of (4.1).

DEFINITION 4.1. The function $\eta\colon \mathbb{R}_+ \times \Omega \times \mathbb{R} \to \mathbb{R}$ is said to be:

(a) Uniformly Lipschitz (UL) if:

(i) $\eta(\cdot,\cdot,x) \in \mathbb{L} = \{$adapted processes, left continuous with right limits$\}$.

(ii) $$\forall\, t, \qquad \sup_{s\le t,\, x\ne y} \frac{|\eta(s,w,x)-\eta(s,w,y)|}{|x-y|} \le k(t,w) < \infty.$$

(b) Uniformly Lipschitz on compacts (ULC) if it satisfies (i) but (ii) is changed to

(ii′)
$$\forall\, t,\ \Lambda(\subseteq \mathbb{R}) \text{ compact}, \qquad \sup_{\substack{s\le t\\ (x,y)\in\Lambda\times\Lambda\\ x\ne y}} \frac{|\eta(s,w,x)-\eta(s,w,y)|}{|x-y|} \le k(t,w,\Lambda) < \infty,$$

where in both case $k(\cdot,\cdot)$ and $k(\cdot,\cdot,\Lambda)$ can be chosen increasing and in $\mathbb{L}$.

(c) Locally bounded if

$$\forall\, t>0,\ \Lambda(\subseteq\mathbb{R}) \text{ compact}, \qquad \sup_{s\le t,\, x\in\Lambda} |\eta(s,w,x)| = k(w) < \infty \quad \text{a.s.}$$

The following lemma is an important tool and can be found in Le Gall [14] and Barlow and Perkins [3] for more general coefficients, but for only one driving semimartingale Z.

LEMMA 4.2. *Assume X^1 and X^2 satisfy*

$$dX_t^i = \sum_j \sigma_j(t, w, X_t^i)\, dZ_t^j + dV_t^i, \qquad i = 1, 2, \tag{4.2}$$

where Z^j are continuous semimartingales, V^j are CFV and σ_i are ULC functions. Then

$$L_t^0(X^1 - X^2) = 0 \quad \text{for all } t \geq 0 \text{ a.s.}$$

This result motivates the following definition (see Barlow and Perkins [3]).

DEFINITION 4.3. Consider (Z^i), a finite family of continuous semimartingales. Let σ_i: $\mathbb{R}_+ \times \Omega \times \mathbb{R} \to \mathbb{R}$ be $\mathscr{P} \otimes B(\mathbb{R})$ measurable (here $\mathscr{P}$ is the predictable σ-field), such that for any continuous process Y, $\sigma_i(t, w, Y_t) \in L(Z^i)$ (it is integrable with respect to Z^i; see Protter [20], Chapter 4), for example, if σ_i is locally bounded or if $\sigma_i(\cdot, \cdot, Y_\cdot) \in \mathbb{L}$. Then we say $\boldsymbol{\sigma} = (\sigma_i, \dots, \sigma_n) \in LT(Z^1, \dots, Z^n)$ iff for any X^1 and X^2 continuous processes satisfying (4.2), $L_t^0(X^1 - X^2) = 0$ for all $t \geq 0$ a.s.

In this context Lemma 4.2 can be expressed in the following way: If $\sigma_i \in$ ULC, then $\boldsymbol{\sigma} \in LT(Z^1, \dots, Z^n)$.

The LT condition has very interesting consequences (see Barlow and Perkins [3, 4]). For us, the most important thing is that (4.1) is stable under maximum and minimum operations, as has been noted in the former cited paper.

LEMMA 4.4. *Let $\boldsymbol{\sigma} \in LT(Z^1, \dots, Z^n)$, $h_k(\cdot, \cdot, \cdot)$ measurable, $h_k(\cdot, \cdot, x)$ adapted and h_k locally bounded. Let X^1 and X^2 be two solutions of (Itô),*

$$\begin{aligned} dX_t &= \sum_j \sigma_j(t, w, X_t)\, dZ_t^j + \sum_k h_k(t, w, X_t)\, dA_t^k, \\ X_0 &= X_0, \end{aligned} \tag{4.3}$$

where Z^j are continuous semimartingales and A^k are CFV. Then $(X^1 \vee X^2)_t = \max\{X_t^1, X_t^2\}$ and $(X^1 \wedge X^2)_t = \min\{X_t^1, X_t^2\}$ are solutions of (4.3).

PROOF. By Tanaka's formula and the LT condition,

$$\begin{aligned} X_t^1 \vee X_t^2 &= X_t^1 + \int_0^t 1_{\{X_s^2 > X_s^1\}}\, d(X_s^2 - X_s^1) + \frac{1}{2} L_t^0(X^1 - X^2) \\ &= X_0 + \sum_j \int_0^t \{\sigma_j(s, w, X_s^1) + 1_{\{X_s^2 > X_s^1\}}\, \Delta\sigma_j(s)\}\, dZ_s^j \\ &\quad + \sum_k \int_0^t \{h_k(s, w, X_s^1) + 1_{\{X_s^2 > X_s^1\}}\, \Delta h_k(s)\}\, dA_s^k, \end{aligned}$$

where $\Delta\sigma_j(s) = \sigma_j(s, w, X_s^2) - \sigma_j(s, w, X_s^1)$ (analogously for Δh_k). Thus

$$X_t^1 \vee X_t^2 = X_0 + \sum_j \int_0^t \sigma_j(s, w, X_s^1 \vee X_s^2)\, dZ_s^j$$
$$+ \sum_k \int_0^t h_k(s, w, X_s^1 \vee X_s^2)\, dA_s^k,$$

from which the result holds. □

The next result is without doubt the most important tool we use to prove existence for (4.1). This kind of result is called a comparison lemma, and it is used frequently in the theory of ODE and also in stochastic differential equations (see Barlow and Perkins [3]).

We need to introduce the following partial order.

DEFINITION 4.5. Let $\eta, \gamma: \mathbb{R}_+ \times \Omega \times \mathbb{R} \to \mathbb{R}$ be two functions; we denote $\eta <^* \gamma$ if

$\forall\, (s, w, x)\ \exists\ \delta > 0$ such that if $|x - y| < \delta$, $|x - y'| < \delta$ and $s \le s' \le s + \delta$,

then

$$\eta(s', w, y) \le \gamma(s', w, y').$$

LEMMA 4.6. *Let (M^i) be a finite family of continuous local martingales and (A^k) be a finite family of continuous increasing processes. Assume $\sigma = (\sigma_1, \dots, \sigma_n) \in LT(M^1, \dots, M^n)$ and σ_i are locally bounded, (h_k^j), $j = 1, 2$, are measurable, adapted and locally bounded with $h_k^1 <^* h_k^2$. If X^j satisfies*

$$dX_t^j = \sum_i \sigma_i(t, w, X_t^j)\, dM_t^i + \sum_k h_k^j(t, w, X_t^j)\, dA_t^k, \qquad X_0^j = X_0^j,$$

and $X_0^1 \le X_0^2$, then $X_t^1 \le X_t^2$ for all $t \ge 0$ a.s.

Lemma 4.6 is a minor improvement of Proposition 3.5 in Barlow and Perkins [3], and its proof is essentially the same plus a localization argument.

COROLLARY 4.7. *Assume M^i are continuous local martingales, A^k are CFV (with total variation $|dA^k|$), σ_i are UL, h_k are measurable adapted and there exists a UL function f, such that $|h_k| <^* f$. If X, X^U and X^L satisfy*

$$dX_t = \sum_i \sigma_i(t, w, X_t)\, dM_t^i + \sum_k h_k(t, w, X_t)\, dA_t^k, \qquad X_0 = X_0,$$

$$(U)\quad dX_t^U = \sum_i \sigma_i(t, w, X_t^U)\, dM_t^i + f(t, w, X_t^U) \sum_k |dA_t^k|, \qquad X_0^U = X_0^U,$$

$$(L)\quad dX_t^L = \sum_i \sigma_i(t, w, X_t^L)\, dM_t^i - f(t, w, X_t^L) \sum_k |dA_t^k|, \qquad X_0^L = X_0^L,$$

and $X_0^L \le X_0 \le X_0^U$, then $X_t^L \le X_t \le X_t^U$ for all $t \ge 0$ a.s.

PROOF. Note that equations (U) and (L) have unique solutions, because the coefficients are Lipschitz. To prove the corollary, it is enough to notice that X satisfies

$$dX = \sum_i \sigma_i \, dM^i + \sum h_k \, d(A^k)^+ - \sum h_k \, d(A^k)^-$$

and

$$|h_k| <^* f \Rightarrow -f <^* \pm h_k \text{ and } \pm h_k <^* f. \qquad \square$$

An interesting implication of the comparison lemma is the nonexistence of explosion times for (4.1) under linear growth, which we shall prove in Corollary 4.11.

DEFINITION 4.8. A measurable function $h = \mathbb{R}_+ \times \Omega \times \mathbb{R} \to \mathbb{R}$ is said to have linear growth if there exist processes K and H such that:

(i) $H, K \in \mathbb{L}$, K is increasing, and
(ii) $|h(s,w,x)| \le K(s,w)|x| + H(s,w)$.

DEFINITION 4.9. We say the Itô equation (Z_k are continuous semimartingales and A^k are CFV)

$$(4.4) \qquad dX = \sum \sigma_i(t,w,X_t)\, dZ_t^i + \sum h_k(t,w,X_t)\, dA_t^k, \qquad X_0 = X_0,$$

has an explosion time, if there exists a process X taking values in $\overline{\mathbb{R}}$ such that if $S_n = \inf\{t \ge 0: |X_t| \ge n\}$, then $X_{t \wedge S_n}$ has continuous paths, satisfies

$$X_{t \wedge S_n} = X_0 + \sum_i \int_0^t \sigma_i(s,w,X_s) 1_{|X_s| \le n} \, dZ_s^i + \sum_k \int_0^t h_k(s,w,X_s) 1_{|X_s| \le n} \, dA_s^k$$

and $P[\lim_n S_n < \infty] > 0$. $S = \lim_n S_n$ is called an explosion time for (4.4).

Notice that (4.1) can be written in the form of (4.4). In fact, (4.1) is equivalent to

$$(4.5) \qquad dX_t = \sum_i \sigma_i \, dZ^i + \frac{1}{2} \sum_{i,j} \sigma_i \frac{\partial \sigma_j}{\partial x} d[Z^i, Z^j] + \sum_k h_k \, dA^k.$$

THEOREM 4.10. *If in* (4.4), $\sigma_i \in UL$, h_k *are measurable, adapted and they have linear growth, then* (4.4) *does not have an explosion time.*

PROOF. Without loss of generality we can assume that Z^i are continuous local martingales. Since (h_k) have linear growth, then there exist $K \in \mathbb{L}$ and $H \in \mathbb{L}$ such that

$$\forall\, k, \qquad |h_k(s,w,x)| \le K(s,w)|x| + H(s,w).$$

Define $f(s,w,x) = (K(s,w)+1)|x| + H(s,w) + 1$. It is easy to see that $|h_k| <^* f$ and $f \in UL$. By Corollary 4.7, $X_{t \wedge S_n}^L \le X_{t \wedge S_n} \le X_{t \wedge S_n}^U$ (we have taken $X_0^L = X_0 = X_0^U$), where X and S_n are as in Definition 4.9.

Since equation (U) and (L) have coefficients in UL, they do not have an explosion time (see Protter [20], Chapter 5, Theorem 7). Thus (4.4) does not have an explosion time, either.

Using the relation between the Stratonovich equation (4.1) and the Itô equation (4.5), we get the following corollary.

COROLLARY 4.11. *The Stratonovich equation* (4.1) *does not have an explosion time if* $\sigma_i \in \mathscr{U}\mathscr{A}\mathscr{D}$, $\partial\sigma_i/\partial x(s, w, \cdot\,)$ *are bounded and* h_k *are measurable, adapted and have linear growth.*

If $\sigma_i \in \mathscr{U}\mathscr{A}\mathscr{D}$, but $\partial\sigma_i/\partial x$ are not bounded, the previous result does not hold in general as is shown by the following example. For $\varepsilon > 0$ consider $dX_t = X_t^{1+\varepsilon} \circ dB_t$, $X_0 = 1$, where B is a one-dimensional Brownian motion, $B_0 = 0$. A solution of this equation (up to the explosion time) is, by Itô's formula,

$$X_t = (1 - \varepsilon B_t)^{-1/\varepsilon}, \qquad t < S = \inf\{t > 0: B_t = \varepsilon^{-1}\}.$$

Since, $P(S < \infty) = 1$, this equation has an explosion time.

Now, we shall concentrate on the main theorem of this section, which gives us sufficient conditions for the existence of a solution for (4.1). For that matter the following lemma, which can be found in Barlow and Perkins [3], is very useful.

LEMMA 4.12 (Approximation lemma). *Let* $h: \mathbb{R} \to \mathbb{R}$ *be a lower semicontinuous function bounded by* M. *Then if* $f_n(x) = \inf_y\{h(y) + n|x - y|\} - 2^{-n}$, *we have the following*:

(i) $\forall\, x, n,\ |f_n(x)| \le M + 1$.
(ii) $f_n(x) \nearrow h(x)$.
(iii) $\forall\, x, y,\ |f_n(x) - f_n(y)| \le n|x - y|$ (*Lipschitz*).
(iv) $f_n <^* f_{n+1}$.
(v) *If* h *is continuous at* x *and* $x_n \to x$, *then* $f_n(x_n) \to h(x)$.

Notice that we do not assume that h is left continuous.

PROOF. Only part (v) is not proved in the cited paper.
By definition, $f_n(x_n) \le h(x_n) \to h(x)$, then $\limsup f_n(x_n) \le h(x)$.
On the other hand, $\forall\ n\ \exists\ y_n$ such that

$$f_n(x_n) \ge h(y_n) + n|x_n - y_n| - 2^{-n} - n^{-1}.$$

Since f_n are uniformly bounded, we conclude $\limsup_n n|x_n - y_n| < \infty \Rightarrow y_n \to x$.

Finally,

$$\liminf f_n(x_n) \geq \liminf h(y_n) = h(x),$$

from which $\lim f_n(x_n) = h(x)$. □

The last definition we need in this section is the following.

DEFINITION 4.13. A measurable subset D of $\mathbb{R}^2$ is said to have content 0 if $D_y = \{s\colon (s,y) \in D\}$ has null measure for every nonatomic measure on $\mathbb{R}$, except for a set of Lebesgue measure 0 of $y \in \mathbb{R}$.

Any countable subset D of $\mathbb{R}^2$ has content 0. Also if A is a countable subset of $\mathbb{R}$, then $D = \mathbb{R} \times A$ has content 0, but $\mathbb{R} \times A$ is not countable.

THEOREM 4.14. *Consider the Stratonovich equation* (4.1), *where* (Z^i) *are continuous semimartingales*, (A^k) *are CFV and*

(a.1) $\sigma_i \in \mathscr{U}\mathscr{A}\mathscr{D}$, $i = 1, \dots, n$.
(a.2) *If* $\partial\sigma_i/\partial x(s, w, \cdot)$ *is the cadlag version, then*

$$\forall\, t, \qquad \sup_{s \leq t,\, x \in \mathbb{R}} \left| \frac{\partial \sigma_i}{\partial x}(s, w, x) \right| = \gamma_i(t, w) < \infty \quad a.s.$$

(a.3) $D_i(w) = \{(s, x)\colon \partial\sigma_i/\partial x(s, w, \cdot)$ *is discontinuous at* $x\}$ *has content* 0 *a.s. in* w.
(b) $h_k(\cdot, \cdot, x)$ *is adapted*, $h_k(s, w, \cdot)$ *is continuous and* h_k *has linear growth*.
(c) $[Z^i, Z^j] = 0$, $i \neq j$.

Then (4.1) *has a unique minimal strong solution* $\underline{X}$ (*resp. maximal* $\overline{X}$). *That is, if* Y *is any strong solution of* (4.1), *then* $\underline{X}_t \leq Y_t$ *for all* $t \geq 0$ *a.s.* ($Y_t \leq \overline{X}_t$). *Moreover*, $\underline{X}$ *and* $\overline{X}$ *are strict solutions.*

PROOF. To avoid cumbersome notation, we shall denote, for example, $\sigma_i(t, w, X_t)$ by $\sigma_i(X_t)$, where it is convenient and there is no possible misunderstanding. By part (c), (4.1) can be written as

$$dX_t = \sum_i \sigma_i(X_t)\, dM_t^i + \frac{1}{2} \sum_i \left(\sigma_i \frac{\partial \sigma_i}{\partial x} \right)(X_t)[M^i, M^i]_t$$
$$+ \sum_i \sigma_i(X_t)\, dC_t^i + \sum_k h_k(X_t)\, dA_t^k,$$

where $Z^i = M^i + C^i$ is the canonical decomposition of Z^i into a local continuous martingale and a CFV process C^i. Since $\pm\sigma_i$, $\pm h_k$ satisfy part (b), we can restrict our study to the following Itô equation:

$$dX = \sum \sigma_i\, dM^i + \frac{1}{2} \sum_i \sigma_i \frac{\partial \sigma^i}{\partial x} d[M^i, M^i] + \sum b_k\, dF^k, \tag{4.6}$$

where now F^k are continuous increasing and adapted processes, and b_k satisfy part (b).

Define the following version of $\partial\sigma_i/\partial x$:

$$\frac{\partial\tilde{\sigma}_i}{\partial x}(s,w,x) = \begin{cases} \dfrac{\partial\sigma_i}{\partial x}(s,w,x-) \wedge \dfrac{\partial\sigma_i}{\partial x}(s,w,x), & \text{if } \sigma_i(s,w,x) \geq 0, \\ \dfrac{\partial\sigma_i}{\partial x}(s,w,x-) \vee \dfrac{\partial\sigma_i}{\partial x}(s,w,x), & \text{if } \sigma_i(s,w,x) < 0. \end{cases}$$

It is easy to see that $\partial\tilde{\sigma}_i/\partial x(s,w,\cdot\,)$ has left and right limits, and

$$A_i(w) = \left\{(s,x)\colon \frac{\partial\tilde{\sigma}_i}{\partial x}(s,w,x) \neq \frac{\partial\sigma_i}{\partial x}(s,w,x)\right\} \subseteq D_i(w).$$

Define

$$\rho_i(s,w,x) = \frac{1}{2}\sigma_i(s,w,x)\frac{\partial\tilde{\sigma}_i}{\partial x}(s,w,x).$$

Since $\sigma_i = g_i \cdot A$, we have

$$\frac{\partial\sigma_i}{\partial x}(s,w,x) = \int_0^s \frac{\partial g_i}{\partial x}(u,w,x)\,dA_u,$$

$$\frac{\partial\sigma_i}{\partial x}(s,w,x-) = \int_0^s \frac{\partial g_i}{\partial x}(u,w,x-)\,dA_u.$$

From this, $\rho_i(\cdot,w,x)$ is continuous at s when $\sigma_i(s,w,x) \neq 0$, but if $\sigma_i(s,w,x) = 0$, then $\rho_i(\cdot,w,x)$ is obviously continuous at s. It is not difficult to see that $\rho_i(s,w,\cdot\,)$ is lower semicontinuous and moreover $\liminf_{y\to x}\rho_i(s,w,y) = \rho_i(s,w,x)$.

In the first part of the proof, we shall consider (4.6) where we have replaced $\partial\sigma_i/\partial x$ by $\partial\tilde{\sigma}_i/\partial x$ and we made the extra assumption:

(4.7) $\quad M^i, [M^i,M^i], F^k, \sigma_i, b_k, \dfrac{\partial\tilde{\sigma}_i}{\partial x}$ are uniformly bounded by C.

As a consequence, $\forall\ s \geq 0,\ x \in \mathbb{R},\ |\rho_i(s,w,x)| \leq C^2/2$. Consider $\rho_i^r(s,w,x) = \inf_y\{\rho_i(s,w,y) + r|Y-x|\} - 2^{-r}$ and $b_k^r(s,w,x)$ in the analogous way. By the approximation lemma and the definition of ρ_i^r (resp. b_k^r), we have the following:

1. $\forall\ s,x$, $|\rho_i^r(s,w,x)| \leq C'$ and $|b_k^r(s,w,x)| \leq C'$; $C' = (C^2/2 \vee C) + 1$.
2. $\rho_i^r(s,w,\cdot\,) \nearrow \rho_i(s,w,\cdot\,)$; $b_k^r(s,w,\cdot\,) \nearrow b_k(s,w,\cdot\,)$ as $r \to \infty$.
3. $\rho_i^r(\cdot,\cdot,x)$ are adapted and $|\rho_i^r(s,w,x) - \rho_i^r(s,w,y)| \leq r|x-y|$ (the same is true for b_k^r).
4. $\rho_i^r <^* \rho_i^{r+1}$ and $b_k^r <^* b_k^{r+1}$.
5. If $\rho_i(s,w,\cdot\,)$ is continuous at a [in particular, if $\sigma_i(s,w,a) = 0$] and $x_r \to a$, then $\rho_i^r(s,w,x_r) \to \rho_i(s,w,a)$.

Since $b_k(s, w, \cdot\,)$ is continuous, we have

$$b_k^r(s, w, x_r) \to b_k(s, w, a).$$

Consider the equation

$$\begin{aligned} dY_t &= \sum \sigma_i(t, w, Y_t)\, dM_t^i + \sum \rho_i^r(t, w, Y_t)\, d[M^i, M^i]_t \\ &\quad + \sum_k b_k^r(t, w, Y_t)\, dF_t^k, \qquad (4.8)\\ Y_0 &= X_0. \end{aligned}$$

Since all the coefficients of (4.8) are Lipschitz, this equation has a unique strong solution Y_t^r, which turns out to be strict by Picard's iterations.

By the comparison lemma, for fixed t, Y_t^r is increasing in r. Let us call $\underline{X}_t = \lim_r Y_t^r \le \infty$. Take $Z_t^r = Y_t^r - Y_0$, which is also increasing in r. By Doob's maximal inequality, we have

$$E\left(\sup_t \left|\int_0^t \sigma_i(s, w, Y_s^r)\, dM_s^i\right|^2\right) \le 4E\left(\int_0^\infty \sigma_i^2(Y_s^r)\, d[M^i, M^i]_s\right) \le 4C^3.$$

Also,

$$E\left(\sup_t \left|\int_0^t \rho_k^r(s, w, Y_s^r)\, dF_s^k\right|^2\right) \le (C')^2 C^2$$

and

$$E\left(\sup_t \left|\int_0^t \rho_i^r(s, w, Y_s^r)\, d[M^i, M^i]_s\right|^2\right) \le (C')^2 C^2,$$

from which

$$E\left(\sup_t |Z_t^r|^2\right) \le C'' < \infty \qquad (C'' \text{ independent of } r).$$

We conclude, by the monotone convergence theorem, that

$$E\left(\sup_t [(\underline{X}_t - X_0) \vee 0]^2\right) < \infty,$$

from which $\underline{X}_t < \infty$ for all t a.s.

The next step is to prove that $\underline{X}$ is a solution of (4.6) with $\partial\sigma_i/\partial x$ replaced by $\partial\tilde{\sigma}_i/\partial x$. Again, by Doob's maximal inequality and the DCT,

$$E\left(\sup_t \left|\int_0^t \sigma_i(s, w, Y_s^r)\, dM_s^i - \int_0^t \sigma_i(s, w, \underline{X}_s)\, dM_s^i\right|^2\right) \xrightarrow[r\to\infty]{} 0.$$

Then, for some subsequence, which we assume is the whole sequence,

$$\sup_t \left|\int_0^t \sigma_i(s, w, Y_s^r)\, dM_s^i - \int_0^t \sigma_i(s, w, \underline{X}_s)\, dM_s^i\right| \xrightarrow[r\to\infty]{} 0 \quad \text{a.s.}$$

By the DCT, applied w by w, we have

$$\sup_t \left| \int_0^t b_k^r(s, w, Y_s^r)\, dF_s^k - \int_0^t b_k(s, w, \underline{X}_s)\, dF_s^k \right|$$
$$\leq \int_{-\infty}^{\infty} |b_s^r(s, w, Y_s^r) - b_k(s, w, \underline{X}_s)|\, dF_s^k \xrightarrow[r \to \infty]{} 0 \quad \text{a.s.}$$

The only thing left to prove is

$$\sum_i \int_0^t \rho_i^r(s, w, Y_s^r)\, d[M^i, M^i]_s \to \sum_i \int_0^t \rho_i(s, w, \underline{X}_s)\, d[M^i, M^i]_s.$$

We have

$$Y_t^r = X_0 + \sum_i \int_0^t \sigma_i(Y_s^r)\, dM_s^i + \sum_k \int_0^t b_k^r(Y_s^r)\, dF_s^k + \sum_i \int_0^t \rho_i^r(Y_s^r)\, d[M^i, M^i]_s$$
$$= X_0 + S_1 + S_2 + S_3.$$

Since Y^r, S_1 and S_2 converge, then S_3 has to converge. Let us call G_t that limit. Then if $t' < t$,

$$|G_t - G_{t'}| = \left| \lim_r \sum_i \int_{t'}^t \rho_i^r(s, w, Y_s^r)\, d[M^i, M^i]_s \right| \leq C' \sum_i \int_t^{t'} d[M^i, M^i]_s,$$

from which we conclude that G_t is continuous and has paths of bounded variation. So far we have proved

$$\underline{X}_t = X_0 + \sum_i \int_0^t \sigma_i(\underline{X}_s)\, dM_s^i + \sum_k \int_0^t b_k(\underline{X}_s)\, dF_s^k + G_t. \tag{4.9}$$

Then $\underline{X}_t$ is a continuous semimartingale. By the monotonicity of Y_t^r in r, we can conclude that $Y_t^r \to \underline{X}_t$ uniformly on compact sets (Dini's theorem) for every w.

Define $\tilde{D}_i^\alpha = \{(s, x)\colon \rho_i(s, w, \cdot\,)$ is discontinuous at x and $|\sigma_i(s, w, x)| \geq \alpha > 0\}$. Since $\tilde{D}_i^\alpha \subseteq D_i$, we conclude that $\tilde{D}_i^\alpha$ has content 0.

From now on, we denote $\tilde{D}_i^\alpha$ as $\tilde{D}^\alpha$ and D_i as D. In particular, $\tilde{D}_s^\alpha = \{x\colon \rho_i(s, w, \cdot\,)$ is discontinuous at x and $|\sigma_i(s, w, x)| \geq \alpha > 0\}$. Let $\tilde{D} = \bigcup_{\alpha > 0} \tilde{D}^\alpha$; then $\tilde{D}_s = \bigcup_{\alpha > 0} \tilde{D}_s^\alpha = \{x\colon \rho_i(s, w, x-) \neq \rho_i(s, w, x+)$ and $\sigma_i(s, w, x) \neq 0\}$ is just the set of $x \in \mathbb{R}$, where $\rho_i(s, w, \cdot\,)$ is discontinuous at x. We have $\tilde{D} \subset D$. Let

$$I = E\left(\int_0^t 1_{\tilde{D}_s^\alpha}(\underline{X}_s)\, d[M^i, M^i]_s \right) \leq \left(\frac{1}{\alpha} \right)^2 E\left(\int_0^t 1_{\tilde{D}_s^\alpha}(\underline{X}_s) \sigma_i^2(\underline{X}_s)\, d[M^i, M^i]_s \right).$$

By part (c) and (4.9), we have $\sigma_i(\underline{X}_s)\, d[M^i, M^i]_s = d[\underline{X}, M^i]_s$. Then, by the Kunita–Watanabe inequality,

$$I \leq \frac{C^{1/2}}{\alpha^2} E\left(\int_0^\infty 1_{\tilde{D}_s^\alpha}(\underline{X}_s) \sigma_i^2(s, w, \underline{X}_s)\, d[\underline{X}, \underline{X}]_s \right)^{1/2}.$$

If we use the occupation–time formula (Lemma 3.1), we get

$$I \le \frac{C^{1/2}}{\alpha^2}\left\{E\left(\int_{\mathbb{R}}\int_0^\infty 1_{\tilde{D}_s^\alpha}(a)\sigma_i^2(s,w,a)L(a,ds)\,da\right)\right\}^{1/2},$$

where $L(a,s) = L_s^a(\underline{X})$ is the local time of $\underline{X}$. Thus

$$I \le \frac{C^{3/2}}{\alpha^2}\left(E\int_{\mathbb{R}}\int_0^\infty 1_{D_s}(a)L(a,ds)\,da\right)^{1/2}.$$

Since $L(a,ds)$, as a measure on s, is diffuse for every a and D has content 0, we have

$$\int_{\mathbb{R}}\int_0^\infty 1_{D_s}(a)L(a,ds)\,da = 0 \quad \text{a.s., so } I = 0.$$

In particular, $P\{w: \mu_i\{s: \underline{X}_s \in \tilde{D}_s^\alpha\} = 0\} = 1$, where μ_i is the random measure associated with $[M^i, M^i]$, from which we conclude $P\{w: \mu_i\{s: \underline{X}_s \in \tilde{D}_s\} = 0\} = 1$. So there exists a set $\mathscr{A} \in \mathscr{F}$ with $P(\mathscr{A}) = 0$ and for every $w \in \mathscr{A}^c$,

$$\rho_i^r(s,w,Y_s^r) \xrightarrow[r\to\infty]{} \rho_i(s,w,\underline{X}_s) \quad d\mu_i(s)\text{-a.e.}$$

By the DCT we have

$$\int_0^t \rho_i^r(s,w,Y_s^r)\,d[M^i,M^i]_s \to \int_0^t \rho_i(s,w,\underline{X}_s)\,d[M^i,M^i]_s.$$

Then we have proved that $\underline{X}$ satisfies (4.6) with $\partial\tilde{\sigma}_i/\partial x$ instead of $\partial\sigma_i/\partial x$. We shall prove now that $\underline{X}$ is a solution of (4.6).

For this to happen it is enough to prove that for fixed t,

$$J_t = \left|\int_0^t \left(\sigma_i\frac{\partial\tilde{\sigma}_i}{\partial x}\right)(s,w,\underline{X}_s) - \left(\sigma_i\frac{\partial\sigma_i}{\partial x}\right)(s,w,\underline{X}_s)\,d[M^i,M^i]_s\right| = 0 \quad \text{a.s.},$$

because of the path continuity of J_t. By the Kunita–Watanabe inequality, we have

$$\begin{aligned} J_t &= \left|\int_0^t\left(\frac{\partial\tilde{\sigma}_i}{\partial x} - \frac{\partial\sigma_i}{\partial x}\right)(s,w,\underline{X}_s)\,d[\underline{X},M^i]_s\right| \\ &\le ([M^i,M^i]_t)^{1/2}\left(\int_0^t\left(\frac{\partial\tilde{\sigma}_i}{\partial x} - \frac{\partial\sigma_i}{\partial x}\right)^2(s,w,\underline{X}_s)\,d[\underline{X},\underline{X}]_s\right)^{1/2} \\ &= ([M^i,M^i]_t)^{1/2}\left(4C^2\int_{\mathbb{R}}\int_0^t 1_{D_s}(a)L(a,ds)\,da\right)^{1/2} = 0 \quad \text{a.s.} \end{aligned}$$

Assume Y is any solution of (4.6). Then it is a solution of (4.6) where $\partial\sigma_i/\partial x$ is replaced by $\partial\tilde{\sigma}_i/\partial x$. By the comparison lemma, $Y_t^r \le Y_t$ for all $t \ge 0$, from which $\underline{X}_t \le Y_t$ for all $t \ge 0$. Then $\underline{X}$ is the minimal solution of (4.6), and since it is the limit of strict solutions it is also strict.

To relax condition (4.7), we proceed, as usual, by localization.

The maximal solution is obtained in an analogous way, by approximating ρ_i and b_k from above (which can be done by approximating, from below, $-b_k$ and a suitable version of $-\rho_i$). □

REMARK. Condition (a3) is automatically satisfied when σ_i does not depend on s (homogeneous case) or when $\sigma_i(s, w, \cdot\,)$ is C^1. Moreover, we have the following corollary.

COROLLARY 4.15. *Consider* (4.1), *where* σ_i *satisfies* (a1) *and* (a2) *and* $\partial\sigma_i/\partial x(s, w, \cdot\,)$ *is continuous, and* h_k *satisfies part* (*b*). *Then* (4.1) *has a unique minimal solution* $\underline{X}$ (*resp. unique maximal solution* $\overline{X}$) *among the strong solutions, and* $\underline{X}$, $\overline{X}$ *are strict.*

PROOF. Now (4.1) can be written as

$$dX_s = \sum \sigma_i(X_s)\, dZ_s^i + \frac{1}{2}\sum_i \left(\sigma_i \frac{\partial \sigma_i}{\partial x}\right)(X_s)\, d[Z^i, Z^i]_s$$
$$+ \frac{1}{2}\sum_{i \neq j} \left(\sigma_i \frac{\partial \sigma_j}{\partial x}\right)(X_s)\, d[Z^i, Z^j]_s + \sum_k h_k(X_s)\, dA_s^k.$$

This is of the form of (4.6) in the previous proof. Then, by the same approximation used in that proof, the result holds. □

5. Uniqueness. Our goal is to give conditions for strong uniqueness. We shall attack this problem from two different sides: the first one via Girsanov's theorem, and the second one through a change of variables on the state space to remove the drift $\sigma\sigma'$. Let us consider the stochastic differential equation:

$$\begin{aligned} dX_t &= \sum_i \sigma_i(t, w, X_t)\, dZ_t^i + \sum_k b_k(t, w, X_t)\, dA_t^k, \\ X_0 &= X_0, \end{aligned} \tag{5.1}$$

on a certain filtered space $(\Omega, \mathscr{F}, (\mathscr{F}_t), P)$. We assume Z^i are continuous semimartingales, A^k are CFV, $b_k(\cdot\,,\cdot\,,\cdot\,)$ are measurable functions and σ_i are $\mathscr{P} \otimes \mathscr{B}(\mathbb{R})$ measurable functions. We shall assume that $b_k(\cdot\,, w, \cdot\,)$ and $\sigma_i(\cdot\,, w, \cdot\,)$ are locally bounded. Remember that h is locally bounded if

$$\forall\, t, \forall\, \Lambda (\subseteq \mathbb{R}) \text{ compact}, \qquad \sup_{0 \le s \le t,\, x \in \Lambda} |h(s, w, x)| < \infty.$$

DEFINITION 5.1. We say that (5.1) satisfies the strong uniqueness property if for any two continuous processes X_t and Y_t defined on Ω, adapted to $(\mathscr{F}_t)$ and verifying (5.1), $X = Y$ holds (i.e., X and Y are indistinguishable).

Since we are interested mainly in strong solutions, we introduce the following weak version of weak uniqueness.

DEFINITION 5.2. We say that (5.1) satisfies the weak* uniqueness property if for any two continuous processes X_t and Y_t defined on Ω, adapted to $(\mathscr{F}_t)$ and verifying (5.1), $\mathscr{L}(X) = \mathscr{L}(Y)$ holds, where $\mathscr{L}(X)$ is the law that X induces on path space $C([0,\infty),\mathbb{R})$.

THEOREM 5.3. *If* $\sigma = (\sigma_1, \dots, \sigma_n) \in LT(Z^1, \dots, Z^n)$, *then weak* uniqueness and strong uniqueness are equivalent for* (5.1).

PROOF. It is clear that strong uniqueness implies weak* uniqueness. For the converse assume that X and Y are two (continuous) solutions of (5.1). The LT condition implies that $X \vee Y$ and $X \wedge Y$ are also solutions; then $\mathscr{L}(X \vee Y) = \mathscr{L}(X \wedge Y)$ and so $X \vee Y = X \wedge Y$, from which $X = Y$. □

In the previous section we have obtained strict solutions for the Stratonovich equation. For that reason the following definition is useful for us.

DEFINITION 5.4. We say that (5.1) satisfies the strict uniqueness property if there is at most one strict solution.

LEMMA 5.5. *If* (5.1) *has minimal and maximal solutions* (*among the strong solutions*) *which are strict, then the three notions of uniqueness are equivalent.*

PROOF. We shall prove weak* ⇒ strong ⇒ strict ⇒ strong ⇒ weak*.

Denote by $\overline{X}$ and $\underline{X}$ the maximal and minimal solutions. If weak* uniqueness holds, then $\mathscr{L}(\overline{X}) = \mathscr{L}(\underline{X})$, which implies $\overline{X} = \underline{X}$, and then there is at most one strong solution. The second implication always holds, because the set of strong solutions contains the set of strict solutions.

Let us prove now that strict ⇒ strong. If X is any strong solution, we have $\underline{X} \le X \le \overline{X}$, but $\underline{X} = \overline{X}$ because strict uniqueness holds. Thus strong uniqueness holds. The last implication is always true. □

When $(Z^1, \dots, Z^n)$ is an n-dimensional Brownian motion, we can consider another notion of uniqueness.

DEFINITION 5.6. Let B be an n-dimensional Brownian motion and $A = (A^1, \dots, A^m)$ a vector of CFV processes. Consider the Itô equation

$$(5.2)\qquad \begin{aligned} dX_t &= \sum_i \sigma_i(t, X_t)\, dB_t^i + \sum_k b_k(t, X_t)\, dA_t^k, \\ X_0 &= X_0, \end{aligned}$$

where σ_i, h_k are $\mathscr{B}(\mathbb{R}^2)$ measurable (they do not depend on w), and (X_0, A) is independent of B. We say that (5.2) satisfies the weak uniqueness property if given that $(\Omega, \mathscr{F}, (\mathscr{F}_t), P, B, A, X_0, X)$ and $(\Omega', \mathscr{F}', (\mathscr{F}_t'), P', B', A', X_0', X)$ are two solutions with $\mathscr{L}(X_0, A) = \mathscr{L}(X_0', A')$, then $\mathscr{L}(X) = \mathscr{L}(X')$.

LEMMA 5.7. *If σ_i and h_k are nonrandom ULC, then weak uniqueness holds for* (5.2).

PROOF. Assume that $(\Omega, \mathscr{F}, (\mathscr{F}_t), P, B, X_0, X)$ and $(\Omega', \mathscr{F}', (\mathscr{F}_t'), P', B', X_0', X')$ are two solutions. Take φ a $C_0^\infty(\mathbb{R})$ function with support on $[-(n+1), (n+1)]$ such that $\varphi = 1$ on $[-n, n]$. Consider $g_i(s, x) = \sigma_i(s, x)\varphi(x)$ and $h_k(s, x) = b_k(s, x)\varphi(x)$. Then

$$(5.3) \qquad dZ_s = \sum_i g_i(s, Z_s)\, dB_s^i + \sum_k h_k\, dA^k, \qquad Z_0 = X_0,$$

has a unique strong solution Z on $(\Omega, \mathscr{F}, (\mathscr{F}_t), P, B, X_0)$ and a unique strong solution Z' on $(\Omega', \mathscr{F}', (\mathscr{F}_t), P', B', X_0')$ (see Protter [20], Chapter 5, Theorem 7), because the coefficients are uniformly Lipschitz. In particular, Picard's iterations converge to the solution of (5.3). Consider

$$L(W)_t = X_0 + \sum_i \int_0^t g_i(s, W_s)\, dB_s^i + \int_0^t h_k(s, W_s)\, dA_s^k.$$

If $W^{(0)} = X_0$, $W^{(n)} = L(W^{(n-1)})$, we have $W^{(n)} \to_{\text{UCP}} Z$. By induction it is easy to prove that $\mathscr{L}(W^{(n)}) = \mathscr{L}(W'^{(n)})$ and so $\mathscr{L}(Z) = \mathscr{L}(Z')$. We conclude that $\mathscr{L}(X^{T_n}) = \mathscr{L}(X'^{T_n'})$, where $T_n = \inf\{t\colon |X_t| \geq n\}$ (similarly for T_n'). By assumption $T_n \nearrow_{n\to\infty} \infty$ a.s. (respectively $T_n' \nearrow \infty$), because X is a solution of (5.2). Thus $\mathscr{L}(X) = \mathscr{L}(X')$. □

REMARK. Obviously, weak uniqueness implies weak* uniqueness. The next result is a uniqueness theorem for the Stratonovich equation in the Brownian case.

THEOREM 5.8 (Uniqueness). *Let $B = (B^1, \dots, B^n)$ be a n-dimensional Brownian motion defined on the filtered space $(\Omega, \mathscr{F}, (\mathscr{F}_t), P)$. Let σ_i be a nonrandom function in $\mathscr{U}\mathscr{A}\mathscr{D}$ and b_k a nonrandom function which satisfy*:

$$(5.4) \qquad \forall\, t, \forall \Lambda (\subseteq \mathbb{R}) \text{ compact}, \qquad \sup_{s \leq t,\, x \in \Lambda} \left| \frac{\partial \sigma_i}{\partial x}(s, x) \right| = \gamma_i(t) < \infty,$$

and

$$(5.5) \qquad b_k \text{ are ULC}.$$

Consider the Stratonovich equation:

$$(5.6) \qquad dX_t = \sum_i \sigma_i(t, X_t) \circ dB_t^i + \sum_k b_k(t, X_t)\, dA_t^k, \qquad X_0 = X_0,$$

where $(X_0, A^1, \dots, A^m)$ is independent of B.
Then strong and weak uniqueness hold.

PROOF. Since $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_n) \in LT(B^1, \dots, B^n)$, then weak* uniqueness and strong uniqueness are equivalent. Moreover, weak uniqueness implies weak* uniqueness, so it is enough to prove weak uniqueness. By conditioning on $(X_0, A^1, \ldots, A^m)$, we can assume that $(X_0, A^1, \ldots, A^m)$ are nonrandom.

Assume $(\Omega, \mathscr{F}, (\mathscr{F}_t), B, P, X)$ is a solution of (5.6). Then

$$Z_t = \exp\left\{ -\sum_i \int_0^t \frac{\partial \sigma_i}{\partial x}(s, X_s)\, dB_s^i - \frac{1}{2}\int_0^t \sum_i \left(\frac{\partial \sigma_i}{\partial x}(s, X_s)\right)^2 ds \right\}$$

is a martingale (see Karatzas and Shreve [13], Corollary 3.5.13). Fix $t > 0$ and define

$$\tilde{P}_t^X(A) = E(1_A Z_t) \quad \text{for all } A \in \mathscr{F}_s,\ s \le t.$$

By Girsanov's theorem, the process defined by

$$W_s(X) = \left(W_s^1(X), \ldots, W_s^n(X)\right),$$

where

$$W_s^i(X) = B_s^i + \int_0^s \frac{\partial \sigma_i}{\partial x}(u, X_u)\, du, \qquad s \le t,$$

is an n-dimensional Brownian motion on $(\Omega, \mathscr{F}_t, (\mathscr{F}_s)_{s \le t}, \tilde{P}_t^X)$. Under $\tilde{P}_t^X$, X satisfies

$$dU_s = \sum_i \sigma_i(s, U_s)\, dW_s^i(X) + \sum_k b_k(s, U_s)\, dA_s^k, \qquad s \le t,\ U_0 = X_0.$$

If we repeat this procedure with another solution $(\Omega', \mathscr{F}', (\mathscr{F}_t'), P', B', X')$, we end up with the existence of another Brownian motion $(W_s(X'))_{s \le t}$ defined on $(\Omega', \mathscr{F}', (\mathscr{F}_s')_{s \le t}, \tilde{P}_t^{X'})$ such that under $\tilde{P}_t^{X'}$, X' satisfies

$$dV_s = \sum_i \sigma_i(s, V_s)\, dW_s^i(X') + \sum_k b_k(s, V_s)\, dA_s^k, \qquad s \le t,\ V_0 = X_0.$$

We have that $(\Omega, \mathscr{F}, (\mathscr{F}_s)_{s \le t}, \tilde{P}_t^X, W(X), X)$ and $(\Omega', \mathscr{F}', (\mathscr{F}_s')_{s \le t}, \tilde{P}_t^{X'}, W(X'), X')$ satisfy the same Itô equation, which has ULC coefficients. Then by Lemma 5.7, the law induced by X (under $\tilde{P}_t^X$) is the same as the one induced by X'. Since Z^{-1} is $\sigma(X, W(X))$ [resp. Z'^{-1} is $\sigma(X', W(X'))$] measurable, we conclude $\forall\ t$, $\mathscr{L}((X_s)_{s \le t}) = \mathscr{L}((X_s')_{s \le t})$, from which the result holds. □

We summarize the existence and uniqueness results in the following theorem.

THEOREM 5.9 (Existence and uniqueness). *Let B a n-dimensional Brownian motion. Assume that, in* (5.2), *σ_i are nonrandom and*:

(a) $\sup_{s \le t,\, x \in \mathbb{R}} |\partial \sigma_i / \partial x(s, x)| = \gamma_i(t) < \infty$.
(b) $D_i = \{(s, x)\colon \partial \sigma_i / \partial x(s, \cdot\,)$ *is discontinuous at* $X\}$ *has content* 0.
(c) b_k *are UL*.
(d) $(X_0, A^1, \ldots, A^m)$ *is independent of* B.

Then there is only one strong solution which is also strict.

The proof of Theorem 5.8 was based on Girsanov's theorem which allows us to remove the drift $\sigma_i\, \partial\sigma_i/\partial x$, but this can be done only in the context of an n-dimensional Brownian motion. To obtain a general result on uniqueness, we shall remove the drift $\sigma_i\, \partial\sigma_i/\partial x$ by a change of variables on the state space $\mathbb{R}$. To this end we write the following definitions.

A function f defined on $I = [a, b]$ is said to be in $\mathscr{A}\mathscr{D}(I)$ iff

$$\tilde{f}(x) = \begin{cases} f(x), & \text{if } x \in [a, b], \\ f(a), & x < a, \\ f(b), & x > b, \end{cases} \qquad \text{is in } \mathscr{A}\mathscr{D}.$$

In a similar way f is said to be Lipschitz on I, if $\tilde{f}$ is Lipschitz.

If $a < b < c$ and $f \in \mathscr{A}\mathscr{D}([a, b])$, $g \in \mathscr{A}\mathscr{D}[b, c]$ and $f(b) = g(b)$, then it is not difficult to prove that

$$h(x) = \begin{cases} f(x), & x \in [a, b], \\ g(x), & x \in [b, c], \end{cases}$$

belong to $\mathscr{A}\mathscr{D}([a, c])$. A similar result holds for Lipschitz functions.

DEFINITION 5.10. We say that $\sigma \in \mathscr{A}\mathscr{D}$ satisfies hypothesis A iff for every $\zeta \in \mathscr{Z} = \{x: \sigma(x) = 0\}$ there exist: $\delta > 0$, a function F defined on $I = [\zeta - \delta, \zeta + \delta]$ and a constant $c > 0$ such that:

(A.1) $F' \in \mathscr{A}\mathscr{D}(I)$ and $F'(x) \geq c$ on I.

(A.2) If F'' is the cadlag version of the second derivative of F, then

$$h = F''\sigma^2 + \sigma\sigma' F' \text{ is Lipschitz on } I.$$

Note that F: $I \to F(I)$ is one to one. $(F^{-1})' \in \mathscr{A}\mathscr{D}(I)$ and $0 \leq (F^{-1})' \leq c^{-1}$. In particular, F^{-1} is Lipschitz on I and it is a difference of two convex functions (on I).

In Appendix A we shall give some sufficient conditions for hypothesis A. Roughly speaking, hypothesis A is a regularity condition for σ on the set of zeros of σ. This regularity will be enough to guarantee uniqueness for the Stratonovich equation:

$$(5.7) \qquad dX_t = \sigma(X_t) \circ dZ_t + \sum_k b_k(X_t)\, dA_t^k, \qquad X_0 = X_0,$$

where as usual Z is a continuous semimartingale, b_k are Lipschitz and A^k are CFV.

REMARK. If $\sigma > 0$ (or $\sigma < 0$), then σ satisfies hypothesis A. This condition appears in several works when uniqueness of SDE is studied; for example, see Le Gall [14]. Also, see the work of Mackevicius [15] in this direction, in the Brownian case.

LEMMA 5.11. *Let S be a finite stopping time* [*i.e.*, $P(S < \infty) = 1$] *and* $\mathscr{A} \in \mathscr{F}_S$ *with* $P(\mathscr{A}) > 0$. *Consider the Stratonovich equation*:

$$X_t = Y + \int_S^{S+t} \sigma(X_u) \circ dZ_u + \sum_k \int_S^{S+t} b_k(X_u)\, dA_u^k, \tag{5.8}$$

where $Y \in \mathscr{F}_S$, $\sigma \in \mathscr{A}\mathscr{D}$ *with* σ' *bounded*, b_k *are Lipschitz and* A^k *are CFV.*

If $\overline{X}_t$ *and* $\underline{X}_t$ *are the maximal and minimal solutions of* (5.8) *under* P, *then they are also the maximal and minimal solutions under* $R(\cdot) = P(\cdot\,|\mathscr{A})$.

PROOF. Notice that under R, Z is still a continuous semimartingale and (A^k) are CFV. Also, $\overline{X}$ and $\underline{X}$ are solutions of (5.8) under R. Assume $\overline{V}$ is the maximal solution of (5.8) under R (the existence of $\overline{V}$ is guaranteed by Theorem 4.14).

Take $W_t = \overline{V}_t 1_{\mathscr{A}} + \overline{X}_t 1_{\mathscr{A}^c}$. Since $\mathscr{A} \in \mathscr{F}_S$ we have under P:

$$\overline{V}_t 1_{\mathscr{A}} = Y 1_{\mathscr{A}} + \int_S^{S+t} \sigma(\overline{V}_u) 1_{\mathscr{A}} \circ dZ_u + \sum_k \int_S^{S+t} b_k(\overline{V}_u) 1_{\mathscr{A}}\, dA_u^k$$

and

$$\overline{X}_t 1_{\mathscr{A}} = Y 1_{\mathscr{A}^c} + \int_S^{S+t} \sigma(\overline{X}_u) 1_{\mathscr{A}^c} \circ dZ_u + \sum_k \int_S^{S+t} b_k(\overline{X}_u) 1_{\mathscr{A}^c}\, dA_u^k.$$

Thus W_t is a solution of (5.8) under P. So we can conclude $\overline{V}_t \le \overline{X}_t$ R-a.s., and then $\overline{X}$ is the maximal solution of (5.8) under R. In a similar way $\underline{X}$ is the minimal solution of (5.8) under R. □

LEMMA 5.12. *Assume the conditions of the previous lemma hold. Take* $T = \inf\{t > 0: \overline{X}_t > \underline{X}_t\}$ *and* $\mathscr{Q} = \{x: \sigma(x) = 0\}$. *Then*:

(i) $T > 0$ *a.s. on* $\{Y \notin \mathscr{Q}\}$;

(ii) *if* σ *satisfies hypothesis* A, *then* $T > 0$ *a.s. on* $\{Y \in \mathscr{Q}\}$.

PROOF. (i) Notice that $\{Y \notin \mathscr{Q}\} \in \mathscr{F}_S$. If $P(Y \notin \mathscr{Q}) = 0$, then part (i) holds trivially. If $P(Y \notin \mathscr{Q}) > 0$, we can apply Lemma 5.11 to get that $\overline{X}$ and $\underline{X}$ are the maximal and minimal solutions of (5.8) under $R = P(\cdot\,/Y \notin \mathscr{Q})$. Take $\eta = d(Y, \mathscr{Q}) = \inf_{\zeta \in \mathscr{Q}} |Y - \zeta| > 0$ R-a.s., and $\delta = \eta/3$. Define $\tau = \inf\{t > 0: |\overline{X}_t - Y| > \delta$ or $|\underline{X}_t - Y| > \delta\}$. By continuity we get that $\tau > 0$ R-a.s. Also $d(\overline{X}_t, \mathscr{Q}) > \delta$ for $t \le \tau$ and $d(\underline{X}_t, \mathscr{Q}) > \delta$ for $t \le \tau$.

Consider

$$F(w, x) = \begin{cases} \displaystyle\int_{Y(w)}^{x} \frac{1}{\sigma(u)}\, du, & Y(w) - \delta \le x \le Y(w) + \delta, \\ F(w, Y(w) + \delta), & x > Y(w) + \delta, \\ F(w, Y(w) - \delta), & x < Y(w) - \delta. \end{cases}$$

Then $\partial F(w, x)/\partial x = 1/\sigma(x)$ has only one sign on $I = [Y - \delta, Y + \delta]$, so it is one to one. Also in that interval σ^{-1} is Lipschitz, from which $F(w, \cdot\,)$ is the difference of two convex functions. Also, $F(\cdot, \cdot)$ is $\mathscr{F}_S \otimes B$ measurable, and

$$\frac{\partial^2 F}{\partial x^2}(w, x)\sigma^2(x) + \sigma(x)\sigma'(x)\frac{\partial F}{\partial x}(w, x) = 0.$$

In what follows we denote $F(x)$ instead of $F(w, x)$.

By Itô's formula we get

$$\overline{W}_t = F(\overline{X}_t) = F(Y) + \int_S^{S+t} dZ_u + \sum_k \int_S^{S+t} \frac{b_k(F^{-1}(\overline{W}_u))}{\sigma(F^{-1}(\overline{W}_u))}\, dA_u^k \quad \text{for } t \le \tau. \tag{5.9}$$

Also $\underline{W}_t = F(\underline{X}_t)$ satisfies (5.9). Since this equation has coefficients locally Lipschitz, we have $\underline{W}_t = \overline{W}_t$, $0 \le t \le \tau$, from which $\underline{X}_t = \overline{X}_t$, $0 \le t \le \tau$, and so $T \ge \tau > 0$ R-a.s.

(ii) Assume $P(Y \in \mathscr{D}) > 0$. Since $C_n = \mathscr{D}(\cap[-n, n])$ is compact and σ satisfies hypothesis A, we have for every $\zeta \in C_n$ there exists $\delta(\zeta) > 0$ such that σ satisfies certain properties on $I_\zeta = [\zeta - \delta(\zeta), \zeta + \delta(\zeta)]$. By compactness of C_n there exist $\zeta_1, \dots, \zeta_p \in C_n$ such that

$$C_n \subseteq \bigcup_{i=1}^{p} I_i, \quad \text{where } I_i = I_{\zeta_i}.$$

If $P(Y \in I_i) > 0$, define $R_i = P(\cdot / Y \in I_i)$. By Lemma 5.11, $\overline{X}$ and $\underline{X}$ are the maximal and minimal solutions of (5.8) under R_i. Let δ_i, F_i, c_i, h_i be given by hypothesis A, and define $\rho_i = (\delta_i \wedge \delta(\zeta_i))/2$ and consider

$$\tau_i = \inf\{t > 0 \colon |\overline{X}_t - Y| \ge \rho_i/3 \text{ or } |\underline{X} - Y| \ge \rho_i/3\}.$$

If $\overline{W}_t = F_i(\overline{X}_t)$, $\underline{W}_t = F_i(\underline{X}_t)$, $0 \le t \le \tau_i$, then they satisfy

$$U_t = F(Y) + \int_S^{S+t} g(U_s)\, dZ_s + \sum_k \int_S^{S+t} f_k(U_s)\, dA_s^k + \int_S^{S+t} l_i(U_s)\, d[Z, Z]_s,$$

where $g(x) = F'(F^{-1}x)\sigma(F^{-1}x)$, $f_k(x) = F'(F^{-1}x)b_k(F^{-1}x)$ and $l_i(x) = h_i(F^{-1}x)$. These coefficients are Lipschitz on $[\zeta_i - \rho_i, \zeta_i + \rho_i]$ and so $\overline{W}_t = \underline{W}_t$, $0 \le t \le \tau$, from which the result holds. □

REMARK. If $\overline{X}$ and $\underline{X}$ are the maximal and minimal solutions of $dX_t = \sigma(X_t) \circ dZ_t + \Sigma_k b_k(X_t)\, dA_t^k$, $X_0 = X_0$, and S is a stopping time such that $\overline{X}_S = \underline{X}_S$ a.s., then $\overline{X}_{S+t}$ and $\underline{X}_{S+t}$ are the maximal and minimal solutions of

$$X_t = Y + \int_S^{S+t} \sigma(X_u) \circ dZ_u + \sum_k \int_S^{S+t} b_k(X_u)\, dA_u^k.$$

THEOREM 5.13. *Assume $\sigma \in \mathscr{AD}$, σ satisfies hypothesis* A, *σ' is bounded and b_k are Lipschitz. Then* (5.7) *has a unique strong solution which is also strict.*

PROOF. By Theorem 4.14, (5.7) has a maximal and a minimal solution $\overline{X}$ and $\underline{X}$ respectively (which are also strict). Consider $\tau = \inf\{t > 0: \overline{X}_t > \underline{X}_t\}$, and define $S_n = \tau \wedge n$. Then $\overline{X}_{S_n} = \underline{X}_{S_n}$. By Lemma 5.12 and the remark preceding this theorem, we conclude that $T = \inf\{t > 0: \overline{X}_{S_{n+t}} > \underline{X}_{S_{n+t}}\} > 0$ a.s., which implies that $\tau \geq n$ a.s. Thus $\tau = \infty$ a.s. and so $\overline{X} = \underline{X}$ is the unique strong solution of (5.7), which is also strict. □

Our purpose now is to generalize the previous result to the equation

$$dX_t = \sum_i \sigma_i(t, w, X_t) \circ dZ_t^i + \sum_k b_k(t, w, X_t)\, dA_t^k, \quad X_0 = X_0, \tag{5.10}$$

where we shall assume that b_k are ULC, (A^k) are CFV, (Z^i) are continuous semimartingales, which satisfy the extra assumption

$$[Z^i, Z^j]_t = \begin{cases} \alpha_i(w) G_t(w), & i = j, \\ 0, & i \neq j, \end{cases} \tag{5.11}$$

where G_t is an increasing continuous process and $\alpha_i \geq 0$ a.s.

DEFINITION 5.14. Assume $\sigma_i \in \mathscr{UAD}$. We say that $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_n) \in UA(Z^1, \dots, Z^n)$ if there exists a measurable function F: $\mathbb{R}_+ \times \Omega \times \mathbb{R} \to \mathbb{R}$, such that

$$\frac{\partial F}{\partial x}(s, w, x) \in \mathscr{UAD}, \qquad \frac{\partial F}{\partial x} > 0, \tag{5.12}$$

$$h = \frac{\partial F}{\partial x} \sum_i \alpha_i \sigma_i \frac{\partial \sigma_i}{\partial x} + \frac{\partial^2 F}{\partial x^2} \sum_i \alpha_i \sigma_i^2 \in \text{ULC}. \tag{5.13}$$

An important case where $\boldsymbol{\sigma} \in UA(Z^1, \dots, Z^n)$ is when $\sum_i \alpha_i(w) \sigma_i^2(s, w, x) > 0$, which is the analog of the case $\sigma^2 > 0$. In this situation take

$$F(t, w, x) = -\int_0^t \int_0^x \frac{\sum_i \alpha_i(w) g_i(u, w, z) \sigma_i(u, w, z)}{\left(\sum_i \alpha_i(w) \sigma_i^2(u, w, z)\right)^{3/2}}\, dz\, dA_u(w),$$

where $\sigma_i = g_i \cdot A$. It is not difficult to verify that $h \equiv 0$.

THEOREM 5.15. *If, in* (5.10), *$\sigma_i \in UA(Z^1, \dots, Z^n)$, b_k are ULC and (Z_i) satisfy* (5.11), *then strong uniqueness holds.*

PROOF. Since $\partial F/\partial x \in \mathscr{AUD}$ we have

$$\frac{\partial F}{\partial x}(s, w, x) = \int_0^s c(u, w, x)\, dD_u,$$

where D is increasing continuous process. If X is a solution of (5.10), then $Y_t = F(t, w, X_t)$ satisfies (by the generalized Itô formula Theorem 3.2)

$$Y_t = F(0, w, X_0) + \int_0^t \mathscr{K}(s, w, X_s)\, dD_s + \int_0^t \frac{\partial F}{\partial x}(s, w, X_s)\, dX_s + \frac{1}{2}\int_0^t \frac{\partial^2 F}{\partial x^2}(s, w, X_s)\, d[X, X]_s,$$

where

$$F(t, w, x) = \int_0^t \mathscr{K}(s, w, x)\, dD_s$$

and

$$\mathscr{K}(s, w, x) = \int_0^x C(s, w, y)\, dy.$$

Now,

$$\begin{aligned} Y_t = F(0, w, X_0) &+ \int_0^t \mathscr{K}(s, w, X_s)\, dD_s \\ &+ \int_0^t \frac{\partial F}{\partial x}\left\{\sum_i \sigma_i\, dZ_s^i + \frac{1}{2}\sum_i \sigma_i \frac{\partial \sigma_i}{\partial x}\alpha_i\, dG_s + \sum_k b_k\, dA_s^k\right\} \\ &+ \frac{1}{2}\int_0^t \frac{\partial^2 F}{\partial x^2}\sum_i \alpha_i \sigma_i^2\, dG_s, \end{aligned} \tag{5.14}$$

$$Y_t = F(0, w, X_0) + \sum_i \int_0^t \gamma_i(s, w, Y_s)\, dZ_s^i + \frac{1}{2}\int_0^t \eta(s, w, Y_s)\, dG_s + \int_0^t \rho(s, w, Y_s)\, dD_s + \sum_k \int_0^t \lambda_k(s, w, Y_s)\, dA_s^k,$$

where

$$\gamma_i(s, w, y) = \left(\frac{\partial F}{\partial x}\sigma_i\right)(s, w, F^{-1}(s, w, y)), \eta(s, w, y) = h(s, w, F^{-1}(s, w, y)),$$

$$\rho(s, w, y) = \mathscr{K}(s, w, F^{-1}(s, w, y)),$$

and

$$\lambda_k(s, w, y) = \left(\frac{\partial F}{\partial x}b_k\right)(s, w, F^{-1}(s, w, y))$$

for $y \in F(s, w, \mathbb{R})$. $F^{-1}(s, w, \cdot\,)$ is the inverse function of $F(s, w, \cdot\,)$, which exists because of (5.12), and h is given by (5.13).

Since F^{-1}, $\partial F/\partial x$, σ_i, h, $\mathscr{K}$ and b_k are ULC, the Itô stochastic differential equation (5.14) has coefficients which are ULC and so strong uniqueness holds for it. Then (5.10) has at most one strong solution [here we have used the fact that $F(s, w, \cdot\,)$ is one to one]. □

COROLLARY 5.16. *If, in addition,* (σ_i) *and* (b_k) *satisfy the hypotheses of Theorem* 4.14, *then* (5.10) *has a unique strong solution, which is also strict.*

COROLLARY 5.17. *If* $\sigma \in \mathscr{U}\mathscr{A}\mathscr{D}$ *has linear growth and satisfies*:

(a) $D = \{(s, x)\colon \partial\sigma/\partial x(s, w, \cdot\,)$ *is discontinuous at* $x\}$ *has content* 0,
(b) $\sigma^2 > 0$, *and*
(c) b_k *are UL, then*

$$dX = \sigma(t, w, X_t) \circ Z_t + \sum_k b_k(t, w, X_t)\, dA_t^k,$$

$$X_0 = X_0,$$

has a unique strong solution, and this solution is strict.

COROLLARY 5.18. *Let* $B = (B^1, \dots, B^n)$ *be an n-dimensional Brownian motion and* (A^k) *a finite family of CFV processes. Assume* $\sigma_i \in \mathscr{U}\mathscr{A}\mathscr{D}$ *and they have linear growth. Also assume that* $D_i = \{(s, x)\colon \partial\sigma_i/\partial x(s, w, \cdot\,)$ *is discontinuous at* $x\}$ *has content* 0, *and* b_k *are UL. If* $\sum_i \sigma_i^2(s, w, x) > 0$, *then*

$$dX_t = \sum_i \sigma_i(t, w, X_t) \circ dB_t^i + \sum_k b_k(t, w, X_t)\, dA_t^k, \qquad X_0 = X_0,$$

has a unique strong solution, which is strict.

APPENDIX

We shall give sufficient conditions for hypothesis A, which is important in the uniqueness problem.

DEFINITION A.1. Let $\sigma \in \mathscr{A}\mathscr{D}$. We say σ satisfies hypothesis B iff $\forall\ \zeta \in \mathscr{Z} = \{x\colon \sigma(x) = 0\}$ there is a $\delta > 0$ such that if $I_1 = [\zeta, \zeta + \delta]$ and $I_2 = [\zeta - \delta, \zeta]$, then for $i = 1, 2$:

(a.1) $\sigma\sigma'$ is Lipschitz on I_i, or
(a.2) There exists a Lipschitz function g_i defined on I_i such that

$$\rho_i(x) = \frac{\int_\zeta^x g_i(u)\, du}{\sigma(x)}$$

is well defined, $\rho_i \in \mathscr{A}\mathscr{D}(I_i)$ and $\rho_i(x) \ge c > 0$ on I_i. Without loss of generality we shall assume $\rho_i(\zeta) = 1$. Note that $g(\zeta) = 0$.

Examples of functions satisfying hypothesis B:

I. If σ' is locally Lipschitz on $\mathcal{Z}$, that is, for every $\zeta \in \mathcal{Z}$ there exists a $\delta > 0$ such that σ' is Lipschitz on $[\zeta - \delta, \zeta + \delta]$.

II. If $\forall\, \zeta \in \mathcal{Z}$ there is a $\delta > 0$ such that for $i = 1, 2$, $\sigma\sigma'$ is Lipschitz on I_i or there is a Lipschitz function g_i such that:

(a.3) $\forall\, x \in I_i$, $|g(x) - \sigma'(x)| \le |\sigma(x)|\phi(x)$;

(a.4) If $r(x) = \sigma(x) - \int_\zeta^x g_i(u)\,du$, then $\forall\ x \in I_i$, $|r(x)| \le (\sigma(x))^2\phi(x)$, where ϕ is bounded on I_i and $\forall\, x \in \mathcal{Z} \cap I_i$, $\phi(z) = \lim_{x \to z} \phi(x) = 0$.

Case II is more general than case I. In fact, it is enough to consider σ such that $\sigma(0) = 0$ and σ' satisfying:

$$\sigma'(x) = \begin{cases} x, & x \in [0,1] \setminus \bigcup_{n \ge 0} [2^{-n} - b_n, 2^{-n}), \\ x + x^4, & x \in \bigcup_{n \ge 0} [2^{-n} - b_n, 2^{-n}), \\ 1, & x \ge 0, \\ 0, & x < 0, \end{cases}$$

where b_n satisfies $\int_{2^{-n}-b_n}^{2^{-n}} (u + u^4)\,du = 2^{-6n}$. Such a σ' is discontinuous in any neighborhood of 0, so it does not satisfy case I, but it is not difficult to prove that σ satisfies case II.

A sufficient condition to have (a.3) and (a.4) is the following: For any $\zeta \in \mathcal{Z}$ assume there is a $\delta > 0$ such that σ' is continuous on $I = [\zeta - \delta, \zeta + \delta]$, $\sigma'(\zeta) \ne 0$ and there is a Lipschitz function g such that

$$\lim_{x \to \zeta} \frac{|\sigma'(x) - g(x)|}{|x - \zeta|} = 0.$$

For example, if σ is twice differentiable at ζ and $\sigma'(\zeta) \ne 0$, then the previous condition holds with $g(x) = \sigma'(\zeta) + \sigma''(\zeta)(x - \zeta)$.

PROPOSITION A.2. *Hypothesis* B $\Rightarrow$ *hypothesis* A.

PROOF. Let $\zeta \in \mathcal{Z}$ and take δ given by hypothesis B. If $\sigma\sigma'$ is Lipschitz on I_i, take $\rho_i = 1$, otherwise take

$$\rho_i(x) = \frac{\int_\zeta^x g_i(u)\,du}{\sigma(x)} \in \mathcal{AD}.$$

Define

$$\rho(x) = \begin{cases} \rho_1(x), & x \in I_1, \\ \rho_2(x), & x \in I_2. \end{cases}$$

Since $\rho_i(\zeta) = \rho_2(\zeta) = 1$, then $\rho \in \mathcal{AD}(I)$ where $I = I_1 \cup I_2$. Consider $F(x) = \int_\zeta^x \rho(u)\,du$, then F is C^1, $F' = \rho \in \mathcal{AD}(I)$ and $F' \ge c > 0$ on I. Let F'' be the cadlag version of the second derivative of F. Define

$$h = F''\sigma^2 + F'\sigma\sigma'.$$

If $\sigma\sigma'$ is Lipschitz on I_1 but not on I_2, we have

$$h = \begin{cases} \sigma\sigma', & \text{on } I_1, \\ g_2, & \text{on } I_2. \end{cases}$$

Since $g_2(\zeta) = \sigma\sigma'(\zeta) = 0$, h is Lipschitz on I (remember that g_2 is Lipschitz). If $\sigma\sigma'$ is Lipschitz on I, then $h = \sigma\sigma'$ and again h is Lipschitz on I. Finally, assume that $\sigma\sigma'$ is not Lipschitz on I_1 and I_2. In this case

$$h = \begin{cases} g_1, & \text{on } I_1, \\ g_2, & \text{on } I_2, \end{cases} \quad \text{where } g_i \text{ is Lipschitz on } I_i.$$

Since $g_1(\zeta) = g_2(\zeta) = 0$, then h is Lipschitz on I. In any case h is Lipschitz and so hypothesis A is verified. □

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