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BROWNIAN SURVIVAL AMONG GIBBSIAN TRAPS

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We consider Brownian motion evolving among killing traps. We develop a technique of "enlargement of obstacles." This technique allows us to replace given trap configurations by configurations of enlarged traps, when deriving upper estimates on the probability that Brownian motion survives. Applied in a context of random obstacles, this reduces the complexity of the description for the environment seen by Brownian motion. We apply the method to the case where traps are distributed according to a fairly general Gibbs measure and obtain a result in the spirit of Donsker–Varadhan's theorem on Wiener sausage asymptotics.

0. Introduction. We study here the long time survival probability of a Brownian motion $Z_.$ on $\mathbb{R}^d$, $d \geq 1$, moving among random obstacles constructed by translating a model nonpolar compact set K at the points of a Gibbs point process independent of $Z_.$. We assume that the law $\mathbb{P}$ of this point process satisfies the Dobrushin–Lanford–Ruelle (DLR) equations relative to an activity number $\nu > 0$ and a suitable translation invariant pair potential $V(x-y)$, $V(\cdot)$ symmetric, compactly supported. Precise assumptions on $\mathbb{P}$ are given in Section 2. Let us simply mention at this point that we do not require translation invariance of $\mathbb{P}$, nor uniqueness for the solution of the DLR equations.

If T stands for the entrance time of $Z_.$ into the obstacles, and P_0 for standard Wiener measure, we show that,

$$\lim_{t\to\infty} t^{-d/(d+2)} \log(\mathbb{P} \otimes P_0[T > t]) = -c(d,p), \tag{0.1}$$

with

$$c(d,p) = (p\omega_d)^{2/(d+2)} \left(\frac{d+2}{2}\right)\left(\frac{2\lambda_d}{d}\right)^{d/(d+2)}. \tag{0.2}$$

Here ω_d, λ_d stand, respectively, for the volume of the unit ball of $\mathbb{R}^d$ and the principal Dirichlet eigenvalue of $-(1/2)\Delta$ in the unit ball of $\mathbb{R}^d$, and $p \in (0,\infty)$ is the pressure:

$$p = \lim_{N\to\infty} N^{-d} \log\Big(Z\big([0,N]^d\big)\Big), \tag{0.3}$$

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where for Λ a bounded Borel subset of $\mathbb{R}^d$,

$$Z(\Lambda) = \sum_{n \geq 0} \frac{\nu^n}{n!} \int_{\Lambda^n} dz_1 \cdots dz_n \exp\left\{ -\sum_{i<j} V(z_i - z_j) \right\}. \tag{0.4}$$

Our assumptions on $V(\cdot)$ ensure the existence of such a p. The constant $c(d, p)$ arises in fact from the minimization problem

$$c(d,p) = \inf_U \{p|U| + \lambda(U)\}, \tag{0.5}$$

where U runs over the bounded open sets in $\mathbb{R}^d$ with negligible boundary. In (0.5), $|U|$ stands for the volume of U and $\lambda(U)$ for the principal Dirichlet eigenvalue of $-(1/2)\Delta$ in U.

When $V = 0$, we are in the case of a Poisson cloud with intensity ν, and in this case $p = \nu$. The asymptotic result (0.1) in this context can be found in Donsker and Varadhan [2], when K is a ball of arbitrary radius, and in [11] when K is an arbitrary nonpolar compact set.

Our results include, for instance, Poisson point processes with exclusion at arbitrary activity $\nu > 0$ (see Gallavotti and Miracle–Sole [3], Mürmann [6], Preston [8] and Ruelle [10]). In this case V is the hard core potential which is infinite on $\overline{B}(0, a)$ and 0 elsewhere. If K is precisely $\overline{B}(0, a)$, we have the case of nonoverlapping traps, at arbitrary activity ν, and (0.1) recovers a result which was argued by Kayser and Hubbard [5] on physical grounds. We also treat cases of potentials which are sufficiently repulsive at the origin but may take negative values.

There seemed to be some questions on whether or not the asymptotics (0.1) would be influenced by the occurrence of a "phase transition" (nonuniqueness of the solution of the DLR equations). We work here with possibly large values of ν where such nonuniqueness of $\mathbb{P}$ is expected.

Let us give some ideas of the proof of (0.1). The main difficulty lies in the proof of the upper bound part of (0.1). The existing proofs in the case of a Poissonian cloud (see [2] and [11]) crucially involve a step in which one dominates the survival probability $\mathbb{P} \cap P_0[T > t]$ by a similar quantity, where now $\mathbb{R}^d$ is replaced by a torus. This step creates a strong rigidity of the proof, and does not seem to be available in the present situation.

Here we build up on the ideas of [11, 12, 13] and bypass the projection argument. The heart of the matter is a technique developed in Section 1. This technique, when applied to our problem, allows one to work with much bigger obstacles modeled on a ball $\overline{B}(0, b)$ instead of K, and to restrict our attention to what happens in a certain open subset of diameter $\sim$ const. $t^{1/(d+2)}$, inside a cubic box of size $t^{(d+1)/(d+2)}$ centered at the origin, instead of the whole of $\mathbb{R}^d$. Thanks to this reduction one has a good control on the "number of possibilities" for this modified obstacle environment.

Let us briefly describe the "enlargement technique" of Section 1. We start with a deterministic cloud configuration of "true traps" of size $a\varepsilon$ (ε small; in the application to our problem $\varepsilon = t^{-1/(d+2)}$), and with a possibly unbounded subset $\mathscr{T}$ of $\mathbb{R}^d$.

First we replace at every point of the cloud the true traps by enlarged obstacles which are balls of radius $b\varepsilon$. Then we chop $\mathbb{R}^d$ into cubes of unit side. We say that a point of the cloud x_i in such a cube is "good" if the fraction of volume occupied by enlarged obstacles in successive concentric balls $B(x_i, 10^l b\varepsilon)$, $l \geq 0$, $10^l b\varepsilon < 1$, intersected with the cube, is nonvanishing. It turns out that thanks to a covering argument, one has a good control on the volume of enlarged traps at bad points.

The next step is to determine which cubes of $\mathbb{R}^d$ are of "clearing type" or of "forest type." This is done purely in terms of enlarged traps, by picking a small number r and asking that in a box of forest type the total volume left unoccupied by enlarged obstacles sitting at good points is smaller than the volume of a ball of radius r. The other boxes are said to be of "clearing type."

Now one introduces Θ_b, the open set obtained by considering the neighborhood of size 1, of the union of "clearing boxes" in $\mathscr{T}$, and deleting from it enlarged obstacles sitting at good points. The real open set under study is Θ, the complement in $\mathscr{T}$ of true traps. The main point is that by making r small, one can pick ε sufficiently small so that uniformly on the cloud configuration and the open set $\mathscr{T}$, the bottom of the Dirichlet spectrum of $-(1/2)\Delta$ in Θ_b is not really bigger than the bottom of the Dirichlet spectrum of $-(1/2)\Delta$ in Θ, provided the bottom of the spectrum in Θ has a reasonable value.

This enables us to detect clearings of unit size left open by true obstacles in $\mathscr{T}$ by restricting to a neighborhood of size 1 of the "clearing boxes" in $\mathscr{T}$ and by working with enlarged traps.

In fact one can enlarge at a faster rate. If ε denotes the size of enlarged obstacles and $\varepsilon'(\varepsilon) \leq \varepsilon$ the size of true obstacles, the critical condition when $d \geq 2$ is that the "capacity of the true trap per volume of the enlarged trap" ratio tends to infinity, that is,

$$\lim_{\varepsilon \to 0} [\log(1/\varepsilon')]^{-1}/\varepsilon^2 = \infty, \quad \text{when } d = 2,$$

$$\lim_{\varepsilon \to 0} (\varepsilon')^{d-2}/\varepsilon^d = \infty, \quad \text{when } d = 3,$$

(when $d = 1$, the true obstacles can be points).

1. Some uniform exponential and eigenvalue estimates. The object of this section is to derive uniform estimates which allow us to replace small obstacles or traps, for instance in a bounded open set of $\mathbb{R}^d$, by obstacles of a larger size without noticeably raising the principal Dirichlet eigenvalue of the part of the open set occupied by the obstacles, provided the principal Dirichlet eigenvalue for the initial configuration has a reasonable value.

The methods we use are inspired by the techniques developed in [11, 12, 13]. The main improvement here shows how to get rid of the compactness assumptions present in [11, 12]. When we come back in Section 2 to the trapping questions described in the introduction, we do not need to dominate the initial problem on $\mathbb{R}^d$ by a problem on a torus, as required in [11] or [12].

Let us first describe our setting. We denote by Ω the set of simple pure point Radon measures in $\mathbb{R}^d$. An element $\omega = \sum_i \delta_{x_i}$ of Ω, will describe the cloud of points, where obstacles fall on $\mathbb{R}^d$. The obstacles will be the translates of εK, where K is a nonpolar compact set of $\mathbb{R}^d$ and ε a positive number, at the points x_i of the support of ω.

We shall study the effect of these obstacles in a nonempty possibly unbounded open set $\mathscr{T}$ of $\mathbb{R}^d$. This open set should not be viewed as being fixed. As a matter of fact, our controls will be uniform on $\mathscr{T}$, and for the application to the original trapping problem, we shall pick $\mathscr{T} = (-N[t^{d/(d+2)}], N[t^{d/(d+2)}])^d$, and $\varepsilon = t^{-1/(d+2)}$, for t going to infinity.

For each multiindex $m \in \mathbb{Z}^d$, C_m stands for the cube

$$C_m = \{z \in \mathbb{R}^d,\ m_i \le z_i < (m_i + 1),\ i = 1, \dots, d\}. \tag{1.1}$$

We also set (K is the model for the obstacles),

$$a = \sup\{|z|,\ z \in K\}. \tag{1.2}$$

The enlarged obstacles are obtained in the following fashion. We have two numbers, b and δ, where $b > a$ and $0 < \delta < 1$. We say that a point $x_i \in C_m$, in the support of ω, is good, see [11], if for all closed balls $C = \overline{B}(x_i, 10^{l+1}\varepsilon b)$, $0 \le l$ and $10^{l+1}\varepsilon b < 1/2$,

$$\left| C_m \cap C \cap \left(\bigcup_{x_j \in C_m} \overline{B}(x_j, b\varepsilon) \right) \right| \ge \frac{\delta}{3^d} |C_m \cap C|. \tag{1.3}$$

Here $|\cdot|$ denotes Lebesgue volume. We let Good(m) be the set of good points in C_m and G be the union $\bigcup_m$ Good(m). From (2.4) in [13], by a covering argument, we know that the union of balls of radius $b\varepsilon$ centered at "bad points" of C_m covers a small fraction of the volume of C_m, namely,

$$\left| C_m \cap \left(\bigcup_{x_i \in \mathrm{Bad}(m)} \overline{B}(x_i, b\varepsilon) \right) \right| \le \delta |C_m| = \delta, \tag{1.4}$$

if Bad(m) stands for the bad points of C_m. We also chop identically each segment $[k, k+1]$ into at most $[\sqrt{d}/b\varepsilon] + 1$ intervals of length $b\varepsilon/\sqrt{d}$ each, except perhaps the "last one." This yields closed boxes of diameter less than $b\varepsilon$, with union $\overline{C}_m$.

We can now decide whether a cube C_m is of "clearing type" or "forest type" as follows. We introduce a number $r > 0$, and set Cl_m to be the event, "there is a clearing of size r in the cube C_m:"

$$Cl_m = \{\omega,\ |\tilde{U}_m(\omega)| \ge 2^{-d}|B(0,r)| = 2^{-d}\omega_d r^d\}, \tag{1.5}$$

if $\tilde{U}_m(\omega)$ is the open subset of $\mathring{C}_m$ obtained by taking the complement in the interior of C_m of the closed boxes where a good point of C_m [$x_i \in$ Good(m)] falls. We then set $A(\omega)$ to be the closed set union of all closed cubes $\overline{C}_m$ in $\mathbb{R}^d$

where there is a clearing of size r:

$$1_{A(\omega)}(z) = \sum_m 1_{\overline{C}_m}(z) \cdot 1_{Cl_m}(\omega). \tag{1.6}$$

We now define A^1 as the open set of points at distance less than 1 from A. If A is empty, so is A^1.

At this stage, let us mention that the closed set $A(\omega)$ as well as the open sets $\tilde{U}_m(\omega)$, $m \in \mathbb{Z}^d$, have been defined purely in terms of the enlarged obstacles without any reference to the "true obstacles" built on εK. Following [13], we introduce the successive excursions of the canonical process $Z_.(w)$ on $C(\mathbb{R}_+, \mathbb{R}^d)$ at distance 1 from $A(\omega)$:

$$D_1 = \inf\left\{v \geq 0,\ Z_v \in (A^1)^c\right\} \leq \infty,$$

$$R_1 = \inf\{v \geq D_1,\ Z_v \in A\} = H_A \circ \theta_{D_1} + D_1 \leq \infty,$$

where H_A is the entrance time in A and θ the canonical shift. By induction, for $n \geq 1$, we set

$$D_{n+1} = H_{(A^1)^c} \circ \theta_{R_n} + R_n \leq \infty,$$

$$R_{n+1} = H_A \circ \theta_{D_{n+1}} + D_{n+1} \leq \infty.$$

If U is a nonempty open set, and T_U denotes the entrance time of $Z_.$ in U^c, we set

$$r_U(t,x,y) = p_t(x,y) P^t_{x,y}[T_U > t], \qquad t > 0, \tag{1.7}$$

where $p_t(\cdot,\cdot)$ is the Brownian transition density and $P^t_{x,y}$ the Brownian bridge measure in time t from x to y. If x or y is not in U, $r_U(t,x,y) = 0$. In what follows, we will tend to use the letter "H" to denote entrance times in closed sets and "T" to denote exit time from open sets. Of course it is in the nature of things that the exit time from an open set is also the entrance time in the complement.

Formula (1.7) defines the symmetric kernel of a C_0 self adjoint contraction semigroup in $L^2(U, dx)$, and $\lambda(U)$ will stand for the bottom (≥ 0) of the spectrum of the generator of the semigroup. If U is bounded, the semigroup is of trace class and $\lambda(U)$ is the principal Dirichlet eigenvalue of $-(1/2)\Delta$ in U. If U is empty, we set $\lambda(U) = \infty$. Finally, we introduce Θ_b, the open set complement in $\mathcal{T} \cap A^1$ of $\bigcup_{x_i \in G} \overline{B}(x_i, b\varepsilon)$, and $\tilde{T} = T_{\mathcal{T}} \wedge T$, where $T_{\mathcal{T}}$ is the entrance time in $\mathcal{T}^c$ and T the entrance time in the true obstacle set $\mathcal{K} = \bigcup_{x_i}(x_i + \varepsilon K)$.

We are now ready to derive exponential estimates for the exit time $\tilde{T}$ of $\Theta = \mathcal{T} \setminus \mathcal{K}$, which as a byproduct will show that when ε is small, $\lambda_b =_{\text{def}} \lambda(\Theta_b)$ is not significantly bigger than $\lambda(\Theta)$, provided $\lambda(\Theta)$ is not too large. In what follows, P_z stands for Wiener measure starting from z.

THEOREM 1.1. *Pick* $M > 0$, $\rho > 0$.

$$\limsup_{r\to 0} \sup_{b>a, K, 0<\delta<1} \limsup_{\varepsilon\to 0} \sup_{z,\omega,\mathscr{T}} E_z\big[\exp\{(\lambda_b \wedge M - \rho)\tilde{T}\}\big] \le 1 + \tfrac{8}{3}C(d, M, \rho), \tag{1.8}$$

where $C(d, M, \rho)$ *is defined in* (1.9).

For the reader puzzled by the statement of Theorem 1.1, it is may be helpful to give the translation of (1.8) in quantifier language. It means that for any $M > 0$, $\rho > 0$, $\eta > 0$, we can find $r_0 > 0$, such that for $r \le r_0$, for any K and $b > a$ [$= a(K)$, see (1.2)], $0 < \delta < 1$, there is an $\varepsilon_0 > 0$, such that for $0 < \varepsilon \le \varepsilon_0$, and any $z, \omega, \mathscr{T}$, $E_z[\exp\{(\lambda_b \wedge M - \rho)\tilde{T}\}] \le 1 + (8/3)C(d, M, \rho) + \eta$. In fact we shall see that we can pick $\eta = 0$ in the previous statement. It should also be mentioned that we could assume $z = 0$ without any loss of generality, since the cloud configuration ω is arbitrary.

PROOF. We start with the following lemma.

LEMMA 1.2. *For any* $M > 0$, $\rho > 0$,

$$\sup_{U\subset R^d \text{ open}, z} E_z\big[\exp\{(M \wedge \lambda(U) - \rho)T_U\}\big] \vee 1 =_{\text{def}} C(d, M, \rho) < \infty. \tag{1.9}$$

PROOF. With no loss of generality, we assume $(M \wedge \lambda(U) - \rho) > 0$, and $U \ne \varnothing$:

$$\begin{aligned} &E_z\big[\exp\{(M \wedge \lambda(U) - \rho)T_U\}\big] \\ &\quad = 1 + (\lambda(U) \wedge M - \rho)\int_0^\infty ds \int_{R^d} dy\, e^{(\lambda(U)\wedge M-\rho)s} r_U(s, z, y). \end{aligned} \tag{1.10}$$

Now if R_t denotes the semigroup with kernel $r_U(t, \cdot, \cdot)$ on $L^2(U, dx)$ for $t > 2$ and $x \in U$,

$$\begin{aligned} r_U(t, x, x) &= \big(r_U(1, x, \cdot), R_{t-2}(r_U(1, x, \cdot))\big)_{L^2(U,dx)} \\ &\le \exp\{-\lambda(U)(t-2)\}\|r_U(1, x, \cdot)\|^2_{L^2(U,dx)} \\ &\le (2\pi)^{-d/2}\exp\{-\lambda(U)(t-2)\}, \end{aligned}$$

since r_U is dominated by the Brownian transition density. As follows classically from Cauchy–Schwarz inequality and the Chapman–Kolmogorov relation, applied at time $t/2$,

$$r_U(t, x, y) \le \sup_z r_U(t, z, z).$$

It follows that for $t > 2$, x, y in $\mathbb{R}^d$:

$$r_U(t,x,y) \le \left((2\pi)^{-d/2}\exp\{-\lambda(U)(t-2)\}\right) \wedge \left((2\pi t)^{-d/2}\exp\left\{-\frac{(x-y)^2}{2t}\right\}\right). \tag{1.11}$$

For convenience, let us write $\lambda = \lambda(U) \wedge M$. From (1.10) and (1.11) we see that with $\sigma_{d-1} = \mathrm{vol}(S^{d-1})$:

$$\begin{aligned} E_z&[\exp\{(\lambda-\rho)T_U\}] \\ &\le 1 + 2(M-\rho)e^{2(M-\rho)} \\ &\quad + \lambda\int_2^\infty ds\int_0^\infty dr\,\sigma_{d-1}r^{d-1}\Big[\left((2\pi)^{-d/2}e^{-\lambda(s-2)}\right) \\ &\qquad \wedge\left((2\pi s)^{-d/2}e^{-r^2/2s}\right)\Big]e^{(\lambda-\rho)s}. \end{aligned} \tag{1.12}$$

The last term of (1.12) is bounded by

$$\begin{aligned} &\lambda\int_0^\infty ds\int_{\sqrt{2\lambda}\,s}^\infty dr\,\sigma_{d-1}r^{d-1}(2\pi s)^{-d/2}\exp\left\{-\frac{r^2}{2s}+(\lambda-\rho)s\right\} \\ &\quad + \lambda\omega_d\left(\frac{\lambda}{\pi}\right)^{d/2}\int_0^\infty s^d e^{-\rho s+2M}\,ds \\ &= \lambda\int_0^\infty\int_{\sqrt{\lambda}}^\infty \sigma_{d-1}\left(\frac{s}{\pi}\right)^{d/2} r^{d-1}\exp\{-(r^2-\lambda+\rho)s\}\,dr\,ds \\ &\quad + \lambda e^{2M}\omega_d d!\cdot\rho^{-(d+1)}\left(\frac{\lambda}{\pi}\right)^{d/2}. \end{aligned}$$

Setting $r-\sqrt{\lambda} = u$, and using $r^2-\lambda \ge (r-\sqrt{\lambda})^2$, when $r \ge \sqrt{\lambda}$, the last expression is smaller than

$$\begin{aligned} &\lambda\int_0^\infty\int_0^\infty \sigma_{d-1}(s/\pi)^{d/2}\left(u+\sqrt{\lambda}\right)^{d-1}\exp\{-(u^2+\rho)s\}\,du\,ds \\ &\quad + e^{2M}\omega_d d!\rho^{-(d+1)}(M/\pi)^{d/2}\cdot M \\ &\le M\sigma_{d-1}\Gamma\left(\frac{d}{2}+1\right)\pi^{-d/2}\int_0^\infty\frac{\left(u+\sqrt{M}\right)^{d-1}}{\left(u^2+\rho\right)^{d/2+1}}\,du \\ &\qquad + Me^{2M}\omega_d d!\rho^{-(d+1)}(M/\pi)^{d/2} < \infty. \end{aligned}$$

This, together with (1.12), proves our claim. □

Let us now prove Theorem 1.1. We define

$$c_1(d) = \tfrac{1}{2}\inf_{F\in\mathscr{C}} P_0[H_F < H_3] > 0, \tag{1.13}$$

where $\mathscr{C}$ is the class of compact subsets F in $\overline{B}(0,2)$ such that $|F| \geq 2^{-d}(1 - 2^{-d})|B(0,2)|$ and $H_3 = \inf\{u \geq 0, |Z_u - Z_0| \geq 3\}$ (and similarly for H_c, $c > 0$). We will also need

$$(1.14) \qquad \beta(d, M, \rho) = 1/[24C(d, M, \rho) + 2],$$

so that $3\beta/(1 - 2\beta) = [8C(d, M, \rho)]^{-1}$.

Now we pick $0 < r < 1/4$ small enough so that

$$(1.15) \qquad (1 - c_1)^{[1/(4\sqrt{r})]} \leq \beta,$$

$$(1.16) \qquad E_z\big[\exp\{MH_{\sqrt{r}}\}\big] \leq 1 + \beta, \qquad z \in \mathbb{R}^d.$$

We also set

$$(1.17) \qquad \begin{aligned} \alpha(\delta, b, K, d) = &\inf_{|z| \leq 1,\, F \in \mathscr{C}'} P_z[H_F < H_{10}] \\ &\times \inf_{|z| \leq b} P_z\big[H_K < H_{B(0,3b)^c}\big] > 0, \end{aligned}$$

where $\mathscr{C}'$ is the class of compact subsets of $\overline{B}(0,1)$ with relative volume no smaller than $\delta/6^d$. We then introduce $m(M, \rho, \delta, b, K, d)$, the smallest integer such that

$$(1.18) \qquad (1 - \alpha)^m \leq \big[8C(d, M, \rho)\big]^{-1}.$$

Suppose now ε is small enough, which we will assume from now on, so that

$$(1.19) \qquad 10^{m+1}b\varepsilon + b\varepsilon < r < 1/4.$$

Then exactly as in [13], (2.19)–(2.22), if z is at distance smaller or equal to $b\varepsilon$ of a good point x_i,

$$(1.20) \qquad \begin{aligned} P_z\big[H_r > \tilde{T}\big] \geq P_z\big[H_{10^{m+1}b\varepsilon + b\varepsilon} \geq \tilde{T}\big] &\geq 1 - (1 - \alpha)^m \\ &\geq 1 - \big[8C(d, M, \rho)\big]^{-1} \geq 1/2, \end{aligned}$$

as follows by looking at the successive times of escape of the process $Z_.$ from the balls $\overline{B}(x_i, 10^{l+1}\varepsilon b)$, $1 \leq l \leq m$. Moreover, when (1.19) holds, if $z \in \overline{C_m} \cap A^c$ for some $m \in \mathbb{Z}^d$, then the intersection of $B(z, 2r)$ with the union of (closed) subboxes in $\overline{C_m}$ containing some good point has volume bigger than

$$|B(z, 2r) \cap C_m| - 2^{-d}|B(0, r)| \geq (1 - 2^{-d})2^{-d}|B(0, 2r)|,$$

since $r < 1/4$. It now follows as in [13], (2.16), that $P_z[H_{\overline{C_m} \setminus \tilde{U}_m} < H_{3r}] \geq 2c_1$. Combining with (1.20), this yields under (1.19):

$$(1.21) \qquad \text{for } z \in A^c, \quad P_z\big[\tilde{T} < H_{4r}\big] \geq c_1.$$

We will now prove the following lemma.

LEMMA 1.3.

$$(1.22) \quad \textit{For } z \in \mathbb{R}^d, \quad E_z\Big[\exp\big\{M\big(H_A \wedge \tilde{T}\big)\big\}\Big] \leq 1 + \big[8C(d, M, \rho)\big]^{-1}.$$

PROOF. If $z \in A \cup \mathscr{T}^c \cup \mathscr{K}$ ($\mathscr{K}$, we recall is $\bigcup_i (x_i + K)$, the true obstacle set), (1.22) is immediate. Otherwise, denote by H^i, $i \geq 0$ the successive times of travel of $Z_.$ at distance $\sqrt{r}$:

$$H^0 = 0, \qquad H^{i+1} = H^i + H_{\sqrt{r}} \circ \theta_{H^i}, \qquad i \geq 0.$$

We have

$$\begin{aligned} E_z\Big[\exp\Big\{M\big(H_A \wedge \tilde{T}\big)\Big\}\Big] &= \sum_{k \geq 0} E_z\Big[H^k < H_A \wedge \tilde{T} \leq H^{k+1}, e^{M(H_A \wedge \tilde{T})}\Big] \\ &\leq \sum_{k \geq 0} E_z\Big[H^k < H_A \wedge \tilde{T}, e^{\{MH^k\}}\Big] E_0\big[e^{MH_{\sqrt{r}}}\big]. \end{aligned}$$

Now when $k \geq 1$,

$$\begin{aligned} & E_z\Big[H^k < H_A \wedge \tilde{T}, e^{MH^k}\Big] \\ & \quad = E_z\Big[H^{k-1} < H_A \wedge \tilde{T}, e^{MH^{k-1}} E_{Z_{H^{k-1}}}\Big[\Big(H_{\sqrt{r}} < H_A \wedge \tilde{T}\Big) e^{MH_{\sqrt{r}}}\Big]\Big] \\ (1.23) & \quad \leq E_z\Big[H^{k-1} < H_A \wedge \tilde{T}, e^{MH^{k-1}}\Big(E_{Z_{H^{k-1}}}\Big[H_{\sqrt{r}} < H_A \wedge \tilde{T}\Big] \\ & \qquad\qquad + E_{Z_{H^{k-1}}}\big[e^{MH_{\sqrt{r}}}\big] - 1\Big)\Big]. \end{aligned}$$

Now on the set $H^{k-1} < H_A \wedge \tilde{T}$, $k \geq 1$, $Z_{H^{k-1}}$ belongs to A^c. Using (1.21) together with the strong Markov property at the successive times of travel at distance $4r$, we find

$$E_{Z_{H^{k-1}}}\Big[H_{\sqrt{r}} < H_A \wedge \tilde{T}\Big] \leq (1 - c_1)^{[\sqrt{r}/4r]} \leq \beta(d, M, \rho)$$

on $H^{k-1} < H_A \wedge \tilde{T}$ [see (1.15)]. If we now use this last inequality together with (1.16) in (1.23), we obtain for $k \geq 1$:

$$\begin{aligned} E_z\Big[H^k < H_A \wedge \tilde{T}, e^{MH^k}\Big] &\leq E_z\Big[H^{k-1} < H_A \wedge \tilde{T}, e^{MH^{k-1}}\Big](2\beta) \\ &\leq (2\beta)^k, \quad \text{by induction}. \end{aligned}$$

It then follows that

$$E_z\Big[\exp\Big\{M\big(H_A \wedge \tilde{T}\big)\Big\}\Big] \leq (1 + \beta) \sum_{k \geq 0} (2\beta)^k = \frac{1 + \beta}{1 - 2\beta} = 1 + \frac{3\beta}{1 - 2\beta},$$

which proves our claim, thanks to the choice (1.14) of β. □

We shall now give a bound on $E_z[\exp\{(\lambda_b \wedge M - \rho)\tilde{T}\}]$, z in $\mathbb{R}^d$. It is enough to study the case where $\lambda_b \wedge M - \rho > 0$. In this case we introduce the stopping time

$$\begin{aligned} \tau &= \inf\{u \geq 0, |Z_u - Z_0| \geq r\} \wedge \tilde{T}, \quad \text{when } Z_0 \in A^1 \\ &= H_A \wedge \tilde{T}, \quad \text{when } Z_0 \notin A^1. \end{aligned}$$

T_b will stand for the exit time from Θ_b:

$$T_b = \inf\left\{u \geq 0,\, Z_u \notin A^1 \cap \mathscr{T} \text{ or } Z_u \in \bigcup_{x_i \in G} \overline{B}(x_i, b\varepsilon)\right\}.$$

Similarly as in [12], we define

$$S_0 = 0, \qquad S_1 = \tau \circ \theta_{T_b} + T_b, \qquad S_{k+1} = S_k + S_1 \circ \theta_{S_k}, \qquad k \geq 1.$$

Under the assumption that $\lambda =_{\text{def}} (\lambda_b \wedge M - \rho) > 0$, the S_k are easily seen to be finite, thanks to Lemma 1.2 and 1.3. We set $J = \inf\{k \geq 0,\, Z_{S_J} \in \mathscr{T}^c \cup \mathscr{K}\}$; then $S_J \geq \tilde{T}$ and from what follows next, J is finite almost surely. Indeed, for $k \geq 1$,

$$E_z\left[Z_{S_0}, \dots, Z_{S_k} \notin \mathscr{T}^c \cup \mathscr{K},\, e^{\lambda S_k}\right] = E_z\Big[Z_{S_0}, \dots, Z_{S_{k-1}} \notin \mathscr{T}^c \cup \mathscr{K},$$
$$e^{\lambda S_{k-1}} E_{Z_{S_{k-1}}}\left[e^{\lambda T_b} E_{Z_{T_b}}\left[e^{\lambda\tau} 1\{Z_\tau \notin \mathscr{T}^c \cup \mathscr{K}\}\right]\right]\Big].$$

Now one has

$$E_{Z_{S_{k-1}}}\left[e^{\lambda T_b} E_{Z_{T_b}}\left[e^{\lambda\tau} 1\{Z_\tau \notin \mathscr{T}^c \cup \mathscr{K}\}\right]\right]$$
$$\leq E_{Z_{S_{k-1}}}\left[e^{\lambda T_b}\left(E_{Z_{T_b}}[e^{\lambda\tau}] - 1 + E_{Z_{T_b}}[Z_\tau \notin \mathscr{T}^c \cup \mathscr{K}]\right)\right].$$

Using (1.16) or Lemma 1.3, as well as the definition of $C(d, M, \rho)$ in Lemma 1.2, this is smaller than

$$\tfrac{1}{8} + E_{Z_{S_{k-1}}}\left[e^{\lambda T_b} E_{Z_{T_b}}\left[Z_\tau \notin \mathscr{T}^c \cup \mathscr{K}\right]\right]. \tag{1.24}$$

Now Z_{T_b} belongs to $(A^1)^c \cup \mathscr{T}^c$ or $\bigcup_G \overline{B}(x_i, b\varepsilon)$. Consider the expression $E_{Z_{T_b}}[Z_\tau \notin \mathscr{T}^c \cup \mathscr{K}]$. If $Z_{T_b} \in \mathscr{T}^c$, it is zero. If Z_{T_b} belongs to $(A^1)^c \cap \mathscr{T}$, then $\tau = H_A \wedge \tilde{T}$ and the expression is

$$P_{Z_{T_b}}\left[H_A < \tilde{T}\right] \leq (1 - c_1)^{[1/4r]} \leq \left[8C(d, M, \rho)\right]^{-1}, \tag{1.25}$$

using the strong Markov property at the successive times of travel of $Z_.$ at distance $4r$, as well as (1.21) and (1.15). Now if $Z_{T_b} \in \bigcup_G \overline{B}(x_i, b\varepsilon) \cap A^1$, the expression is

$$P_{Z_{T_b}}\left[H_r < \tilde{T}\right] \leq P_{Z_{T_b}}\left[H_{10^{m+1}b\varepsilon + b\varepsilon} < \tilde{T}\right] \leq (1 - \alpha)^m \leq \left[8C(d, M, \rho)\right]^{-1},$$

thanks to (1.20). In any case, we see that the expression in (1.24) is smaller than $1/4$. From this we deduce that

$$E_z\left[Z_{S_0}, \dots, Z_{S_k} \notin \mathscr{T}^c \cup \mathscr{K},\, e^{\lambda S_k}\right] \leq (1/4)^k, \qquad k \geq 0,$$

which also yields the finiteness of J, since $\lambda > 0$. We can now write

$$
\begin{aligned}
E_z\big[\exp\{\lambda \tilde{T}\}\big] &\le E_z\big[\exp\{\lambda S_J\}\big] \\
&\le 1 + \sum_{k=0}^{\infty} E_z\Big[Z_{S_0}, \dots, Z_{S_k} \notin \mathscr{T}^c \cup \mathscr{K}, \\
&\qquad\qquad e^{\lambda S_{k+1}} 1\big(Z_{S_{k+1}} \in \mathscr{T}^c \cup \mathscr{K}\big)\Big] \\
&\le 1 + \sum_{k=0}^{\infty} E_z\Big[Z_{S_0}, \dots, Z_{S_k} \notin \mathscr{T}^c \cup \mathscr{K}, e^{\lambda S_k} E_{Z_{S_k}}\big[e^{\lambda T_b} E_{Z_{T_b}}[e^{\lambda \tau}]\big]\Big].
\end{aligned}
$$

Using (1.16) or Lemma 1.3, and then Lemma 1.2, this is smaller than

$$
1 + 2C(d, M, \rho) \sum_{k=0}^{\infty} \left(\tfrac{1}{4}\right)^k = 1 + \tfrac{8}{3} C(d, M, \rho),
$$

which completes the proof of Theorem 1.1. □

Theorem 1.1 will now be applied to show that when r is small, for sufficiently small ε the bottom of the Dirichlet spectrum of $-(1/2)\Delta$ is not significantly raised when replacing Θ by Θ_b, provided the bottom of the spectrum in Θ has a reasonable value. The proof is very similar to that of [11], except for the fact that Θ and Θ_b in our present setting can be unbounded and we cannot use eigenfunction expansions any more.

COROLLARY 1.4. *For $M > 0$,*

$$
\lim_{r \to 0} \sup_{b > a, K, 0 < \delta < 1} \limsup_{\varepsilon \to 0} \sup_{\omega, \mathscr{T}} \big(\lambda(\Theta_b) \wedge M - \lambda(\Theta) \wedge M\big)_+ = 0.
$$

PROOF. To prove the claim, it is enough to check that for $\rho > 0$, $\sup_z E_z[\exp\{(\lambda_b \wedge M - \rho)\tilde{T}] < \infty$ implies $\lambda_b \wedge M - \rho \le \lambda(\Theta)$.

Suppose $\lambda_b \wedge M - \rho > \lambda(\Theta)\,(\ge 0)$. Then we can find an L^2 unit continuous function f with compact support in Θ and in $H^1(\mathbb{R}^n)$ such that

$$
\lambda(\Theta) \le \int_0^{\infty} \mu \, d(E_\mu f, f) < \lambda_b \wedge M - \rho, \tag{1.26}
$$

if E_μ is a resolution of the identity corresponding to the self adjoint Dirichlet semigroup in $L^2(\Theta, dx)$. Since the Dirichlet form decreases under absolute values, we can assume $f \ge 0$. Then, setting $\lambda = \lambda_b \wedge M - \rho$,

$$
\begin{aligned}
\infty > \|f\|_1 \|f\|_\infty \sup_z E_z\big[\exp\{\lambda \tilde{T}\}\big] &\ge \int_0^{\infty} ds\, \lambda e^{\lambda s} \int_{\Theta \times \Theta} dx\, dy\, r_\Theta(s, x, y) f(x) f(y) \\
= \int_0^{\infty} ds\, \lambda e^{\lambda s} \int_0^{\infty} e^{-\mu s} d(E_\mu f, f) &\ge \int_0^{\infty} ds\, \lambda \exp\left\{\left[\lambda - \int_0^{\infty} \mu\, d(E_\mu f, f)\right] s\right\},
\end{aligned}
$$

using Jensen's inequality in the last step. But the last inequality is impossible in view of (1.26). This proves our claim. □

Although this will not be needed for our application in Section 2, one can refine Theorem 1.1 and Corollary 1.3 in the following way. The "true obstacles" can be picked of a much smaller size $\varepsilon' K$ than εK. The natural condition in order to be able to replace at good points the obstacles built on $\varepsilon' K$ by obstacles built on the larger $\varepsilon \overline{B}(0, b)$ is that the ratio of "the capacity of the true obstacle" to the volume of the enlarged obstacle goes to infinity. That is $(\varepsilon')^{d-2}/\varepsilon^d \to \infty$ when $d \geq 3$, $[\log(1/\varepsilon')]^{-1}/\varepsilon^2 \to \infty$ when $d = 2$. In dimension 1, Theorem 1.1 already includes the case of $K = \{0\}$, that is, "$\varepsilon' = 0$." If this condition is violated, one has the following counterexample.

EXAMPLE 1.5. Pick ω to be the sum of Dirac masses sitting at points of $\varepsilon\mathbb{Z}^d$, $d \geq 2$. The obstacles are made of balls ε' centered at these points, where $\varepsilon' = \varepsilon^{d/(d-2)}$, $d \geq 3$, $\varepsilon' = \exp\{-1/\varepsilon^2\}$, $d = 2$. Pick $b = \sqrt{d}$ and $\mathscr{T} = B(0,1)$. The enlarged obstacles are made of balls $\overline{B}(m\varepsilon, \sqrt{d}\,\varepsilon)$, $m \in \mathbb{Z}^d$, and they cover $\mathscr{T}$ (and $\mathbb{R}^d$). Now every point of the cloud is a good point (for any $0 < \delta < 1$), there are no clearing cubes and $\lambda(\Theta_b) = \infty$. However we are, as far as the true obstacles are concerned, in the "constant capacity regime," and it is known (see Cioranescu and Murat [1] and Ozawa [7]) that $\lambda(\Theta)$ converges to a finite value, $\lambda_2 + \pi$ when $d = 2$ and $\lambda_d + \operatorname{cap}(\overline{B}(0,1))$ when $d \geq 3$, as ε goes to zero. Here $\operatorname{cap}(\overline{B}(0,1))$ stands for the capacity of $\overline{B}(0,1)$ relative to $(1/2)\Delta$, and λ_d is the principal Dirichlet eigenvalue of $-(1/2)\Delta$ in $B(0,1)$. □

We now assume that $d \geq 2$, and that the obstacles are translates at points of the support of ω of $\varepsilon' K$, where K is a nonpolar compact subset of $\overline{B}(0,1)$ and $\varepsilon' = f(\varepsilon) \leq a\varepsilon$, with

$$(1.27)\qquad \begin{aligned} &\lim_{\varepsilon \to 0} (\varepsilon')^{d-2}/\varepsilon^d = \infty, \quad \text{when } d \geq 3,\\ &\lim_{\varepsilon \to 0} (\log 1/\varepsilon')^{-1}/\varepsilon^2 = \infty, \quad \text{when } d = 2. \end{aligned}$$

THEOREM 1.6. *Under* (1.27), *for any* $M > 0$, $\rho > 0$,

$$(1.28)\qquad \begin{aligned} &\limsup_{r \to 0} \sup_{b > 2a,\, K,\, 0<\delta<1,\, f(\cdot)} \limsup_{\varepsilon \to 0} \sup_{z,\, \omega,\, \mathscr{T}} E_z\big[\exp\{(\lambda(\Theta_b) \wedge M - \rho)\tilde{T}\}\big]\\ &\qquad\qquad \leq 1 + \tfrac{8}{3} C(d, M, \rho),\\ &\limsup_{r \to 0} \sup_{b > 2a,\, K,\, 0<\delta<1,\, f(\cdot)} \limsup_{\varepsilon \to 0} \sup_{\omega,\, \mathscr{T}} \big(\lambda(\Theta_b) \wedge M - \lambda(\Theta) \wedge M\big)_+ = 0. \end{aligned}$$

PROOF. The proof follows exactly that of Theorem 1.1, except that we now pick α in a different way from (1.17). Indeed, from [12], Lemma 1.3 and its proof, we know that given $b > 2a$ and $0 < \delta < 1$, there is a constant $\alpha(d, \delta) > 0$, such that for any $\tilde{r} < 1/20$,

$$(1.29)\qquad \liminf_{\varepsilon \to 0} \inf_{\sim\sim} P_z\big[T < H_{B(y,\, 10\gamma)^c}\big] \geq 2\alpha,$$

where $\inf_{\sim\sim}$ means that the infimum is taken over ω in Ω, $\gamma \in (\tilde{r}, 1/20)$, z and

y, such that $|z - y| = \gamma$ and

$$\left| \bigcup_i \bar{B}(x_i, b\varepsilon) \cap \bar{B}(y, \gamma) \right| \geq \delta \cdot 6^{-d} |\bar{B}(y, \gamma)|.$$

Now we pick $0 < r < 1/20$ satisfying (1.15) and (1.16).

We then pick $m \geq 1$ according to (1.18) and apply (1.29) with $\tilde{r} = (r/2)10^{-(m+2)}$, so that we have an ε_0 which can be picked so that $10b\varepsilon_0 < 10^{-(m+2)}r/2$, and for $\varepsilon \leq \varepsilon_0$,

$$P_z\left[T < H_{B(x_i, 10^{l+2}b\varepsilon)^c}\right] \geq \alpha,$$

whenever $|z - x_i| = 10^{l+1}b\varepsilon \in (10^{-(m+2)}r/2, 1/20)$ and x_i is a good point of the support of ω. There are now at least $(m + 1)$ points of the form $10b\varepsilon 10^l$, $l \geq 0$ in the interval $(10^{-(m+2)}r/2, r/2)$, from which we see, as in (1.20),

$$P_z\left[H_r > \tilde{T}\right] \geq 1 - (1 - \alpha)^m \geq 1 - \left[8C(d, M, \rho)\right]^{-1},$$

where $\varepsilon \leq \varepsilon_0$, and z is at distance less than or equal to $b\varepsilon$ from some good point of ω. The condition $\varepsilon \leq \varepsilon_0$ also implies (1.19) and from then on the proof follows identically. □

2. Brownian survival among Gibbsian traps. We are now going to apply the results of Section 1 to the trapping problem described in the introduction. One canonical Brownian motion $Z_.$, under the law P_0, will now move among random traps in $\mathbb{R}^d$, obtained by translating a nonpolar compact set K of $\mathbb{R}^d$ at the points of an independent Gibbs point process.

More precisely, we have a symmetric measurable function $V(\cdot)$ on $\mathbb{R}^d$, with values in $(-\infty, \infty]$, which is bounded below by $-M$ ($M > 0$), and compactly supported in $B(0, l)$. We assume that $V(\cdot)$ is a stable potential (see Ruelle [9], page 33), that is, there is a $B > 0$ such that for any $z_1, \dots, z_n$ in $\mathbb{R}^d$,

$$\sum_{i<j} V(z_i - z_j) \geq -nB.$$

We also consider a number $\nu > 0$ and set for any bounded measurable set A in $\mathbb{R}^d$:

$$(2.1) \qquad Z(A) =_{\text{def}} \sum_{n \geq 0} \frac{\nu^n}{n!} \int_{A^n} dz_1 \cdots dz_n \exp\left\{ - \sum_{i<j} V(z_i - z_j) \right\} < \infty.$$

We let $p_A(d\omega)$ stand for $e^{\nu|A|}$ times the Poisson point measure on A with intensity ν. We require that $\mathbb{P}$, the law of the Gibbs point process on Ω, satisfy the DLR equations (after Dobrushin–Lanford–Ruelle), namely, for any bounded measurable A, the conditional distribution of $\omega_A =_{\text{def}} 1_A \cdot \omega$ given

ω_{A^c} is the law

$$p_A(\omega_{A^c}, d\omega) = Z(\omega_{A^c}, A)^{-1} \times \exp\Big\{-\int_{A^c}\int_A \omega_{A^c}(dz')\omega(dz)V(z'-z) - \tfrac{1}{2}\int\int_{z'\neq z} \omega(dz')\omega(dz)V(z-z')\Big\} p_A(d\omega), \tag{2.2}$$

where $Z(\omega_{A^c}, A)$ is the normalizing constant, which is finite thanks to our assumptions on V. For A, B bounded measurable we have

$$1 + \nu|A| \le Z(A), \tag{2.3}$$

$$Z(A \setminus B^l)Z(B) \le Z(A \cup B), \tag{2.4}$$

if B^r denotes the r neighborhood of B. We also assume that there is a number $p \in (0, \infty)$ (the pressure) such that

$$\lim_{L_1, \dots, L_d \to \infty} |\Lambda_L|^{-1} \log Z(\Lambda_L) = p, \tag{2.5}$$

if $\Lambda_L = [0, L_1] \times \cdots \times [0, L_d]$, $L_i \ge 0$.

We also require that $-p$ govern the logarithmic rate of the large deviation probability that a large ball receives no points of the cloud,

$$\lim_{R\to\infty} |B(0,R)|^{-1} \log(\mathbb{P}[\omega(B(0,R)) = 0]) = -p. \tag{2.6}$$

Since $\mathbb{P}[\omega(A) = 0] = E^{\mathbb{P}}[Z(\omega_{A^c}, A)^{-1}]$, this a type of assumption, in view of (2.5), that the interaction at the boundary is "well behaved."

Let us mention that we do not require $\mathbb{P}$ to be invariant under translations. Let us give some examples.

Example 1. If $V \ge 0$ (repulsive interaction), with the special case of hard cores $V = \infty$ on $\overline{B}(0, a)$, 0 elsewhere (Poisson with exclusion) and $V = 0$ (Poisson of intensity ν). In this case Gibbs measures are known to exist (see for instance Preston [8], Chapter 6, also Mürmann [6] for Poisson point processes with exclusion). Moreover, when $V \ge 0$, for A, B bounded measurable disjoint,

$$Z(A \cup B) \le Z(A)Z(B), \tag{2.7}$$

$$\mathbb{P}[\omega(A) = 0] = E^{\mathbb{P}}\big[Z(\omega_{A^c}, A)^{-1}\big] \le Z(A_l)^{-1}, \tag{2.8}$$

if $A_l = A \setminus (A^c)^l$ denote the set of points at distance at least l from A^c, [(2.8) holds even when V changes sign]. Our assumptions (2.5) and (2.6) now follow by classical arguments (see Ruelle [9], page 181). This class of examples contains the situation of nonoverlapping traps, where $C = \overline{B}(0, a)$ and V is infinite on $\overline{B}(0, a)$ and 0 elsewhere.

Example 2. In the case where $V(z) \ge \psi(|z|)$, $0 \le |z| \le \tilde{l}$, with $\psi \ge 0$, decreasing, $\int_0^{\tilde{l}} \psi(u)u^{d-1}\,du = \infty$ (and of course V compactly supported in $B(0, l)$, bounded below by $-M$), one can construct Gibbs measures $\mathbb{P}$ for which there exist $\alpha > 0$, $\beta > 0$ such that

$$\mathbb{P}\left[\sum_{m \in \mathscr{R}} \omega(B_m)^2 \geq N^2 \operatorname{card} \mathscr{R}\right] \leq \exp\{-(\alpha N^2 - \beta)\operatorname{card} \mathscr{R}\},$$

whenever $\mathscr{R}$ is a finite subset of $\mathbb{Z}^d$, and B_m is the cube centered at lm, with sides of length l (see Ruelle [10], Corollary 2.8 and Preston [8], page 108). Assumption (2.5) now follows from Theorem 3.3 in Ruelle [10], and (2.6) is proved by using very similar arguments as in Section 3 of [10]; see also Israel [4], Appendix B in the case of hard cores and Ruelle [9], Theorem 3.4.6.

Let us recall that T stands for the entrance time of $Z_.$ into the obstacles and $c(d, p)$ is the constant introduced in (0.2). Our main result is:

THEOREM 2.1.

$$\lim_{t \to \infty} t^{-d/(d+2)} \log \mathbb{P} \otimes P_0[T > t] = -c(d, p). \tag{2.9}$$

PROOF. The lower bound part of (2.9) is classical (see [2] or [11]). One simply writes that for $R > 0$,

$$\mathbb{P} \otimes P_0[T > t] \geq \mathbb{P}\left[\omega(B(0, Rt^{1/(d+2)} + a)) = 0\right] P_0\left[T_{B(0, Rt^{1/(d+2)})} > t\right],$$

the model obstacle K being included in $\overline{B}(0, a)$. Using (2.6), one gets

$$\liminf_{t \to \infty} t^{-d/(d+2)} \log \mathbb{P} \otimes P_0[T > t] \geq -\{p|B(0, R)| + \lambda(B(0, R))\};$$

optimizing over R one finds precisely $-c(d, p)$ in the right member of the last inequality.

Let us now prove the upper bound part of (2.9). First it is convenient to adopt $t^{1/(d+2)}$ and $t^{2/(d+2)}$ as new space and time units, so that we now study standard Brownian motion until time $s =_{\text{def}} t^{d/(d+2)}$ among obstacles which are translates of $t^{-1/(d+2)}K =_{\text{def}} \varepsilon K$ at the points x_i of the cloud governed by the law $\mathbb{P}_s$, obtained by rescaling $\mathbb{P}$ with the factor $\varepsilon = t^{-1/(d+2)}$.

We now pick for $s \geq 1$,

$$\mathscr{T} = (-N[s], N[s])^d,$$

where $N \geq 1$ is an integer large enough (see (2.10) in [13]) so that

$$\limsup_{s \to \infty} s^{-1} \log P_0[T_{\mathscr{T}} \leq s] \leq -(c(d, p) + 1). \tag{2.10}$$

Using the notations of Section 1, let us pick $r > 0$, $b > (2\sqrt{d}\, l) \vee a$, $0 < \delta < 1$. As a result of (1.4) (see (2.14) in [13]), for m in $\mathbb{Z}^d$,

$$|\tilde{U}_m| \leq |U_m| + \delta, \tag{2.11}$$

where we recall that $U_m(\omega)$ is the complement in the interior of C_m of the closed subboxes where a point of C_m falls (for $\tilde{U}_m$, where a good point of C_m

falls). It follows that

$$\begin{aligned} Cl_m &= \left\{|\tilde{U}_m| \geq 2^{-d}|B(0,r)|\right\} \\ &\subseteq \left\{|U_m \cap \tilde{C}_m| \geq 2^{-d}|B(0,r)| - \delta - d\big(b/\sqrt{d}\big)s^{-1/d}\right\}, \end{aligned} \tag{2.12}$$

if $\tilde{C}_m$ denotes the complement in C of the "last layer of subboxes" of C_m, namely,

$$\tilde{C}_m = \left\{z \in \mathbb{R}^d,\ m_i \leq z_i < m_i + \left[\sqrt{d}/b\varepsilon\right]b\varepsilon/\sqrt{d},\ i = 1,\dots,d\right\}. \tag{2.13}$$

Setting $\tilde{\mathscr{T}} = (-N[s] - 1,\ N[s] + 1)^d$, we have:

LEMMA 2.2. *For any $r > 0$,*

$$\limsup_{n_0\to\infty}\ \limsup_{b\to\infty,\,\delta\to 0}\ \limsup_{s\to\infty} s^{-1}\log\mathbb{P}_s\left[|A \cap \tilde{\mathscr{T}}| \geq n_0\right] = -\infty. \tag{2.14}$$

PROOF. We have

$$\mathbb{P}_s\left[|A \cap \tilde{\mathscr{T}}| \geq n_0\right] \leq (2Ns + 2)^{dn_0} \sup_{|\mathscr{M}|=n_0} \mathbb{P}_s\left[\bigcap_{m\in\mathscr{M}} \left\{|U_m| \geq 2^{-d}|B(0,r)| - \delta\right\}\right],$$

where $\mathscr{M}$ runs over subsets of $[-N[s] - 1, N[s]]^d$ with n_0 elements. For such an $\mathscr{M}$, we have

$$\mathbb{P}_s\left[\bigcap_{m\in\mathscr{M}} \left\{|U_m| \geq 2^{-d}|B(0,r)| - \delta\right\}\right] \leq 2^{n_0(\sqrt{d}s^{1/d}/b+1)^d} \sup_U \mathbb{P}_s[\omega(U) = 0],$$

where U runs over the complement of union of subboxes in $\bigcup_{m\in\mathscr{M}}\mathring{C}_m$ with volume bigger than $n_0(2^{-d}|B(0,r)| - \delta)$. For such a U, a repeated use of (2.4) [see (2.8)] shows that

$$\begin{aligned} \mathbb{P}_s[\omega(U) = 0] &\leq \mathbb{P}_s\left[\omega\Big(\bigcup_{m\in\mathscr{M}} \big(\tilde{C}_m \cap U\big)\Big) = 0\right] \\ &\leq Z\big(b/\sqrt{d} - 2l\big)^{-n_0 s(2^{-d}|B(0,r)| - \delta - d(b/\sqrt{d})s^{-1/d})\cdot(\sqrt{d}/b)^d}, \end{aligned}$$

where $Z(u)$ stands for $Z([0,u]^d)$, $u \geq 0$. It now follows that

$$\begin{aligned} &\limsup_{s\to\infty} s^{-1}\log\mathbb{P}_s\left[|A \cap \tilde{T}| \geq n_0\right] \\ &\quad\leq n_0\log 2\big(\sqrt{d}/b\big)^d - n_0\frac{\log Z\big(b/\sqrt{d} - 2l\big)}{\big(b/\sqrt{d}\big)^d}\big(2^{-d}|B(0,r)| - \delta\big). \end{aligned}$$

Now using (2.5), we get

$$\limsup_{b\to\infty,\,\delta\to 0}\ \limsup_{s\to\infty} s^{-1}\log\mathbb{P}_s\left[|A \cap \tilde{T}| \geq n_0\right] \leq -n_0 p 2^{-d}|B(0,r)|,$$

which now yields our claim.

In view of our choice of N in (2.10) and of Lemma 2.2, Theorem 2.1 will follow if we show that,

$$\limsup_{r\to 0}\ \limsup_{n_0\to\infty}\ \limsup_{b\to\infty,\,\delta\to 0}\ \limsup_{s\to\infty} s^{-1} \times \log\Big(\mathbb{P}\otimes P_0\big[\tilde{T} > s,\ |A\cap\tilde{\mathscr{T}}| \le n_0\big]\Big) \le -c(d,p). \tag{2.15}$$

If we apply Theorem 1.1 to the bound

$$E_0\big[\exp\{(\lambda_b \wedge M - \rho)_+ \tilde{T}\}\big] \ge \exp\{(\lambda_b \wedge M - \rho)_+ s\} P_0[\tilde{T} > s],$$

where $\lambda_b = \lambda(\Theta_b)$ and $\Theta_b = \mathscr{T}\cap A^1 \setminus \bigcup_{x_i\in G}\overline{B}(x_i, b\varepsilon)$, we see that for any $M > 0$ and $\rho > 0$, the left member of (2.15) is smaller than

$$\limsup_{r\to 0}\ \limsup_{n_0\to\infty}\ \limsup_{b\to\infty,\,\delta\to 0}\ \limsup_{s\to\infty} s^{-1}\log\Big(\mathbb{E}\big[\exp\{-(\lambda_b\wedge M-\rho)_+ s\},\ |A\cap\tilde{\mathscr{T}}| \le n_0\big]\Big). \tag{2.16}$$

Observe now that $\mathscr{T}\cap A^1 \subseteq D =_{\text{def}} \mathscr{T}\cap(\bigcup_{\overline{C}_m\cap A\neq\varnothing} C_m)^0$; in other words, D is obtained by deleting in $\mathscr{T}$ all closed boxes $\overline{C}_m$ which are not neighbors of A.

We can also introduce U (respectively $\tilde{U}$), the complement in D of the union over $m\in\mathbb{Z}^d$ of closed subboxes intersecting $\mathring{C}_m$ and containing a point of C_m (respectively, a good point of C_m). We now have: $\Theta_b \subset \tilde{U}$, and thanks to (1.4) and $|A\cap\tilde{\mathscr{T}}| \le n_0$,

$$U \subset \tilde{U} \subset D, \qquad |\tilde{U}| \le |U| + \delta|D| \le |U| + \delta n_0 3^d. \tag{2.17}$$

Observe that the number of possibilities for D grows at most polynomially in s for fixed n_0, and that for fixed D the number of possibilities for U and $\tilde{U}$ is smaller than $2^{2n_0 3^d(\sqrt{d}\,s^{1/d}/b+1)^d}$. It follows that the expression in (2.16) is smaller than

$$\limsup_{r\to 0}\ \limsup_{n_0\to\infty}\ \limsup_{b\to\infty,\,\delta\to 0}\left(\limsup_{s\to\infty}\ \sup_{D,U,\tilde{U}}\Big\{-\big(\lambda(\tilde{U})\wedge M-\rho\big)_+ + s^{-1}\log(\mathbb{P}_s[\omega(U)=0])\Big\} + 2n_0 3^d \log 2(\sqrt{d}/b)^d\right), \tag{2.18}$$

where $U, \tilde{U}, D$ satisfy the constraints (2.17) in the supremum which appears in (2.18), and the dependence on r has in fact disappeared. On the other hand,

$$\begin{aligned}\mathbb{P}_s[\omega(U)=0] &\le \exp\Big\{-s\big(|U| - 3^d n_0 d(b/\sqrt{d})s^{-1/d}\big)/(b/\sqrt{d})^d \\ &\qquad \times\log Z(b/\sqrt{d} - 2l)\Big\} \\ &\le \exp\Big\{-s\big(|\tilde{U}| - 3^d n_0\big(\delta + d(b/\sqrt{d})s^{-1/d}\big)\big) \\ &\qquad \times\log Z(b/\sqrt{d} - 2l)/(b/\sqrt{d})^d\Big\},\end{aligned}$$

thanks to (2.17).

As a result, (2.18) is smaller than

$$\limsup_{n_0\to\infty}\ \limsup_{b\to\infty,\,\delta\to 0}\Big(-\inf_{\tilde U}\Big\{\big(\lambda(\tilde U)\wedge M-\rho\big)_+ + |\tilde U|\log Z\big(b/\sqrt{d}-2l\big)/\big(b/\sqrt{d}\big)^d\Big\} + 3^d n_0\delta\log\big(Z(b/\sqrt{d}-2l)\big)/\big(b/\sqrt{d}\big)^d\Big) \tag{2.19}$$

and now $\tilde U$ runs over all bounded open subsets of $\mathbb{R}^d$ with negligible boundary and volume smaller than $3^d n_0$. For such a $\tilde U$ we have

$$\begin{aligned}\big(\lambda(\tilde U)\wedge M-\rho\big)_+ &+ |\tilde U|\log\big[Z(b/\sqrt{d}-2l)\big]/\big(b/\sqrt{d}\big)^d\\ &\ge \lambda(\tilde U)\wedge M+p|\tilde U|-\rho-3^d n_0\Big|p-\log\big[Z(b/\sqrt{d}-2l)\big]/\big(b/\sqrt{d}\big)^d\Big|\\ &\ge \big(\lambda(\tilde U)+p|\tilde U|\big)\wedge M-\rho-3^d n_0\Big|p-\log\big[Z(b/\sqrt{d}-2l)\big]/\big(b/\sqrt{d}\big)^d\Big|.\end{aligned}$$

It follows, thanks to (2.6), that (2.19) is smaller than

$$-\inf_{\tilde U}\big(\lambda|\tilde U|+p|\overline{U}|\big)\wedge M+\rho,$$

$\tilde U$ running over bounded open subsets of $\mathbb{R}^d$ with negligible boundary. Letting ρ go to zero and M go to infinity, we see that (2.15) is smaller than

$$-\inf_{\tilde U}\big(\lambda(\tilde U)+p|\tilde U|\big).$$

Now using the isoperimetric inequality, Donsker and Varadhan [2] showed that the infimum is attained when $\tilde U$ is a ball of radius $R_0 = (2\lambda_d/dp\omega_d)^{1/(d+2)}$, and that for such a $\tilde U$, $\lambda(\tilde U)+p|\tilde U| = c(d,p)$. This completes the proof of the theorem. □

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REFERENCES

[1] Cioranescu, D. and Murat, F. (1982). Un terme étrange venu d'ailleurs. In *Nonlinear P.D.E. and Their Applications. Séminaire du Collège de France* (H. Brézis and J. L. Lions, eds.) **2** 98–138; **3** 154–178. Pitman, New York.

[2] Donsker, M. D. and Varadhan, S. R. S. (1975). Asymptotics for the Wiener sausage. *Comm. Pure Appl. Math.* **28** 525–565.

[3] Gallavotti, G. and Miracle-Sole, S. (1968). A variational principle for the equilibrium of hard sphere systems. *Ann. Inst. H. Poincaré Ser. A* **3** 287–299.

[4] Israel, R. B. (1979). *Convexity in the Theory of Lattice Gases*. Princeton Univ. Press.

[5] Kayser, R. F. and Hubbard, J. B. (1978). Reaction diffusion in a medium containing a random distribution of nonoverlapping traps. *J. Chem. Phys.* **80** 3.

[6] Mürmann, M. G. (1978). Poisson point processes with exclusion. *Z. Wahrsch. Verw. Gebiete* **43** 23–37.

[7] OZAWA, S. (1983). Point interaction potential approximation for $(-\Delta + U)^{-1}$ and eigenvalues of the Laplacian in wildly perturbed domains. *Osaka J. Math.* **20** 923–937.
[8] PRESTON, C. (1976). *Random Fields. Lecture Notes in Math.* **534**. Springer, Berlin.
[9] RUELLE, D. (1969). *Statistical Mechanics*. Benjamin, New York.
[10] RUELLE, D. (1970). Superstable interactions in classical statistical mechanics. *Comm. Math. Phys.* **18** 127–159.
[11] SZNITMAN, A. S. (1990). Lifschitz tail and Wiener sausage I. *J. Funct. Anal.* **94** 223–246.
[12] SZNITMAN, A. S. (1990). Long time asymptotics for the shrinking Wiener sausage. *Comm. Pure Appl. Math.* **43** 809–820.
[13] SZNITMAN, A. S. (1991). On long excursions of Brownian motion among Poissonian obstacles. In *Proc. Durham Conf., 1990. London Math. Soc. Lecture Notes Series*. (M. Barlow and N. Bingham, Eds.) 353–375. To appear.

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