The Annals of Probability
1993, Vol. 21, No. 1, 433–452

TRANSIENCE / RECURRENCE AND CENTRAL LIMIT THEOREM BEHAVIOR FOR DIFFUSIONS IN RANDOM TEMPORAL ENVIRONMENTS

By Mark Pinsky and Ross G. Pinsky

Northwestern University and Technion–Israel Institute of Technology

Let $\sigma(t)$ be an ergodic Markov chain on a finite state space E and for each $\sigma \in E$, define on $\mathbb{R}^d$ the second-order elliptic operator

$$L_\sigma = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(x;\sigma)\frac{\partial^2}{\partial x_i\,\partial x_j} + \sum_{i=1}^{d} b_i(x;\sigma)\frac{\partial}{\partial x_i}.$$

Then for each realization $\sigma(t) = \sigma(t,\omega)$ of the Markov chain, $L_{\sigma(t)}$ may be thought of as a time-inhomogeneous diffusion generator. We call such a process a diffusion in a random temporal environment or simply a random diffusion. We study the transience and recurrence properties and the central limit theorem properties for a class of random diffusions. We also give applications to the stabilization and homogenization of the Cauchy problem for random parabolic operators.

1. Introduction and statement of results. Let $\sigma(t)$ be an ergodic Markov chain on a finite state space E with generator G and for each $\sigma \in E$, define on $\mathbb{R}^d$ the second-order elliptic operator

$$L_\sigma = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(x;\sigma)\frac{\partial^2}{\partial x_i\,\partial x_j} + \sum_{i=1}^{d} b_i(x;\sigma)\frac{\partial}{\partial x_i}. \tag{1.1}$$

Then for each realization $\sigma(t) = \sigma(t,\omega)$ of the Markov chain, $L_{\sigma(t)}$ may be though of as a time-inhomogeneous diffusion generator [with coefficients $a_{ij}(x;\sigma(t,\omega))$ and $b_i(x;\sigma(t,\omega))$]. Without further mention, we will always assume that $a(x;\sigma) = \{a_{ij}(x;\sigma)\}_{i,j=1}^d$ is continuous in x and positive definite and that $b(x;\sigma) = \{b_i(x;\sigma)\}_{i=1}^d$ is bounded on compacts and measurable. It then follows from [9] that $L_{\sigma(t)}$ generates a unique time-inhomogeneous diffusion process $X(t) = X(t;\sigma(\cdot))$. We will call such a process a diffusion in a random temporal environment or simply a random diffusion. Note that in fact $X(t)$ may be thought of as the first component of the Markov process $(X(t),\sigma(t)) \in \mathbb{R}^d \times E$ generated by $L_\sigma + G$. This is a special case of a class of processes that has been studied recently by Freidlin and Eizenberg (see, e.g., [1] and [2]).

In this paper we study the transience and recurrence properties and the central limit theorem properties for a certain class of random diffusions. Denote by μ the invariant probability measure for the Markov chain $\sigma(t)$ and,

Received January 1990; revised July 1991.

AMS 1991 *subject classifications*. Primary 60J60; secondary 60H25.

Key words and phrases. Diffusion processes, random environment, transience and recurrence, central limit theorem, random parabolic operators.

for $f \in \mathscr{B}(E)$, let $\langle f \rangle = \sum_{\sigma \in E} f(\sigma)\mu_\sigma$. Also let $r = |x|$ and $\phi = x/|x| \in S^{d-1}$. We will consider random diffusions satisfying the following hypothesis.

HYPOTHESIS A. *$L_\sigma = \frac{1}{2}\Delta + V$ for large $|x|$ on $\mathbb{R}^d$, $d \ge 2$, where $V = |x|^\delta \hat{b}(x/|x|;\sigma) \cdot \nabla$, $\delta \in [-1,1)$, $\hat{b}(\phi;\sigma) \not\equiv 0$, $\hat{b}(\phi;\sigma) \in C^1(S^{d-1})$ for each $\sigma \in E$ and $\langle \hat{b}(\phi;\cdot)\rangle = 0$ for each $\phi \in S^{d-1}$.*

Hypothesis A states that for $|x|$ sufficiently large, the drift field is homogeneous of degree $\delta \in [-1,1)$ and mean zero. Actually the proof of our main result will reveal that it still holds when the conditions on the coefficients are relaxed somewhat. These weaker conditions appear as Hypothesis A′ at the end of this section. At the end of this section we also discuss what occurs if $\delta \in \mathbb{R} - [-1,1)$. It turns out that the cases $\delta < -1$ and $\delta > 1$ are trivial and uninteresting; on the other hand, the case $\delta = 1$ remains an open problem.

Now if the drift were identically 0, then $X(t)$ would of course be recurrent for $d = 2$ and transient for $d \ge 3$. With the introduction of the asymptotically homogeneous mean zero drift field, transience or recurrence will depend not only on d but also on δ and on $\hat{b}(\phi;\sigma)$.

Before describing our results, we recall several facts concerning the generator G. We have $\mu G = G1 = 0$ and, by the ergodicity assumption, 0 is a simple eigenvalue for G. By the Fredholm alternative, the equation $Gu = v$ is solvable if and only if $\langle v \rangle = 0$ in which case $u = G^{-1}v$ is unique up to the addition of a multiple of the vector 1. When we write $vG^{-1}v$, we mean the product of the two functions $v(\sigma)$ and $G^{-1}v(\sigma)$ and not the quadratic form in v. That is, $vG^{-1}v = (vG^{-1}v)(\sigma) = v(\sigma)(G^{-1}v)(\sigma)$. Since $\langle v \rangle = 0$, it follows that $\langle vG^{-1}v\rangle$ is well defined, that is, is independent of the arbitrary multiple of 1 appearing in $G^{-1}v$. Finally, we have

(1.2) $\langle vG^{-1}v\rangle \le 0$ for all v satisfying $\langle v \rangle = 0$ and equality holds if and only if $v \equiv 0$ [4].

We first consider the problem of transience and recurrence. It turns out that the transience and recurrence properties of random diffusions satisfying Hypothesis A may be described entirely in terms of the behavior of the generator

$$\hat{L} = \begin{cases} \frac{1}{2}\Delta, & \text{if } -1 \le \delta < 0, \\ \frac{1}{2}\Delta - \langle VG^{-1}V\rangle, & \text{if } \delta = 0, \\ -\langle VG^{-1}V\rangle, & \text{if } 0 < \delta < 1. \end{cases} \tag{1.3}$$

Substituting V as given in Hypothesis A into (1.3) and using (1.2) and the fact that differentiation with respect to x_i of a function depending only on $\phi \in S^{d-1}$ yields the product of $1/r$ and another function depending only on ϕ, one finds that $\hat{L}$ may be written in the general form

$$\hat{L} = r^\gamma\left[c_1(\phi)\frac{\partial^2}{\partial r^2} + \frac{c_2(\phi)}{r}\frac{\partial}{\partial r} + \frac{1}{r}\frac{\partial}{\partial r}D_{S^{d-1}} + \frac{1}{r^2}L_{S^{d-1}}\right], \tag{1.4}$$

where

$$\gamma = \begin{cases} 0, & -1 \le \delta \le 0, \\ 2\delta, & \text{if } 0 < \delta < 1, \end{cases}$$

$c_1(\phi) \ge 0$, $D_{S^{d-1}}$ is a first-order operator on S^{d-1} and $L_{S^{d-1}}$ is a (possibly degenerate) diffusion generator on S^{d-1}.

From here on, when considering the problem of transience and recurrence, we shall assume the following hypothesis.

HYPOTHESIS B. *Either $\hat{L}$ is nondegenerate or, in representation* (1.4), $c_1(\phi) > 0$ *for all* $\phi \in S^{d-1}$ *and* $(c_2/c_1)(\phi)$ *is constant.*

REMARK. From (1.2) and (1.3), it follows that $\hat{L}$ is always nondegenerate if $-1 \le \delta \le 0$. If $0 < \delta < 1$, then it is easy to see from (1.3) that the nondegeneracy of $\hat{L}$ is equivalent to the nondegeneracy of the matrix

$$\left\{\langle \hat{b}_i(\phi;\cdot)G^{-1}\hat{b}_j(\phi;\cdot)\rangle\right\}_{i,j=1}^{d}$$

for all $\phi \in S^{d-1}$.

If $\hat{L}$ is nondegenerate, then so too is $L_{S^{d-1}}$ in which case the diffusion it generates possesses a unique invariant probability density on S^{d-1}. Let $\nu(\phi)$ denote this invariant probability density with respect to normalized Lebesgue measure (which we denote by $d\phi$). Define

$$\rho = \begin{cases} \int_{S^{d-1}} (c_2(\phi) - c_1(\phi))\nu(\phi)\, d\phi, & \text{if } L_{S^{d-1}} \text{ is nondegenerate}, \\ \dfrac{c_2}{c_1} - 1, & \text{if } L_{S^{d-1}} \text{ is degenerate and } \dfrac{c_2}{c_1} \text{ is constant}. \end{cases}$$

The probabilistic import of ρ is as follows:

PROPOSITION 1. *Let $\hat{X}(t)$ denote the diffusion generated by $\hat{L}$.*

(i) *If $\rho > 0$, then starting from outside the unit ball, $\hat{X}(t)$ hits the unit ball with probability less than* 1.

(ii) *If $\rho = 0$, then starting from outside the unit ball, $\hat{X}(t)$ hits the unit ball with probability* 1 *and reaches the origin with probability* 0.

(iii) *If $\rho > 0$, then starting from outside the unit ball, $\hat{X}(t)$ reaches the origin with probability* 1.

REMARK. In case (i), $\hat{X}(t)$ is transient to ∞ and in case (ii), $\hat{X}(t)$ is null recurrent. In case (iii), the question of whether $\hat{X}(t)$ must remain at 0 for all $t \ge \tau_0$ or whether it can pass through 0 is a delicate one. This question was investigated by Williams [10] in the special case $\hat{L} = \frac{1}{2}\Delta + \mu(\theta)/2r\,\partial/\partial r$. She identified a value $\rho_0 < 0$ such that if $\rho \le \rho_0$, then $\hat{X}(t)$ must remain at 0 for all $t \geq \tau_0$, while if $\rho_0 < \rho < 0$, then one can define $\hat{X}(t)$ so that it reaches 0 with probability 1 but spends Lebesgue measure zero time there. Presumably a similar phenomenon occurs for $\hat{L}$ as in (1.4). Thus, in case (iii), $\hat{X}(t)$ is null recurrent or transient according to whether it can pass through 0 or must remain there. (However, if we were to amend the radial part of the drift of $\hat{L}$ in a neighborhood of the origin so that the resulting operator, call it $\hat{L}'$, had a bounded radial drift, then $\hat{L}'$ would generate a transient process if $\rho > 0$ and a recurrent one if $\rho \leq 0$.)

We can now state the following theorem.

THEOREM 1. *Let $X(t) = X(t; \sigma(\cdot))$ be a diffusion in a random temporal environment generated by L_σ and assume that Hypotheses* A *and* B *are satisfied. Then for almost every realization $\sigma(\cdot)$, $X(t)$ is recurrent or transient according to whether $\rho \leq 0$ or $\rho > 0$.*

There are certain interesting particular cases in which Theorem 1 may be used to give a very simple criterion for transience or recurrence.

COROLLARY 1 (Negative homogeneity). *If $-1 \leq \delta < 0$, then the random diffusion is recurrent if $d = 2$ and transient if $d \geq 3$.*

REMARK. Thus, if $-1 \leq \delta < 0$, the homogeneous, mean zero drift is sufficiently weak so as not to affect the transience or recurrence of the random diffusion. In contrast, in the nonrandom case, one needs $\delta < -1$ to guarantee that the drift does not influence transience or recurrence.

COROLLARY 2 (Radial case). *Let the vector field V be radial, that is $V = r^\delta \bar{b}(\phi; \sigma)\partial/\partial r$, where $\bar{b}(\phi; \sigma)$ is a scalar. If $0 < \delta < 1$, assume that $\langle \bar{b}(\phi, \cdot)G^{-1}\bar{b}(\phi, \cdot)\rangle \neq 0$ for each $\phi \in S^{d-1}$.*

(i) *If $\delta = 0$, then the random diffusion is recurrent if $d \leq 2 + 2c$ and transient if $d > 2 + 2c$, where $c = -\int_{S^{d-1}} \langle \bar{b}(\phi, \cdot)G^{-1}\bar{b}(\phi, \cdot)\rangle \, d\phi > 0$.*

(ii) *If $0 < \delta < 1$, then the random diffusion is recurrent in all dimensions $d \geq 2$.*

REMARK 1. Corollary 2 shows that in the radial case, the randomness introduced through the homogeneous, mean zero drift yields recurrence in higher dimensions than one would normally expect.

REMARK 2. In the radial case, one can check that Theorem 1 holds even if $\delta = 1$. Thus the random diffusion is recurrent for all $d \geq 2$ if $\delta = 1$.

COROLLARY 3 [Vector field divergence free up to the homogeneity term and Markov chain $\sigma(t)$ reversible]. *Assume that the Markov chain $\sigma(t)$ is reversible and let the vector field $V = r^\delta \hat{b}(\phi; \sigma) \cdot \nabla$ satisfy $\nabla \cdot \hat{b}(\phi, \sigma) \equiv 0$. Also,* *in Hypothesis* B, *assume that* $\hat{L}$ *is nondegenerate* (*which is always true if* $\delta = 0$).

(i) *If* $\delta = 0$, *then the random diffusion is recurrent if* $d = 2$ *and transient if* $d \geq 3$.

(ii) *If* $0 < \delta < 1$, *then the random diffusion is transient in all dimensions* $d \geq 2$.

REMARK 1. It is interesting to note that in the case of positive homogeneity, the vector fields of Corollary 3 (along with the reversibility assumption) yield transience in all dimensions, whereas the radial vector fields of Corollary 2 yield recurrence in all dimensions.

REMARK 2. Corollary 3 covers, in particular, the case of the "gradient vector field" in which $V = r^{\delta}\hat{b}(\sigma) \cdot \nabla$.

REMARK 3. Consider the case $-1 < \delta \leq 0$, $d = 2$ and $V = r^{\delta}\hat{b}(\sigma) \cdot \nabla$ so that there is no ϕ-dependence in $\hat{b}$. If $\delta = 0$, also assume that $\sigma(t)$ is reversible. Then it is trivial to check that for each fixed σ, L_{σ} generates a transient diffusion. However, this transience is cancelled out and by Corollary 1 (in the case $-1 < \delta < 0$) or Corollary 3 (in the case $\delta = 0$), it follows that the random diffusion is almost surely recurrent. It is natural then to ask whether the random diffusion must necessarily be recurrent if L_{σ} is a recurrent generator for all $\sigma \in E$. It turns out that even on the half-line, where the topology plays no role, one can construct a random diffusion which is almost surely transient but for which each L_{σ} is positive recurrent. For this result and other related ones, see [7].

Now, in the general case, one would ideally like the transience or recurrence criterion of Theorem 1 to be given explicitly in terms of the following objects: the statistics $\langle \hat{b}_i(\phi; \cdot) G^{-1} \hat{b}_j(\phi; \cdot) \rangle$, the dimension d and the homogeneity parameter δ. It is easy to see that $c_1(\phi)$ and $c_2(\phi)$ are given in terms of the above objects; however $\nu(\phi)$, which solves the adjoint equation $\tilde{L}_{S^{d-1}}\nu = 0$, cannot be written down explicitly in general, except in the case $d = 2$. When $d = 2$, we have the following proposition.

PROPOSITION 2. *Let* $L_{S^1} = c_3(\phi)\partial^2/\partial\phi^2 + c_4(\phi)\partial/\partial\phi$ *be a nondegenerate diffusion generator on* S^1. *Then the corresponding invariant probability density is given by*

$$\nu(\phi) = \frac{k}{c_3(\phi)} \exp\left(\int_0^{\phi} \frac{c_4}{c_3}(s)\, ds\right) \left[\int_0^{\phi} \exp\left(-\int_0^t \frac{c_4}{c_3}(s)\, ds\right) dt \right.$$
$$\left. + \exp\left(\int_0^{2\pi} \frac{c_4}{c_3}(s)\, ds\right) \int_{\phi}^{2\pi} \exp\left(-\int_0^t \frac{c_4}{c_3}(s)\, ds\right) dt \right],$$

where k *is the appropriate normalization constant.*

PROOF. We leave the calculation to the reader. □

Thus, in the two-dimensional case, Theorem 1 gives a criterion (albeit a rather complicated one) for transience or recurrence explicitly in terms of the above-mentioned objects.

We now turn to the central limit theorem. The operator $\hat{L}$ which determined transience or recurrence of the random diffusion will also determine the limiting process in the central limit theorem. Let $X_x(t;\sigma(\cdot))$ denote the random diffusion starting from $x \in \mathbb{R}^d - \{0\}$ and define

$$(1.5)\quad (X_x^\varepsilon(t),\sigma^\varepsilon(t)) = \begin{cases} \varepsilon^{1/2} X_{\varepsilon^{-1/2}x}\left(\dfrac{t}{\varepsilon};\sigma\left(\dfrac{t}{\varepsilon}\right)\right), & \text{if } -1 \le \delta \le 0, \\ \varepsilon^{1/2(1-\delta)} X_{\varepsilon^{-1/2(1-\delta)}x}\left(\dfrac{t}{\varepsilon};\sigma\left(\dfrac{t}{\varepsilon}\right)\right), & \text{if } 0 < \delta < 1. \end{cases}$$

The scaled random diffusion can now be written as $X_x^\varepsilon(t;\sigma^\varepsilon(\cdot))$.

Let $\Omega = C([0,\infty),\mathbb{R}^d)$ denote the space of continuous functions, $\omega(\cdot)$, from $[0,\infty)$ to $\mathbb{R}^d$ with the usual σ-field $\mathscr{F}$ and filtration $\mathscr{F}_t$, $t \ge 0$. Let $\tau_0 = \inf\{t \ge 0: \omega(t) = 0\}$ and let $\mathscr{F}_{\tau_0^-}$ be the σ-algebra up to time τ_0^-. Denote by P_x^ε the probability measure on $(\Omega,\mathscr{F})$ induced by the random diffusion $X_x^\varepsilon(t;\sigma^\varepsilon(\cdot))$ and let $P_x^\varepsilon|_{\mathscr{F}_{\tau_0^-}}$ denote its restriction to $(\Omega,\mathscr{F}_{\tau_0^-})$. Since the drift of the original generator $L_{\sigma(t)}$ is bounded on compacts, it follows that $P_x^\varepsilon(\tau_0 = \infty) = 1$. Thus, the P_x^ε-completions of $\mathscr{F}_{\tau_0^-}$ and $\mathscr{F}$ coincide and the restriction above is in name only. Let $\hat{X}_x(t)$ denote the process generated by $\hat{L}$ for $x \in \mathbb{R}^d - \{0\}$. Recall that by Proposition 1, $\hat{X}_x(t)$ reaches 0 with probability 0 or 1 according to whether $\rho \ge 0$ or $\rho < 0$. If $\rho \ge 0$, let $\hat{P}_x$ denote the measure on $(\Omega,\mathscr{F})$ induced by the process $\hat{X}_x(t)$. If $\rho < 0$, then as noted in the remark following Proposition 1, the process may or may not become stuck once it reaches 0. For our purposes, however, the behavior after time τ_0 will be irrelevant. For $\rho < 0$, let $\hat{P}_x|_{\mathscr{F}_{\tau_0^-}}$ denote the measure induced by $\hat{X}_x(t)$ on $(\Omega,\mathscr{F}_{\tau_0^-})$.

For the central limit theorem, we replace Hypothesis B by the following hypothesis.

HYPOTHESIS C. *For each starting point $x \in \mathbb{R}^d$, there is a unique solution to the martingale problem for $\hat{L}$ up to time τ_0.*

THEOREM 2. *Assume that Hypotheses* A *and* C *hold. Let $x \in \mathbb{R}^d - \{0\}$, let $X_x^\varepsilon(t)$ denote the scaled random diffusion obtained from* (1.5) *and let $\hat{X}_x(t)$ denote the diffusion generated by $\hat{L}$ as in* (1.3).

(i) *If $\rho \ge 0$, then* $\text{w-lim}_{\varepsilon\to 0} P_x^\varepsilon = \hat{P}_x$. *That is,*

$$\{X_x^\varepsilon(t), 0 \le t < \infty\} \xrightarrow[\varepsilon\to 0]{\text{w}} \{\hat{X}_x(t), 0 \le t < \infty\};$$

(ii) *If $\rho < 0$, then* $\text{w-lim}_{\varepsilon\to 0} P_x^\varepsilon|_{\mathscr{F}_{\tau_0^-}} = P_x^\varepsilon|_{\mathscr{F}_{\tau_0^-}}$ *That is,*

$$\{X_x^\varepsilon(t), 0 \le t < \tau_0\} \xrightarrow[\varepsilon\to 0]{\text{w}} \{\hat{X}_x(t), 0 \le t < \tau_0\}.$$

REMARK 1. Note that if $-1 \le \delta < 0$, the scaling is classical and the limit is Gaussian; if $\delta = 0$, the scaling is classical but the limit is non-Gaussian; if $0 < \delta < 1$, the scaling is nonclassical and the limit is non-Gaussian. In the case $0 < \delta < 1$, the diffusion in a random temporal environment grows more quickly than in the classical central limit theorem. In contrast to this, in the case of diffusion in a random spatial environment, the diffusion grows more slowly than in the classical case—in the random spatial environment case, the scaling is $X(t/\varepsilon)/(\log \varepsilon)^2$ [8].

REMARK 2. The reason that we can obtain a central limit theorem for the process only up until time τ_0^- is that Hypothesis A holds only for large $|x|$. If we require that Hypothesis A hold for all $x \in \mathbb{R}^d$, then if $-1 < \delta < 1$ and possibly if $\delta = -1$, the random diffusion will have a positive probability of becoming stuck at 0.

We now give an application of transience and recurrence for random diffusions to the stabilization of the Cauchy problem for the random parabolic equation $u_t = L_{\sigma(t)}u$ in an exterior domain D. First, recall that if L is a strictly elliptic homogeneous diffusion generator, then the solution to the Cauchy problem

$$\begin{aligned} w_t &= Lw \quad \text{in } D \times (0,\infty), \\ w(x,0) &= f(x) \quad \text{in } D, \\ w(y,t) &= g(y), \qquad y \in \partial D,\, t > 0, \end{aligned}$$

where f is bounded, can be represented probabilistically as

$$w(x,t) = E_x\big(f(X(t));\, \tau_D > t\big) + E_x\big(g(X(\tau_D));\, \tau_D \le t\big),$$

where $X(t)$ is the process generated by L. Now if $X(t)$ is recurrent, then $P_x(\tau_D < \infty) = 1$ and $\overline{w}(x) = E_x g(X(\tau_D)) = \lim_{t\to\infty} w(x,t)$ exists and is the unique bounded solution of the exterior Dirichlet problem $L\overline{w} = 0$ in D and $\overline{w} = g$ on ∂D. On the other hand, if $X(t)$ is transient, then $P(\tau_D = \infty) > 0$ and $\lim_{t\to\infty} w(x,t)$ will not exist in general and in any case will depend on the initial data. (For the limit to exist, the behavior of f at ∞ must be compatible with the Martin boundary at ∞ for L.)

Now consider the Cauchy problem for the random parabolic operator $\partial/\partial t - L_{\sigma(t)}$ in an exterior domain D:

$$\begin{aligned} u_t &= L_{\sigma(t)}u \quad \text{in } D \times (0,\infty), \\ u(x,0) &= f(x) \quad \text{in } D, \\ u(y,t) &= g(y), \qquad y \in \partial D,\, t > 0, \end{aligned} \tag{1.6}$$

where f is bounded. Because $L_{\sigma(t)}$ is a time-inhomogeneous operator, $\lim_{t\to\infty} u(x,t)$ will never exist, so the type of stabilization described above will never occur. However, an "ergodic stabilization" will occur if the random diffusion is almost surely recurrent.

THEOREM 3. *Let L_σ be as in* (1.1), *assume that $\sigma(t)$ is reversible and assume that the random diffusion generated by $L_{\sigma(t)}$ is almost surely recurrent. Then for each $x \in D$, the solution $u(x,t) = u(x,t;\sigma(\cdot))$ satisfies*

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t u(x,s)\,ds = w(x),$$

for almost every environment $\sigma(\cdot)$, where $w(x) = \sum_{\sigma\in E}\mu_\sigma(x,\sigma)$, μ is the invariant probability measure for $\sigma(t)$ and v is the unique bounded solution of the exterior Dirichlet problem

$$(1.7)\qquad \begin{aligned} (L_\sigma + G)v &= 0 \quad \text{in } D\times E,\\ v(y,\sigma) &= g(y) \quad \text{on } \partial D \times E. \end{aligned}$$

Using Theorem 2, we can prove a homogenization result for the solution to the Cauchy problem for the random parabolic equation $u_t = L_{\sigma(t)}u$ in all of $\mathbb{R}^d$. Let f be a bounded continuous function on $\mathbb{R}^d$. Let $\sigma^\varepsilon(t) = \sigma(t/\varepsilon)$ denote the rapidly fluctuating environment and let $u^\varepsilon(x,t) = u^\varepsilon(x,t;\sigma^\varepsilon(\cdot))$ denote the solution to

$$(1.8)\qquad \begin{aligned} u_t^\varepsilon &= L_{\sigma^\varepsilon(t)}u^\varepsilon \quad \text{in } \mathbb{R}^d\times(0,\infty),\\ u^\varepsilon(x,0) &= \begin{cases} f(\varepsilon^{1/2}x), & \text{if } -1\le\delta\le 0,\\ f(\varepsilon^{1/2(1-\delta)}x), & \text{if } 0<\delta<1. \end{cases} \end{aligned}$$

THEOREM 4. *Let L_σ be as in Theorem* 2, *let $\hat{L}$ be as in* (1.3) *and assume that $\sigma(t)$ is reversible and that $\rho \ge 0$. Let $\hat{w}(x,t)$ denote the solution to*

$$\begin{aligned} \hat{w}_t &= \hat{L}\hat{w} \quad \text{in } \mathbb{R}^d\times[0,\infty),\\ \hat{w}(x,0) &= f(x). \end{aligned}$$

The solution $u^\varepsilon(x,t) = u^\varepsilon(x,t;\sigma^\varepsilon(\cdot))$ of (1.8) *has the following behavior*:

(i) *If* $-1\le\delta\le 0$, *then for each* $(x,t)\in(\mathbb{R}^d - \{0\})\times[0,\infty)$, $\lim_{\varepsilon\to 0} u^\varepsilon(\varepsilon^{-1/2}x, \varepsilon^{-1}t) = \hat{w}(x,t)$ *for a.e. environment $\sigma^\varepsilon(\cdot)$*;

(ii) *If* $0<\delta<1$, *then for each* $(x,t)\in\mathbb{R}^d\times[0,\infty)$, $\lim_{\varepsilon\to 0} u^\varepsilon(\varepsilon^{-1/2(1-\delta)}x, \varepsilon^{-1}t) = \hat{w}(x,t)$ *for a.e. environment $\sigma^\varepsilon(\cdot)$.*

We now turn to the proofs of the corollaries to Theorem 1.

PROOF OF COROLLARY 1.

$$\hat{L} = \frac{1}{2}\Delta = \frac{1}{2}\frac{\partial^2}{\partial r^2} + \frac{d-1}{2r}\frac{\partial}{\partial r} + \frac{1}{2}\Delta_{S^{d-1}};$$

thus $\rho = d-2$. □

PROOF OF COROLLARY 2. We have

$$-\langle VG^{-1}V\rangle = -\langle \bar{b}(\phi;\cdot)G^{-1}\bar{b}(\phi;\cdot)\rangle r^{2\delta}\left[\frac{\partial^2}{\partial r^2} + \frac{\delta}{r}\frac{\partial}{\partial r}\right].$$

If $0 < \delta < 1$, then $\hat{L} = -\langle VG^{-1}V\rangle$ and thus $(c_2/c_1)(\phi)$ is constant and, by assumption, $c_1(\phi) > 0$ for all $\phi \in S^{d-1}$. Thus $\rho = c_2/c_1 - 1 = \delta - 1$. If $\delta = 0$, then $\hat{L} = \frac{1}{2}\Delta - \langle VG^{-1}V\rangle$. Thus $c_1 = \frac{1}{2} - \langle \bar{b}(\phi;\cdot)G^{-1}\bar{b}(\phi;\cdot)\rangle$, $c_2 = (d-1)/2$ and $\nu(\phi) = 1$. The positivity of c in the statement of the corollary follows from (1.2) and the fact that, by Hypothesis A, $\hat{b}(\phi;\sigma) \not\equiv 0$. □

PROOF OF COROLLARY 3. To prove Corollary 3, we need to express $\hat{L}$ explicitly in terms of Euclidean coordinates. Making a straightforward calculation with the appropriate algebraic manipulations to insure that the lead order term is in divergence form with respect to a symmetric matrix, one finds that if $0 \le \delta < 1$, then

$$\begin{aligned}\hat{L} = r^{2\delta}\Bigg[\nabla\cdot a(\phi)\nabla &+ \frac{1}{2}\sum_{j=1}^{d}\Big(\langle\nabla\cdot\hat{b}(\phi;\cdot)G^{-1}\hat{b}_j(\phi;\cdot)\rangle \\ &\qquad + \langle\hat{b}_j(\phi;\cdot)G^{-1}\nabla\cdot\hat{b}(\phi;\cdot)\rangle\Big)\frac{\partial}{\partial x_j} \\ &+\frac{1}{2}\sum_{j=1}^{d}\left(\sum_{i=1}^{d}\left(\left\langle\frac{\partial\hat{b}_j}{\partial x_i}(\phi;\cdot)G^{-1}\hat{b}_i(\phi;\cdot)\right\rangle\right.\right. \\ &\qquad\left.\left.-\left\langle\hat{b}_i(\phi;\cdot)G^{-1}\frac{\partial\hat{b}_j}{\partial x_i}(\phi;\cdot)\right\rangle\right)\right)\frac{\partial}{\partial x_j} + \nabla(\delta\log r)\hat{a}(\phi)\nabla\Bigg],\end{aligned}\tag{1.9}$$

where

$$a_{ij}(\phi) = \begin{cases} -\frac{1}{2}\big(\langle\hat{b}_i(\phi;\cdot)G^{-1}b_j(\phi;\cdot)\rangle + \langle\hat{b}_j(\phi;\cdot)G^{-1}\hat{b}_i(\phi;\cdot)\rangle\big) + \frac{1}{2}I, & \text{if } \delta = 0,\\ -\frac{1}{2}\big(\langle\hat{b}_i(\phi;\cdot)G^{-1}\hat{b}_j(\phi;\cdot)\rangle + \langle\hat{b}_j(\phi;\cdot)G^{-1}\hat{b}_i(\phi;\cdot)\rangle\big), & \text{if } 0 < \delta < 1,\end{cases}$$

and

$$\hat{a}_{ij}(\phi) = \begin{cases} -\langle\hat{b}_i(\phi;\cdot)G^{-1}\hat{b}_j(\phi;\cdot)\rangle + \frac{1}{2}I, & \text{if } \delta = 0,\\ -\langle\hat{b}_i(\phi;\cdot)G^{-1}\hat{b}_j(\phi;\cdot)\rangle, & \text{if } 0 < \delta < 1.\end{cases}$$

Now since $\sigma(t)$ is reversible, G is symmetric with respect to the inner product $\langle\cdot,\cdot\rangle$ and thus so is G^{-1}. Consequently, the terms on the right-hand side of (1.9) containing $\partial\hat{b}_j/\partial x_i$ cancel, and also $a = \hat{a}$. This, together with the fact that $\nabla \cdot \hat{b}(\phi;\sigma) \equiv 0$, yields

$$\hat{L} = r^{2\delta}[\nabla \cdot a(\phi)\nabla + \nabla(\delta \log r)a(\phi)\nabla] = r^{\delta}\nabla \cdot a(\phi)r^{\delta}\nabla. \tag{1.10}$$

Now, by Proposition 1, ρ is positive or nonpositive according to whether the process $\hat{X}(t)$ generated by $\hat{L}$ and starting outside the unit ball hits the unit ball with probability less than 1 or probability 1. In light of (1.10), it suffices to consider this property for the process $\hat{Y}(t)$ generated by $r^{-\delta}\nabla \cdot a(\sigma)r^{\delta}\nabla$, since this process is obtained from $\hat{X}(t)$ via a time change. By the nondegeneracy assumption on $\hat{L}$ and the continuity of $a(\phi)$ (this latter following from Hypothesis A), it follows that $mI \le a(\phi) \le MI$ for all $\phi \in S^{\delta-1}$ and constants $0 < m \le M < \infty$. By the work of Ichihara [3], it then follows that starting from outside the unit ball, $\hat{Y}(t)$ will hit the unit ball with probability 1 if and only if the process $Z(t)$ does so, where $Z(t)$ is generated by $r^{-\delta}\nabla \cdot Ir^{\delta}\nabla = \Delta + (\delta/r)\partial/\partial r$. But $Z(t)$ is a Bessel process with parameter $d - 1 + \delta$. Thus, starting from outside the unit ball, $Z(t)$ will hit the unit ball with probability 1 if and only if $d - 2 + \delta \le 0$. We conclude that $\rho \le 0$ if and only if $d - 2 + \delta \le 0$. This proves the corollary. □

We now indicate what happens for other values of the parameter δ. If $\delta < -1$, then the drift vector field V is weaker than the vector field $((d-1)/2r)\partial/\partial r$ contributed by the term $\frac{1}{2}\Delta$. It is easy to check then that V has no bearing on transience or recurrence even if it is not assumed to be mean 0; thus $X(t)$ is recurrent for $d = 2$ and transient for $d \ge 3$. Similarly, one can check that the central limit theorem goes through as in the case $-1 \le \delta < 0$ even if V is not necessarily mean 0. Now consider the case $\delta > 1$. If $\hat{b}(\phi;\sigma)$ satisfies $x/r \cdot \hat{b}(\phi;\sigma) = 0$, for all $\phi \in S^{d-1}$ and all $\sigma \in E$, then the vector field V has no radial component —it is always tangential to S^{d-1}. In this case, it is easy to see that V has no bearing on transience or recurrence; thus $X(t)$ is recurrent for $d = 2$ and transient for $d \ge 3$. On the other hand, if V has a nonzero radial component, then by continuity and the mean zero assumption, there exists an open set $U \subset S^{d-1}$, a $\sigma_0 \in E$ and an $\varepsilon > 0$ such that $x/r \cdot \hat{b}(\phi;\sigma_0) > \varepsilon$ for all $\phi \in U$. Using this and the fact that $\delta > 1$, it is easy to show that $X(t)$ explodes to ∞ with probability 1 for almost every realization $\sigma(\cdot)$; thus $X(t)$ is transient for all $d \ge 2$. By the same reasoning, if V has a nonzero radial component, then as $\varepsilon \to 0$, the process $X_x^{\varepsilon}(t, \sigma^{\varepsilon}(\cdot))$ converges to the process which is at x for $t = 0$ and at ∞ for all $t > 0$. The case $\delta = 1$ is an open problem both for transience and recurrence and for the central limit theorem. (But see Remark 2 following Corollary 2.)

The proofs of Theorems 1 and 2 will reveal that they still hold with the same operator $\hat{L}$ if Hypothesis A is replaced by the following weaker hypothesis.

HYPOTHESIS A′.

$$L_\sigma = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(x;\sigma)\frac{\partial^2}{\partial x_i\,\partial x_j} + \sum_{i=1}^{d} b_i(x;\sigma)\frac{\partial}{\partial x_i},$$

where a and b are given as follows:

(i)(a) *If* $\delta > 0$, $mI \le a(x;\sigma) \le MI$, *for constants* $0 < m \le M < \infty$;
(b) *If* $\delta \le 0$ *and* $\rho \ne 0$, $a(x;\sigma) = I + o(1)$ *as* $|x| \to \infty$;
(c) *If* $\delta \le 0$ *and* $\rho = 0$, $a\ (x;\sigma) = I + o(1/\log|x|)$ *as* $|x| \to \infty$;
(ii) *With* $\hat{b}(\phi;\sigma)$ *and* δ *as in Hypothesis* A,
(a) *If* $\rho \ne 0$, $b(x;\sigma) = |x|^{\delta}\hat{b}(x/|x|;\sigma) + o(|x|^{-1+|\delta|})$ *as* $|x| \to \infty$;
(b) *If* $\rho = 0$, $b(x;\sigma) = |x|^{\delta}\hat{b}(x/|x|;\sigma) + o(|x|^{-1+|\delta|}/\log|x|)$ *as* $|x| \to \infty$.

Theorems 1, 2 and 3 are proved in Sections 2, 3 and 4 respectively. We have omitted the proof of Theorem 4 as it is similar to that of Theorem 3. The proof of that part of Proposition 1 concerning the hitting of the unit ball will follow from Lemma 1 which is used in the proof of Theorem 1. The proof of the part concerning the reaching of the origin is proved similarly and has been omitted.

2. Proof of Theorem 1. We consider $X(t)$ as the first component of the Markov process $Z(t) = (X(t), \sigma(t))$ generated by $A = L_{\sigma} + G$. Since $\sigma(t)$ is ergodic and has a compact state space, it is easy to see that $X(t)$ is recurrent (transient) for almost every realization $\sigma(\cdot)$ if and only if $Z(t)$ is recurrent (transient). Thus it suffices to consider the transience or recurrence of the Markov process $Z(t)$. We will investigate this by the method of Lyapunov functions:

(2.1) A sufficient condition for the recurrence of $Z(t)$ is the existence of a function $f \in C^2(\mathbb{R}^d \times E)$ such that $Af(x,\sigma) \le 0$ for $|x|$ sufficiently large and $\lim_{|x|\to\infty} f(x,\sigma) = \infty$.

(2.2) A sufficient condition for the transience of $Z(t)$ is the existence of a function $f \in C^2(\mathbb{R}^d \times E)$ such that $Af(x,\sigma) \le 0$ and $f(x,\sigma) > 0$ for $|x|$ sufficiently large and $\lim_{|x|\to\infty} f(x,\sigma) = 0$.

Conditions (2.1) and (2.2) appear in [5], Section 3, where diffusion processes were treated. An almost identical proof works for the generator A. The first step toward constructing an appropriate Lyapunov function is the following lemma.

LEMMA 1.

(i) *If* $\rho > 0$, *there exist an* $\varepsilon > 0$, *a constant* $k > 0$ *and a strictly positive function* $g \in C^2(S^{d-1})$ *such that the function* $f(r,\phi) = r^{-k}g(\phi)$ *satisfies* $\hat{L}f \le -\varepsilon k r^{-k-2+\gamma}$.

(ii) *If* $\rho < 0$, *there exist an* $\varepsilon > 0$ *and a function* $g \in C^2(S^{d-1})$ *such that the function* $f(r,\phi) = \log r + g(\phi)$ *satisfies* $\hat{L}f \le -\varepsilon r^{-2+\gamma}$.

(iii) *If* $\rho = 0$, *there exist an* $\varepsilon > 0$ *and functions* $g, h \in C^2(S^{d-1})$ *such that the function*

$$f(r,\phi) = \log\log r + \frac{g(\phi)}{\log r} + \frac{h(\phi)}{(\log r)^2}$$

satisfies

$$\hat{L}f \le -\frac{\varepsilon r^{-2+\gamma}}{(\log r)^2}, \quad \textit{for large } r.$$

REMARK. This lemma in conjunction with [5], Section 3, gives a proof of that part of Proposition 1 concerning the hitting of the unit ball. Actually, to prove Proposition 1 in the case $\rho = 0$, the rather long and delicate proof of the lemma is unnecessary—rather one can find a $g \in C^2(S^{d-1})$ such that $\hat{L}(\log r + g(\phi)) \equiv 0$ and use the Lyapunov function $f(r, \phi) = \log r + g(\phi)$.

PROOF. First assume $\rho > 0$. Let $f(r, \phi) = r^{-k} g(\phi)$, with $k > 0$ and g as yet undetermined. From (1.4), we have

$$\hat{L}f = r^{-k-2+\gamma}\left[k(k+1)c_1 g - kc_2 g - kD_{S^{d-1}}g + L_{S^{d-1}}g\right].$$

We look for an appropriate g in the form $g = 1 + kg_1$. Substituting, we obtain $\hat{L}f = kr^{-k-2+\gamma}[c_1 - c_2 + L_{S^{d-1}}g_1 + O(k)]$ as $k \to 0$. Thus if we can solve

$$L_{S^{d-1}}g_1 \le c_2 - c_1 - 2\varepsilon, \tag{2.3}$$

then by picking k sufficiently small we would have g strictly positive and $\hat{L}f \le -\varepsilon k r^{-k-2+\gamma}$ as desired. If $L_{S^{d-1}}$ is degenerate, then by Hypothesis B, c_2/c_1 is constant and c_1 is strictly positive. Since in this case $\rho = c_2/c_1 - 1 > 0$, (2.3) holds for small enough ε and $g_1 = 1$. On the other hand, if $L_{S^{d-1}}$ is nondegenerate, then, by the Fredholm alternative, (2.3) is solvable if and only if $\rho - 2\varepsilon = \int_{S^{d-1}}(c_2 - c_1 - 2\varepsilon)(\phi)\,\nu(\phi)\,d\phi \ge 0$.

Now assume that $\rho < 0$. Let $f(r, \phi) = \log r + g(\phi)$, with g as yet undetermined. Then, by (1.4), we have $\hat{L}f = r^{-2+\gamma}[-c_1 + c_2 + L_{S^{d-1}}g]$. If $L_{S^{d-1}}$ is degenerate, then again by Hypothesis B, c_2/c_1 is constant and c_1 is strictly positive. Since $\rho < 0$, picking $g = \text{const.}$ gives $L_{S^{d-1}}g - c_1 + c_2 \le -\varepsilon$ for some $\varepsilon > 0$. On the other hand, if $L_{S^{d-1}}$ is nondegenerate then, by the Fredholm alternative, $L_{S^{d-1}} \le c_1 - c_2 - \varepsilon$ can be solved if and only if $-\rho - \varepsilon = \int_{S^{d-1}}(c_1 - c_2 - \varepsilon)(\phi)\nu(\phi)\,d\phi \ge 0$.

Finally, assume that $\rho = 0$. Let

$$f(r, \phi) = \log\log r + \frac{g(\phi)}{\log r} + \frac{h(\phi)}{(\log r)^2}$$

with g and h as yet undetermined. By (1.4), we have

$$\hat{L}f = r^{-2+\gamma}\Big[(L_{S^{d-1}}g - c_1 + c_2)(\log r)^{-1} + (L_{S^{d-1}}h + c_1 g - c_2 g - D_{S^{d-1}}g - c_1)(\log r)^{-2} + O\big((\log r)^{-3}\big)\Big],$$

as $r \to \infty$. Since $\rho = 0$, by the same reasoning as above, we can choose g such that $L_{S^{d-1}}g = c_1 - c_2$. Thus, to prove the proposition, it suffices to find an h and an $\varepsilon > 0$ such that

$$L_{S^{d-1}}h \le (c_2 - c_1)g + c_1 + D_{S^{d-1}}g - 2\varepsilon. \tag{2.4}$$

If $L_{S^{d-1}}$ is degenerate, then by Hypothesis B, $c_1 = c_2$. Thus $g = \text{const.}$ and $D_{S^{d-1}} = 0$, so (2.4) reduces to $L_{S^{d-1}}h \le c_1 - 2\varepsilon$. Since, by Hypothesis B, c_1 is strictly positive, this will hold with $h = \text{const.}$ and ε sufficiently small. On the other hand, if $L_{S^{d-1}}$ is nondegenerate, then (2.4) may be solved for h and some $\varepsilon > 0$ if and only if

$$\langle (c_2 - c_1)g + c_1 + D_{S^{d-1}}g \rangle > 0. \tag{2.5}$$

[We note that if the term $D_{S^{d-1}}g$ were missing, then (2.5) would follow easily. Indeed, one can show, analogous to (1.2), that $-\langle fL_{S^{d-1}}^{-1}f \rangle \ge 0$ for all f with $\langle f \rangle = 0$ [6]. Thus

$$\langle (c_2 - c_1)g \rangle = -\langle (c_2 - c_1)L_{S^{d-1}}^{-1}(c_2 - c_1) \rangle \ge 0$$

and $c_1 > 0$ by the nondegeneracy of $L_{S^{d-1}}$.]

To show (2.5), we proceed as follows. The function $m(r, \phi) = \log r + g(\phi)$, where g is as above, satisfies $\hat{L}m = 0$. Thus, from [5], Theorem 3, the process generated by $\hat{L}$, starting outside the unit ball, will hit the unit ball with probability 1. Now let

$$\hat{m}(r, \phi) = \frac{1}{\log r} - \frac{g(\phi)}{(\log r)^2} + \frac{h(\phi)}{(\log r)^3},$$

with h as yet undetermined. We have

$$\hat{L}\hat{m} = r^{-2}\Big[\big(L_{S^{d-1}}h + 2c_1 + 2D_{S^{d-1}}g + 2(c_2 - c_1)g\big)(\log r)^{-3} + O\big((\log r)^{-4}\big)\Big] \quad \text{as } r \to \infty.$$

Now if instead of (2.5), we had $\langle (c_2 - c_1)g + c_1 + D_{S^{d-1}}g \rangle < 0$, then, by the Fredholm alternative, we could solve

$$L_{S^{d-1}}h \le -2\big((c_2 - c_1)g + c_1 + D_{S^{d-1}}g\big) - \varepsilon,$$

for sufficiently small $\varepsilon > 0$. But then we would have $\hat{L}\hat{m} \le 0$ for large r and, from [5], Section 3, it would follow that the process generated by $\hat{L}$, starting from outside the unit ball, will hit the unit ball with probability strictly less than 1. This is a contradiction. We conclude then that

$$\langle (c_2 - c_1)g + c_1 + D_{S^{d-1}}g \rangle \ge 0. \tag{2.6}$$

To complete the proof of (2.5), we must show that equality cannot hold in (2.6). Note that for fixed c_1, c_2 and $L_{S^{d-1}}$, (2.6) holds for any first-order operator $D_{S^{d-1}}$ on S^{d-1} for which the corresponding operator $\hat{L}$ in (1.4) is elliptic and nondegenerate. Thus, by the nondegeneracy assumption and compactness, we have

$$\langle (c_2 - c_1)g + c_1 + (1 + t)D_{S^{d-1}}g \rangle \ge 0 \quad \text{for sufficiently small } t > 0. \tag{2.7}$$

Now, assume that equality holds in (2.6). Then $\langle D_{S^{d-1}}g \rangle = -\langle (c_2 - c_1)g \rangle - \langle c_1 \rangle < 0$, by the parenthetical remark following (2.5). Substituting this into (2.7) gives a contradiction. Thus, in fact, (2.5) holds. This completes the proof of the lemma. □

In the sequel,

$$f_0(x) = f_0(r,\phi) = \begin{cases} r^{-k}g(\phi), & \text{if } \rho > 0, \\ \log\log r + \dfrac{g(\phi)}{\log r} + \dfrac{h(\phi)}{(\log r)^2}, & \text{if } \rho = 0, \\ \log r + g(\phi), & \text{if } \rho < 0, \end{cases}$$

will always refer to the Lyapunov function constructed in Lemma 1. Also, in order to treat all values of ρ simultaneously, we define $k = 0$ in the case $\rho \le 0$. Note the following important fact which will be used extensively in the sequel:

$$\text{(2.8)} \quad \text{For } m \ge 1, \qquad \frac{\partial^m f_0}{\partial x_{i_1}\partial x_{i_2}\cdots\partial x_{i_m}} = \begin{cases} O(r^{-k-m}), & \text{if } \rho \ne 0, \\ O\left(\dfrac{r^{-k-m}}{\log r}\right), & \text{if } \rho = 0. \end{cases}$$

We will now prove the theorem by constructing an appropriate Lyapunov function satisfying (2.1) if $\rho \le 0$ and (2.2) if $\rho > 0$. We must treat different ranges of the parameter δ separately. We will treat the case $\rho \ne 0$. If $\rho = 0$, the proof is amended by replacing every term of the form $O(r^{-l})$ by $O(r^{-l}/\log r)$.

Case 1: $\delta = 0$. We look for a Lyapunov function in the form $f(r,\phi,\sigma) = f_0(r,\phi) + f_1(r,\phi,\sigma) + f_2(r,\phi,\sigma)$, with f_1 and f_2 as yet undetermined. Since f_0 is independent of $\sigma \in E$, $Gf_0 = 0$. Thus, for large r,

$$\text{(2.9)} \qquad Af = \tfrac{1}{2}\Delta f_0 + \tfrac{1}{2}\Delta f_1 + \tfrac{1}{2}\Delta f_2 + Vf_0 + Vf_1 + Vf_2 + Gf_1 + Gf_2.$$

Note that $\frac{1}{2}\Delta f_0 = O(r^{-k-2})$ and $Vf_0 = 0(r^{-k-1})$. In order to eliminate this latter term, we choose f_1 so that

$$\text{(2.10)} \qquad Gf_1 = -Vf_0.$$

This is possible by the Fredholm alternative since $\langle Vf_0\rangle = 0$ (which follows from $\langle V\rangle = 0$ and the fact that f_0 is independent of σ). Now Vf_0 is a linear combination of functions of the form $w(r)z(\phi,\sigma)$, where $\lim_{r\to\infty} w(r) = 0$. Thus $f_1 = -G^{-1}Vf_0$ will be a linear combination of functions of the form $G^{-1}w(r)z(\phi,\sigma) = w(r)G^{-1}z(\phi,\sigma)$. Now $G^{-1}z(\phi,\sigma)$ is uniquely defined up to the addition of an arbitrary function depending only on r and ϕ. We will always choose $G^{-1}z(\phi,\sigma)$ to be independent of r so that the decay rates of $w(r)$ and $G^{-1}w(r)z(\phi,\sigma)$ coincide.

From (2.8) and the above discussion, it is clear that

$$\text{(2.11)} \qquad f_1(r,\phi,\sigma) = O(r^{-k-1}),\ Vf_1 = O(r^{-k-2}) \quad\text{and}\quad \tfrac{1}{2}\Delta f_1 = O(r^{-k-3}).$$

We now want to pick f_2 so that

$$\text{(2.12)} \qquad Gf_2 = -\tfrac{1}{2}\Delta f_0 - Vf_1 + \hat{L}f_0.$$

This is possible if and only if $\langle -\frac{1}{2}\Delta f_0 - Vf_1 + \hat{L}f_0\rangle = 0$, that is, if and only if

$\hat{L}f_0 = \frac{1}{2}\Delta f_0 + \langle Vf_1 \rangle = \frac{1}{2}\Delta f_0 - \langle VG^{-1}V \rangle f_0$. This is how we were led to define $\hat{L}$ as we did in (1.3). By the same reasoning as above, we may select f_2 so that

$$(2.13)\qquad f_2 = O(r^{-k-2}),\ Vf_2 = 0(r^{-k-3}) \quad\text{and}\quad \tfrac{1}{2}\Delta f_2 = 0(r^{-k-4}).$$

Thus we have $f = f_0 + O(r^{-k-1})$, and, substituting (2.10)–(2.13) into (2.9), we have $Af = \hat{L}f_0 + O(r^{-k-3})$. This proves the theorem in light of (2.1) and (2.2) and Lemma 1.

CASE 2: $0 < \delta < 1$. Again we look for a Lyapunov function in the form $f(r,\phi,\sigma) = f_0(r,\sigma) + f_1(r,\phi,\sigma) + f_2(r,\phi,\sigma)$ and thus Af is again as given in (2.9). We have

$$(2.14)\qquad \tfrac{1}{2}\Delta f_0 = O(r^{-k-2}).$$

We choose f_1 so that

$$(2.15)\qquad Gf_1 = -Vf_0,$$

which is possible since $\langle Vf_0 \rangle = 0$. As in the previous case, we may select $f_1 = -G^{-1}Vf_0$ so that

$$(2.16)\qquad f_1 = O(r^{-k-1+\delta}),\ Vf_1 = O(r^{-k-2+2\delta}) \quad\text{and}\quad \tfrac{1}{2}\Delta f_1 = O(r^{-k-3+\delta}).$$

Now pick f_2 so that

$$(2.17)\qquad Gf_2 = -Vf_1 + \hat{L}f_0.$$

This is possible if and only if $\langle -Vf_1 + \hat{L}f_0 \rangle = 0$, that is, if and only if $\hat{L}f_0 = \langle Vf_1 \rangle = -\langle VG^{-1}V \rangle f_0$, which is how we defined $\hat{L}$ in (1.3). As before, we may select f_2 so that

$$(2.18)\qquad f_2 = 0(r^{-k-2+2\delta}),\ Vf_2 = 0(r^{-k-3+3\delta}) \quad\text{and}\quad \tfrac{1}{2}\Delta f_2 = O(r^{-k-4+2\delta}).$$

Thus, we have $f = f_0 + O(r^{-k-1+\delta})$ and, substituting (2.14)–(2.18) into (2.9) gives $Af = \hat{L}f_0 + O(r^q)$, where $q = \max(-k-2, -k-3+3\delta)$. As before, the theorem now follows from (2.1) and (2.2) and Lemma 1.

CASE 3: $-1 < \delta < 0$. This time it suffices to look for a Lyapunov function in the form $f(r,\phi,\sigma) = f_0(r,\phi) + f_1(r,\phi,\sigma)$. Thus

$$(2.19)\qquad Af = \tfrac{1}{2}\Delta f_0 + \tfrac{1}{2}\Delta f_1 + Vf_0 + Vf_1 + Gf_1.$$

We have

$$(2.20)\qquad V_0 f_0 = O(r^{-k-1+\delta}).$$

Choose f_1 so that

$$(2.21)\qquad Gf_1 = -Vf_0,$$

which is possible since $\langle Vf_0 \rangle = 0$. As before, we may select f_1 so that

$$(2.22)\qquad f_1 = O(r^{-k-1+\delta}),\ Vf_1 = O(r^{-k-2+2\delta}) \quad\text{and}\quad \tfrac{1}{2}\Delta f_1 = O(r^{-k-3+\delta}).$$

Thus we have $f = f_0 + O(r^{-k-1+\delta})$ and $Af = \frac{1}{2}\Delta f_0 + O(r^{-k-2+2\delta})$. As before, the theorem now follows from (2.1) and (2.2) and Lemma 1. □

3. Proof of Theorem 2. Without loss of generality, assume that in Hypothesis A, $L_\sigma = \frac{1}{2}\Delta + V$ for $|x| \geq 1$. Then the generator of the process $(X_x^\varepsilon(t), \sigma^\varepsilon(t))$ is given by

$$A_\varepsilon = \begin{cases} \frac{1}{2}\Delta + \varepsilon^{(-1-\delta)/2}V + \dfrac{1}{\varepsilon}G, & \text{on } |x| \geq \varepsilon \text{ if } -1 \leq \delta < 0, \\ \dfrac{\varepsilon^{\delta/(1-\delta)}}{2}\Delta + \varepsilon^{-1/2}V + \dfrac{1}{\varepsilon}G, & \text{on } |x| \geq \varepsilon \text{ if } 0 \leq \delta < 1. \end{cases}$$

Let $f_0 \in C_0^\infty(\mathbb{R}^d)$ and fix $0 < \gamma < 1$. It is possible to pick two bounded functions f_1 and f_2 on $\{|x| \geq \gamma\} \times E$ such that

(3.1)
(a) $A_\varepsilon\big(f_0 + \varepsilon^{(1-\delta)/2}f_1 + \varepsilon^{1-\delta}f_2\big) = \hat{L}f_0 + O(\varepsilon^{(1-\delta)/2})$ uniformly on $|x| \geq \gamma$, if $-1 \leq \delta \leq 0$.

(b) $A_\varepsilon\big(f_0 + \varepsilon^{1/2}f_1 + \varepsilon f_2\big) = \hat{L}f_0 + O(\varepsilon^{1/2} \vee \varepsilon^{\delta/(1-\delta)})$ uniformly on $|x| \geq \gamma$, if $0 < \delta < 1$.

Indeed this follows by two applications of the Fredholm alternative almost identical to the Fredholm alternative argument in the proof of Theorem 1.

Recall the notation introduced prior to Theorem 2: $\Omega = C([0,\infty), \mathbb{R}^d)$, $\mathscr{F}$ is the usual σ-field on Ω, $\mathscr{F}_t$ is the usual filtration and P_x^ε is the measure on $(\Omega, \mathscr{F})$ induced by $X_x^\varepsilon(t)$. Now let Ω' denote the space of functions from $[0,\infty)$ to $\mathbb{R}^d \times E$ which are continuous in the first variable $x \in \mathbb{R}^d$ and right continuous in the second variable $\sigma \in E$, and let $\mathscr{F}'$ and $\mathscr{F}_t'$ denote the usual σ-algebra and filtration. Let Q_x^ε denote the measure on Ω' induced by the process $(X_x^\varepsilon(t), \sigma^\varepsilon(t))$; of course, P_x^ε is the first marginal of Q_x^ε. Denote elements of Ω' by $(X(t), \sigma(t))$ and elements of Ω by $X(t)$. Let

$$\tilde{f}_\varepsilon = \begin{cases} f_0 + \varepsilon^{(1-\delta)/2}f_1 + \varepsilon^{1-\delta}f_2, & \text{if } -1 \leq \delta < 0, \\ f_0 + \varepsilon^{1/2}f_1 + \varepsilon f_2, & \text{if } 0 \leq \delta < 1. \end{cases}$$

Fix a positive integer n and, for $\varepsilon \leq \gamma$, define $H_\varepsilon(t) = H_\varepsilon(t; X(\cdot), \sigma(\cdot))$ on Ω' by

$$H_\varepsilon(t) = \tilde{f}_\varepsilon\big(X(t \wedge \tau_\gamma \wedge \tau_n), \sigma(t \wedge \tau_\gamma \wedge \tau_n)\big) - \int_0^{t \wedge \tau_\gamma \wedge \tau_n} A_\varepsilon \tilde{f}_\varepsilon(X(s), \sigma(s))\, ds. \tag{3.2}$$

It follows that

(3.3) $H_\varepsilon(t)$ is a Q_x^ε-martingale.

Now define $H_0(t)$ on Ω by

$$H_0(t) = f_0\big(X(t \wedge \tau_\gamma \wedge \tau_n)\big) - \int_0^{t \wedge \tau_\gamma \wedge \tau_n} \hat{L}f_0(X(s))\, ds.$$

From (3.1), the fact that f_1 and f_2 are bounded and the fact that $\varepsilon \le \gamma$, it follows that

$$(3.4)\qquad H_\varepsilon(t) = \begin{cases} H_0(t) + O(\varepsilon^{(1-\delta)/2}), & \text{if } -1 \le \delta \le 0, \\ H_0(t) + O(\varepsilon^{\delta/(1-\delta)} \vee \varepsilon^{1/2}), & \text{if } 0 < \delta < 1, \end{cases}$$

uniformly over Ω'.

From (3.2) and (3.4) and the fact that $\mathscr{F}_t \subset \mathscr{F}_t'$, it follows that for each $f_0 \in C_0^\infty(\mathbb{R}^d)$ there exists a constant C_{f_0} and a family of functions $f_\varepsilon \in C_b^2(\mathbb{R}^d)$, $\varepsilon > 0$, which converge to f_0 uniformly as $\varepsilon \to 0$ such that $f_\varepsilon(y + X(t \wedge \tau_\gamma \wedge \tau_n)) + C_{f_0}t$ is a P_x^ε-submartingale for all $y \in \mathbb{R}^d$ and all $\varepsilon \in (0, \gamma]$. Now define $Y(t) = X(t \wedge \tau_\gamma \wedge \tau_n)$ and let $\mathbb{R}_x^\varepsilon$ denote the measure on $(\Omega, \mathscr{F})$ induced by $Y(t)$ under P_x^ε. It follows directly that $f_\varepsilon(y + X(t)) + C_{f_0}t$ is an $\mathbb{R}_x^\varepsilon$-submartingale for all $y \in \mathbb{R}^d$ and all $\varepsilon \in (0, \gamma]$. This condition on $\{\mathbb{R}_x^\varepsilon\}$, $\varepsilon > 0$, is reminiscent of the sufficiency condition for tightness in Theorem 1.4.6 of Stroock and Varadhan [9] which we now recall: If for all $f_0 \in C_0^\infty(\mathbb{R}^d)$, there exists a constant C_{f_0} such that $f_0(y + X(t)) + C_{f_0}t$ is a submartingale for all $y \in \mathbb{R}^d$ and all $\varepsilon \in (0, \gamma]$, then $\{\mathbb{R}_x^\varepsilon\}$ is a tight family on $(\Omega, \mathscr{F})$ as $\varepsilon \to 0$, that is, for any sequence $\{\varepsilon_n\}_{n=1}^\infty$ satisfying $\lim_{n\to\infty} \varepsilon_n = 0$, the sequence $\{\mathbb{R}_x^{\varepsilon_n}\}_{n=1}^\infty$ has a convergent subsequence. Stroock and Varadhan proved tightness by showing that their condition guaranteed that

$$\lim_{\delta\to 0} \inf_{\varepsilon \in (0,\gamma]} \mathbb{R}_x^\varepsilon\left(\sup_{\substack{0 \le s \le t \le T \\ t-s \le \delta}} |X(t) - X(s)| \le \rho \right) = 1$$

for all $T > 0$ and $\rho > 0$. Now if one replaces the Stroock–Varadhan condition with our condition above and chases through the Stroock–Varadhan proof, one finds that our condition guarantees that

$$\lim_{\gamma\to 0} \lim_{\delta\to 0} \inf_{\varepsilon \in (0,\gamma]} \mathbb{R}_x^\varepsilon\left(\sup_{\substack{0 \le s \le t \le T \\ t-s \le \delta}} |X(t) - X(s)| \le \rho \right) = 1$$

for all $T > 0$ and $\rho > 0$. This is clearly enough to conclude that $\{\mathbb{R}_x^\varepsilon\}$ is a tight family as $\varepsilon \to 0$. Equivalently, $\{P_x^\varepsilon\}$ is a tight family when restricted to $(\Omega, \mathscr{F}_{\tau_\gamma \wedge \tau_n})$.

For $0 \le s < t$, it follows from (3.3) that

$$(3.5)\qquad E^{Q_x^\varepsilon}(H_\varepsilon(t); B) = E^{Q_x^\varepsilon}(H_\varepsilon(s); B), \quad \text{for all } B \in \mathscr{F}_s.$$

From (3.4) and (3.5), we have

$$(3.6)\qquad E^{P_x^\varepsilon}(H_0(t); B) = \begin{cases} E^{P_x^\varepsilon}(H_0(s); B) + O(\varepsilon^{(1-\delta)/2}), & \text{if } -1 \le \delta \le 0, \\ E^{P_x^\varepsilon}(H_0(s); B) + O(\varepsilon^{\delta/(1-\delta)} \vee \varepsilon^{1/2}), & \text{if } 0 < \delta < 1, \end{cases}$$

for all $B \in \mathscr{F}_s$.

Now, if $\overline{P}_{n,\gamma}$ is an accumulation point of P_x^ε on $(\Omega, \mathscr{F}_{\tau_\gamma \wedge \tau_n})$ as $\varepsilon \to 0$, then from (3.6) we conclude that

$$E^{\overline{P}_{n,\gamma}}(H_0(t);B) = E^{\overline{P}_{n,\gamma}}(H_0(s);B), \quad \text{for all } B \in \mathscr{F}_s.$$

In other words,

$$f_0\big(X(t \wedge \tau_\gamma \wedge \tau_n)\big) - \int_0^{t\wedge\tau_\gamma\wedge\tau_n} \hat{L} f_0(X(s))\, ds$$

is a $\overline{P}_{n,\gamma}$-martingale. Also, $\overline{P}_{n,\gamma}(X(0) = x) = 1$. But Hypothesis C states that there is a unique solution to the martingale problem for $\tilde{L}$ up to time τ_0. Thus we conclude that $\overline{P}_{n,\gamma} = \hat{P}_x$ on $(\Omega, \mathscr{F}_{\tau_\gamma \wedge \tau_n})$.

Thus, in fact, $\lim_{\varepsilon \to 0} P_x^\varepsilon = \hat{P}_x$ on $(\Omega, \mathscr{F}_{\tau_\gamma \wedge \tau_n})$. Using Lemma 11.1 in [9] in the case $\hat{P}_x(\tau_0 = \infty) = 1$ and a slight variation of it if $\hat{P}_x(\tau_0 = \infty) < 1$, it follows that

$$\lim_{\varepsilon\to 0} P_x^\varepsilon = \begin{cases} \hat{P}_x \text{ on } (\Omega, \mathscr{F}), & \text{if } \hat{P}_x(\tau_0 = \infty) = 1, \\ \hat{P}_x \text{ on } (\Omega, \mathscr{F}_{\tau_0 -}), & \text{if } \hat{P}_x(\tau_0 = \infty) < 1. \end{cases}$$

This completes the proof of the theorem. □

4. Proof of Theorem 3. Let $\sigma^T(t) = \sigma(T - t)$, $0 \le t \le T$. For convenience, define $\sigma^T(t) = \sigma(0)$ for $t > T$. Define $X^T(t) = X^T(t;\sigma(\cdot)) \equiv X(t;\sigma^T(\cdot))$ and let $\tau_D^T = \inf\{t \ge 0: X^T(t) \in \partial D\}$. It is well known that the solution to the time-inhomogeneous problem (1.6) may be represented probabilistically as

$$u(x,t) = u(x,t;\sigma(\cdot)) = E_{x,0}\big(f(X^t(t)); \tau_D^t > t\big) + E_{x,0}\big(g\big(X^t(\tau_D^t)\big); \tau_D^t \le t\big),$$

where $E_{x,0}$ indicates that the time-inhomogeneous diffusion $X^t(s)$ generated by $L_{\sigma^t(s)}$ is almost surely at x at time 0. For any $N \le t$, we may rewrite this as

$$\begin{aligned} u(x,t) = {} & E_{x,0}\big(g\big(X^t(\tau_D^t)\big); \tau_D^t \le N\big) + E_{x,0}\big(g\big(X^t(\tau_D^t)\big); N < \tau_D^t \le t\big) \\ & + E_{x,0}\big(f(X^t(t)); \tau_D^t > t\big). \end{aligned} \tag{4.1}$$

Now $E_{x,0}(g(X^N(\tau_D^N)); \tau_D^N \le N)$ is measurable with respect to $\{\sigma(s), 0 \le s \le N\}$. Let

$$H_N(\sigma(\cdot)) = H_N(\sigma(s), 0 \le s \le N) \equiv E_{x,0}\big(g\big(X^N(\tau_D^N)\big); \tau_D^N \le N\big). \tag{4.2}$$

Then it is easy to see that for $t > N$,

$$E_{x,0}\big(g\big(X^t(\tau_D^t)\big); \tau_D^t \le N\big) = H_N(\sigma(t - N + \cdot)). \tag{4.3}$$

Let $M_{x,\mu}$ denote the expectation with respect to the process $Z(t) = (X(t), \sigma(t))$ generated by $L_\sigma + G$ with initial distribution $\delta_x \times \mu$ and let $\tau_D = \inf\{t \ge 0: X(t) \in \partial D\}$. Recall that μ is the invariant probability measure for $\sigma(t)$. By the reversibility assumption, $\{\sigma(t), 0 \le t \le N\}$ with initial distribution μ has the same distribution as $\{\sigma^N(t), 0 \le t \le N\}$. From this, (4.2), (4.3) and the ergodic theorem, it follows that for each $x \in D$,

$$(4.4)\quad \lim_{t\to\infty} \frac{1}{t}\int_N^t E_{x,0}\big(g(X^s(\tau_D^s));\ \tau_D^s \le N\big)\,ds = M_{x,\mu}\big(g(X(\tau_D));\ \tau_D \le N\big),$$

for almost all environments $\sigma(\cdot)$.

For $t \ge N$, the last two terms on the right-hand side of (4.1) satisfy

$$(4.5)\quad \begin{aligned} &|E_{x,0}\big(g(X^t(\tau_D^t));\ N \le \tau_D^t < t\big) + E_{x,0}\big(f(X^t(t));\ \tau_D^t > t\big)| \\ &\quad \le cP_{x,0}\big(\tau_D^t > N\big), \end{aligned}$$

for some constant $c > 0$.

Now $P_{x,0}(\tau_D^N > N)$ is measurable with respect to $\{\sigma(s),\ 0 \le s \le N\}$. Let $K_N(\sigma(\cdot)) = K_N(\sigma(s),\ 0 \le s \le N) = P_{x,0}(\tau_D^N > N)$. Then it is easy to see that for $t > N$,

$$(4.6)\quad P_{x,0}\big(\tau_D^t > N\big) = K_N\big(\sigma(t - N + \cdot)\big).$$

From (4.5), (4.6), respectively and the ergodic theorem, we obtain for each $x \in D$, analogous to (4.4),

$$(4.7)\quad \lim_{t\to\infty} \frac{1}{t}\int_N^t P_{x,0}(\tau_D^s > N)\,ds = M_{x,\mu}(\tau_D > N),$$

for almost every environment $\sigma(\cdot)$.

By assumption, the random diffusion is almost surely recurrent and, as noted at the beginning of Section 2, the almost sure recurrence of the random diffusion is equivalent to the recurrence of $Z(t) = (X(t), \sigma(t))$. Thus

$$(4.8)\quad \lim_{N\to\infty} M_{x,\mu}(\tau_D > N) = 0$$

and

$$(4.9)\quad \lim_{N\to\infty} M_{x,\mu}\big(g(X(\tau_D));\ \tau_D \le N\big) = M_{x,\mu}g\big(X(\tau_D)\big).$$

From (4.1), (4.4), (4.5) and (4.7)–(4.9), we obtain for each $x \in D$,

$$(4.10)\quad \lim_{t\to\infty} \frac{1}{t}\int_0^t u(x,s)\,ds = M_{x,\mu}g\big(X(\tau_D)\big).$$

Now, $M_{x,\mu}g(X(\tau_D)) = \Sigma_{\sigma\in E}\mu_\sigma M_{x,\sigma}g(X(\tau_D))$, where $M_{x,\sigma}$ denotes the expectation with respect to $Z(t) = (X(t), \sigma(t))$ with initial distribution $\delta_x \times \delta_\sigma$. The proof is completed by observing that $M_{x,\sigma}g(X(\tau_D))$ solves (1.7) (see [2], pages 381–390). □

Acknowledgment. The authors wish to thank the referee for catching an error in the original version of the proof of the central limit theorem.

REFERENCES

[1] EIZENBERG, A. and FREIDLIN, M. (1990). On the Dirichlet problem for a class of second order pde systems with small parameter. *Stochastics Stochastics Rep.* **33** 111–148.

[2] FREIDLIN, M. (1985). *Functional Integration and Partial Differential Equations*. Princeton Univ. Press.

[3] ICHIHARA, K. (1978). Some global properties of symmetric diffusion processes. *Publ. Res. Inst. Math. Sci.* 441–486.

[4] IOSIFESCU, M. (1980). *Finite Markov Processes and Their Applications*. Wiley, New York.

[5] PINSKY, R. G. (1987). Recurrence, transience and bounded harmonic functions for diffusions in the plane. *Ann. Probab.* **15** 954–984.

[6] PINSKY, R. G. (1988). Hypercontractivity estimates for nonselfadjoint diffusion semigroups. *Proc. Amer. Math. Soc.* **104** 532–536.

[7] PINSKY, R. G. and SCHEUTZOW, M. (1992). Some remarks and examples concerning the transience and recurrence of random diffusion. *Ann. Inst. H. Poincaré Probab. Statist.* To appear.

[8] SINAI, Y. G. (1982). The limiting behavior of a one-dimensional random walk in a random medium. *Theory Probab. Appl.* **27** 256–268.

[9] STROOCK, D. W. and VARADHAN, S. R. S. (1979). *Multidimensional Diffusion Processes*. Springer, New York.

[10] WILLIAMS, R. J. (1985). Brownian motion with polar drift. *Trans. Amer. Math. Soc.* **292** 225–246.

DEPARTMENT OF MATHEMATICS
NORTHWESTERN UNIVERSITY
EVANSTON, ILLINOIS 60208

DEPARTMENT OF MATHEMATICS
TECHNION–ISRAEL INSTITUTE OF TECHNOLOGY
32000 HAIFA
ISRAEL