The Annals of Probability
1993, Vol. 21, No. 1, 340–361

AN EMBEDDING OF COMPENSATED COMPOUND POISSON PROCESSES WITH APPLICATIONS TO LOCAL TIMES[1]

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We present a Brownian embedding for a broad class of compensated compound Poisson processes. Applications of this method are discussed for a problem of level crossings, as well as Donsker's invariance type of principles. In particular, we give a central limit theorem for local times.

1. Introduction. Consider an integrable compound Poisson process $\{P(t); t \geq 0\}$ with representation $P(t) \equiv \sum_{i=1}^{N(t)} X_i$, where X_j are i.i.d. nonnegative random variables with a common cumulative distribution function F and $\{N(t); t \geq 0\}$ is a Poisson process with mean arrival rate of 1, which is totally independent of $\sigma\{X_j;\ j \geq 1\}$. Then $Z(t) \equiv P(t) - \mu t$ is an L^1-compensated compound Poisson process for $\mu \equiv EX_1$. In the next section, we propose an embedding of the stochastic process Z in Brownian motion, that is, we shall prove that on a probability space big enough to carry a Brownian motion B, there exists a time-change σ_t such that one has the representation $Z(t) = B(\sigma_t)$. Moreover, and this is the main motivation of this work, our embedding yields strong estimates on the local times of a large class of pure-jump Lévy processes. In Khoshnevisan (1990), we use these estimates to solve an open problem on crossings of empirical processes.

Other authors have considered Brownian embeddings for Lévy processes. See, for example, Monroe (1972), who develops a different embedding of Lévy processes, which he uses to investigate the γ-variation of sample paths. However, none of these other embeddings seem to give sufficiently strong estimates on local times to be useful for our applications.

The particular case where the X_i's are simply equal to one (i.e., a compensated Poisson process) is of significant interest. Results of Révész (1982) imply that in this case

$$\left\{ \frac{\#\{s \leq t: Z(ns) = 0\}}{\sqrt{n}}; 0 \leq t \leq 1 \right\} \Rightarrow \left\{ L_t^0(B); 0 \leq t \leq 1 \right\}, \tag{1.1}$$

where $\Rightarrow$ denotes weak convergence in the usual Skorohod space $D(0,1)$ and L^0 is the local time of Brownian motion at zero. The same author uses a path decomposition argument to obtain a similar result for zero-crossing of a linear

Received September 1989; revised May 1991.

[1]Work based in part on the author's doctoral dissertation at the University of California, Berkeley, with partial support by NSF 87-01426.

AMS 1991 *subject classifications.* Primary 60J55, 60F15; secondary 60J75, 60F10, 60F17.

Key words and phrases. Compensated compound Poisson processes, Brownian motion, invariance principles, local times.

empirical process converging to local time of Brownian bridge at 0. The question of whether or not the convergence in (1.1) can be extended uniformly in all the levels was left open in Révész (1982).

As a corollary to some of our results in this direction, we prove that under a mild moment condition on F (and hence on the underlying Lévy measure ν)

$$\left(Z_n, \frac{\gamma}{\mu\sqrt{n}} C(Z_n)\right) \Rightarrow (B, L(B)),$$

where

$$\gamma^2 = EX_1^2;$$

$$Z_n = \{Z_n(t); t \geq 0\} = \left\{\frac{Z(nt)}{\gamma\sqrt{n}}; t \geq 0\right\};$$

$$C(Z_n) = \{C_t^x(Z_n); (x,t) \in \mathbb{R} \times [0,1]\};$$

$$C_t^x(Z_n) = \#\{s \leq t: Z_n(t) = x\};$$

$$\#A = \text{cardinality of set } A;$$

$$L(B) = \{L_t^x(B); (x,t) \in \mathbb{R} \times [0,1]\} = \text{local time of } B;$$

$$\Rightarrow \text{denotes weak convergence.}$$

This, in particular, can be used to solve the corresponding problem for empirical processes. The latter has been an open question in this area. See Shorack and Wellner (1986), for example. The solution to this problem will appear in Khoshnevisan (1990).

Borodin (1986) has proved similar results (with slightly worse rates) for random walks, using Skorohod embedding coupled with analytical methods. Our probabilistic method together with Skorohod embedding can be applied in the random walk case, requiring less restrictive conditions on the moments and yielding better rates of convergence.

This paper is divided into three sections. The next section describes the basics of the embedding which are crucial to our strong invariance principle for local times. The strong invariance principle shall appear in Section 3.

Finally a few words about the notation: Throughout this paper, we use a generic constant, C (which may vary from line to line, and sometimes even within a line,) when the constant value in question is independent of anything interesting.

2. The embedding.

2A. *The Poisson case—heuristics.* In this part of Section 2, we shall heuristically present the idea of the embedding for the Poisson process and the rigorous (as well as more general) treatment proceeds in the next subsection. Not only do we hope that this arrangement makes the material more easily understood, but also the case of the Poisson process is the most important one, mainly due to its applicability to the empirical process.

Let $\{B(t): t > 0\}$ denote a linear Brownian motion. Define the maximal process: $M(t) = \sup_{s \le t} B(s)$. The idea is to embed the jump points of a scaled Poisson process in B and then use a time-change to embed the whole path. To do this, let $\tau_1 = \inf\{s: M(s) = B(s) + 1\}$. Having defined τ_n, let $\tau_{n+1} = \inf\{s > \tau_n: \sup_{\tau_n \le u \le s} B(u) = B(s) + 1\}$. Define

$$Z(t) = \sup_{\tau_k \le u \le t} B(u) \quad \text{if } \tau_k \le t \le \tau_{k+1}.$$

It then follows that $Z(t)$ is an increasing process with downward jumps of size 1. So if one could manage to time-change Z, using the fact that $\tau_1, \tau_2 - \tau_1, \dots$ are i.i.d. exponential random variables of rate one, it should not come as a surprise that for some increasing process $\sigma(t)$, $-B(\sigma(t))$ is a compensated Poisson process. Furthermore, since all the level crossings of $-B(\sigma(t))$ are also those of $-B$, it then allows us in Section 3 to study the local time for $-B(\sigma(t))$ through that of B. The main idea here is to use the i.i.d. structure of large excursions of B from level sets, much like Révész (1981), and then compare them to those of $B(\sigma)$.

2*B*. *The general case.* Let ν be a Radon measure on $(0,\infty)$ such that $\nu(\{0\}) = 0$, $\nu(\mathbb{R}) = \nu(\mathbb{R}^+) < \infty$ and

$$\text{(A)} \qquad \mu \equiv \int_0^\infty xF(dx) < \infty,$$

$$\text{(B)} \qquad \int_0^\infty x^2F(dx) \equiv \gamma^2 < \infty,$$

$$\text{(C)} \qquad \int_0^\infty x^4F(dx) < \infty,$$

where $F(t) = \nu((0,t])/\nu(\mathbb{R})$.

Indeed in the current section, we only need to assume (A), but since (B) and (C) are used heavily in the next few sections, we have stated them along with equation (A). Now enlarge the probability space (if need be) so that we can have the following all in one probability space: (i) Let $V_1, V_2, \dots$ be an i.i.d. sequence or random variables, having distribution F defined above. Denote by $\{\Sigma_j;\ j = 1, 2, \dots\}$ the corresponding filtration, that is, $\Sigma_n \equiv \sigma\{V_1, \dots, V_n\}$. (ii) Let $\{B(t);\ t \ge 0\}$ be a Brownian motion process totally independent of the V's, that is, for $\Theta_t \equiv \sigma\{B(s);\ s \le t\}$: Θ_∞ is independent of Σ_∞.

We are ready to describe the embedding. Define the stopping times:

$$\tau_0 \equiv 0,$$

$$\tau_{j+1} \equiv \inf\left\{s \ge \tau_j: \sup_{\tau_j \le r \le s} B_r - B_s = V_{j+1}\right\}.$$

Also, define the auxilliary process $\tilde{S}$:

$$\tilde{S}(t) \equiv \sum_{j=0}^{\infty} \sup_{\tau_j \le s \le t} B_s 1\{\tau_j \le t < \tau_{j+1}\}.$$

We now start with some preliminary lemmas. Some of the ideas in the first lemma have already been encountered in Williams (1977).

2.1 LEMMA. *$\{\tilde{S}(\tau_{j+1}-)-\tilde{S}(\tau_j);\ j\geq 0\}$ is an i.i.d. sequence of exponential random variables with rate μ. Equivalently $\{\mu[\tilde{S}(\tau_{j+1}-)-\tilde{S}(\tau_j)];\ j\geq 0\}$ is an i.i.d. sequence of exponential random variables with mean* 1.

PROOF. Define, temporarily, the stopping times T_t as

$$T_t \equiv \inf\{s>0\colon B(s)>t\}.$$

Since $\tilde{S}(\tau_1-)=\sup_{0\leq s\leq\tau_1}B(s)$, strong Markov property implies

$$\begin{aligned}
&P\big[\tilde{S}(\tau_1-)>t+s\big]\\
&\quad=P\Big[\sup_{0\leq s\leq\tau_1}B(s)>t+s\Big]=P\Big[T_t<\tau_1,\sup_{0\leq s\leq\tau_1-T_t}B(s)\circ\theta(T_t)>s\Big]\\
&\quad=P[T_t<\tau_1]P[T_s<\tau_1]=P\Big[\sup_{0\leq s\leq\tau_1}B(s)>t\Big]P\Big[\sup_{0\leq s\leq\tau_1}B(s)>s\Big]\\
&\quad=P\big[\tilde{S}(\tau_1-)>t\big]P\big[\tilde{S}(\tau_1-)>s\big].
\end{aligned}$$

Hence $\tilde{S}(\tau_1-)$ is exponentially distributed. Moreover, it is well-known that there exists another Brownian motion $\beta(t)$ such that $\beta(0)=0$ and [see Revuz and Yor (1989)]

$$|\beta(t)|=\sup_{0\leq s\leq t}B(s)-B(t).$$

Furthermore, $\tau_1=\inf\{s>0\colon|\beta(s)|=V_1\}$. This gives us the distribution of $\tilde{S}(\tau_1-)$ completely, since by the above, for each n fixed,

$$E\Big[\sup_{0\leq s\leq\tau_1\wedge n}B(s)\Big]=E\big[|\beta(\tau_1\wedge n)|\big]+E\big[B(\tau_1\wedge n)\big]=E\big[|\beta(\tau_1\wedge n)|\big]$$

by optional stopping. Here $a\wedge b$ denotes the minimum of the two numbers a and b. This implies that $E[\sup_{0\leq s\leq\tau_1}B(s)]=\lim_{n\uparrow\infty}E[\sup_{0\leq s\leq\tau_1\wedge n}B(s)]=\lim_{n\uparrow\infty}E[|\beta(\tau_1\wedge n)|]=E[|\beta(\tau_1)|]=\mu$, where the last equality follows since τ_1 is also the first hitting time of $\pm V_1$ for β, and hence $|\beta(\tau_1\wedge n)|\leq V_1$, so that the use of dominated convergence theorem is justified.

This implies that $\tilde{S}(\tau_1-)$ is exponential of rate μ. As to the rest of the statement, the following Markovian argument (and a standard induction method) proves the independence:

$$\begin{aligned}
&P\big[\tilde{S}(\tau_2-)-\tilde{S}(\tau_1)\geq m_2,\tilde{S}(\tau_1-)\geq m_1\big]\\
&\quad=P\big[\tilde{S}(\tau_1-)\geq m_1,\tilde{S}(\tau_2-\tau_1-)\circ\theta(\tau_1)\geq m_1\big]\\
&\quad=E\Big(1\big\{\tilde{S}(\tau_1)\geq m_1\big\}P\big[\tilde{S}(\tau_2-\tau_1-)\circ\theta(\tau_1)\geq m_2|\Theta_{\tau_1}\big]\Big)\\
&\quad=E\Big(1\big\{\tilde{S}(\tau_1-)\geq m_1\big\}P_{B(\tau_1)}\big[\tilde{S}(\tau_2-)\geq m_2\big]\Big)\\
&\quad=P\big[\tilde{S}(\tau_1-)\geq m_1\big]P\big[\tilde{S}(\tau_2-)-\tilde{S}(\tau_1)\geq m_2\big].
\end{aligned}$$

The statement about the identical distributions follows from stationarity of the increments of Brownian motion. □

Defining

$$\Psi_0 \equiv 0, \qquad \Psi_n \equiv \sum_{i=1}^{n} V_j, \qquad \tilde{N}(t) \equiv \sum_{j=0}^{\infty} \Psi_j 1\{\tau_j \le t < \tau_{j+1}\}$$

and

$$A(t) \equiv \tilde{N}(t) + \tilde{S}(t), \tag{2.1}$$

we prove the following elementary lemma:

2.2 LEMMA. *The following both hold, with probability* 1: (i) *The map* $t \to A(t)$ *is increasing*. (ii) $t \to A(t)$ *is continuous*.

PROOF. For (ii), observe that the only times when A can possibly be discontinuous are τ_j's, since those are the jump times of both $\tilde{N}$ and $\tilde{S}$. However for any k, $\tilde{N}(\tau_k) = \Psi_k = \Sigma_{i=1}^{k} V_i$ and $\tilde{S}(\tau_k) = B(\tau_k)$, and finally, $\tilde{S}(\tau_k -) = \sup_{\tau_{k-1} \le s \le \tau_k} B(s)$. Since from definition $\tilde{S}(\tau_k -) - \tilde{S}(\tau_k) = V_k$, continuity of A follows. Part (i) follows easily from (ii). In other words, (i) says that A compensates $\tilde{N}$. □

Let σ be the right continuous inverse of $A(t)$, that is,

$$\sigma(t) \equiv \sigma_t = \inf\{s \colon A(s) = \mu t\}. \tag{2.2}$$

Now define the process $Z(t)$ as

$$Z(t) \equiv \tilde{S}(\sigma_t).$$

Observe that: (i) $t \to Z(t)$ has jumps at times $(1/\mu)A(\tau_j)$, $j = 1, 2, \dots,$ so that the interarrival times are exponential with mean 1. (ii) Between its jumps, $t \to Z(t)$ has slope μ. Hence, replacing Z by $-Z$ and B by $-B$, we have:

2.3 PROPOSITION. *The process* $P(t) \equiv Z(t) - \mu t$ *is a compound Poisson process with positive jumps of size according to the Lévy measure* ν.

This gives an embedding of an $L^1(P)$-compensated compound Poisson process Z with expected arrival rate 1 and Lévy measure ν, in Brownian motion, in terms of a time-inhomogeneous time change; this process can be written as

$$Z(t) \equiv P(t) + \mu t$$

for $\{P(t);\, t \ge 0\}$ as in the above proposition.

3. A strong invariance principle. In this section, we show that under mild conditions, the embedded process of the previous section is close to the underlying Brownian motion process. This is well-defined through the means of weak convergence. Furthermore, we shall get rates of convergence that are similar to Skorohod-type of embeddings. To do this, we shall need some preliminary lemmas:

3.1 LEMMA. *For the construction of Section* 2,

$$\sup_{0\le t\le 1} |\tau_n - nt\gamma^2| = O\left(\sqrt{n\log\log n}\right) \quad a.s.$$

PROOF. First of all, note that the arrival times, $\tau_1, \tau_2 - \tau_1, \dots$ are i.i.d. random variables. Also, since $\{t^2 - 6tB(t)^2 + B(t)^4;\ t \ge 0\}$ is a mean-zero martingale [see, for example, Revuz and Yor (1989)], for each fixed positive k:

$$\begin{aligned} E\tau_1^2 1\{\tau_1 \le k\} + EB(\tau_1)^4 1\{\tau_1 \le k\} &= 6E\tau_1 B(\tau_1)^2 1\{\tau_1 \le k\} \\ &= E\tau_1^2 1\{\tau_1 \le k\} + EV_1^4 1\{\tau_1 \le k\} \\ &\le 6\sqrt{E\tau_1^2 1\{\tau_1 \le k\} EV_1^4} \end{aligned}$$

by an application of the Cauchy–Schawrz inequality. So by solving for $E\tau_1^2 1\{\tau_1 \le k\}$ and letting $k \to \infty$,

$$E\tau_1^2 \le 36EV_1^4 < \infty.$$

At this point, the law of the iterated logarithm implies

$$\tau_n - nE\tau_1 = O\left(\sqrt{n\log\log n}\right).$$

Let β be the Brownian motion process introduced in the proof of Lemma 2.1. Then since $\beta^2(t) - t$ is a mean zero martingale, the same stopping time argument as in the proof of Lemma 2.1 shows that

$$E\tau_1 = E|\beta(\tau_1)|^2 = E\left[\sup_{0\le s\le \tau_1} B(s) - B(\tau_1)\right]^2 = E\left[V_1^2\right] = \gamma^2.$$

Since with probability 1, $\sup_{0\le s\le\tau_1} B(s) - B(\tau_1) = V_1$, by definition. The lemma follows without the supremum over t, from the usual law of the logarithm. To put the supremum, just use Polya's theorem, since everything is increasing, and the identity map $t \to t$ is continuous. □

From now on, we shall implicitly assume the construction of the previous section, unless specifically mentioned.

3.2 LEMMA. *With probability* 1:

$$\sup_{0\le t\le 1} |A(nt) - \mu\gamma^{-2}nt| = O\left(\sqrt{n\log\log n}\right).$$

PROOF. Since $A(\tau_n)$ is a sum of n i.i.d. exponential random variables of mean μ, it follows that

$$A(\tau_n) - n\mu = O\left(\sqrt{n\log\log n}\right). \tag{3.1}$$

Define the right-continuous inverse ρ_n of τ_n as usual, that is, define

$$\rho_n \equiv \inf\{s > 0: \tau_n \geq s\}.$$

From Lemma 3.1,

$$n - \rho_n \gamma^2 = O\big(\sqrt{n \log\log n}\,\big). \tag{3.2}$$

Therefore, applying this to (3.1),

$$A(n) - \rho_n \mu = O\big(\sqrt{\rho_n \log\log \rho_n}\,\big) = O\big(\sqrt{n \log\log n}\,\big).$$

This and (3.2) together prove

$$A(n) - n\mu\gamma^{-2} = O\big(\sqrt{n \log\log n}\,\big),$$

which is the lemma without the uniformity. By usual arguments, this completes the proof. □

3.3 REMARK. Since A increases only when B does, it follows that $t \to \tilde{S}(\sigma_t)$ increases only when $t \to B_t$ does, and hence

$$Z(t) = \tilde{S}(\sigma_t) = B(\sigma_t).$$

3.4 LEMMA. *With probability* 1:

$$\sup_{t \leq n} \big|\tilde{S}(t\gamma^{-2}) - Z(t)\big| = O\big(n^{1/4}(\log\log n)^{1/4}\sqrt{\log n}\,\big).$$

PROOF. Simply notice that by Remark 3.3, Lemma 3.2 and the uniform modulus of continuity of Brownian motion [see Revuz and Yor (1989), for example],

$$\begin{aligned}\sup_{0 \leq t \leq 1} \big|Z(\mu^{-1}A(nt\gamma^2)) - Z(nt)\big| &= \sup_{0 \leq t \leq 1} \big|B(nt\gamma^2) - B(\sigma_{nt})\big| \\ &= O\Big(\sqrt{|\gamma^{-2}\sigma_n - n|\log|\gamma^{-2}\sigma_n - n|}\,\Big) \\ &= O\big(n^{1/4}(\log\log n)^{1/4}\sqrt{\log n}\,\big),\end{aligned}$$

since by Lemma 3.2 and definition of σ,

$$\gamma^{-2}\sigma(n) - n = O\big(\sqrt{n \log\log n}\,\big). \tag{3.3}$$

However, by definition for all t, $Z(\mu^{-1}A_t) = \tilde{S}(t)$, so the lemma is proved. □

3.5 LEMMA. *For any* $\beta > 1$, *with probability* 1,

$$\lim_{n \to \infty} n^{-1/4}(\log\log n)^{-\beta/4} \max_{j \leq n} V_j = 0.$$

PROOF. To simplify the notation, define $M_n = \max_{j \leq n} V_j$ and $r(u) = u^{1/4} \log\log u^{\beta/4}$, $u \geq 4$. Here $\beta > 1$ is fixed. Then

$$P\{M_n \geq x\} = 1 - P\{V_1 \leq x\}^n \leq 1 - \big[1 - x^{-4}EV_1^4\big]^n.$$

Therefore for large n,

$$P\{M_n \geq r(n)\} \leq C(\log\log n)^{-\beta}.$$

Fix $\rho > 1$ and let $t_n = \rho^{\rho^n}$. Then for n large,

$$P\{M_n \geq r(t_n)\} \leq C_\rho n^{-\beta},$$

which sums, since $\beta > 1$. Hence with probability 1,

$$M_n \leq r(t_n) \quad \text{ultimately}.$$

Now let $t_n \leq t \leq t_{n+1}$. So for any $\varepsilon > 0$, as $t \to \infty$,

$$M(t) \leq M(t_{n+1}) \leq r(t_n)\rho^{\rho^{1/4}}\left[\frac{\log(\rho^n \log\rho + \rho\log\rho)}{\log(\rho^n \log\rho)}\right]$$

$$\leq (1-\varepsilon)r(t)\rho^{\rho^{1/4}} \quad \text{ultimately}.$$

This proves the lemma since $\rho > 1$ and $\beta > 1$ are arbitrary. □

We now proceed to prove a strong approximation theorem for our construction of the compound Poisson process.

3.6 PROPOSITION. *For our above construction, assuming equations* (A)–(C):

$$\limsup_{n\to\infty} \sup_{0\leq t\leq 1} \frac{|Z(nt) - B(\gamma^2 nt)|}{n^{1/4}(\log\log n)^{1/4}\sqrt{\log n}} < \infty \quad a.s.$$

PROOF. In view of Lemma 3.4, it is enough to show that with probability 1,

$$\sup_{0\leq t\leq n} |\tilde{S}(t) - B(t)| = O\left(n^{1/4}(\log\log n)^{1/4}\sqrt{\log n}\right). \tag{3.4}$$

But notice that

$$\sup_{0\leq t\leq \tau_n} |\tilde{S}(t) - B(t)| = \max_{1\leq j\leq n} V_j.$$

By Lemma 3.5,

$$\max_{j\leq n} V_j = o\left(n^{1/4}(\log\log n)^{1/4}\sqrt{\log n}\right).$$

Therefore with probability 1,

$$\sup_{0\leq t\leq \tau_n} |\tilde{S}(t) - B(t)| = O\left(n^{1/4}(\log\log n)^{1/4}\sqrt{\log n}\right) \quad \text{ultimately}.$$

Furthermore, by properties of $\{B(t);\, t \geq 0\}$ and by Lemma 3.1,

$$\sup\{|B(t)|: \min(\tau_n, \gamma^2 n) \leq t \leq \max(\tau_n, \gamma^2 n)\} = O\left(n^{1/4}(\log\log n)^{1/4}\sqrt{\log n}\right)$$

and

$$\begin{aligned}&\sup\{\tilde{S}(t): \min(\tau_n, \gamma^2 n) \le t \le \max(\tau_n, \gamma^2 n)\}\\ (3.5)\qquad &= \sup\{B(t): \min(\tau_n, \gamma^2 n) \le t \le \max(\tau_n, \gamma^2 n)\}\\ &= O\Big(n^{1/4}(\log\log n)^{1/4}\sqrt{\log n}\Big).\end{aligned}$$

Thus

$$\begin{aligned}\sup_{0\le t\le n\gamma^2}\big|\tilde{S}(t) - B(t)\big| &\le \sup_{0\le t\le \tau_n}\big|\tilde{S}(t) - B(t)\big| + O\Big(n^{1/4}(\log\log n)^{1/4}\sqrt{\log n}\Big)\\ &= O\Big(n^{1/4}(\log\log n)^{1/4}\sqrt{\log n}\Big).\end{aligned}$$

By (3.4), this completes the proof. □

The above proposition readily implies the following two well-known corollaries:

3.7 COROLLARY. *Let Z be a compensated compound Poisson process, with Lévy measure ν satisfying equations* (A)–(C). *Then for a Brownian motion B and for $Z_n(t) \equiv (Z(nt)/\gamma\sqrt{n})$, $Z_n \Rightarrow B$. Here $\Rightarrow$ denotes weak convergence in* $D([0,1])$ *equipped with Skorohod topology.*

PROOF. Let Z and B be as in our construction. Then we have shown that

$$n^{1/2}\sup_{0\le t\le 1}\big|Z(nt) - B(\gamma^2 nt)\big| \to 0 \quad \text{a.s.}$$

But for each fixed n, $\{(B(\gamma^2 nt)/\gamma\sqrt{n}); t\ge 0\}$ is a Brownian motion. Hence the result follows. □

3.8 COROLLARY. *Define $\mathbf{K}$ to be the set of absolutely continuous functions f on $[0,1]$ such that $f(0) = 0$ and $\int_0^1 \dot{f}(t)^2\,dt \le 1$, where $\dot{f}$ is the derivative of f. Then assuming equations* (A) *and* (B), $\mathbf{K}$ *is exactly the set of accumulation points for the function sequence* $\{(\log\log n)^{-1/2}Z_n(t);\ 0\le t\le 1\}$ *in* $C([0,1])$ *endowed with the compact open topology.*

PROOF. To derive this result, we need to adapt the proof of Proposition 3.6 to the case where (C) need not hold any more. We shall briefly sketch the argument:

STEP 1. Lemma 3.1 becomes $\sup_{0\le t\le 1}|\tau_{nt} - nt\gamma^2| = o(n)$ a.s.

STEP 2. Lemma 3.2 becomes $\sup_{0\le t\le 1}|A(nt) - \dot{\mu}\gamma^{-2}nt| = o(n)$ a.s.

STEP 3. Proposition 3.6 becomes

$$\lim_{n\uparrow\infty}\frac{|Z(nt) - B(\gamma^2 nt)|}{\sqrt{n\log\log n}} = 0 \quad \text{a.s.}$$

Now the result follows from Strassen's law of the iterated logarithm [see Revuz and Yor [1989], for example]. □

The above two corollaries are well known and we wrote them this way as easy consequences of the embedding. There are no claims to their originality on the part of the author.

4. The problem of level crossings. We shall use the construction of the previous section to give a formulation of strong invariance principles for the level crossings of compensated compound Poisson processes. To this end, define:

4.1 DEFINITION.
(a) For any stochastic process $X \equiv \{X(t);\ t \geq 0\}$, define (when it exists) the *crossing process* of X as

$$C(X) \equiv \{C_t^x(X);\ (x,t) \in \mathbb{R} \times [0,1]\},$$

where $C_t^x(X) \equiv \#\{s \leq t\colon\ X(s) = x\}$.
(b) For any stochastic process $X \equiv \{X(t);\ t \geq 0\}$, define (when it exists) the *local time* of X as $L(X) \equiv \{L_t^x(X);\ (x,t) \in \mathbb{R} \times [0,1]\}$, where $L_t^x(X) \equiv \lim_{\varepsilon \downarrow 0}(2\varepsilon)^{-1}\int_0^t 1_{[x-\varepsilon,\, x+\varepsilon]}(X(s))\, ds$ almost surely. [See Revuz and Yor (1989).]

It is well known that many continuous semimartingales have local times in the sense of our definition, and their properties are well understood; an important example of them is Brownian motion [see Revuz and Yor (1989), Chapter IV]. The following lemma shows that compensated compound Poisson processes also possess local times, and this fact is linked to their crossing process in a fundamental manner.

4.2 LEMMA. *Let Z be any compensated compound Poisson process with Lévy measure ν and expected arrival rate* 1. *Then for*

$$Z_n(t) \equiv \frac{Z(nt)}{\gamma\sqrt{n}},$$

the process after scaling, there is a simple relation between its local time and crossing process, namely:

$$L(Z_n) = \frac{\gamma}{\mu\sqrt{n}}C(Z_n).$$

PROOF. This is a special case of results in Chapter 2 of Adhikari (1987) [see also the proof of Lemma 3.2.4(c) in Khoshnevisan (1989)]. □

For typographical simplicity, introduce the following notation:

$$L(x,t) = L_t^x(B),$$
$$C(x,t) = C_t^x(Z),$$

where B and Z are as in the previous sections. Also define

$$T(x) \equiv \inf\{s > 0\colon Z(s) = x\}. \tag{4.1}$$

Then using this, iteratively define

$$T_0(x) \equiv 0,$$
$$T_{j+1}(x) = T_j(x) + T(x) \circ \theta\big(T_j(x)\big)$$

for $j = 1, 2, \ldots$. We shall also abbreviate $T_j(0)$ to T_j for all j, and $T(0)$ to T. Define

$$\Delta_j(x) \equiv L\big(x, \sigma\big(T_{j+1}(x)\big)\big) - L\big(x, \sigma\big(T_j(x)\big)\big). \tag{4.2}$$

Then the following lemma holds:

4.3 LEMMA. *For each x, the sequence, $\{\Delta_j(x);\ j \geq 1\}$ is an i.i.d. sequence of exponential random variables, whose distribution is independent of x.*

PROOF.

STEP 1. We show that $\{\Delta_j(x);\ j \geq 1\}$ is an independent sequence of random variables. This is so, simply because of

$$\Delta_j(x) \in \sigma\big\{B(s)\colon \sigma\big(T_j(x)\big) \leq s \leq \sigma\big(T_{j+1}(x)\big)\big\}$$

and since Brownian motion is a strong Markov process and has independent and identically distributed increments.

STEP 2. We now show that Δ_j's have identical distributions.

$$\begin{aligned} P[\Delta_1 \geq \alpha] &= P\big[L(0, \sigma(T_2)) - L(0, \sigma(T_1)) \geq \alpha\big] \\ &= E\big\{P\big[L(0, \sigma(T_2)) - L(0, \sigma(T_1)) \geq \alpha \big| B(s);\ s \leq \sigma(T_1)\big]\big\} \\ &= E\big\{P_{B(\sigma(T_1))}\big[L(0, \sigma(T)) \geq \alpha\big]\big\} = P[\Delta_0 \geq \alpha]. \end{aligned}$$

STEP 3. To get the result for $\Delta_j(x)$ for any x, observe that all of the arguments in the above two steps go through, if instead of B, one looks at the Brownian motion $\{B(t + \zeta_x) - B(\zeta_x);\ t \geq 0\}$, where $\zeta_x \equiv \inf\{s > 0\colon B(s) = x\}$.

STEP 4. The last step in the proof is to check that the distributions are exponential. But for all $a, b \geq 0$ and for $\zeta^x \equiv \inf\{s > 0\colon L(0, s) > x\}$,

$$\begin{aligned} P[\Delta_0 > a + b] &= P\big[L(0, \sigma(T_1)) > a + b\big] = P\big[\Delta_0 \circ \theta(\zeta^a) > b, \zeta < \sigma(T_1)\big] \\ &= P[\Delta_0 > b]P\big[\zeta^a < \sigma(T_1)\big] = P[\Delta_0 > a]P[\Delta_0 > b], \end{aligned}$$

where the third equality is due to the fact that B is zero on the points of increase of L. This ends the proof. □

The next result shows that our construction is very close to Brownian motion, not only in the sense of the supremum norm, but also in the sense of local times.

4.4 PROPOSITION. *If* (A)–(C) *hold, then, for our construction and for each fixed* $x \in \mathbb{R}$, *with probability* 1 *as* $n \to \infty$,

$$\sup_{0 \le t \le 1} \left| L_t^{x/\sqrt{n}}(Z_n) - L_t^{x/\sqrt{n}}(B_n) \right| = O\left(n^{-1/4}(\log\log n)^{3/4}\right) \quad \textit{ultimately}.$$

In particular,

$$\sup_{t \le 1} \left| L_t^0(Z_n) - L_t^0(B_n) \right| = O\left(n^{-1/4}(\log\log n)^{3/4}\right).$$

PROOF. Fix $x \in \mathbb{R}$. Then since

$$L\left(x, \sigma\left(T_{C(x,nt)}(x)\right)\right) = \sum_{j=1}^{C(x,nt)} \Delta_j(x)$$

and since $C(x, nt) \to \infty$ with probability 1, by the law of the iterated logarithm, there exists a positive finite κ such that

$$\sum_{j=1}^{C(x,nt)} \Delta_j(x) - \kappa C(x,nt) = O\left(\sqrt{C(x,nt)\log\log C(x,nt)}\right). \tag{4.3}$$

Rewrite (4.2) as

$$\frac{\sum_{j=1}^{C(x,nt)} \Delta_j(x)}{C(x,nt)} \times \frac{C(x,nt)}{\sqrt{n}\,\log\log n} = \frac{L\left(x, \sigma\left(T_{C(x,nt)}(x)\right)\right)}{\sqrt{n}\,\log\log n}. \tag{4.4}$$

But by the modulus of local times and (3.3), $\sup_{0 \le t \le 1} |L(x, \gamma^2 nt) - L(x, \sigma_{nt})|$ is almost surely

$$O\left(\sqrt{|\sigma_n - \gamma^2 n| \log|\sigma_n - \gamma^2 n|}\right) = O\left(n^{1/4}(\log\log n)^{1/4}\sqrt{\log n}\right). \tag{4.5}$$

Observe that by Kesten's law of the iterated logarithm [see Kesten (1965); also see Khoshnevisan (1989) for an elementary proof, and some related results]

$$L(x, \gamma^2 nt) = O\left(\sqrt{n}\,\log\log n\right) \quad \text{ultimately}.$$

Hence by (4.5),

$$L(x, \sigma_{nt}) = O\left(\sqrt{n}\,\log\log n\right) \quad \text{ultimately}.$$

Since L and σ are increasing in time with probability 1 and since $T_{C(x,nt)}(x) \le nt$ almost surely, it follows that

$$L\left(x, \sigma\left(T_{C(x,nt)}(x)\right)\right) = O\left(\sqrt{n}\,\log\log n\right). \tag{4.6}$$

On the other hand, it is an elementary fact that $\lim_{n\to\infty} C(x, nt) = \infty$, hence the strong law of large numbers implies that with probability 1,

$$(4.7)\qquad \frac{\sum_{j=1}^{C(x,nt)}\Delta_j(x)}{C(x,nt)} \to \kappa.$$

Therefore, equations (4.4), (4.6) and (4.7) together imply

$$(4.8)\qquad \sup_{0\le t\le 1} C(x,nt) = C(x,n) = O\left(\sqrt{n \log\log n}\right) \text{ ultimately.}$$

Therefore, equations (4.3) and (4.8) imply

$$(4.9)\quad L\left(x, \sigma\left(T_{C(x,nt)}(x)\right)\right) - \kappa C(x,nt) = O\left(n^{1/4}(\log\log n)^{3/4}\right) \text{ ultimately.}$$

But the monotonicity of local times implies that

$$L\left(x, \sigma\left(T_{C(x,nt)}(x)\right)\right) \le L(x, \sigma_{nt}) \le L\left(x, \sigma\left(T_{C(x,nt)+1}(x)\right)\right).$$

Therefore we have shown that there exists a finite positive constant κ such that

$$(4.10)\qquad L(x, \gamma^2 nt) - \kappa C(x,nt) = O\left(n^{1/4}(\log\log n)^{3/4}\right).$$

But

$$(4.11)\qquad L(x,\gamma^2 nt) = L^x_{\gamma^2 nt}(B) = \gamma\sqrt{n}\, L_t^{x/\gamma\sqrt{n}}(B_n),$$

where we had $B_n(t) \equiv (B(nt\gamma^2)/\gamma\sqrt{n}$. A counting argument yields

$$(4.12)\qquad C(x,nt) = C^x_{nt}(Z) = \gamma\sqrt{n}\, C_t^{x/\gamma\sqrt{n}}(Z_n).$$

At this point, equations (4.10), (4.11) and (4.12) together prove the result if we can compute the value of κ. This would be rather difficult to do directly. However, we fortunately have an easy way out, since for all Borel A, as $n \to \infty$,

$$\int_0^t 1\{Z_n(s) \in A\}\, ds = \int_A L_t^x(Z_n) \to_d \int_A L_t^x(B_n) = \int_0^t 1\{B_n(s) \in A\}\, ds$$

(the right-hand side being independent of n in distribution). Therefore, by Lemma 4.2, $\kappa = \gamma/\mu$. □

At this point, one is led to the following question: Can one show that the convergence in the above proposition holds uniformly in the space variable x? We are ready to state the main theorem of this section:

4.5 THEOREM. *There exists a suitable probability space on which one can put a sequence of Brownian motions B_n and a sequence of compensated compound Poisson processes Z_n with Lévy measure ν satisfying equations* (A)–(C), *such that for any compact interval I and all $\varepsilon > 0$,*

$$(4.13)\quad \lim_{n\uparrow\infty} n^{1/4}(\log n)^{-3/4-\varepsilon} \sup_{t\in I}\sup_{x\in R} \left|L_t^x(Z_n) - L_t^x(B_n)\right| = 0 \quad a.s.$$

If (A)–(C) *hold*,

$$\limsup_{n\uparrow\infty} n^{1/4}(\log\log n)^{-1/4}(\log n)^{-1/2}\sup_{t\in I}|Z_n(t)-B_n(t)|<\infty \quad a.s. \tag{4.14}$$

Notice that (4.14) is a restatement of Proposition 3.6; the following proof concentrates on proving 4.13, which is a uniform version of Proposition 4.4.

PROOF. Without loss of generality, set $I \equiv [0,1]$. Since the proof is rather long, we shall divide the proof into two main parts.

PART 1. For any sequence of (nonrandom) finite sets S_n such that

$$\operatorname{card}(S_n)=O(n^k)$$

for some positive integer k, and for any positive ε, with probability 1 the following ultimately holds:

$$\sup_{x\in S_n}\sup_{0\le t\le 1}\left|L(x,\gamma^2 nt)-\frac{\gamma}{\mu}C(x,nt)\right| = o\big(n^{1/4}(\log n)^{3/4+\varepsilon}\big).$$

PROOF. As in the proof of Proposition 4.4, it is enough to show

$$\sup_{x\in S_n}\left|L\big(x,\sigma\big(T_{C(x,n)}(x)\big)\big)-\frac{\gamma}{\mu}C(x,n)\right| = o\big(n^{1/4}(\log\log n)^{3/4+\varepsilon}\big).$$

However, we shall first find a slower rate of convergence:

$$\begin{aligned}
&P\left\{\left|L\big(x,\sigma\big(T_{C(x,n)}(x)\big)\big)-\frac{\gamma}{\mu}C(x,n)\right|\ge\delta\sqrt{n\log n}\right\}\\
&\quad=P\left\{\left|\sum_{j=1}^{C(x,n)}\left(\Delta_j(x)-\frac{\gamma}{\mu}\right)\right|\ge\delta\sqrt{n\log n}\right\}\\
&\quad\le P\left\{\max_{j\le\beta\sqrt{n}\log n}\left|\sum_{i=1}^{j}\left(\Delta_j(x)-\frac{\gamma}{\mu}\right)\right|\ge\delta\sqrt{n\log n}\right\}\\
&\qquad+P\{C(0,n)>\beta\sqrt{n}\log n\}.
\end{aligned}\tag{4.15}$$

But $t\to C(0,t)$ is an additive functional with respect to the filtration of the Lévy process in question. Furthermore, Proposition 4.4 readily implies

$$\frac{\gamma}{\mu}\frac{C(0,n)}{\sqrt{n}}\to_d L(0,1).$$

Therefore, using a subadditivity argument, one can show that there exist constants c_1 and c_2 such that for all k, n,

$$P\{C(0,n)\ge k\sqrt{n}\}\le c_1e^{-c_2k}.$$

In fact much more can be said about the tail distribution of $n^{-1/2}C(0,n)$. Namely, that with the appropriate choice of c_1, the constant c_2 can be taken to be arbitrarily large. We shall, however, have no need for this fact, and therefore refrain from giving a proof. Putting $\log n$ for k in the above, it follows that

$$P\{C(0,n) \geq \beta\sqrt{n}\log n\} \leq c_1 n^{-\beta c_2}. \tag{4.16}$$

In particular, the above probability sums for large enough β. Going back to (4.15), we see that

$$\begin{aligned} &P\left\{\left|L\big(x, \sigma\big(T_{C(x,nt)}(x)\big)\big) - \frac{\gamma}{\mu}C(x,nt)\right| \geq \delta\sqrt{n\log n}\right\} \\ &\quad\leq c_1 n^{-\beta c_2} \\ &\qquad + \beta\sqrt{n}\log n \max_{j \leq \beta\sqrt{n}\log n} P\left\{\left|\sum_{i=1}^{j}\left(\Delta_j(x) - \frac{\gamma}{\mu}\right)\right| \geq \delta\sqrt{n\log n}\right\}. \end{aligned} \tag{4.17}$$

Define a new set of random variables $\Delta_{j,n}(x)$ as

$$\Delta_{j,n}(x) \equiv \Delta_j(x) 1\left\{\left|\Delta_j(x) - \frac{\gamma}{\mu}\right| \leq n^{1/16}\right\}.$$

Then Lemma 4.3, and the fact that we have already seen that $\Delta_j(x)$'s have mean (γ/μ), together imply that for any $K > 0$, there exists $C = C_K$ such that for all n,

$$\begin{aligned} &P\{\text{There exist } i \leq \beta\sqrt{n}\log n \colon \Delta_{i,n}(x) \neq \Delta_i(x)\} \\ &\quad\leq \beta\sqrt{n}\log n P\{\Delta_{1,n}(x) \neq \Delta_1(x)\} \\ &\quad= \beta\sqrt{n}\log n P\left\{\left|\Delta_i(x) - \frac{\gamma}{\mu}\right| \geq n^{1/16}\right\} \leq Cn^{-K}. \end{aligned} \tag{4.18}$$

By possibly enlarging the above constant C, notice that for all $K > 0$, there is a constant C_K such that for all $j \leq \beta\sqrt{n}\log n$ and all n,

$$\begin{aligned} &\left|\sum_{i=1}^{j}\left(E\Delta_{i,n}(x) - E\Delta_i(x)\right)\right| \\ &\quad\leq \beta\sqrt{n}\log n E|\Delta_{1,n}(x) - E\Delta_1(x)| \\ &\quad\leq \beta\sqrt{n}\log n \sqrt{E|\Delta_1|^2 P\{\Delta_1 \neq \Delta_{1,n}\}} \leq Cn^{-K}. \end{aligned} \tag{4.19}$$

Now we apply the well-known Bernstein's inequality, using (4.18) and (4.19): For all $K > 0$, there is a $C = C_K$ such that for all $j \leq \beta\sqrt{n}\log n$, $n \geq 1$, and all $\alpha > 0$:

$$\begin{aligned} &P\left\{\left|\sum_{i=1}^{j}\left(\Delta_i(x) - \frac{\gamma}{\mu}\right)\right| \geq \alpha\sqrt{n\log n}\right\} \\ &\quad\leq Cn^{-K} + 2\exp\left\{-\frac{\alpha^2 n\log n}{2j\gamma/\mu + 2/3\alpha n^{9/16}(\log n)^{1/2}}\right\} \leq Cn^{-K}. \end{aligned} \tag{4.20}$$

Going over the proof that has so far led us to (4.20), we see that the reason that the rate of convergence in (4.20) is $\sqrt{n \log n}$ and not $n^{1/4}(\log n)^{3/4+\varepsilon}$ is that we were not able to obtain very sharp (i.e., Gaussian-type) estimates on the tail distribution of $C(0, n)$ in (4.16). Here is how we can use (4.20) to sharpen to our satisfaction. From now on, fix an arbitrary $K > 0$. Then

$$P\{C(0,n) \geq \beta\sqrt{n \log n}\,, |\sigma_n - n\gamma^2| \leq n\varepsilon\}$$

$$\leq P\left\{L(0, n\gamma^2) \geq \frac{\gamma}{\mu}\frac{\beta}{2}\sqrt{n \log n}\right\} \tag{4.21}$$

$$+ P\left\{\left|C(0,n) - \frac{\mu}{\gamma}L(0, n\gamma^2)\right| \geq \frac{\beta}{2}\sqrt{n \log n}\,, |\sigma_n - n\gamma^2| \leq n\varepsilon\right\}.$$

The first term on the right-hand side of inequality (4.21) is bounded above by Cn^{-K} for some $C = C_K$ by the fact that $\{L(0,t);\ t \geq 0\}$ has the same finite-dimensional distributions as $\{\sup_{s \leq t} B(s);\ t \geq 0\}$ [see Revuz and Yor (1989)], which has tails of form e^{-cx^2}. The second term on the right-hand side of (4.21) is, by the technique in Proposition 4.4, bounded above by

$$Cn^{-K} + P\left\{|L(0, \sigma_n) - L(0, n\gamma^2)| \geq \frac{\gamma\beta}{2\mu}\sqrt{n \log n}\,, |\sigma_n - n\gamma^2| \leq n\varepsilon\right\}$$

$$\leq Cn^{-K} + P\left\{L(0, n\varepsilon) \geq \frac{\gamma\beta}{2\mu}\sqrt{n \log n}\right\} \leq Cn^{-K}$$

for β large enough (and/or ε small enough).

We are ready to complete the proof of the first part of the theorem. Write

$$P\left\{\left|\sum_{i=1}^{j}\left(\Delta_i(x) - \frac{\gamma}{\mu}\right)\right| \geq \delta n^{1/4}(\log n)^{3/4+\varepsilon}, |\sigma_n - n\gamma^2| \leq n\varepsilon\right\}$$

$$\leq Cn^{-K} + \beta\sqrt{n \log n} \max_{j \leq \beta\sqrt{n \log n}} P\left\{\left|\sum_{i=1}^{j}\left(\Delta_i(x) - \frac{\gamma}{\mu}\right)\right| \geq \delta n^{1/4}(\log n)^{3/4+\varepsilon}\right\}.$$

The above is, by Bernstein's inequality and the truncation argument of (4.18)–(4.20), less than or equal to

$$Cn^{-K} + \beta\sqrt{n \log n} \max_{j \leq \beta\sqrt{n \log n}} 2\exp\left\{-\frac{\delta^2 n^{1/2}(\log n)^{3/2+2\varepsilon}}{2j\gamma/\mu + 2/3\delta n^{5/16}(\log n)^{3/4+\varepsilon}}\right\}$$

$$\leq Cn^{-K} + \beta c_1\sqrt{n \log n}\, \exp\{-c_2(\log n)^{1+2\varepsilon}\} \leq Cn^{-K}$$

for β large enough. Therefore, for S_n as stated in the statement of the problem, since all calculations were made independently of t, by a standard argument (on t's), for any $K, \varepsilon > 0$, there is a constant $C = C_{\varepsilon, K}$ such that for all $\alpha > 0$,

$$P\left\{\sup_{x\in S_n}\sup_{t\in[0,1]}\left|\frac{\gamma}{\mu}C(x,nt) - L(x,\gamma^2 nt)\right| \geq \delta n^{1/4}(\log n)^{3/4+\varepsilon},\ |\sigma_n - n\gamma^2| \leq n\varepsilon\right\} \leq Cn^{-K}. \tag{4.22}$$

Since K can be picked arbitrarily large, by summing (4.22) over n and using the Borel–Cantelli lemma, it follows that for each $\varepsilon, \delta > 0$,

$$P\left\{\sup_{x\in S_n}\sup_{t\in[0,1]} D_n(x,t) \geq \delta n^{1/4}(\log n)^{3/4+\varepsilon},\ |\sigma_n - n\gamma^2| \leq n\varepsilon \text{ i.o.}\right\} = 0, \tag{4.23}$$

where $D_n(x,t) \equiv |(\gamma/\mu)C(x,nt) - L(x,\gamma^2 nt)|$. However, σ_n is a sum of i.i.d. L^2 random variables mean γ^2, so by the strong law of large numbers, with probability 1,

$$|\sigma_n - n\gamma^2| \leq n\varepsilon \quad \text{ultimately}.$$

In view of (4.23), the first (and the hard) part of the proof of the theorem is complete.

The second part of the theorem involves a softer argument than the former part:

PART 2. In this part, we shall complete the proof of the theorem. To do so, we use some information on the modulus of continuity of Brownian local time in the space variable as in McKean (1962). Actually, all we need is that for all $\varepsilon > 0$ there exists a constant $C = C_\varepsilon$ such that as $\delta \downarrow 0$:

$$\sup_{0\leq t\leq 1}\sup_{|x-y|\leq\delta}|L(x,t) - L(y,t)| \leq C\delta^{1/2-\varepsilon}.$$

As stated, the above fact can be found in Revuz and Yor [(1989), Chapter 6]. At this point define for $k > 2$ the subdivision

$$S_n \equiv \{\pm jn^{-k}: 1 \leq j \leq n^{k+1}/2\}.$$

Then S_n satisfies the assumptions of the first part of the proof. Hence

$$\sup_{|j|\leq n^{k+1}/2}\left|L(x_j, n\gamma^2) - \frac{\gamma}{\mu}C(x_j, n)\right| = o\left(n^{1/4}(\log n)^{3/4+\varepsilon}\right), \tag{4.24}$$

where for $|j| \leq (n^{k+1}/2)$,

$$x_j \equiv x_{j,n} = jn^{-k}.$$

Without loss of generality, restrict attention to those j's that are nonnegative.

Then by Brownian scaling,

$$\sup_{j \le n^{k+1}/2} \sup_{x_j \le x \le x_{j+1}} \left| L(x, n\gamma^2) - L(x_j, n\gamma^2) \right|$$

has the same distribution as

$$\sqrt{n} \sup_{j \le n^{k+1}/2} \sup_{x_j \le x \le x_{j+1}} \left| L(n^{-1/2}x, \gamma^2) - L(n^{-1/2}x_j, \gamma^2) \right|,$$

which, by the modulus of continuity result cited earlier, is less than or equal to

$$\sup_{j \le n^{k+1}/2} C\sqrt{n} \left(n^{-1/2} |x_j - x_{j+1}| \right)^{1/2-\varepsilon} = Cn^{(1-2k)/4+\varepsilon}.$$

By continuity arguments for Brownian local times,

$$\sup_{j \le n^{k+1}/2} \sup_{x_j \le x \le x_{j+1}} \left| L(x, n\gamma^2) - L(x_j, n\gamma^2) \right| = o(n^{1/4}).$$

This, together with (4.24), implies

$$\sup_{j \le n^{k+1}/2} \left| C(x_j, n) - C(x_{j+1}, n) \right| = o\left(n^{1/4} (\log\log n)^{3/4} \right). \tag{4.25}$$

We claim that with probability 1,

$$\sup_{j < n^{k+1}/2} \sup_{x_j \le x \le x_{j+1}} \left| C(x_j, n) - C(x, n) \right| = o\left(n^{1/4} (\log n)^{3/4+\varepsilon} \right). \tag{4.26}$$

Leaving the proof of (4.26) aside for the time being, we see that for large k,

$$\begin{aligned} \sup_{n \ge x \ge 0} \left| \frac{\gamma}{\mu} C(x, n) - L(x, n\gamma^2) \right| &\le \sup_{j \le n^{k+1}/2} \sup_{x_j \le x \le x_{j+1}} \left| C(x, n) - C(x_{j+1}, n) \right| \frac{\gamma}{\mu} \\ &\quad + \sup_{j \le n^{k+1}/2} \left| \frac{\gamma}{\mu} C(x_j, n) - L(x_j, n\gamma^2) \right| \\ &\quad + \sup_{j \le n^{k+1}/2} \sup_{x_j \le x \le x_{j+1}} \left| L(x_j, n\gamma^2) - L(x, n\gamma^2) \right| \\ &= o\left(n^{1/4} (\log n)^{3/4+\varepsilon} \right) \quad \text{ultimately}. \end{aligned} \tag{4.27}$$

However, by the law of the iterated logarithm, with probability 1,

$$\sup_{x \ge \sqrt{3n \log\log n}} L(x, n) = 0 \quad \text{ultimately}$$

and

$$\sup_{x \ge \sqrt{3n \log\log n}} C(x, n) = 0 \quad \text{ultimately},$$

which, in conjunction with (4.27), gives the desired result. So it remains to give:

PROOF OF (4.26). Let $t(1), t(2), \dots$ be the jump times of Z in ascending order. In other words, if we let $t(0) = 0$ and temporarily define $t = \inf\{s > 0: Z(s) < Z(s+1)\}$, then for all $i = 0, 1, 2, \dots$,

$$t(i+1) \equiv t_{i+1} \equiv t(i) + t \circ \theta_{t(i)}.$$

Recall the definition of $T_j(x)$ from (4.1). Based on this, define

$$S_j(x) \equiv \min\{k \geq 1: t_k \geq T_j(x)\};$$

$$U_j(x) \equiv (S_j(x)).$$

Recall that we are restricting our attention to $x_i \geq 0$, without any loss of generality. With this in mind, consider the number of times before t that Z hits x_{i+1} but jumps prior to hitting x_i, that is,

$$\begin{aligned} N_t(x_i, x_{i+1}) &= \sum_{j=0}^{C(x_{i+1}, t)+1} 1\{Z(U_j(x_i) -) > x_i\} \\ &= \sum_{j=0}^{C(x_{i+1}, t)+1} 1\{Z(T_j(x_{i+1})) - Z(U_j(x_i) -) < n^{-k}\}. \end{aligned}$$

This follows from the fact that $x_{i+1} - x_i = n^{-k}$ and $Z(T_j(x_{i+1})) = x_{i+1}$. By the strong Markov property, for each i and n, $\{Z(T_j(x_{i+1})) - Z(U_j(x_i) -);\ j \geq 0\}$ is an i.i.d. sequence of exponentials with mean μ^{-1}. Therefore, proceeding as in the argument leading to (4.22), we obtain an estimate on $N_n(x_i, x_{i+1})$ as follows: For every K, α and $\varepsilon > 0$, there exists a constant $C = C_{\varepsilon, K, \alpha}$ such that for all $n \geq 1$,

$$\begin{aligned} &P\Big\{\sup_i \big|N_n(x_i, x_{i+1}) - (1 - \exp\{-\mu n^{-k}\})(C(x_{i+1}, n) + 2)\big| \\ &\qquad\qquad > \alpha n^{1/4}(\log n)^{3/4+\varepsilon},\ |\sigma_n - n\gamma^2| \leq n\varepsilon\Big\} \qquad (4.28) \\ &\quad \leq Cn^{-K}. \end{aligned}$$

But by (4.24) and Kesten's law of the iterated logarithm [see Kesten (1965)], with probability 1,

$$\sup_i C(x_i, n) = O\left(\sqrt{n \log\log n}\right). \qquad (4.29)$$

Since k in (4.28), that is, the mesh-size, is by assumption greater than 2, (4.28) and (4.29) together with (3.1) imply that with probability 1,

$$\sup_i N_n(x_i, x_{i+1}) = o\left(n^{1/4}(\log n)^{3/4+\varepsilon}\right) \quad \text{ultimately}.$$

In other words, with probability 1, for all i,

$$C(x_{i+1}, n) \le \inf_{x_i \le x \le x_{i+1}} C(x, n) + o\big(n^{1/4}(\log n)^{3/4+\varepsilon}\big). \tag{4.30}$$

Here the $o(\cdots)$ is independent of i.

On the other hand, $\{Z(t_j) - Z(t_{j+1}-);\ j \ge 0\}$ is an i.i.d. sequence of exponential random variables with mean μ^{-1}. Therefore, by an easy argument, with probability 1,

$$\min_{j \le n}\big(Z(t_j) - Z(t_{j+1}-)\big) \ge n^{-k+1} \quad \text{ultimately}.$$

This means that, up to the jth jump, every time Z goes through a point x, it must either have been through all points of the form, $x - \eta$, $0 \le \eta \le n^{-k+1}/2$, or all points of the form $x + \eta$, $0 \le \eta \le n^{-k+1}/2$. In particular, with probability 1, for all j,

$$\sup_{x_j \le x \le x_{j+1}} C(x, t_n) \le \max\{C(x_j, t_n), C(x_{j+1}, t_n)\} \quad \text{ultimately}.$$

By (4.25) and the strong law of large numbers, this means that with probability 1, for all j,

$$\sup_{x_j \le x \le x_{j+1}} C(x, t_n) \le C(x_{j+1}, t_n) + o\big(n^{1/4}(\log n)^{3/4+\varepsilon}\big) \quad \text{ultimately}. \tag{4.31}$$

Here the term $o(\cdots)$ is independent of j. Some renewal theory, together with (4.31) and (4.30), implies (4.26). Hence the proof of the theorem is complete. □

Actually, pushing these arguments a little further, one obtains a stronger result. However, we first generalize assumption (C) to

$$\int_0^\infty x^{2M} F(dx) < \infty \quad \text{for some } M \ge 2. \tag{D}$$

Then the following weak deviation theorem can be proved with little more difficulty than Theorem 4.5.

4.6 THEOREM. *If the conditions of the previous theorem are met, and if we further assume equation* (D), *then for all* $\alpha > 0$,

$$\limsup_{n\to\infty} n^M P\left\{\sup_{(x,t)\in R\times[0,1]} |L_t^x(Z_n) - L_t^x(B_n)| \ge \alpha n^{1/4}(\log n)^{3/4+\varepsilon}\right\} < \infty.$$

PROOF.

STEP 1. First get probability bounds on the modulus of continuity of L [see Revuz and Yor (1989) or McKean (1962)].

STEP 2. Proceed exactly as in the proof of Theorem 4.5, explicitly bounding the tail probability of σ_n, that is,

$$P\{|\sigma_n - n\gamma^2| \geq n\varepsilon\} \geq Cn^{-M}.$$

Indeed, it is the above that determines the rate of decrease of our deviation probabilities. □

4.7 COROLLARY. *Under the assumptions of Theorem* 4.5, *the following hold, with probability* 1:

(i) *For each fixed* $y \in \mathbb{R}$: $\limsup_{n\to\infty}(\log\log n)^{-1/2}L_1^x(Z_n) = \sqrt{2}$.

(ii) $\limsup_{n\to\infty}\sup_{x\in R}(\log\log n)^{-1/2}L_1^x(Z_n) = \sqrt{2}$.

(iii) $\liminf_{n\to\infty}\sup_{x\in R}(\log\log n)^{1/2}L_1^x(Z_n) = \sqrt{2}\rho_0$, *where* ρ_0 *is the smallest positive root of the Bessel function with index* 0.

(iv) *The function sequence* $\{x \to \sqrt{\log\log n}\, L_1^{(\log\log n)^{-1/2}x}(Z_n);\ n \geq 1\}$ *is relatively compact in* $C((-\infty,\infty))$ *equipped with the compact-open topology. Moreover, the set of its limit points is precisely* (*up to a null set*)

$$\left\{ f \in C((-\infty,\infty)) \colon \int \frac{(\dot{f})^2}{f} \leq 8, \int f \leq 1 \right\}.$$

PROOF. Statements (i) and (ii) are consequences of Theorem 4.5 and the first half of Kesten's law of the iterated logarithm for Brownian local times [see Kesten (1965) or see Khoshnevisan (1989) for an elementary proof]. Statement (iii) holds by Theorem 4.5 and the second half of Kesten's law of the iterated logarithm. The constant $\sqrt{2}\rho_0$ was determined in Csaki and Földes (1986). The final statement is from Theorem 4.5 and the result of Donsker and Varadhan (1977). □

We will state the following version of Theorem 4.5 which can be obtained by a more careful revision of our estimates:

4.8 THEOREM. *With everything as in Theorem* 4.5, *almost surely*

$$\limsup_{n\to\infty} n^{1/4}(\log n)^{-3/4} \sup_{t\in I}\sup_{x\in R} |L_t^x(Z_n) - L_t^x(B_n)| \leq 2^{1/4}.$$

In light of Proposition 4.4, this rate seems at least nearly optimal.

REMARK. The above proof is influenced by Révész (1981).

Acknowledgments. The author wishes to thank his dissertation advisor, Professor P. W. Millar, for his encouragement during the course of this work. Also we are grateful to Professor J. W. Pitman for his constant and invaluable help. Finally many thanks are due to the anonymous referee. As well as several suggestions and corrections, his insight helped remove an unnecessary condition in Theorem 4.5.

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