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LOCAL TIMES, OPTIMAL STOPPING AND SEMIMARTINGALES

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Let X be a semimartingale, and S its Snell envelope. Under the assumption that X and S are continuous semimartingales in H^1, this article obtains a new, maximal, characterisation of S, and gives an application to the optimal stopping of functions of diffusions. We present a counterexample to the standard assertion that S is just "a martingale on the go-region and X on the stop-region."

1. Introduction. It is well known that under suitable integrability conditions on the process X, living on the usual filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t), \mathbb{P})$, the process S, defined by

$$S_t = \operatorname*{ess\,sup}_{t \le \tau} \mathbb{E}[X_\tau | \mathscr{F}_t],$$

is the minimal supermartingale which dominates X. In this article we explore the possibility of finding another characterisation of S when X is a semimartingale.

Under the assumption that X and S are continuous semimartingales in H^1 [see Jacod (1979) for a definition of H^1], we obtain a new, maximal characterisation of S in terms of an "anticipative" SDE involving the local time of $S - X$ at 0 (Theorem 5), and give an application to the optimal stopping of continuous functions of diffusions, establishing new, sufficient conditions for the so-called smooth pasting condition to hold.

Finally, it is a standard assertion that S is just "a martingale on the go-region and X on the stop-region"—we take this to mean (at least in the case where X is a semimartingale) that

$$S_t = S_0 + \int_0^t 1_{((S-X)_{t-} > 0)}\, dM_s + \int 1_{((S-X)_{t-} = 0)}\, dX_s,$$

where M is a martingale. In Section 5 we give a counterexample to this assertion, exhibiting a process X—which is a continuous function of a Brownian motion—such that $S - X$ develops a nontrivial local time at 0.

2. Some preliminary results. Fix a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t), \mathbb{P})$ satisfying the usual conditions and a (special) semimartingale X living on $(\Omega, \mathscr{F}, (\mathscr{F}_t), \mathbb{P})$ with $X \in H^1$.

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We take the canonical decomposition of X:

$$X = M + A,$$

where M is a martingale and A is a predictable process of integrable variation, with $A_0 = 0$, and fix $T < \infty$.

Define $S = S^T$, where

$$S_t^T \stackrel{\text{def}}{=} \begin{cases} \operatorname*{ess\,sup}_{t \le \tau \le T} \mathbb{E}[X_\tau | \mathscr{F}_t]: t \le T, \\ X_T: t > T. \end{cases}$$

It is well known [see Dellacherie and Meyer (1980), Appendix] that S is the minimal supermartingale which dominates $(X_t^T) =_{\text{def}} (X_{t \wedge T})$, and it is obvious that, since $X \in H^1$, $S \in H^1$ and hence is a special semimartingale with decomposition

$$S = N + B$$

with B a predictable decreasing process, with $B_0 = 0$.

We shall assume from now on that

(1) $$A,\ B \text{ and } N' \stackrel{\text{def}}{=} N - M \text{ are all continuous.}$$

Note that if A is continuous, then the continuity of B and N' is assured if the filtration $(\mathscr{F}_t)$ is quasi-left-continuous. We show this as follows: It is not hard to see that there is a version of S such that for any stopping time $\sigma \le T$:

$$S_\sigma = M_\sigma + \operatorname*{ess\,sup}_{\sigma \le \tau \le T} \mathbb{E}[A_\tau | \mathscr{F}_\sigma] \quad \text{a.s.},$$

while if σ is a predictable stopping time and is announced by the sequence T_n:

$$\begin{aligned} S_{\sigma-} &= M_{\sigma-} + \lim_{n \to \infty} \operatorname*{ess\,sup}_{T_n \le \tau \le T} \mathbb{E}\left[A_\tau | \mathscr{F}_{T_n}\right] \\ &= M_{\sigma-} + \operatorname*{ess\,sup}_{\sigma \le \tau \le T} \mathbb{E}[A_\tau | \mathscr{F}_\sigma] \quad \text{a.s.}, \end{aligned}$$

by continuity of A and the quasi-left-continuity of the filtration. It follows from Meyer's predictable section theorem [Dellacherie and Meyer (1980)] that (there is a version of S such that)

$$S = M + S',$$

where S' is a continuous supermartingale, which thus has a decomposition

$$S' = N' + B,$$

with N' and B continuous.

We now prove that the "standard" optimal stopping times are indeed optimal.

DEFINITION. For each stopping time $\tau \ge 0$ and each $\varepsilon \ge 0$, define

$$\sigma_\tau^\varepsilon = \inf\{s \ge \tau: S_s \le X_s + \varepsilon\},$$

and set

$$\sigma_\tau = \sigma_\tau^0.$$

LEMMA 1. *Under condition* (1), *for each fixed* $t \in [0, T]$ σ_t *is t-optimal, that is,*

$$S_t = \mathbb{E}[X_{\sigma_t} | \mathscr{F}_t];$$

and so, since $S - X$ *is continuous,*

$$S_t = \mathbb{E}[S_{\sigma_t} | \mathscr{F}_t].$$

PROOF. First we show that if τ is any stopping time with $t \le \tau \le T$, then

$$\mathbb{E}[X_{\tau \wedge \sigma_t} | \mathscr{F}_t] \ge \mathbb{E}[X_\tau | \mathscr{F}_t]; \tag{2}$$

in other words, $\tau \wedge \sigma_t$ is at least as good a stopping time as τ.

Now

$$\begin{aligned}
&\mathbb{E}[X_{\tau \wedge \sigma_t} | \mathscr{F}_t] - \mathbb{E}[X_\tau | \mathscr{F}_t] \\
&\quad = \mathbb{E}[X_{\tau \wedge \sigma_t} - X_\tau | \mathscr{F}_t] \\
&\quad = \mathbb{E}[(X_{\sigma_t} - X_\tau) 1_{(\sigma_t \le \tau)} | \mathscr{F}_t] \\
&\quad \ge \mathbb{E}[(S_{\sigma_t} - S_\tau) 1_{(\sigma_t \le \tau)} | \mathscr{F}_t] \quad (\text{since } S \ge X, \text{ while } S_{\sigma_t} = X_{\sigma_t}) \\
&\quad = \mathbb{E}[S_{\tau \wedge \sigma_t} - S_\tau | \mathscr{F}_t] \\
&\quad \ge 0 \quad (\text{by the optional sampling theorem}),
\end{aligned}$$

establishing (2).

To prove the lemma, take a sequence (T_n) of $1/n$ optimal stopping times for S_t—so that

$$\mathbb{E}[X_{T_n} | \mathscr{F}_t] \ge S_t - 1/n \quad \text{and} \quad t \le T_n \le T, \qquad \forall\, n \ge 1,$$

and note that, from (2), we may assume wlog that $T_n \le \sigma_t$. Now

$$1/n \ge \mathbb{E}[S_t - X_{T_n} | \mathscr{F}_t] \ge \mathbb{E}[S_{T_n} - X_{T_n} | \mathscr{F}_t]$$

and

$$S_{T_n} \ge X_{T_n},$$

so it follows from Markov's inequality that

$$\mathbb{P}(S_{T_n} - X_{T_n} \ge n^{-1/2}) \le n^{-1/2}.$$

But

$$(T_n < \sigma_t^{n^{-1/2}}) \subseteq (S_{T_n} - X_{T_n} \ge n^{-1/2})$$

so

$$\mathbb{P}(\sigma_t^{n^{-1/2}} \le T_n \le \sigma_t) \overset{n \to \infty}{\to} 1.$$

Moreover, it follows from the continuity of $S - X$ that

$$\sigma_t^{n^{-1/2}} \to \sigma_t \quad \text{a.s.},$$

so

(3) $$T_n \xrightarrow{p} \sigma_t.$$

Noting that A^T is uniformly integrable, because $X \in H^1$, and that

$$A^T_{T_n} \xrightarrow{p} A^T_{\sigma_t}$$

[by virtue of (3)], we see that

$$\mathbb{E}\big[S_t - X_{\sigma_t} | \mathscr{F}_t \big] = \mathbb{E}\big[S'_t - A^T_{\sigma_t} | \mathscr{F}_t \big] = \lim_{n \to \infty} \mathbb{E}\big[S'_t - A^T_{T_n} | \mathscr{F}_t \big] \le 0 \quad \text{a.s.} \qquad \square$$

LEMMA 2. *Under condition* (1), *the process* $\tilde{N}$ *given by*

$$\tilde{N}_t = \int_0^{t \wedge T} 1_{((S-X)_{s-} > 0)} \, dS_s$$

is a martingale.

PROOF. Recall that $S = N + B$. It follows from Lemma 1 and the fact that N is a martingale that for any stopping time $\tau \le T$:

$$\begin{aligned} B_\tau &= S_\tau - N_\tau \\ &= \mathbb{E}\big[S_{\sigma_\tau} - N_{\sigma_\tau} | \mathscr{F}_\tau \big] \\ &= \mathbb{E}\big[B_{\sigma_\tau} | \mathscr{F}_\tau \big] \quad \text{a.s.} \end{aligned}$$

Since B is decreasing this means that

(4) $$B_\tau = B_{\sigma_\tau} \quad \text{a.s.}$$

Since B is predictable and, for each t, σ_t is a stopping time (and hence optional), we see that

$$\big\{ (\omega, t) \colon B_t(\omega) \ne B_{\sigma_t(\omega)}(\omega) \big\} \text{ is an optional set}$$

and so it follows from Meyer's optional section theorem and (4) that

$$\mathbb{P}\big(B_t = B_{\sigma_t}, \forall\, t \le T \big) = 1.$$

We deduce that B is a.s. constant on the maximal components of the open set

$$\{ t > 0 \colon (S - X)_{t-} > 0 \}.$$

Therefore,

$$\int_0^{t \wedge T} 1_{((S-X)_{s-} > 0)} \, dB_s = 0 \quad \text{a.s.},$$

and it follows that

$$\tilde{N}_t = \int_0^{t \wedge T} 1_{((S-X)_{s-} > 0)} \, dS_s$$
$$= \int_0^{t \wedge T} 1_{((S-N)_{s-} > 0)} \, dN_s \quad \text{a.s.},$$

so that $\tilde{N}$ is a local martingale. Finally, since $S \in H^1$, $\tilde{N} \in H^1$ so that $\tilde{N}$ is a martingale. □

From now on in this section we shall work with the pair (S', A^T), rather than the pair (S, X). All results may be translated into equivalent ones for (S, X) by recalling that

$$S_t = S'_t + M_t \quad \text{and} \quad X_t = A_t + M_t.$$

DEFINITION. In what follows we denote by A^- the decreasing component of A. We denote by $L^0_t(S' - A)$, or just L^0_t, the local time of $(S' - A)$ at 0, and we define

$$\mu_t \overset{\text{def}}{=} \frac{1}{2} \frac{dL^0_t}{dA^-_t}.$$

REMARK. Since A and L^0 are continuous, we can (and shall) take the unique version of μ which is predictable.

THEOREM 3. *The local time of $(S' - A)$ at 0, L^0, is, as a measure, absolutely continuous with respect to A^-, and μ satisfies*

$$0 \le \mu_t \le 1_{((S'-A)_t = 0)} \quad (A^- \; a.e.).$$

PROOF. Consider the process $Z =_{\text{def}} S' - A^T$. Since $S' \ge A^T$, $Z \ge 0$, so that, since Z is continuous:

$$Z_t = Z_0 + \int_0^t 1_{(Z_s > 0)} \, dZ_s + \int_0^t 1_{(Z_s = 0)} \, dZ_s$$
$$= Z_0 + \int_0^t 1_{(Z_s > 0)} \, dZ_s + \tfrac{1}{2} L^0_t(Z)$$

(from Tanaka's formula [see Azéma and Yor (1978)]).

Thus

$$S'_t = Z_t + A^T_t$$
$$(5) \qquad = S'_0 + \int_0^t 1_{(Z_s > 0)} \, dS'_s + \int_0^t 1_{(Z_s = 0)} \, dA^T_s + \tfrac{1}{2} L^0_t(Z)$$
$$= S'_0 + \tilde{N}_t + \int_0^t 1_{(Z_s = 0)} \, dA^T_s + \tfrac{1}{2} L^0_t(Z),$$

where, from Lemma 2, $\tilde{N}$ is a continuous martingale. But S' is a special semimartingale, so the right-hand side of (5) is its canonical decomposition and we may conclude that $N' = \tilde{N} + S'_0$ and

$$\int_0^t 1_{(Z_s=0)}\, dA_s^T + \tfrac{1}{2} L_T^0(Z)$$

is the decreasing process B. But $L_t^0(Z)$ is an increasing process so $-\int_0^t 1_{(Z_s=0)}\, dA_s^T$ must be an increasing process and

$$\tfrac{1}{2} L_\cdot^0(Z) \ll -\int_0^\cdot 1_{(Z_s=0)}\, dA_s^T,$$

with Radon-Nikodym derivative $\mu \le 1$. □

COROLLARY 4. *There is a predictable process μ as in Theorem* 3, *with* $0 \le \mu \le 1$ *such that* $\mu = 0$ *except on* $\operatorname{supp}(A^-)$, *and, for* $0 \le t \le T$,

$$(6) \qquad S'_t = A_t + \mathbb{E}\left(\int_t^T 1_{((S'-A^T)_s > 0)}\, dA_s + \int_t^T \mu_s 1_{((S'-A^T)_s = 0)}\, dA_s \middle| \mathcal{F}_t \right).$$

PROOF. From (5) and Theorem 3,

$$Z_t = Z_0 + \tilde{N}_t - \int_0^t 1_{(Z_s>0)}\, dA_s^T - \int_0^t \mu_s 1_{(Z_s=0)}\, dA_s^T,$$

but $Z_T = 0$ so

$$\begin{aligned} Z_t &= \mathbb{E}[Z_t - Z_T | \mathcal{F}_t] \\ &= \mathbb{E}\left(\tilde{N}_t - \tilde{N}_T + \int_t^T 1_{(Z_s>0)}\, dA_s^T + \int_t^T \mu_s 1_{(Z_s=0)}\, dA_s^T \middle| \mathcal{F}_t \right). \end{aligned}$$

Equation (6) now follows on recalling that $\tilde{N}$ is a martingale. □

REMARKS. 1. It is (6) which we term an anticipative SDE. An associate editor informs us that such equations also arise in finance theory and filtering theory, where they are termed "forward–backward" equations.

2. The proof of Corollary 4 also establishes that

$$\begin{aligned} \tilde{N}_t &= \mathbb{E}\left(\int_0^T 1_{(Z_s>0)}\, dA_s^T + \int_0^T \mu_s 1_{(Z_s=0)}\, dA_s^T \middle| \mathcal{F}_t \right) - Z_0 \\ &= \mathbb{E}\left(\int_0^T 1_{(Z_s>0)}\, dA_s^T - \tfrac{1}{2} L_T^0 \middle| \mathcal{F}_t \right) - Z_0. \end{aligned}$$

3. A maximal characterisation of S and a uniqueness result. We now consider solution pairs $(\tilde{S}, \tilde{\mu})$ to the discontinuous version of the anticipative SDE (6).

THEOREM 5. *Suppose that* (W, ν) *are a solution pair to the discontinuous version of* (6)*—that is,* ν *is a predictable process and*

$$(7)\qquad W_t = A_t + \mathbb{E}\left(\int_t^T 1_{((W-A^T)_{s-}>0)}\,dA_s + \int_t^T \nu_s 1_{((W-A^T)_{s-}=0)}\,dA_s \Big| \mathscr{F}_t\right)$$

for all $t \in [0, T]$*—then* [*by virtue of* (7)] *we can* (*and shall*) *assume that* W *is right-continuous and*:

(i) $W \geq A^T$ *a.s.*,
(ii) $W \leq S'$ *a.s.*,
(iii) *if* W *is a supermartingale, then* $W = S'$ *a.s.*,
(iv) *if*

$$(8)\qquad 0 \leq \nu_s \leq 1 \quad \text{and} \quad \{s: (W - A^T)_{s-} = 0\} \subseteq \operatorname{supp}(A^-),$$

then $W = S'$ *a.s.*

In other words, any solution of (7) *dominates* A^T; S' *is the maximal solution of* (7) *and the unique supermartingale solution of* (7); *and* (S', μ) *is the unique solution pair of* (7) *which also satisfies* (8).

PROOF. (i) Given $t \in [0, T]$, define $\tau_t = \inf\{s \geq t\colon W_s \geq A_s\}$; notice that $\tau_t \leq T$ (because $W_T = A_T$) and that, since W is cadlag, $W_{\tau_t} \geq A_{\tau_t}$. Now on $[t, \tau_t)$, $W_. < A_.$, so, since A is continuous,

$$\begin{aligned} W_t &= A_t + \mathbb{E}\left(\int_{\tau_t}^T 1_{((W-A^T)_{s-}>0)}\,dA_s + \int_{\tau_t}^T \nu_s 1_{((W-A^T)_{s-}=0)}\,dA_s \Big| \mathscr{F}_t\right) \\ &= A_t + \mathbb{E}\left[W_{\tau_t} - A_{\tau_t} | \mathscr{F}_t\right] \geq A_t \quad \text{a.s.} \end{aligned}$$

Right-continuity of W and A then implies that

$$\mathbb{P}[W_t \geq A_t, \forall\, t \in [0, T]] = 1.$$

(ii) If we now define $\tau_t = \inf\{s \geq t\colon W_s = A_s\}$, then $\tau_t \leq T$ and by right-continuity $W_{\tau_t} = A_{\tau_t}$, while we see from (i) that, on $[t, \tau_t)$, $W_. > A_.$, so, by continuity of A:

$$\begin{aligned} W_t &= A_t + \mathbb{E}\left(\int_t^{\tau_t} dA_s + \int_{\tau_t}^T 1_{((W-A^T)_{s-}>0)}\,dA_s + \int_{\tau_t}^T \nu_s 1_{((W-A^T)_{s-}=0)}\,dA_s \Big| \mathscr{F}_t\right) \\ &= \mathbb{E}\left[W_{\tau_t} | \mathscr{F}_t\right] = \mathbb{E}\left[A_{\tau_t} | \mathscr{F}_t\right]. \end{aligned}$$

Thus for every $t \in [0, T]$ there is a stopping time $\tau_t \in [t, T]$ with $W_t = \mathbb{E}[A_{\tau_t} | \mathscr{F}_t]$, so, by the definition of S' and right-continuity of S and W, $W \leq S'$ a.s.

(iii) It follows from (i) and (ii) that

$$(9)\qquad A^T \leq W \leq S',$$

but Snell's criterion tells us that if W is a supermartingale satisfying (9), then $W = S'$.

(iv) Suppose (W, ν) satisfy both (7) and (8), then for all $t \in [0, T]$:

$$(10)\quad \begin{aligned} W_t = A_t &- \left(\int_0^t 1_{((W-A^T)_{s-}>0)}\, dA_s + \int_0^t \nu_s 1_{((W-A^T)_{s-}=0)}\, dA_s \right) \\ &+ \mathbb{E}\left(\int_0^T 1_{((W-A^T)_{s-}>0)}\, dA_s + \int_0^T \nu_s 1_{((W-A^T)_{s-}=0)}\, dA_s \middle| \mathscr{F}_t \right). \end{aligned}$$

So

$$W_t = A_0 + \tilde{M}_t + \int_0^t (1 - \nu_s) 1_{((W-A^T)_{s-}=0)}\, dA_s$$

[where $(\tilde{M}_t)$ is the martingale given by the second term on the right-hand side of (10)], but, from (8), $1_{((W-A^T)_{s-}=0)} \geq 0$ and is 0 unless $s \in \operatorname{supp}(A^-)$, and so

$$W_t = A_0 + \tilde{M}_t - \int_0^t (1 - \nu_s) 1_{((W-A^T)_{s-}=0)}\, dA_s^-.$$

It follows that, since $\nu \leq 1$, W is a supermartingale and hence, by (iii), $W = S'$ a.s. □

4. Remarks on the foregoing and some corollaries.

4.1. *Remarks on Theorem* 5. 1. The process A^T itself satisfies (7) with $\nu = 0$, and it follows from Theorem 5(i) that it is the *minimal* solution. It is interesting to ask under what conditions (if any are needed) are all semimartingales Y satisfying $A^T \leq Y \leq S'$ of the form

$$Y_t = \mathbb{E}\left[A^T_{\tau_t} \middle| \mathscr{F}_t \right] \quad \text{for some increasing collection of stopping times } \{\tau_t\}.$$

2. It should be possible to construct a process A such that $W_t =_{\text{def}} \sup_{s \in [t,T]} \mathbb{E}[A_s | \mathscr{F}_t]$ satisfies (7) but $A^T \neq W \neq S'$.

3. It is possible to show that if (W, ν) satisfy (7), then

$$\nu_s 1_{((S'-A^T)_{s-}=0)} \leq \mu_s 1_{((S'-A^T)_{s-}=0)} \quad (A^- \text{ a.e.}).$$

4.2. *Convergence as* $T \uparrow \infty$. If $X \in H^1$, then A is closed on the right as a continuous process of integrable variation, so by mapping $[0, \infty]$ to $[0, 1]$ we can see that $\tilde{S}$, given by

$$\tilde{S}_t = \operatorname*{ess\,sup}_{t \leq \tau \leq \infty} \mathbb{E}[A_\tau | \mathscr{F}_t],$$

satisfies (7) (with $T = \infty$) and is a supermartingale; more generally, we may apply the results of Theorems 3 and 5 and Corollary 4 to $\tilde{S}$. Moreover, since $S'(A^T)$ increases with T and is bounded above by $\tilde{S}$, we see that $S^\infty =_{\text{def}} \lim_{T \uparrow \infty} S'(A^T)$ is a supermartingale which dominates A and hence $\tilde{S} = S^\infty$.

4.3. *Some corollaries for Markov processes.* Suppose that $(\xi_t;\ t \geq 0)$ is a diffusion in $\mathbb{R}^d$ and $X_t = e^{-\alpha t} g(\xi_t, T - t)$, where g is a continuous function. It is clear that

$$S_t = e^{-\alpha t} f(\xi_t, T - t)$$

for a suitable continuous function f, with $f \geq g$ [see, e.g., Krylov (1980)]. Because f and g are continuous, it is apparent that $L^0(S-X)$ will only increase when $\xi_t \in \partial D$, where

$$D \stackrel{\text{def}}{=} \{(x,s)\colon f(x,s) > g(x,s)\},$$

and hence we may deduce the following theorem.

THEOREM 6. *If ξ, f and g are as above and there exist (deterministic) measures m_1 and m_2 such that:*

(i) *ξ has a density ρ with respect to m_1,*
(ii) *$dA^- \ll dm_2$,*
(iii) *$m_1 \otimes m_2(\partial D) = 0$,*

then $L^0(S-X)$ is indistinguishable from 0.

PROOF. It follows from the strong Markov property that μ [as in (6)] and $K =_{\text{def}} dA_s^-/dm_2(s)$ are of the form $\mu_s = \mu(\xi_s, T-s)$ and $K_s = e^{-\alpha t}k(\xi_s, T-s)$ respectively—so denoting the t-section of ∂D by $(\partial D)_t$,

$$\begin{aligned}
\mathbb{E}L_t^0(S-X) &= \mathbb{E}\left(\int_0^t \mu_s 1_{((S-X)_s>0)}\, dA_s^-\right)\\
&= \mathbb{E}\left(\int_0^t \mu_s 1_{((\xi_s, T-s)\in \partial D)}\, dA_s^-\right)\\
&= \int_0^t\int_{(\partial D)_s} e^{-\alpha s}\mu(a, T-s)\rho(\xi_0, a; s)k(a, T-s)\, dm_1(a)\, dm_2(s)\\
&= \int_{\partial D\cap(\mathbb{R}^d\times[0,t])} e^{-\alpha s}\mu(a,T-s)\rho(\xi_0, a; s)\\
&\qquad\qquad\times k(a, T-s)\, d(m_1\otimes m_2)(a,s)\\
&= 0.
\end{aligned}$$
□

REMARK. Under the conditions of Theorem 6, we may obtain the following representation for f:

$$\begin{aligned}
f(\xi_0, T) &= A_0 + \mathbb{E}\left(\int_0^T 1_{((\xi_s, T-s)\in D)}\, dA_s\right)\\
&= \mathbb{E}\left(A_T - \int_0^T 1_{((\xi_s, T-s)\in D^c)}\, dA_s\right)\\
&= \mathbb{E}\left(e^{-\alpha T}g(\xi_T, 0) + \int_0^T e^{-\alpha s}k(\xi_s, T-s)\, ds\right)\\
&= \int_{\mathbb{R}^d} e^{-\alpha T}\rho(\xi_0, a; T)g(a,0)\, dm_1(a)\\
&\quad + \int_{D^c\cap(\mathbb{R}^d\times[0,T])} e^{-\alpha s}\rho(\xi_0, a; s)k(a, T-s)\, d(m_1\otimes m_2)(a,s).
\end{aligned}$$

The following corollary is then immediate.

COROLLARY 7. *Suppose the conditions of Theorem* 6 *are satisfied. If* ρ *is* C^1 *in* ξ *with derivatives which are uniformly continuous in* $\mathbb{R}^d \times [t_0, t_1]$ *for any* $0 < t_0 < t_1 < \infty$, *then* f *is* C^1 *in* ξ *for all* $t > 0$.

REMARK. This is the smooth pasting condition; see, for example, Krylov (1980) or Jacka and Lynn (1990) for applications.

5. A counterexample. The foregoing analysis might lead one to speculate (as the author did) that $\mathscr{L}_t =_{\text{def}} L_t^0(S' - A)$ is always 0. Unfortunately, the following counterexample shows that this is not so.

5.1. *The counterexample.* Let $(B_t;\ t \geq 0)$ be a Brownian motion on the filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t), \mathbb{P})$ with $\mathscr{F}_t = \sigma(B_s;\ s \leq t)$ and $\mathscr{F} = \mathscr{F}_\infty$. Let $L_t^b =_{\text{def}} L_t^b(B)$ and define

$$A_t = L_t^a - L_t^{-a},$$

for some fixed $a > 0$. As usual,

$$S_t = \operatorname*{ess\,sup}_{t \leq \tau \leq T} \mathbb{E}[A_\tau | \mathscr{F}_t].$$

Since

$$X_t \overset{\text{def}}{=} |B_t - a| - |B_t + a|$$

differs from A_t by the martingale $\int_0^t \operatorname{sgn}(B_s - a)\,dB_s - \int_0^t \operatorname{sgn}(B_s + a)\,dB_s$, the optimal stopping problem is equivalent to that of optimally stopping X_t. Now $X_t = f(B_t)$, where $f(x) = |x - a| - |x + a|$, and so, since f is bounded above by $2a$ and achieves this maximum at any $x \leq -a$, we should always stop A if $B \leq -a$. Conversely, if $B_t > -a$, then, defining $\tau_t = \inf\{s \geq t\colon B_s \leq -a\} \wedge T$,

$$\mathbb{E}[A_{\tau_t} | \mathscr{F}_t] > A_t,$$

since there is a positive probability that L^a will increase on $[t, \tau_t]$, while L^{-a} will not increase on $[t, \tau_t]$ since L^{-a} is continuous and only increases when $B = -a$. It follows that the optimal stopping policy is to stop the first time that $B_t \leq -a$.

We may now evaluate S explicitly, but we do not need to do so in order to realise that $\mathscr{L}_T$ may be strictly positive—to see this, apply (6) at $t = 0$; we obtain

$$\begin{aligned} S_0 &= \mathbb{E}\left(\int_0^T 1_{(S_s - A_s > 0)}\, dA_s - \tfrac{1}{2}\mathscr{L}_T\right) \\ &= \mathbb{E}\left(L_T^a - \tfrac{1}{2}\mathscr{L}_T\right). \end{aligned}$$

But $S_0 = \mathbb{E} L_{\tau_0}^a$ so

$$\mathbb{E}\mathscr{L}_T = 2\mathbb{E}\left[L_T^a - L_{\tau_0}^a\right] > 0.$$

5.2. *Its local time*. As we indicated above it is possible to compute S explicitly—in fact, it is easier to compute $\tilde{S} =_{\mathrm{def}} S(X)$. A little calculation should convince you that $\tilde{S}_t = f(B_t, T - t)$, where

$$f(x,t) = 2\left((2a + x) + (a - x)\Phi\left(\frac{(a-x)}{t^{1/2}}\right) - (3a + x)\Phi\left(\frac{(3a+x)}{t^{1/2}}\right)\right)$$
$$+\left(\frac{t}{2\pi}\right)^{1/2}\left(\exp\left(\frac{-(a-x)^2}{2t}\right) - \exp\left(\frac{(3a+x)^2}{2t}\right)\right)$$

for $x \geq -a$, and

$$f(x,t) = 2a$$

for $x \leq -a$ (Φ is, of course, the standard normal distribution function).

A quick check shows that f is piecewise $C^{2,1}$ but that it has a discontinuity in the first spatial derivative at $x = -a$ of $2 - 4\Phi(2a/t^{1/2})$, while (as we would expect)

$$\frac{1}{2}\frac{\partial^2 f}{\partial x^2} - \frac{\partial f}{\partial t} = 0 \quad \text{for } x \neq -a.$$

It follows, from an easily proved generalisation of Itô's formula, that

$$\tilde{S}_t = \tilde{S}_0 + N_t - \int_0^t 2\left(2\Phi\left(2a/(T-s)^{1/2}\right) - 1\right) dL_s^{-a},$$

where N is a martingale started at 0, so that

$$\mathscr{L}_T = 4\int_0^t \left(1 - \Phi\left(2a/(T-s)^{1/2}\right)\right) dL_s^{-a}.$$

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