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CLUSTERS OF A RANDOM WALK ON THE PLANE

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Let $r(n)$ be the radius of the largest disc covered by $S(1), \dots, S(n)$, where $\{S(k);\ k = 1, 2, \dots\}$ is the simple symmetric random walk on Z^2. The main result tells us that $r(n) \geq n^{1/50}$ a.s. for all but finitely many n.

Dedicated to Professor P. Erdös on the occasion that he is 2^{12} weeks old.

1. Introduction. Let $X_1, X_2, \dots$ be a sequence of independent, identically distributed random vectors taking values from Z^2 with distribution

$$\mathbf{P}\{X_1 = (0,1)\} = \mathbf{P}\{X_1 = (0,-1)\} = \mathbf{P}\{X_1 = (1,0)\}$$
$$= \mathbf{P}\{X_1 = (-1,0)\} = 1/4$$

and let

$$S_0 = 0 = (0,0) \quad \text{and} \quad S_n = S(n) = X_1 + X_2 + \cdots + X_n, \qquad n = 1, 2, \dots,$$

that is, $\{S_n\}$ is the simple symmetric random walk on the plane. Further, let

$$\xi(x,n) = \#\{k\colon 0 < k \leq n,\ S_k = x\}$$

$(n = 1, 2, \dots;\ x = (i,j);\ i, j = 0, \pm 1, \pm 2, \dots)$ be the local time of the random walk. We say that the disc

$$Q(N) = \left\{x = (i,j)\colon \|x\| = \left(i^2 + j^2\right)^{1/2} \leq N\right\}$$

is covered by the random walk in time n if

$$\xi(x,n) > 0 \quad \text{for every } x \in Q(N).$$

Let $R(n)$ be the largest integer for which $Q(R(n))$ is covered in n. We quote the known properties of $R(n)$.

THEOREM A [Erdös and Révész (1988), Révész (1989, 1990) and Auer (1990)]. *For any $0 < \varepsilon < 1$, $C > 0$ and $z \in R^+$, we have*

$$R(n) \leq \exp\left(2(\log n)^{1/2} \log_3 n\right) \quad a.s., \tag{1}$$

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for all but finitely many n,

$$R(n) \geq \exp\left(\frac{1-\varepsilon}{\sqrt{120}} (\log n \log_3 n)^{1/2} \right) \quad i.o.\ a.s., \tag{2}$$

$$R(n) \leq \exp\left(C(\log n)^{1/2} \right) \quad i.o.\ a.s., \tag{3}$$

$$R(n) \geq \exp\left((\log n)^{1/2} (\log_2 n)^{-1/2-e} \right) \quad a.s., \tag{4}$$

for all but finitely many n,

$$\begin{aligned} \exp(-120z) &\leq \liminf_{n\to\infty} \mathbf{P}\left\{ \frac{(\log R(n))^2}{\log n} > z \right\} \\ &\leq \limsup_{n\to\infty} \mathbf{P}\left\{ \frac{(\log R(n))^2}{\log n} > z \right\} \leq \exp\left(-\frac{z}{4} \right). \end{aligned} \tag{5}$$

The last inequality suggests the following.

CONJECTURE 1.

$$\lim_{n\to\infty} \mathbf{P}\left\{ \frac{(\log R(n))^2}{\log n} > z \right\} = \exp(-\lambda z), \qquad 0 \leq z < \infty,$$

with some $1/4 \leq \lambda \leq 120$.

In the present paper we intend to investigate the radius of the largest disc (not necessarily around the origin) covered by the random walk in time n. Formally speaking, let $u = (u_1, u_2) \in Z^2$ and define

$$Q(u, N) = \left\{ x = (i, j) \colon \|x - u\|^2 = (i - u_1)^2 + (j - u_2)^2 \leq N^2 \right\}.$$

Let $r(n)$ be the largest integer for which there exists a random vector $u = u(n) \in Z^2$ such that $Q(u, r(n))$ is covered by the random walk in time n, that is,

$$\xi(x, n) \geq 1 \quad \text{for every } x \in Q(u, r(n)).$$

Then we formulate the following theorem.

THEOREM 1. *We have*

$$r(n) \geq n^{1/50} \quad a.s.,$$

for all but finitely many n.

REMARK 1. The author can also prove that

$$r(n) \leq n^{0.42} \quad \text{a.s.},$$

for all but finitely many n. The proof of this will be published elsewhere.

Theorem 1 together with Remark 1 suggests the following.

CONJECTURE 2. There exists a $1/50 \le q_0 \le 0.42$ such that

$$\lim_{n\to\infty} \frac{\log r(n)}{\log n} = q_0 \quad \text{a.s.}$$

REMARK 2. Theorem A tells us that $R(n)$ is about $\exp((\log n)^{1/2})$. The above theorem claims that $r(n)$ is much bigger than $R(n)$.

Inequality (4) was proved by Auer (1990). In fact, he proved the following stronger theorem.

THEOREM B. *For any* $0 < \varepsilon < 1/2$ *we have*

$$\lim_{n\to\infty} \sup_{\|x\| \le g_\varepsilon(n)} \left| \frac{\xi(x,n)}{\xi(0,n)} - 1 \right| = 0 \quad a.s., \tag{6}$$

where

$$g_\varepsilon(n) = \exp\Big((\log n)^{1/2} (\log_2 n)^{-1/2-\varepsilon}\Big).$$

Note that:

(i) since $\lim_{n\to\infty} \xi(0,n) = \infty$ a.s., (6) is indeed stronger than (4);
(ii) (6) tells us that the disc of radius $g_\varepsilon(n)$ around the origin is "homogeneously" covered.

In the present paper we prove that there exists a "nearly" homogeneously covered disc of radius $n^{1/50}$. In fact, we have the following theorem.

THEOREM 2. *There exist a sequence of random vectors* $u = u(n) \in Z^2$, $n = 1, 2, \ldots,$ *and an* $\varepsilon > 0$ *such that*

$$\limsup_{n\to\infty} \sup_{\|x-u\| \le n^{1/50}} \left| \frac{\xi(x,n)}{\xi(u,n)} - 1 \right| \le 1 - \varepsilon \quad a.s.$$

and

$$\liminf_{n\to\infty} \frac{\xi(u,n)}{\log^2 n} \ge \varepsilon \quad a.s.$$

Observe that Theorem 2 implies Theorem 1.

2. Proof of Theorem 2. At first we present a few known lemmas.

LEMMA A [Erdös and Taylor (1960), (3.6)]. *For any $z > 0$ and $\varepsilon > 0$ there exists an $n_0 = n_0(\varepsilon, z)$ such that*

$$\mathbf{P}\{\xi(0,n) \geq z(\log n)^2\} \geq \frac{\max(-z\pi \log n)}{(\log n)^{2z\pi(1+\varepsilon)}}$$

if $n \geq n_0$.

Introduce the following notation:

$$\begin{aligned}
\rho(0) &= 0,\\
\rho(1) &= \min\{j: j > 0,\ S_j = 0\},\\
\rho(i+1) &= \min\{j: j > \rho(i),\ S_j = 0\}, \qquad i = 1,2,\dots,\\
\alpha_i(x) &= \xi(x,\rho(i)) - \xi(x,\rho(i-1)), \qquad i = 1,2,\dots;\ x \in Z^2,\\
\beta_i(x) &= \alpha_i(x) - 1,\\
p(0 \rightsquigarrow x) &= \mathbf{P}\{\alpha_1(x) > 0\}, \qquad x \in Z^2.
\end{aligned}$$

LEMMA B [Spitzer (1964), P5, page 117, and P3, pages 124 and 125]. *Let $\{x_n\}$ be a sequence in Z^2 with $\lim_{n\to\infty}\|x_n\| = \infty$. Then*

$$\lim_{n\to\infty} (\log\|x_n\|)p(0 \rightsquigarrow x_n) = \frac{\pi}{4}.$$

LEMMA C [Petrov (1975), Theorem 16, page 54]. *Let $X_1, X_2, \dots, X_N$ be independent r.v.'s and put $S_N = X_1 + X_2 + \cdots + X_N$. Suppose that there exist positive constants $g_1, g_2, \dots, g_N$ and T such that*

$$\mathbf{E}\exp(tX_k) \leq \exp\left(\frac{g_k}{2}t^2\right), \qquad k = 1,2,\dots,N;\ |t| \leq T.$$

Then

$$\mathbf{P}\{|S_N| \geq z\} \leq 2\exp\left(-\frac{z^2}{2G}\right),$$

for any $0 \leq z \leq GT$, where $G = g_1 + g_2 + \cdots + g_N$.

LEMMA D [Auer (1990)].

$$\begin{aligned}
\mathbf{P}\{\beta_1(x) = -1\} &= q(x) = 1 - p(0 \rightsquigarrow x),\\
\mathbf{P}\{\beta_1(x) = k\} &= (p(0 \rightsquigarrow x))^2(q(x))^k, \qquad k = 0,1,2,\dots,\\
\mathbf{E}\beta_1(x) &= 0,\\
\sigma^2(x) = \mathbf{E}\beta_1^2(x) &= \frac{2(1 - p(0 \rightsquigarrow x))}{p(0 \rightsquigarrow x)},\\
\mathbf{E}\exp(t\beta_1(x)) &\leq \exp\left(\frac{2t^2}{p(0 \rightsquigarrow x)}\right), \qquad |t| \leq \frac{p(0 \rightsquigarrow x)}{2}.
\end{aligned}$$

Now we prove a few simple consequences of the above lemmas.

LEMMA 1. *Let $S_1(n), S_2(n), \ldots$ be a sequence of independent random walks on Z^2. Let $\xi_i(x, n)$ be the local time of $S_i(n)$. Further, let $\alpha, \beta > 0$. Then for any $u > 0$ and $\varepsilon > 0$ there exists an $n_0 = n_0(\varepsilon, u)$ such that for any $n \geq n_0$ we have*

$$\mathbf{P}\left\{\max_{1\le i\le n^\alpha} \xi_i(0, n^\beta) < u(\log n^\beta)^2\right\} \le \exp\left(-\frac{n^{\alpha - u\pi\beta}}{(\log n^\beta)^{2u\pi(1+\varepsilon)}}\right). \tag{7}$$

PROOF. By Lemma A we have

$$\begin{aligned}\mathbf{P}\left\{\max_{1\le i\le n^\alpha} \xi_i(0, n^\beta) < u(\log n^\beta)^2\right\} &\le \left(1 - \frac{\exp(-u\pi\log n^\beta)}{(\log n^\beta)^{2u\pi(1+\varepsilon)}}\right)^{n^\alpha}\\ &\le \exp\left(-\frac{n^\alpha\exp(-u\pi\log n^\beta)}{(\log n^\beta)^{2u\pi(1+\varepsilon)}}\right)\\ &= \exp\left(-\frac{n^{\alpha-u\pi\beta}}{(\log n^\beta)^{2u\pi(1+\varepsilon)}}\right).\end{aligned}$$

Hence we have the lemma. □

LEMMA 2. *For any $\varepsilon > 0$ there exists a $K_0 = K_0(\varepsilon) > 0$ such that*

$$\mathbf{P}\{|\beta_1(x) + \beta_2(x) + \cdots + \beta_N(x)| \ge \delta\sigma(x)N^{3/4}\} \le \exp\left(-(1-\varepsilon)\frac{\delta^2}{4}N^{1/2}\right)$$

provided that

$$K_0 \le \|x\| \le \exp(\theta N^{1/2}), \qquad \theta > 0, \qquad 0 < \delta < \sqrt{\frac{\pi}{(2+\varepsilon)\theta}}.$$

PROOF. Apply Lemma C with

$$X_k = \beta_k(x), \qquad T = \frac{1}{2}p(0 \rightsquigarrow x), \qquad g_k = \frac{4}{p(0 \rightsquigarrow x)},$$

$$z = \delta\sigma(x)N^{3/4}, \qquad G = \frac{4N}{p(0 \rightsquigarrow x)}.$$

Then by Lemmas B and D, for any $\varepsilon > 0$ we obtain

$$0 \le z = \delta\sigma(x)N^{3/4} = \delta\sqrt{\frac{2(1-p(0 \rightsquigarrow x))}{p(0 \rightsquigarrow x)}}\,N^{3/4}$$

$$\le \delta N^{3/4}\sqrt{\frac{2}{p(0 \rightsquigarrow x)}} \le \delta N\sqrt{\frac{(8+\varepsilon)\theta}{\pi}} \le 2N = GT$$

if

$$\delta\sqrt{\frac{(8+\varepsilon)\theta}{\pi}} \le 2.$$

Further,

$$\frac{z^2}{2G} = \frac{\delta^2\sigma^2(x)N^{3/2}}{2\dfrac{4N}{p(0 \rightsquigarrow x)}} = \frac{\delta^2}{8}2(1-p(0 \rightsquigarrow x))N^{1/2} \ge \frac{\delta^2}{4}(1-\varepsilon)N^{1/2}.$$

Hence Lemma C implies Lemma 2. □

LEMMA 3. *For any $K > 0$, $\delta > 0$, $\varepsilon > 0$, there exist an $L = L(K, \varepsilon) > 0$ and an $N_0 = N_0(K, \delta, \varepsilon)$ such that*

$$\mathbf{P}\{|\beta_1(x) + \beta_2(x) + \cdots + \beta_N(x)| \ge L\delta N^{3/4}\} \le \exp\left(-(1-\varepsilon)\frac{\delta^2}{4}N^{1/2}\right)$$

provided that

$$\|x\| \le K \quad and \quad N \ge N_0.$$

PROOF. Apply again Lemma C with

$$X_k = \beta_k(x), \qquad T = \frac{1}{2}p(0 \rightsquigarrow x), \qquad g_k = \frac{4}{p(0 \rightsquigarrow x)},$$

$$z = \delta L N^{3/4}, \qquad G = \frac{4N}{p(0 \rightsquigarrow x)}.$$

Then

$$0 < z \le 2N = GT, \quad \text{for any } L > 0 \text{ if } N \text{ is large enough,}$$

and

$$\frac{z^2}{2G} \ge (1-\varepsilon)\frac{\delta^2}{4}N^{1/2} \quad \text{if } L \text{ is large enough.}$$

Hence by Lemma C we have Lemma 3. □

LEMMA 4. *For any*

$$\varepsilon > 0, \qquad 0 < \delta < \sqrt{\frac{\pi}{(2+\varepsilon)\theta}}, \qquad \theta > 0,$$

there exist an $N_0 = N_0(\varepsilon, \delta)$ *and an* $L = L(\varepsilon)$ *such that*

$$\mathbf{P}\left\{ \max_{\|x\| \le e^{\theta\sqrt{N}}} \frac{|\beta_1(x) + \beta_2(x) + \cdots + \beta_N(x)|}{\max(\sigma(x), L)} \ge \delta N^{3/4} \right\} \le \exp\left(-\left(\frac{\delta^2}{4}(1-\varepsilon) - 2\theta \right) N^{1/2} \right)$$

provided that $N \ge N_0$.

PROOF. Lemma 4 is a trivial consequence of Lemmas 2 and 3. □

LEMMA 5. *Let* $S_1(n), S_2(n), \dots$ *be an arbitrary sequence of random walks on* Z^2. *Define the sequence* $\beta_{i1}(x), \beta_{i2}(x), \dots,\ i = 1, 2, \dots,$ *via* $S_i(n)$ *in the same way as the sequence* $\beta_1(x), \beta_2(x), \dots$ *was defined via* $S(n)$. *Then for any* $\varepsilon > 0$ *there exist an* $N_0 = N_0(\varepsilon, \delta) > 0$ *and an* $L = L(\varepsilon)$ *such that*

$$\mathbf{P}\left\{ \max_{1 \le i \le e^{\gamma\sqrt{N}}} \max_{\|x\| \le e^{\theta\sqrt{N}}} \left| \frac{\beta_{i1}(x) + \beta_{i2}(x) + \cdots + \beta_{iN}(x)}{\max(\sigma(x), L)} \right| \ge \delta N^{3/4} \right\} \le \exp\left(-\left(\frac{\delta^2}{4}(1-\varepsilon) - 2\theta - \gamma \right) N^{1/2} \right) \tag{8}$$

provided that

$$0 < \delta < \left(\frac{\pi}{(2+\varepsilon)\theta} \right)^{1/2}, \qquad \gamma > 0, \qquad \theta > 0, \qquad N \ge N_0.$$

PROOF. Lemma 5 is a trivial consequence of Lemma 4. □

Define the sequence $\{\rho_i(N);\ N = 1, 2, \dots\}$ via $S_i(n)$ in the same way as the sequence $\{\rho(N);\ N = 1, 2, \dots\}$ was defined via $S(n)$ and let $\xi_i(\cdot, \cdot)$ be the local time of $S_i(\cdot)$. Assume that

$$2\theta + \gamma < \frac{\delta^2}{4} < \frac{\pi}{(8 + 4\varepsilon)\theta}. \tag{9}$$

Since by Lemmas D and B for $\|x\| \le e^{\theta\sqrt{N}}$,

$$\max(\sigma(x), L) \le \max\left(\sqrt{\frac{2}{p(0 \rightsquigarrow x)}}, L \right) \le \sqrt{\frac{8\theta\sqrt{N}}{\pi}(1+\varepsilon)},$$

by Lemma 5 we obtain that for all $1 \le i \le e^{\gamma\sqrt{N}}$ we have

$$\delta \ge \max_{\|x\| \le e^{\theta\sqrt{N}}} \left| \frac{\beta_{i1}(x) + \beta_{i2}(x) + \cdots + \beta_{iN}(x)}{\max(\sigma(x), L) N^{3/4}} \right|$$

$$\ge \max_{\|x\| \le e^{\theta\sqrt{N}}} \left| \frac{\xi_i(x, \rho_i(N))}{\xi_i(0, \rho_i(N))} - 1 \right| \sqrt{\left(\frac{\pi}{8} - \varepsilon\right) \frac{1}{\theta}},$$

that is, for any $\varepsilon > 0$ and for all $1 \le i \le e^{\gamma\sqrt{N}}$, we have

$$\max_{\|x\| \le e^{\theta\sqrt{N}}} \left| \frac{\xi_i(x, \rho_i(N))}{\xi_i(0, \rho_i(N))} - 1 \right| \le \delta(1 + 2\varepsilon) \sqrt{\frac{8\theta}{\pi}} \quad \text{a.s.,} \tag{10}$$

for all but finitely many N, provided that the parameters γ, δ, θ satisfy (9). Assume that

$$\delta \sqrt{\frac{8\theta}{\pi}} < 1,$$

that is, we assume

$$2\theta + \gamma < \frac{\delta^2}{4} < \frac{\pi}{32\theta}. \tag{11}$$

Then as a consequence of (10) we obtain that the radius $R_i(\rho_i(N))$ of the largest disc around the origin nearly homogeneously covered by S_i in $\rho_i(N)$ satisfies the inequality

$$R_i(\rho_i(N)) \ge e^{\theta\sqrt{N}} \quad \text{a.s.,} \tag{12}$$

for all $1 \le i \le e^{\gamma\sqrt{N}}$ and for all but finitely many N.

Let $0 < \alpha < 1$, $\beta = 1 - \alpha$. Then by Lemma 1 for all $n = 1, 2, \dots$ there exists a $1 \le i = i(n) \le n^\alpha$ such that

$$\xi(S(in^\beta), (i + 1)n^\beta) - \xi(S(in^\beta), in^\beta) \ge u(\log n^\beta)^2 \quad \text{a.s.,} \tag{13}$$

for all but finitely many n, provided that

$$\alpha > u\beta\pi. \tag{14}$$

Let

$$u(\log n^\beta)^2 = N, \quad \text{that is, } n = \exp\left(\frac{1}{\beta} \sqrt{\frac{N}{u}} \right),$$

and

$$\exp(\gamma\sqrt{N}) = n^\alpha, \quad \text{that is, } \gamma\sqrt{u}\,\beta = \alpha, \quad \text{that is, } \beta = (1 + \gamma\sqrt{u})^{-1}.$$

Then

$$e^{\theta\sqrt{N}} = n^{\theta\sqrt{u}\,\beta}.$$

Inequalities (12) and (13) combined imply that for all $n = 1, 2, \dots$ there exists a $1 \le i \le n^{\alpha}$ such that with probability 1,

$$\xi\big(S(in^{\beta}), (i+1)n^{\beta}\big) - \xi\big(S(in^{\beta}), in^{\beta}\big) \ge u(\log n^{\beta})^2$$

and around $S(in^{\beta})$ there exists a covered disc of radius $n^{\theta\sqrt{u}\beta}$ for all but finitely many N. [We do not claim that the above two statements hold for all but finitely many n; we only claim that they hold for all but finitely many $n = u(N) = \exp((1/\beta)\sqrt{N/u}\,)$.]

Then we want to choose the parameters $\alpha, \beta, \theta, u, \gamma, \delta$ such that they satisfy the conditions

$$2\theta + \gamma < \frac{\delta^2}{4} < \frac{\pi}{32\theta}, \tag{15}$$

$$\alpha > u\pi\beta, \tag{16}$$

$$\alpha = 1 - \beta = \gamma\sqrt{u}\,\beta, \quad \text{that is, } \beta = \frac{1}{1+\gamma\sqrt{u}}, \tag{17}$$

and $\theta\sqrt{u}\,\beta$ is as large as possible. This is equivalent to finding γ, θ, u for which

$$2\theta + \gamma < \frac{\pi}{32\theta}, \qquad \gamma > \sqrt{u}\,\pi,$$

and

$$\theta\sqrt{u}\,\beta = \frac{\theta\sqrt{u}}{1+\gamma\sqrt{u}}$$

is as large as possible.

Let

$$\gamma_0 = \sqrt{u}\,\pi$$

and

$$2\theta + \gamma_0 = 2\theta + \sqrt{u}\,\pi = \frac{\pi}{32\theta}, \quad \text{that is, } \sqrt{u} = \frac{1}{32\theta} - \frac{2\theta}{\pi}.$$

Then

$$\frac{\theta\sqrt{u}}{1+\gamma_0\sqrt{u}} = \frac{2^5\pi\theta^2 - 2^{11}\theta^4}{2^{12}\theta^4 + 7\pi 2^7\theta^2 + \pi^2} = f(\theta).$$

Since

$$f(2^{-4}\sqrt{\pi}) = \frac{3}{146} > \frac{1}{50}, \tag{18}$$

we can choose the parameters $\alpha, \beta, \theta, u, \gamma, \delta$ such that they satisfy (15), (16) and (17) and $\theta\sqrt{u}\,\beta > 1/50$, that is, we proved that there exists a nearly homogeneously covered disc (in the sense of Theorem 2) of radius

$$\bar{r}(n) \geq n^{1/50} \quad \text{a.s.},$$

for all but finitely many N, where $n = \exp((1/\beta)\sqrt{N/u}\,)$ and u and β are defined so that they satisfy the inequalities (15), (16) and (17).

In case

$$\exp\left(\frac{1}{\beta}\sqrt{\frac{N}{u}}\right) < n < \exp\left(\frac{1}{\beta}\sqrt{\frac{N+1}{u}}\right),$$

we also obtain immediately the statement.

Note that we do not claim that the above chosen parameters are the best possible ones. Very likely more careful work gives a somewhat larger constant instead of $1/50$; even evaluating the exact maximum of $f(\theta)$ instead of using (18) we get a somewhat better constant. However, to find the best constant very likely requires essentially new ideas.

3. The waiting time for a new point: A problem. Consider the simple symmetric random walk $\{S(n);\ n = 0, 1, 2, \dots\}$ in Z^d and let V_n be the smallest integer for which

$$S(n + V_n) \neq S(k), \qquad k = 0, 1, 2, \dots, n,$$

that is, V_n is the waiting time for a new point. Clearly,

$$\liminf_{n \to \infty} V_n = 1 \quad \text{a.s.}, \qquad d = 1, 2, \dots .$$

However, the lim sup behavior of V_n is a much harder question. It is easy to see that in the case $d = 1$ we have

$$\limsup_{n \to \infty} \frac{V_n}{n(\log\log n)^2} = C, \qquad 0 < C < \infty.$$

Our Theorem 1 suggests (cf. also Conjecture 2) that in the case $d = 2$ for any $\varepsilon > 0$,

$$\limsup_{n \to \infty} \frac{V_n}{n^{2q_0}} = \infty \quad \text{and} \quad \limsup_{n \to \infty} \frac{V_n}{n^{2q_0+\varepsilon}} = 0.$$

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