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CENTRAL LIMIT PROPERTIES OF GZH-SEMIGROUPS AND THEIR APPLICATIONS IN PROBABILITY THEORY[1]

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A class of topological semigroups called GZH-semigroups is introduced. Conditions under which they have the property that limits of infinitesimal arrays are infinitely divisible are obtained. The convolution semigroup of all probability measures on a second countable LCA-group or on a real separable Hilbert space as well as the semigroup of all positive definite kernels defined on a countable set with complex values and with norms not greater than 1 are reduced to an extended form of Delphic semigroups.

Introduction. It is well known that the convolution semigroup of all probability distributions on the real line has three fundamental properties:

1. The limit of an infinitesimal array is infinitely divisible.
2. Any distribution without a prime factor is infinitely divisible.
3. Every distribution F has a representation $F = G * E$, where G has no prime factor and E is a countable convolution of prime elements.

There are many semigroups with similar properties. Some of these are the convolution semigroup $M(X_1)$ of all probability measures on a second countable locally compact abelian group [11], the convolution semigroup $M(X_2)$ of all probability measures on a real separable Hilbert space [16], Delphic semigroups [7] (including the semigroup $\mathscr{R}^+$ of all positive renewal sequences [7]), the Kingman semigroup $\mathscr{P}$ of all standard p-functions [7], the semigroup $\mathscr{A}^+$ of all "positive delayed renewal sequence elements" [2] and MD-semigroups with property CLT (including the convolution semigroup $\mathbb{P}$ of all point processes defined on a complete separable metric space, the semigroup R^* of all generalized renewal sequences with first terms equal to 1) [4].

In [7], properties 1–3 were placed for the first time into a context of topological semigroups, after which [3] concentrated on properties 1 and 2. The excellent works [13]–[15] reveal the topological semigroup origin of all three properties and may be used to study the above semigroups and many other semigroups.

Continuing the work of [3] and [4], we define in this paper GZH-semigroups and GMD-semigroups, show that a GMD-semigroup has similar properties and

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obtain sufficient conditions under which a topological semigroup is a GMD-semigroup. Similar conditions can be applied to $M(X_1)$ and $M(X_2)$. Thus we obtain the following: the reduction of $M(X_1)$ to a "strongly Delphic form," which enterprise Davidson regarded as "flogging a dead horse" ([8], page 448). Applying these conditions to positive definite kernels, we obtain property 1 for positive definite kernels and prove that all positive definite kernels defined on a countable set with complex values and with norms not greater than 1 form a GMD-semigroup.

1. Definitions and preliminaries. In this paper we mean by a *semigroup* an abelian semigroup with identity e, and by a *topological semigroup* a Hausdorff topological semigroup.

Let S be a semigroup. If $a, b, c \in S$, $a = bc$, then b is called a factor of a and this is denoted by $b|a$; $F(a)$ denotes the set of all factors of a. The subgroup $F(e)$ is called the group of invertible elements and is denoted by $U(S)$ or U. If $a = bc$ and $b, c \notin U$, then b is called a proper factor of a. An element s is called prime if it does not belong to U and has no proper factor. An element s is called infinitely divisible (i.d.) if for each natural number n there is $t_n \in S$ such that $t_n^n \in sU$. Since the relation $R = \{(a, b): a \in bU\}$ is a congruence, the quotient set $S^* := S/U$ is a semigroup. The natural map $f: S \to S/U$ is defined by $f(s) = sU$.

The semigroup $(\mathbb{R}_+, +)$ of nonnegative real numbers, the semigroup $(\overline{\mathbb{R}}_+, +)$ of nonnegative extended real numbers, the additive group $\mathbb{R}$ of real numbers, the additive group $\mathbb{Z}$ of integers, the quotient group $\mathbb{R}/\mathbb{Z}$ and the multiplicative semigroup $\mathbb{D}$ of all complex numbers with norms not greater than 1 are topological semigroups in their natural topologies.

DEFINITION 1.1. Let S be a topological semigroup. We say that a net (x_n) in S shift-converges to x if for each n there exists $u_n \in U(S)$ such that $\lim x_n u_n = x$.

The following lemma is partly due to [13].

LEMMA 1.2. *Let S be a topological semigroup. Then the following hold*:

(i) *The natural map $f: S \to S^*$ is open.*
(ii) *The composition in S^* is continuous.*
(iii) *If $R := \{(x, y): f(x) = f(y)\}$ is closed in $S \times S$, then S^* is a topological semigroup.*
(iv) *If S is second countable, then so is S^*.*

PROOF. (i) For any open set $V \subset S$ and each $u \in U$, $Vu = \{g \in S: gu^{-1} \in V\}$ is open. Hence $f^{-1}f(V) = VU = \bigcup_{u \in U} Vu$ is open.

(ii) Let $f(x), f(y) \in S^*$, let V^* be open in S^* and let $f(x)f(y) \in V^*$. Then $xy \in f^{-1}(V^*)$. Hence there are open sets $W_1, W_2 \subset S$ such that $x \in W_1$, $y \in W_2$ and $W_1W_2 \subset f^{-1}(V^*)$. Now $f(W_1)$ and $f(W_2)$ are open by (i), $f(x) \in$

$f(W_1)$, $f(y) \in f(W_2)$ and $f(W_1)f(W_2) = f(W_1W_2) \subset V^*$, so the composition is continuous.

(iii) If R is closed, then S^* is a Hausdorff space by (i) and [6], Chapter 3, Theorem 11. Hence S^* is a topological semigroup by (ii).

(iv) Let $V_1, V_2, \dots$ be a countable base of S. We now prove that $f(V_1), f(V_2), \dots$ is a countable base of S^*. Let G^* be open in S^*. Then there is a subset E of $\{1, 2, \dots\}$ such that $f^{-1}(G^*) = \bigcup_{k \in E} V_k$, hence $G^* = ff^{-1}(G^*) = \bigcup_{k \in E} f(V_k)$. □

2. Multiple semigroups.

DEFINITION 2.1. Let S be a topological semigroup. Then $(S; H)$ or S is called a multiple semigroup or an M-semigroup if for $k = 1, 2, \dots$ there are continuous homomorphisms H_k: $S \to \mathbb{D} = \{z \in \mathbb{C}\colon |z| \leq 1\}$ such that the following hold:

(i) $a = e$ if and only if $H_k(a) = 1$ for each k;
(ii) $a \in U$ if and only if $|H_k(a)| = 1$ for each k.

DEFINITION 2.2. Let S be an M-semigroup. For each $k = 1, 2, \dots,$ let $S^{(k)} := \{a \in S\colon H_k(a) \neq 0\}$, and let D_k: $S \to (\overline{\mathbb{R}}_+, +)$ be defined by $D_k(a) = -\log|H_k(a)|$ and A_k: $S^{(k)} \to \mathbb{R}/\mathbb{Z}$ by $A_k(a) = \arg H_k(a)/2\pi$. Let $S_1 := \bigcap_{1 \leq k < \infty} S^{(k)}$. Thus

$$S_1 = \{a \in S\colon H_k(a) \neq 0 \text{ for all } k\} = \{a \in S\colon D_k(a) < \infty \text{ for all } k\}.$$

DEFINITION 2.3. Let S be a topological semigroup. Let $(a_{ij} \in S\colon j = 1, \dots, i;\ i = 1, 2, \dots)$ or (a_{ij}) denote the following array in S:

$$\begin{array}{l} a_{11}, \\ a_{21}, a_{22}, \\ a_{31}, a_{32}, a_{33}, \\ \dots . \end{array}$$

Set $a_i := a_{i1} \cdots a_{ii}$ for each i. We say that (a_{ij}) converges to a or say that (a_{ij}) has limit a if the sequence (a_i) converges to $a \in S$.

DEFINITION 2.4. An array (a_{ij}) in an M-semigroup is called a D-infinitesimal array if $\lim_{i \to \infty} \max_j D_k(a_{ij}) = 0$ for each k.

LEMMA 2.5. *Let S be a topological semigroup. For each $k = 1, 2, \dots, M$, let D_k be a continuous homomorphism from S to $(\overline{\mathbb{R}}_+, +)$, let (a_{ij}) be an array in S converging to $a \in S$ and let $D_k(a) < \infty$ and $\lim_{i \to \infty} \max_j D_k(a_{ij}) = 0$ for each k. Then for each decreasing positive sequence (x_n) converging to zero there is an array (b_{nm}) satisfying the following conditions*:

(i) *For each n there is an $i_n = np_n$, where p_n is a natural number, such that $a_{i_n 1}, \dots, a_{i_n i_n}$ can be divided into n classes, each of them consisting of p_n*

elements, and each b_{nm} being the product of all elements in the mth class (thus $b_n = a_{i_n} \to a$).

(ii) $\max_{m_1, m_2}|D_k(b_{nm_1}) - D_k(b_{nm_2})| < x_n$ *for each n and each $k = 1, 2, \ldots, M$.*

PROOF. (i) Let $M = 1$. For each n select an $i_n = np_n$ such that $\max_j D_1(a_{i_n j}) < x_n$. Without loss of generality, let $D_1(a_{i_n 1}) \geq \cdots \geq D_1(a_{i_n i_n})$. We now divide $a_{i_n 1}, \ldots, a_{i_n i_n}$ into n classes in p_n steps.

In step r, where $1 \leq r \leq p_n$, and for each $m = 1, 2, \ldots, n$, insert $a_{i_n, rn-n+m}$ into class m as the rth element of the class. Let $S_m^{(r)}$ denote the sum of the images of the first r elements in class m under the map D_1. Rearrange the order of the classes so that $S_1^{(r)} \leq \cdots \leq S_n^{(r)}$.

For each m, let b_{nm} denote the product of all elements in class m. Then (b_{nm}) is the desired array.

(ii) Suppose that the lemma is true when $M = N - 1$. Let $M = N$. Let $D := D_1 + \cdots + D_N$. Then $\lim_{i \to \infty} \max_j D(a_{ij}) = 0$. By (i) there is an array (g_{st}) such that $g_s = a_{i_s}$, $i_s = sp'_s$ and

$$\max_{t_1, t_2} \left| D(g_{st_1}) - D(g_{st_2}) \right| < \frac{x_s}{sN}.$$

Hence

$$\begin{aligned} \max_t D_k(g_{st}) &\leq \max_t D(g_{st}) \\ &\leq \max_t \left| \frac{D(g_{st}) - D(g_s)}{s} \right| + \frac{D(g_s)}{s} \\ &\leq \frac{x_s}{sN} + \frac{(D_1(g_s) + \cdots + D_N(g_s))}{s} \\ &\to 0 \quad \text{as } s \to \infty, \text{ for } k = 1, 2, \ldots, N. \end{aligned}$$

By the inductive assumption there is an array (b_{nm}) such that $b_n = g_{s_n}$, $s_n = np''_n$ [thus $b_n = a_{i(n)}$, where $i(n) = i_{s_n} = s_n p'_{s_n} = np''_n p'_{s_n}$, $b_n \to a$] and

$$\max_{m_1, m_2} \left| D_k(b_{nm_1}) - D_k(b_{nm_2}) \right| \leq \frac{x_n}{N}$$

for $k = 1, 2, \ldots, N - 1$. Hence

$$\begin{aligned} &\max_{m_1, m_2} \left| D_N(b_{nm_1}) - D_N(b_{nm_2}) \right| \\ &\quad \leq \max_{m_1, m_2} \left| D(b_{nm_1}) - D(b_{nm_2}) \right| + \sum_{1 \leq k \leq N-1} \max_{m_1, m_2} \left| D_k(b_{nm_1}) - D_k(b_{nm_2}) \right| \\ &\quad \leq \frac{p''_n x_{s_n}}{s_n N} + \frac{(N-1)x_n}{N} \\ &\quad \leq x_n. \end{aligned}$$

□

THEOREM 2.6. *Suppose that a D-infinitesimal array in an M-semigroup S converges to* $a \in S_1$, *and that* (x_n) *is a decreasing positive sequence converging to zero. Then there is a D-infinitesimal array* (b_{nm}) *satisfying condition* (i) *of Lemma* 2.5 *and the following*:

$$\text{(ii}'\text{)} \qquad \max_{m_1, m_2} \left| D_k(b_{nm_1}) - D_k(b_{nm_2}) \right| < x_n$$

for each n and each $k = 1, 2, \ldots, n$.

PROOF. Let $(b_{nm}^{(M)})$ denote the (b_{nm}) in Lemma 2.5 for each M. Let $b_{nm} = b_{nm}^{(n)}$ for each n, m. Then this (b_{nm}) is the desired array. □

REMARK. Theorem 2.6 occurs in fact in [4], but the above proof is much more elementary.

LEMMA 2.7. *Let* d, n, *and* L *be natural numbers*, $s \geq ndL$, $x_1, \ldots, x_s \in [0, 1)$. *Then we can select np elements from* $x_1, \ldots, x_s$, *where* $np/s > 1 - 1/L$, *and the elements are denoted by*

$$\begin{gathered} y_{11}, \ldots, y_{1p}, \\ \vdots \\ y_{n1}, \ldots, y_{np}, \end{gathered}$$

such that

$$\max_{i_1, i_2} \left| \sum_j y_{i_1 j} - \sum_j y_{i_2 j} \right| < \frac{1}{d}.$$

PROOF. Let $n(p + d - 1) \leq s < n(p + d)$. If $x_i \in [(k - 1)/d, k/d)$, then we say that x_i belongs to batch k. If the number of elements in each batch is not greater than $n - 1$, then there are at most $(n - 1)d$ elements. Since $s - n(p - 1) \geq nd > (n - 1)d$, we can select p classes from $x_1, \ldots, x_s$, each class consisting of n elements belonging to the same batch. Denote these classes by

$$\begin{gathered} z_{11}, \ldots, z_{1n}, \\ \vdots \\ z_{p1}, \ldots, z_{pn}. \end{gathered}$$

Without loss of generality, let

$$z_{i1} \geq \cdots \geq z_{in}$$

for $i = 1, 2, \ldots, p$. We now rearrange the elements in these p classes into n classes in p steps.

In step q, where $1 \leq q \leq p$, and for $j = 1, \ldots, n$ insert z_{qj} into class j as the qth element of the class. Let $S_j^{(q)}$ denote the sum of the first q elements in class j. Rearrange the order of these classes so that

$$S_1^{(q)} \leq \cdots \leq S_n^{(q)}.$$

From $ndL \leq s < n(p+d)$, it follows that $1/L > d/(p+d)$, $np/s > np/n(p+d) = 1 - d/(p+d) > 1 - 1/L$. □

THEOREM 2.8. *Let (a_{st}) be an array in an M-semigroup S such that $D_k(a_{st}) \neq 0$ for any fixed k and sufficiently large s. Then for any fixed choice of natural numbers d, n, L, M and N there is a natural number $s \geq N$ such that we can select np elements from $a_{s1}, \ldots, a_{ss}$, where $np/s > 1 - 1/L$, and the elements are denoted by $(b_{ij})_{n\times p}$ such that*

$$\max_{i_1, i_2} \left| \sum_j A_k(b_{i_1 j}) - \sum_j A_k(b_{i_2 j}) \right| < \frac{1}{d}$$

for $k = 1, \ldots, M$.

PROOF. The present theorem holds for $M = 1$ by Theorem 2.7. Suppose that the theorem holds for $M = m - 1$. Then, for $r = 2ndL$, there is a natural number $s \geq N$ such that we can select rq elements from $a_{s1}, \ldots, a_{ss}$, where $rq/s > 1 - 1/2L$, and the elements are denoted by $(c_{ij})_{r\times q}$ such that

$$\max_{i_1, i_2} \left| \sum_j A_k(c_{i_1 j}) - \sum_j A_k(c_{i_2 j}) \right| < \frac{1}{rd}$$

for $k = 1, \ldots, m-1$. Let $c_1 = c_{11} \cdots c_{1q}, \ldots, c_r = c_{r1} \cdots c_{rq}$. Then

$$\max_{i_1, i_2} \left| A_k(c_{i_1}) - A_k(c_{i_2}) \right| < \frac{1}{rd}$$

for $k = 1, 2, \ldots, m-1$. By Lemma 2.7 we can select nh elements from $c_1, \ldots, c_r$, where $nh/r > 1 - 1/2L$, and the elements are denoted by $(\tilde{b}_{ij})_{n\times h}$ such that

$$\max_{i_1, i_2} \left| \sum_j A_m\left(\tilde{b}_{i_1 j}\right) - \sum_j A_m\left(\tilde{b}_{i_2 j}\right) \right| < \frac{1}{d}.$$

We also have

$$\begin{aligned} \max_{i_1, i_2} \left| \sum_j A_k\left(\tilde{b}_{i_1 j}\right) - \sum_j A_k\left(\tilde{b}_{i_2 j}\right) \right| &\leq \sum_j \max_{i_1, i_2} \left| A_k\left(\tilde{b}_{i_1 j}\right) - A_k\left(\tilde{b}_{i_2 j}\right) \right| \\ &\leq h \max_{i_1, i_2} \left| A_k(c_{i_1}) - A_k(c_{i_2}) \right| \\ &< \frac{h}{rd} \leq \frac{1}{d} \end{aligned}$$

for $k = 1, 2, \ldots, m - 1$. Since $p = qh$,

$$\frac{np}{s} = \frac{rq}{s}\frac{nh}{r} > \left(1 - \frac{1}{2L}\right)\left(1 - \frac{1}{2L}\right) > 1 - \frac{1}{L}. \qquad \square$$

THEOREM 2.9. *Let a D-infinitesimal array* (a_{ij}) *in an M-semigroup converge to* $a \in S_1$. *Let* d *be a natural number. Then there is a subsequence* $(a_{i_t}) = (x_t)$ *of* (a_i) *such that* $x_t = z_t y_{t1} \cdots y_{td}$ *for every* t *and*

$$\max_{j_1, j_2} |D_k(y_{tj_1}) - D_k(y_{tj_2})| < \frac{1}{t},$$

$$\max_{j_1, j_2} |A_k(y_{tj_1}) - A_k(y_{tj_2})| < \frac{1}{t},$$

$$D_k(z_t) < \frac{D_k(a) + 2}{t}$$

for $k = 1, \ldots, t$.

PROOF. Let (b_{nm}) be the D-infinitesimal array defined in Theorem 2.6 such that $b_n = a_{i_n}$ and

$$\max_{m_1, m_2} \left|D_k(b_{nm_1}) - D_k(b_{nm_2})\right| < \frac{1}{n^2}$$

for $k = 1, 2, \ldots, n$.

By Theorem 2.8 there are $s_1 < s_2 < \cdots$ such that for each fixed t, $D_k(b_{s_t}) < D_k(a) + 1$ for $k = 1, 2, \ldots, t$, and we can select dp elements from $b_{s_t 1}, \ldots, b_{s_t s_t}$, where $dp/s_t > 1 - 1/t$, and the elements are denoted by $(c_{ij})_{d \times p}$ such that

$$\max_{i_1, i_2} \left| \sum_j A_k(c_{i_1 j}) - \sum_j A_k(c_{i_2 j}) \right| < \frac{1}{t}$$

for $k = 1, \ldots, t$. Let $y_{ti} = c_{i1} \cdots c_{ip}$ for $t = 1, \ldots, d$. Then

$$\max_{i_1, i_2} |A_k(y_{ti_1}) - A_k(y_{ti_2})| < \frac{1}{t},$$

$$\begin{aligned}\max_{i_1, i_2} |D_k(y_{ti_1}) - D_k(y_{ti_2})| &\le p \max_{i_1, i_2} |D_k(c_{i_1 j}) - D_k(c_{i_2 j})| \\ &\le \frac{p}{s_t^2} \\ &\le \frac{1}{s_t} \\ &\le \frac{1}{t}.\end{aligned}$$

Let z_t be the product of the remainder elements. Then

$$\begin{aligned} D_k(z_t) &\le (s_t - dp)\left(\frac{D_k(a)+1}{s_t} + \max_{m_1, m_2}\left|D_k(b_{s_t m_1}) - D_k(b_{s_t m_2})\right|\right) \\ &\le (s_t - dp)\left(\frac{D_k(a)+1}{s_t} + \frac{1}{s_t^2}\right) \\ &< \left(\frac{s_t}{t}\right)\left(\frac{1}{s_t}\right)\left(D_k(a) + 1 + \frac{1}{s_t}\right) \\ &= \frac{D_k(a)+2}{t}. \end{aligned}$$

Let $x_t = b_{s_t} = a_{i(t)}$, where $i(t) = i_{s_t}$. Then $x_t = z_t y_{t1} \cdots y_{td}$. □

3. Generalized ZH-semigroups.

Definition 3.1. Let S be a topological semigroup or an M-semigroup, $A \subset S_1$, $f^{-1}f(A) = A$, where f is the natural map defined in Section 1. We now define the following properties of S.

(a) *H-separability*. Let $x, y \in S$. Then $x = y$ if and only if $H_k(x) = H_k(y)$ for each k.

(b) *Stability* for sequences (nets). Let (x_n) and (y_n) be sequences (nets) in S. If $y_n | x_n$ for each n, $x_n \to x \in S$, then (y_n) has a convergent subsequence (subnet).

(c) *Shift-stability* for sequences (nets). Let (x_n) and (y_n) be sequences (nets) in S. If $y_n | x_n$ for each n, $x_n \to x \in S_1$, then (y_n) has a shift-convergent subsequence (subnet).

(d) *Division compactness* for sequences (nets). Let (x_n), (y_n), and (z_n) be sequences (nets) in S. If $y_n z_n = x_n$ for each n, $x_n \to x \in S_1$, (y_n) is convergent, then (z_n) has a convergent subsequence (subnet).

(e) SLS(A) [SLS′(A)], the shift-limit separability on A for sequences (nets). Let (x_n), (y_n), and (z_n) be sequences (nets) in S. If $y_n z_n | x_n$ for each n, $x_n \to x \in A$, (y_n) shift-converges to y, (z_n) shift-converges to z, $\lim_n |H_k(y_n) - H_k(z_n)| = 0$ for each k, then $f(y) = f(z)$.

An element a of S is called a D-infinitesimal limit if there is a D-infinitesimal array converging to a.

(f) ILID(A). If $a \in A$ is a D-infinitesimal limit, then a is i.d.

Remark 3.2. Let an M-semigroup S be shift-stable and division-compact for sequences (nets), let (x_n), (y_n) and (z_n) be in S, $y_n z_n = x_n$ for each n, and $x_n \to x \in S_1$. Then there are a subsequence (subnet) (y_{n_i}) and a sequence (net) (u_i) in $U(S)$ such that $y_{n_i} u_i \to y$. Moreover, there is a subsequence (subnet) of $(z_{n_i} u_i^{-1})$ converging to z, so $yz = x$, $y \in F(x)$.

THEOREM 3.3. *Let S be an M-semigroup. If S has properties* SLS(A), *shift-stability and division compactness for sequences, or has these properties for nets, then S has property* ILID(A).

PROOF. Let $a \in A$ be a limit of a D-infinitesimal array. By Theorem 2.9, for each fixed d there is a sequence (x_t) converging to a such that $x_t = z_t y_{t1} \cdots y_{td}$ for each t. Hence there are sequences (nets) $(u_n^{(1)}), \ldots, (u_n^{(d)})$ in $U(S)$ such that $u_n^{(1)} y_{t_n 1} \to y_1, \ldots, u_n^{(d)} y_{t_n d} \to y_d$. Then $f(y_i) = f(y_j)$ for $i, j = 1, \ldots, d$. Let a subnet of the net $((u_n^{(1)} \cdots u_n^{(d)})^{-1} z_{t_n})$ converge to z. Then $D_k(z) = 0$ for each k, hence $z \in U(S)$. So $a = zy_1 \cdots y_d$ is i.d. □

LEMMA 3.4. *Let S be an M-semigroup.*

(i) *If S is stable for sequences (nets), then S is shift-stable and division-compact for sequences (nets).*

(ii) *If S is stable for sequences (nets) and H-separable, then S has property* SLS(S_1) [SLS′(S_1)].

PROOF. Statement (i) is obvious. We now prove (ii). Let (x_n), (y_n) and (z_n) be sequences (nets) in S; $y_n z_n | x_n$ for each n and $x_n \to x \in S_1$. Let $\lim_n |H_k(y_n) - H_k(z_n)| = 0$ for each k. Take sequences (nets) (u_n) and (v_n) in U such that $y_n u_n \to y$, $z_n v_n \to z$. Since S is stable for sequences (nets), there are subsequences (subnets) (y_{n_s}), (z_{n_s}), (u_{n_s}), and (v_{n_s}) converging to $\tilde{y}$, $\tilde{z}$, u, and v, respectively. Thus $y = \tilde{y}u$, $z = \tilde{z}v$, $H_k(\tilde{y}) = H_k(\tilde{z})$, $|H_k(u)| = |H_k(v)| = 1$ for each k, $\tilde{y} = \tilde{z}$ and $u, v \in U$. Hence $f(y) = f(\tilde{y}) = f(\tilde{z}) = f(z)$. □

LEMMA 3.5. *Let S be an M-semigroup. If S is H-separable and stable for sequences or nets, then S has property* ILID(S_1).

PROOF. The lemma holds by Lemma 3.4 and Theorem 3.3. □

THEOREM 3.6. *Suppose the same assumption as in Lemma 2.5. If in addition S is stable for sequences or nets, then for each fixed natural number d there are $c_1, \ldots, c_d \in S$ such that $a = c_1 \cdots c_d$ and $D_k(c_1) = \cdots = D_k(c_d)$ for each $k = 1, \ldots, M$.*

PROOF. Let $x_n = 1/n^2$ and let (b_{nm}) be the array determined by Lemma 2.5. Let $c_{ij} = b_{id, i(j-1)+1} \cdots b_{id, ij}$ for $j = 1, \ldots, d$, $i = 1, 2, \ldots$. Then

$$\lim_{i \to \infty} c_{i1} \cdots c_{id} = a, \qquad \max_{j_1, j_2} \left| D_k(c_{ij_1}) - D_k(c_{ij_2}) \right| < \frac{i}{i^2 d^2} = \frac{1}{id^2}$$

for each k. Let subsequences (subnets) $(c_{i_s 1}), \ldots, (c_{i_s d})$ converge to $c_1, \ldots, c_d$, respectively. Then $a = c_1 \cdots c_d$ and $D_k(c_1) = \cdots = D_k(c_d)$ for each k. □

DEFINITION 3.7. Let S be an M-semigroup. S is called a generalized ZH-semigroup or a GZH-semigroup if the following hold:

(i) $S^* = S/U$ is a topological semigroup (recall that by a topological semigroup we mean a Hausdorff topological semigroup);
(ii) for each $s \in S_1$, $F(f(s))$ is a compact set.

REMARK. For ZH-semigroups we can refer to [4] ("ZH" are the first two letters of Zhongshan University).

LEMMA 3.8. *Let S be an M-semigroup satisfying condition* (i) *of Definition* 3.7. *If S is shift-stable and division-compact for sequences, and S^* is second countable, then S is a GZH-semigroup.*

PROOF. Let $a \in S_1$. Then, by Remark 3.2, every sequence in $F(a)$ has a subsequence shift-converging to an element of $F(a)$. Hence $F(f(a)) = f(F(a))$ is sequentially compact. Since S^* is second countable, $F(f(a))$ is compact by [6], Theorem 5 of Chapter 5. □

REMARK 3.9. Let S be a GZH-semigroup, $S_1^* := f(S_1)$. Then we can easily verify that S_1^* is a Hun semigroup ([14], Definition 2.2.2) and S_1 is a Hungarian semigroup ([14], Definition 2.21.1) and that S_1^* and S_1 have no nontrivial idempotent elements.

DEFINITION 3.10. Let S be a GZH-semigroup. Then S is called a generalized multiple Delphic semigroup or a GMD-semigroup if S has property ILID(S_1), S is called a GMD(A)-semigroup if S has property ILID(A).

THEOREM 3.11. *Let S be a first countable GZH-semigroup, $s \in S_1$. Then the following hold*:

(i) *There is a representation $s = s_1 s_2$, where s_1 has no prime factor and s_2 is a countable product of prime elements.*
(ii) *If s has no prime factor, then there is a D-infinitesimal array (s_{ij}) such that $s = s_{i1} \cdots s_{ii}$ for each i.*
(iii) *If in addition S is a GMD(A)-semigroup, $s \in A$ and s has no prime factor, then s is infinitely divisible.*

PROOF. Since S_1 is a Hungarian semigroup by Remark 3.9, (i) follows ([14], Theorem 2.23.3). For (ii), let $s^* := f(s)$. Then $s^* \in S_1^*$; s^* has no prime factor. Since S_1^* is a Hun semigroup, s^* is infinitesimally divisible (by [14], Theorem 2.8.9). Let the map $D_k^*\colon S^* \to (\overline{\mathbb{R}}_+, +)$ be defined by $D_k^*(f(s)) = D_k(s)$

for each k. Then D_k^* is continuous. In fact, for any fixed open set $V \subset \overline{\mathbb{R}}_+$,

$$\begin{aligned}(D_k^*)^{-1}(V) &= \{f(x): D_k^*(f(x)) \in V\}\\ &= \{f(x): D_k(x) \in V\}\\ &= \{f(x): x \in D_k^{-1}(V)\}\\ &= f(D_k^{-1}(V)),\end{aligned}$$

and $f(D_k^{-1}(V))$ is open by Lemma 1.2. Thus we have an array $(t_{ij}^*: j = 1, 2, \dots, n_i;\ i = 1, 2, \dots)$ in S^* such that $s^* = t_{i1}^* \cdots t_{in_i}^*$ for each i and $D_k^*(t_{ij}^*) < 1/i$ for $k = 1, \dots, i$, $j = 1, \dots, n_i$, $i = 1, 2, \dots$. Hence we have an array (s_{ij}^*) of S^* such that $s^* = s_{i1}^* \cdots s_{ii}^*$ for each i and $\lim_{i\to\infty} \max_j D_k^*(s_{ij}^*) = 0$ for all k. So we have $s_{ij} \in f^{-1}(s_{ij}^*)$ for each i, j such that $s = s_{i1} \cdots s_{ii}$ for each i and $\lim_{i\to\infty} \max_j D_k(s_{ij}) = 0$ for each k. Finally, (iii) is an immediate consequence of (ii). □

Let T denote the set of sequences in $\mathbb{D} = \{z \in \mathbb{C}: |z| \le 1\}$. The elements $(a_n), (b_n), \dots$ of T will be written briefly as $a, b, \dots,$ respectively. If the product ab of a and b is defined by $ab := (a_n b_n)$, the metric d for T is defined by $d(a, b) = \sum_{n=1}^{\infty} |a_n - b_n|/2^n$, then T is a second countable topological semigroup.

THEOREM 3.12. *Let S be a closed subsemigroup of T. Suppose that if $(a_n) \in S$ and $|a_n| = 1$ for each n, then $(\bar{a}_n) \in S$. Then S is a GMD-semigroup.*

PROOF. It is obvious that the group of invertible elements is $U(S) := \{a \in S: |a_n| = 1 \text{ for each } n\}$. Let $H_k: S \to \mathbb{D}$ be defined by $H_k(a) = a_k$ for each $k = 1, 2, \dots$. Then $(S; H)$ is an M-semigroup. By Lemma 1.2(iii), S^* is a topological semigroup if $R = \{(x, y): f(x) = f(y)\}$ is closed. Let $(x^{(k)}, y^{(k)})$ be a sequence in R converging to (x, y). Then there is $u^{(k)} \in U$ such that $x^{(k)}u^{(k)} = y^{(k)}$ for each k. Since U is closed and $U \subset \mathbb{D}^\infty$, U is compact. Let a sequence of $u^{(k)}$ converge to $u \in U$. Then $xu = y$, $(x, y) \in R$. Hence R is closed. It is obvious that S is H-separable and stable for sequences; hence S has property ILID(S_1) by Lemma 3.5 and is shift-stable and division-compact for sequences by Lemma 3.4. Since S is second countable, S^* is second countable by Lemma 1.2(iv). Hence S is a GZH-semigroup by Lemma 3.8. Thus S is a GMD-semigroup. □

4. Application to probability measure semigroups. Let X be a complete separable metric group and $\mathscr{B}_X$ the σ-algebra generated by the open subsets of X. Let $M(X)$ be the set of probability measures defined on $\mathscr{B}_X$ and let $M(X)$ be equipped with the weak topology. Then $M(X)$ can be metrized as a complete separable metric space and $M(X)$ is a topological semigroup under the convolution operation. Let δ_a denote the measure degenerate at the point

a. The group U of invertible elements of $M(X)$ consists of all degenerate measures. By Lemma 1.2(iii) and [10], we have the following lemma.

LEMMA 4.1. *$M(X)/U$ is a topological semigroup.*

Henceforth, X_1 is a second countable locally compact abelian group, X_2 a real separable Hilbert space. $M(X_1)$ and $M(X_2)$ are the semigroups of all probability measures on $\mathscr{B}_{X_1}$ and $\mathscr{B}_{X_2}$, respectively; Y_1 is the character group of X_1; Y_2 the dual space of X_2. Furthermore, X, M, Y denote $X_1, M(X_1), Y_1$ or $X_2, M(X_2), Y_2$, respectively. Finally, $\hat{\mu}(\cdot)$ is the characteristic function of $\mu \in M$.

Let $\{y_1, y_2, \dots\}$ be a countable dense subset of Y. Define H_k: $M \to \mathbb{D}$ by $H_k(\mu) = \hat{\mu}(y_k)$ for each k. By Lemmas 1.2, 3.8 and 4.1 and [10], we have the following lemma.

LEMMA 4.2. *The semigroup $(M; H)$ is a GZH-semigroup with the properties shift-stability and division compactness for sequences.*

Let $J := \{\mu \in M\colon \hat{\mu}(y) \neq 0 \text{ for each } y \in Y\}$.

Let (μ_n), (α_n) and (β_n) be sequences in M. Let $\mu_n \to \mu \in J$, $\alpha_n \beta_n | \mu_n$ for each n, $\lim_n |\hat{\alpha}_n(y_k) - \hat{\beta}_n(y_k)| = 0$ for each k and $\alpha_n \delta_{a_n} \to \alpha$, $\beta_n \delta_{b_n} \to \beta$. Then by [10] we conclude that $\hat{\alpha}(\cdot)/\hat{\beta}(\cdot)$ is a positive definite function and is continuous in the compact-open topology or the S-topology, so $\hat{\alpha}(\cdot)/\hat{\beta}(\cdot) = \hat{\delta}_c(\cdot)$ for some $c \in X$, $\alpha = \beta * \delta_c$. Hence we have the following lemma.

LEMMA 4.3. *The semigroup $(M; H)$ has property* SLS(J).

THEOREM 4.4. *The semigroup $(M; H)$ is a* GMD(J)*-semigroup.*

PROOF. The theorem holds by Lemmas 4.2 and 4.3 and Theorem 3.3. □

LEMMA 4.5. *Let $\mu \in M$ have no idempotent factor. Let $V := \{y\colon \hat{\mu}(y) \neq 0\}$ and let H be the subgroup generated by V. Then $H = Y$.*

PROOF. The set V is open, hence H is an open and closed subgroup. First, let $X = X_2$. Since $Y = X_2$ is a connected space, $H = Y$. Second, let $X = X_1$. If $H \neq Y$, then $G = (X, H) := \{x \in X\colon \langle x, y\rangle = 1 \text{ for each } y \in H\} \neq \{I\}$, where I is the identity element of X. Now G is the character group of the discrete group Y/H, so G is compact. The normalized Haar measure on G is an idempotent factor of μ. □

LEMMA 4.6. *Let (μ_{ij}) be in M and $\lim_{i\to\infty} \mu_{i1} * \cdots * \mu_{ii} = \mu$. Then*

$$V := \left\{y\colon \hat{\mu}(y) \neq 0,\ \lim_{i\to\infty} \min_j |\hat{\mu}_{ij}(y)| = 1\right\}$$

is a subgroup of Y.

PROOF. We only verify $V^2 \subset V$. Let $\lambda = \mu * \tilde{\mu}$ and $\lambda_{ij} = \mu_{ij} * \tilde{\mu}_{ij}$ for $1 \le j \le i < \infty$, where $\tilde{\mu}(B) = \mu(B^{-1})$ and $\tilde{\mu}_{ij}(B) = \mu_{ij}(B^{-1})$ for each $B \in \mathscr{B}_X$ and each i, j. Then $\lambda = \lim_{i\to\infty} \lambda_{i1} * \cdots * \lambda_{ii}$, $\hat{\lambda}(\cdot)$ and $\hat{\lambda}_{ij}(\cdot)$ are nonnegative functions and

$$V = \left\{y\colon \hat{\lambda}(y) \ne 0,\ \lim_{i\to\infty} \min_j \hat{\lambda}_{ij}(y) = 1\right\}.$$

Since $2(1 - \cos\alpha)(1 - \cos\beta) + 1 - \cos(\alpha - \beta) \ge 0$,

$$1 - \cos(\alpha + \beta) \le 2(1 - \cos\alpha) + 2(1 - \cos\beta).$$

Thus

$$1 - \operatorname{Re}\langle x, y_1 + y_2\rangle \le 2(1 - \operatorname{Re}\langle x, y_1\rangle) + 2(1 - \operatorname{Re}\langle x, y_2\rangle),$$

$$1 - \hat{\lambda}_{ij}(y_1 + y_2) \le 2\left(1 - \hat{\lambda}_{ij}(y_1)\right) + 2\left(1 - \hat{\lambda}_{ij}(y_2)\right).$$

If $y_1, y_2 \in V$, then $\lim_{i\to\infty} \min_j \hat{\lambda}_{ij}(y_1 + y_2) = 1$ by the above inequality. We can easily verify that if

$$\lim_{i\to\infty} \hat{\lambda}_{i1}(y) \cdots \hat{\lambda}_{ii}(y) = \hat{\lambda}(y) \quad \text{and} \quad \lim_{i\to\infty} \min_j \hat{\lambda}_{ij}(y) = 1,$$

then $\hat{\lambda}(y) \ne 0$ if and only if $\sup_i \Sigma_j(1 - \hat{\lambda}_{ij}(y)) < \infty$. Since $\Sigma_j(1 - \hat{\lambda}_{ij}(y_1 + y_2)) \le 2\Sigma_j(1 - \hat{\lambda}_{ij}(y_1) + 1 - \hat{\lambda}_{ij}(y_2))$,

$$\hat{\lambda}(y_1 + y_2) \ne 0 \quad \text{when } y_1, y_2 \in V. \qquad \square$$

DEFINITION 4.7. An array (μ_{ij}) in M is called infinitesimal if $\lim_{i\to\infty} \min_j |\hat{\mu}_{ij}(y)| = 1$ for each y. An array (μ_{ij}) in M is called uniformly infinitesimal if

$$\lim_{i\to\infty} \max_j \sup_{y\in K} \left|\hat{\mu}_{ij}(y) - 1\right| = 0$$

for each compact subset $K \subset Y$.

LEMMA 4.8. *Suppose that an infinitesimal array (μ_{ij}) converges to $\mu \in M$. If μ has no idempotent factor, then $\mu \in J$.*

PROOF. Let $V := \{y\colon \hat{\mu}(y) \ne 0\}$. Then

$$V = \left\{y\colon \hat{\mu}(y) \ne 0,\ \lim_{i\to\infty} \min_j \left|\hat{\mu}_{ij}(y)\right| = 1\right\}.$$

Hence V is a subgroup by Lemma 4.6 and $V = Y$ by Lemma 4.5. $\square$

LEMMA 4.9. *Let $\mu \in M$. If μ has no idempotent factor, then there are homomorphisms $H_1', H_2', \ldots$ such that $(M; H')$ is a GZH-semigroup and $H_k'(\mu) \ne 0$ for each k. If in addition μ has no prime factor, then $\mu \in J$.*

PROOF. By Lemma 4.5, Y is just the subgroup generated by $V = \{y: \hat{\mu}(y) \neq 0\}$. Hence for each fixed $y \in Y$ there are a countable subset $E = \{y_1, y_2, \dots\}$ of V and a natural number n such that $y = y_1 \cdots y_n$ and the group generated by E is dense in Y.

If $\hat{\lambda}(g) = \int_X \langle x, g\rangle \lambda(dx) = e^{i\alpha}$, $\hat{\lambda}(h) = e^{i\beta}$, then

$$\lambda\{x: \langle x, g\rangle = e^{i\alpha}\} = \lambda\{x: \langle x, h\rangle = e^{i\beta}\} = 1,$$

$$\hat{\lambda}(g+h) = e^{i(\alpha+\beta)}.$$

Hence λ is a degenerate measure if and only if $|\hat{\lambda}(y_k)| = 1$ for each k; $\lambda = \delta_I$, where I is the identity element of X, if and only if $\hat{\lambda}(y_k) = 1$ for each k. Define $H'_k: M \to \mathbb{D}$ by $H'_k(\lambda) = \hat{\lambda}(y_k)$ for each k. Then by the proof of Lemma 4.2, $(M; H')$ is a GZH-semigroup.

If in addition μ has no prime factor, there is by Theorem 3.11(ii) an array (μ_{ij}) in M such that $\mu = \mu_{ij} * \cdots * \mu_{ij}$ for each i and

$$\lim_{i\to\infty} \min_j |\hat{\mu}_{ij}(y_k)| = 1 \quad \text{for each } k.$$

So

$$E \subset V_1 := \left\{y: \hat{\mu}(y) \neq 0,\ \lim_{i\to\infty} \min_j |\hat{\mu}_{ij}(y)| = 1\right\}.$$

Now V_1 is a subgroup by Lemma 4.6, hence $y = y_1 \cdots y_n \in V_1$, $\hat{\mu}(y) \neq 0$. □

We now give new proofs of Theorems 4.5.2, 4.11.2 and 4.11.3, Corollary 6.6.2 and Theorems 6.8.1 and 6.8.2 of [10].

THEOREM 4.10. *Let $\mu \in M$ and μ have no idempotent factor. If μ is a limit of an infinitesimal array, then μ is i.d.*

PROOF. By Lemma 4.8, $\mu \in J$. By Theorem 4.4, μ is i.d. □

THEOREM 4.11. *If a uniformly infinitesimal array in $M(X_1)$ converges to μ, then μ is i.d.*

PROOF. Applying Theorem 4.10, the proof is the same as that of [10], Theorem 4.5.2. □

THEOREM 4.12. *Let $\mu \in M$. If μ has neither idempotent nor prime factor, then μ is i.d.*

PROOF. By Lemma 4.9, $\mu \in J$. By Theorems 4.4 and 3.11(iii), μ is i.d. □

THEOREM 4.13. *Every $\mu \in M$ has a representation $\mu = \lambda_H * \lambda_1 * \lambda_2$, where λ_H is the maximal idempotent factor of μ, λ_1 is i.d. and has neither idempotent nor prime factor and λ_2 is a countable convolution of prime elements.*

PROOF. We have $\mu = \lambda_H * \lambda$, where λ_H is the maximal idempotent factor of μ and λ has no idempotent factor in case $M = M(X_1)$ by [10], Theorem 4.11.1, or λ_H is the identity element of $M(X_2)$ and $\lambda = \mu$ has no idempotent factor in case $M = M(X_2)$ as X_2 has no compact subgroup. By Lemma 4.9 there are homomorphisms $H_1', H_2', \ldots$ such that $(M; H')$ is a GZH-semigroup and $H_k'(\lambda) \neq 0$ for each k. Hence $\lambda = \lambda_1 * \lambda_2$ by Theorem 3.11(i), where λ_2 is a countable convolution of prime elements and λ_1 has neither idempotent nor prime factor. By Theorem 4.12, λ_1 is i.d. □

5. Application to positive definite kernels. For the positive definite kernels we can refer to Berg, Christensen and Ressel [1]. Parthasarathy and Schmidt [12] and Horn [5] discuss some problems related with positive definite kernels and positive definite functions.

In this section, X denotes a nonvoid set, $\mathbb{C}$ is the set of complex numbers and $\mathbb{N}$ is the set of natural numbers.

DEFINITION 5.1. A function a: $X \times X \to \mathbb{C}$ is called a kernel function or a kernel. If $a(y, x) = \overline{a(x, y)}$ for each $(x, y) \in X \times X$, then a is called a Hermitian kernel. If $\sum_{1 \le j, k \le n} c_j \bar{c}_k a(x_j, x_k) \geq 0$ for each $n \in \mathbb{N}$, $(x_1, \ldots, x_n) \subset X$, $(c_1, \ldots, c_n) \subset \mathbb{C}$, then a is called a positive definite kernel. Let $P(X)$ or P denote the set of positive definite kernels defined on $X \times X$.

LEMMA 5.2. (i) *A positive definite kernel is a Hermitian kernel.*

(ii) *Let (a_n) be a net in P. If $a_n(x, y) \to a(x, y)$ for each $(x, y) \in X \times X$, then $a \in P$.*

(iii) *If the product ab of $a, b \in P$ is defined by $(ab)(x, y) = a(x, y)b(x, y)$ for each $(x, y) \in X \times X$, then $ab \in P$.*

(iv) *If P is endowed with the pointwise topology, then P is a topological semigroup.*

(v) *If $a \in P$, then $\bar{a} \in P$ and $|a|^2 \in P$.*

PROOF. Statement (iii) is Theorem 3.1.12 of [1]. The other statements are obvious. □

Let $P'(X)$ or P' be the set of positive definite kernels defined on $X \times X$ with values in $\mathbb{D} = \{z \in \mathbb{C}\colon |z| \leq 1\}$.

LEMMA 5.3. *P' is a compact topological semigroup.*

PROOF. P' is a subset of the compact set $\mathbb{D}^{X \times X}$ and, by Lemma 5.2(ii), P' is closed in $\mathbb{D}^{X \times X}$. □

THEOREM 5.4. *Let (a_{st}) be an array in P' converging to $a \in P'$. If $a(x, y) \neq 0$ and $\lim_{s \to \infty} \min_t |a_{st}(x, y)| = 1$ for all $(x, y) \in X \times X$, then $|a| \in P'$ and $|a|$ is i.d.*

PROOF. P' is a topological semigroup with the stability property for nets by Lemma 5.3. For any fixed $m, n \in \mathbb{N}$, $(x_1, \ldots, x_n) \subset X$, $(c_1, \ldots, c_n) \subset \mathbb{C}$ we defined a continuous homomorphism D_{jk}: $P' \to (\overline{\mathbb{R}}_+, +)$ by $D_{jk}(p) = -\log|p(x_j, x_k)|$ for each $j, k = 1, \ldots, n$. By Theorem 3.6 there are $b_1, \ldots, b_{2m} \in P'$ such that $a = b_1 \cdots b_{2m}$ and $D_{jk}(b_1) = \cdots = D_{jk}(b_{2m})$ for each j, k. Hence $|a^{1/m}(x_j, x_k)| = |b_1(x_j, x_k)|^2$ for each j, k. Now $|b_1|^2$ is a positive definite kernel by Lemma 5.2(v), hence

$$\sum_{1 \le j, k \le n} c_j \bar{c}_k \left| a^{1/m}(x_j, x_k) \right| = \sum_{1 \le j, k \le n} c_j \bar{c}_k \left| b_1(x_j, x_k) \right|^2 \ge 0,$$

$|a^{1/m} \in P'$, $|a| \in P'$ and $|a|$ is i.d. □

THEOREM 5.5. *If X is a countable set, then $P'(X)$ is a GMD-semigroup.*

PROOF. By Lemma 5.2 and Theorem 3.12. □

If X is a set consisting of n elements, we write $P'(X)$ as P'_n, so

$$P'_n = \{(a_{ij})_{n \times n} : (a_{ij})_{n \times n} \text{ is an } n \times n \text{ nonnegative definite matrix, } a_{ij} \in \mathbb{D} \text{ for each } i, j\}.$$

COROLLARY 5.6. *P'_n is a GMD-semigroup.* □

REMARK 5.7. Let $\mathscr{N}_\infty(X)$ denote the closure of all real-valued negative definite kernels on $X \times X$ in the space $(-\infty, \infty]^{X \times X}$. Let $\varphi \ge 0$ be a positive definite kernel on $X \times X$. Then φ is i.d. if and only if $-\log \varphi \in \mathscr{N}_\infty(X)$ ([1], Proposition 3.2.7).

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