The Annals of Probability
1993, Vol. 21, No. 1, 161–167

THE HAUSDORFF METRIC OF σ-FIELDS AND THE VALUE OF INFORMATION

BY TIMOTHY VAN ZANDT[1]

Princeton University

Using a result by Landers and Rogge, it is shown that the Hausdorff metric of σ-fields is uniformly equivalent to the metric induced by the Hausdorff distance between sets of measurable functions. An application is given to the continuity of the value of information with respect to the Hausdorff metric of σ-fields.

Let $\langle \Omega, \Sigma, \mu \rangle$ be a probability space and let $\mathfrak{F}$ be the set of (relatively) complete sub-σ-fields of Σ. Define three metrics on $\mathfrak{F}$:

1. Each complete sub-σ-field is a distinct closed subset of the pseudometric space $\langle \Sigma, \rho \rangle$, where $\rho(F, G) = \mu(F \Delta G)$. Then the Hausdorff distance between closed subsets of $\langle \Sigma, \rho \rangle$ induces the following metric on $\mathfrak{F}$:

$$\delta^1(\mathscr{F}, \mathscr{G}) = \max\left\{ \sup_{F \in \mathscr{F}} \inf_{G \in \mathscr{G}} \mu(F \Delta G),\ \sup_{G \in \mathscr{G}} \inf_{F \in \mathscr{F}} \mu(F \Delta G) \right\}.$$

   This metric was introduced by Boylan (1971), and is referred to as the Hausdorff metric of σ-fields.
2. Each complete sub-σ-field $\mathscr{F}$ can be identified with the expectations operator $E[\cdot|\mathscr{F}]: L_1 \to L_1$. Then uniform convergence on the set Φ of measurable functions taking values in $[0, 1]$ induces the following metric on $\mathfrak{F}$:

$$\delta^2(\mathscr{F}, \mathscr{G}) = \sup_{f \in \Phi} \| E[f|\mathscr{F}] - E[f|\mathscr{G}] \|_1.$$

3. Let $\langle X, d \rangle$ be a separable metric space with more than one element. Let $V(X)$ be the set of equivalence classes of measurable functions from $\langle \Omega, \Sigma, \mu \rangle$ into X. Let θ be the metric defined by

$$\theta(f, g) = \inf\{\varepsilon > 0 | \mu\{\omega \in \Omega | d(f(\omega), g(\omega)) > \varepsilon\} < \varepsilon\},$$

   which induces the topology of convergence in measure on $V(X)$. Each complete sub-σ-field $\mathscr{F}$ can be identified with the set of $\mathscr{F}$-measurable functions into X, which we denote by $\mathfrak{M}(\mathfrak{F})$ and which is a closed subset of $\langle V(X), \theta \rangle$, distinct for each $\mathscr{F} \in \mathfrak{F}$. Then the Hausdorff distance between

Received March 1990; revised April 1991.

[1]This work was done while the author was a postdoctoral member of the technical staff at the Mathematical Sciences Research Center of AT & T Bell Laboratories.

AMS 1991 *subject classifications*. Primary 62B10; secondary 60A10.

Key words and phrases. Hausdorff metric of σ-field, value of information.

closed subsets of $\langle V(X), \theta \rangle$ induces the following metric on $\mathfrak{F}$:

$$\delta^3(\mathscr{F}, \mathscr{G}) = \max\left\{ \sup_{f \in \mathfrak{M}(\mathscr{F})} \inf_{g \in \mathfrak{M}(\mathscr{G})} \theta(f, g), \sup_{g \in \mathfrak{M}(\mathscr{G})} \inf_{f \in \mathfrak{M}(\mathscr{F})} \theta(f, g) \right\}.$$

Rogge (1974) showed that δ^1 and δ^2 are equivalent, and Rogge (1974), Theorem 4, and Landers and Rogge (1986), Corollary 5, established the following uniform bounds:

$$\delta^1(\mathscr{F}, \mathscr{G}) \leq \delta^2(\mathscr{F}, \mathscr{G}) \leq 8\delta^1(\mathscr{F}, \mathscr{G}).$$

These inequalities have made δ^1 useful for studying uniform rates of martingale convergence. Significantly, δ^1 also characterizes uniform convergence of expectations for nonnested sequences of sub-σ-fields.

In this article, I show that δ^1 and δ^3 are also equivalent, with the following uniform bounds:

$$\min\{\gamma/2, \delta^1(\mathscr{F}, \mathscr{G})\} \leq \delta^3(\mathscr{F}, \mathscr{G}) \leq 4\delta^1(\mathscr{F}, \mathscr{G}), \tag{1}$$

where $\gamma = \sup_{x,y \in X} d(x, y)$ is the diameter of $\langle X, d \rangle$. These inequalities are closely related to the bounds on δ^2, since when $X = [0, 1]$, $\mathfrak{M}(\mathscr{F}) = E[\Phi | \mathscr{F}]$. However, these inequalities also relate δ^1 to the measurability of functions even when X is not a convex subset of a linear space or conditional expectations are not defined.

An application to the lattice properties of δ^1 is given, using the duality between the sup (join) of two sub-σ-fields and the sup (sum) of the corresponding subspaces of measurable functions into $\mathbb{R}$. We obtain uniform bounds on the continuity of the join operation that are tighter than those derived by Zbâganu (1986).

The result is also applied, more directly, to the comparison of information structures, including nonnested information structures. When the sub-σ-fields are interpreted as information structures and X is interpreted as a set of available actions, $\mathfrak{M}(\mathscr{F})$ is the set of informationally feasible decision rules given information $\mathscr{F}$. Then, heuristically, our bounds on δ^3 imply that two sub-σ-fields are close according to the Hausdorff metric δ^1 if and only if the corresponding sets of informationally feasible decision rules are close.

The main result of this note is then the following theorem, from which (1) follows immediately.

THEOREM 1. *Let γ be the diameter of $\langle X, d \rangle$. For all $\mathscr{F}, \mathscr{G} \in \mathfrak{F}$:*

$$\min\left\{\gamma/2, \sup_{F \in \mathscr{F}} \inf_{G \in \mathscr{G}} \mu(F \Delta G)\right\} \underset{(a)}{\leq} \sup_{f \in \mathfrak{M}(\mathscr{F})} \inf_{g \in \mathfrak{M}(\mathscr{G})} \theta(f, g) \underset{(b)}{\leq} 4 \sup_{F \in \mathscr{F}} \inf_{G \in \mathscr{G}} \mu(F \Delta G).$$

PROOF. *Proof of inequality* (a): Let $x, y \in X$. For $F \in \mathscr{F}$, let $\xi_F \colon \Omega \to X$ be the function that is equal to x on F and to y on F^c. Then

$$\sup_{F \in \mathscr{F}} \inf_{g \in \mathfrak{M}(\mathscr{G})} \theta(\xi_F, g) \leq \sup_{f \in \mathfrak{M}(\mathscr{F})} \inf_{g \in \mathfrak{M}(\mathscr{G})} \theta(f, g). \tag{2}$$

We show below that for any $F \in \mathscr{F}$ and $g \in \mathfrak{M}(\mathscr{G})$,

$$\min\left\{d(x,y)/2,\ \inf_{G\in\mathscr{G}} \mu(F\Delta G)\right\} \le \theta(\xi_F, g). \tag{3}$$

Taking the infimum of $\theta(\xi_F, g)$ in (3) over $g \in \mathfrak{M}(\mathscr{G})$ and then the supremum of both sides of (3) over $F \in \mathscr{F}$ yields

$$\min\left\{d(x,y)/2,\ \sup_{F\in\mathscr{F}}\inf_{G\in\mathscr{G}} \mu(F\Delta G)\right\} \le \sup_{F\in\mathscr{F}}\inf_{g\in\mathfrak{M}(\mathscr{G})} \theta(\xi_F, g). \tag{4}$$

After combining this with (2) and taking the supremum of $d(x,y)$ over $x, y \in X$, we get (a).

Here is the explanation of (3): Let $F \in \mathscr{F}$ and let $g \in \mathfrak{M}(\mathscr{G})$. Let

$$G = \{\omega \in \Omega | d(x, g(\omega)) < d(x,y)/2\},$$

which belongs to $\mathscr{G}$. By the triangular inequality, $d(y, g(\omega)) > d(x,y)/2$ for all $\omega \in G$. If $\omega \in F$ and $\omega \notin G$, then $\xi_F(\omega) = x$ and $d(x, g(\omega)) \ge d(x,y)/2$. If $\omega \notin F$ and $\omega \in G$, then $\xi_F(\omega) = y$ and $d(y, g(\omega)) > d(x,y)/2$. In either case, $d(\xi_F(\omega), g(\omega)) \ge d(x,y)/2$. Thus

$$\mu\{\omega \in \Omega | d(\xi_F(\omega), g(\omega)) \ge d(x,y)/2\} \ge \mu(F\Delta G).$$

Then, by the definition of θ,

$$\min\{d(x,y)/2, \mu(F\Delta G)\} \le \theta(\xi_F, g). \tag{5}$$

Since $G \in \mathscr{G}$, (3) holds.

Proof of inequality (b): Let $f\colon \Omega \to X$ be $\mathscr{F}$-simple, so that there is an $\mathscr{F}$-measurable partition $\{F_1, \dots, F_n\}$ of Ω, indexed so that $F_i \ne F_j$ for $i \ne j$, such that f is constant on each F_i, for $i = 1, \dots, n$. We will show that

$$\inf_{g\in\mathfrak{M}(\mathscr{G})} \theta(f,g) \le \inf\{\mu\{\omega \in \Omega | f(\omega) \ne g(\omega)\} | g \text{ is } \mathscr{G}\text{-simple}\} \tag{6}$$

$$\le \inf\left\{\mu\left(\bigcup_{i=1}^{n} (F_i \Delta G_i)\right) \middle| G_i \in \mathscr{G} \text{ for } i = 1, \dots, n, \text{ and } G_i \cap G_j = \varnothing \text{ for } i \ne j\right\} \tag{7}$$

$$\le \inf\left\{\mu\left(\bigcup_{i=1}^{n} (F_i \Delta G_i)\right) \middle| G_i \in \mathscr{G} \text{ for } i = 1, \dots, n\right\} \tag{8}$$

$$\le 4 \sup_{F\in\mathscr{F}}\inf_{G\in\mathscr{G}} \mu(F\Delta G). \tag{9}$$

Since $\langle X, d\rangle$ is separable, the $\mathscr{F}$-simple functions are θ-dense in $\mathfrak{M}(\mathscr{F})$. Therefore, the LHS of (6) is less than or equal to the RHS of (9) for all $\mathscr{F}$-measurable functions f, and thus (b) holds.

Here are the explanations of inequalities (6) through (9):

(6): This holds because $\theta(f, g) \le \mu\{\omega \in \Omega | f(\omega) \ne g(\omega)\}$.

(7): Let $G_i \in \mathscr{G}$ for $i = 1, \dots, n$ be such that $G_i \cap G_j = \varnothing$ if $i \ne j$. (Note that for arbitrarily many $i \in \{1, \dots, n\}$, G_i may be empty. Thus, such a collection does exist.) Let $g: \Omega \to X$ be a $\mathscr{G}$-simple function that is equal to $f(F_i)$ on G_i for each $i = 1, \dots, n$. If $f(\omega) \ne g(\omega)$, then $\omega \in F_j \setminus G_j$, where j is the unique element of $\{1, \dots, n\}$ such that F_j contains ω. Thus,

$$\{\omega \in \Omega | f(\omega) \ne g(\omega)\} \subset \bigcup_{i=1}^{n} (F_i \Delta G_i).$$

Since g is $\mathscr{G}$-simple, (7) holds.

(8): Let $G_i \in \mathscr{G}$ for $i = 1, \dots, n$. Let $G_i' = G_i \setminus \bigcup_{j \ne i} G_j$ for each $i = 1, \dots, n$. Let $j \in \{1, \dots, n\}$. Since $G_j' \subset G_j$, $G_j' \setminus F_j$ is a subset of $G_j \setminus F_j$. $F_j \setminus G_j'$ is also a subset of $\bigcup_{i=1}^{n}(F_i \Delta G_i)$, as follows.

> If $\omega \in F_j \setminus G_j'$, then either $\omega \in F_j \setminus G_j$, or there is $k \ne j$ such that $\omega \in G_k$. In the latter case, since $\{F_1, \dots, F_n\}$ is a partition and $\omega \in F_j$, $\omega \in G_k \setminus F_k$.

Thus, $\bigcup_{i=1}^{n}(F_i \Delta G_i') \subset \bigcup_{i=1}^{n}(F_i \Delta G_i)$. Since $G_i' \cap G_j' = \varnothing$ if $i \ne j$, (8) holds.

(9): Note that

$$\begin{aligned}
&\inf\left\{\mu\left(\bigcup_{i=1}^{n} (F_i \Delta G_i)\right)\middle| G_i \in \mathscr{G} \text{ for } i = 1, \dots, n\right\} \\
&\quad\le \inf\left\{\sum_{i=1}^{n} \mu(F_i \Delta G_i)\middle| G_i \in \mathscr{G} \text{ for } i = 1, \dots, n\right\} \\
&\quad= \sum_{i=1}^{n} \inf_{G \in \mathscr{G}} \mu(F_i \Delta G).
\end{aligned}$$

Landers and Rogge (1986), Theorem 1, prove that if $\{F_1, \dots, F_n\} \subset \mathscr{F}$ is disjoint, as assumed here, then this last quantity is less than or equal to $4 \sup_{F \in \mathscr{F}} \inf_{G \in \mathscr{G}} \mu(F \Delta G)$. □

Zbâganu (1986), Proposition 8, has shown that the join operation on sub-σ-fields is uniformly continuous in the metric δ^1, with the following bounds:

$$\delta^1\left(\bigvee_{n=1}^{\infty} \mathscr{F}_n, \bigvee_{n=1}^{\infty} \mathscr{G}_n\right) \le 16 \sum_{n=1}^{\infty} \delta^1(\mathscr{F}_n, \mathscr{G}_n). \tag{10}$$

In the first corollary to Theorem 1, we tighten this bound, replacing 16 by 4 for finite sequences of sub-σ-fields. This is extended to infinite sequences using Zbâganu's result.

COROLLARY 2. *Let* $\langle \mathscr{F}_1, \mathscr{F}_2, \dots \rangle$ *and* $\langle \mathscr{G}_1, \mathscr{G}_2, \dots \rangle$ *be sequences in* $\mathfrak{F}$. *Then*

$$\delta^1\left(\bigvee_{n=1}^{\infty} \mathscr{F}_n, \bigvee_{n=1}^{\infty} \mathscr{G}_n\right) \leq 4 \sum_{n=1}^{\infty} \delta^1(\mathscr{F}_n, \mathscr{G}_n).$$

PROOF. Let $X = \mathbb{R}$. Then for $\mathscr{H}_1$ and $\mathscr{H}_2$ in $\mathfrak{F}$, $\mathfrak{M}(\mathscr{H}_1)$ and $\mathfrak{M}(\mathscr{H}_2)$ are closed linear subspaces of $\langle V(X), \theta \rangle$, and

$$(11) \quad \mathfrak{M}(\mathscr{H}_1 \vee \mathscr{H}_2) = \overline{\mathrm{sp}}(\mathfrak{M}(\mathscr{H}_1) \cup \mathfrak{M}(\mathscr{H}_2)) = \mathrm{cl}(\mathfrak{M}(\mathscr{H}_1) + \mathfrak{M}(\mathscr{H}_2)).$$

Recall also that θ then satisfies the following triangular inequality:

$$(12) \qquad \theta(f_1 + f_2, g_1 + g_2) \leq \theta(f_1, g_1) + \theta(f_2, g_2).$$

Let $\mathscr{F}_1, \mathscr{F}_2, \mathscr{G}_1, \mathscr{G}_2 \in \mathfrak{F}$. Then, as explained below,

$$(13) \quad \begin{aligned} &\sup_{f \in \mathfrak{M}(\mathscr{F}_1 \vee \mathscr{F}_2)} \inf_{g \in \mathfrak{M}(\mathscr{G}_1 \vee \mathscr{G}_2)} \theta(f, g) \\ &= \sup_{f_1 \in \mathfrak{M}(\mathscr{F}_1),\, f_2 \in \mathfrak{M}(\mathscr{F}_2)} \inf_{g_1 \in \mathfrak{M}(\mathscr{G}_1),\, g_2 \in \mathfrak{M}(\mathscr{G}_2)} \theta(f_1 + f_2, g_1 + g_2) \\ &\leq \sup_{f_1 \in \mathfrak{M}(\mathscr{F}_1),\, f_2 \in \mathfrak{M}(\mathscr{F}_2)} \inf_{g_1 \in \mathfrak{M}(\mathscr{G}_1),\, g_2 \in \mathfrak{M}(\mathscr{G}_2)} \theta(f_1, g_1) + \theta(f_2, g_2) \\ &= \sup_{f_1 \in \mathfrak{M}(\mathscr{F}_1)} \inf_{g_1 \in \mathfrak{M}(\mathscr{G}_1)} \theta(f_1, g_1) + \sup_{f_2 \in \mathfrak{M}(\mathscr{F}_2)} \inf_{g_2 \in \mathfrak{M}(\mathscr{G}_2)} \theta(f_2, g_2). \end{aligned}$$

The first equality in (13) follows from (11), the inequality follows from (12), and the last equality is by simple rearrangement.

Because (13) also holds when the roles of $\mathscr{F}_n$ and $\mathscr{G}_n$ are reversed, we can write

$$(14) \qquad \delta^3(\mathscr{F}_1 \vee \mathscr{F}_2, \mathscr{G}_1 \vee \mathscr{G}_2) \leq \delta^3(\mathscr{F}_1, \mathscr{G}_1) + \delta^3(\mathscr{F}_2, \mathscr{G}_2).$$

Since $\bigvee_{n=1}^{\infty} \mathscr{H}_n = (\bigvee_{n=1}^{N} \mathscr{H}_n) \vee (\bigvee_{n=N+1}^{\infty} \mathscr{H}_n)$, we can apply (14) inductively to obtain

$$(15) \quad \delta^3\left(\bigvee_{n=1}^{\infty} \mathscr{F}_n, \bigvee_{n=1}^{\infty} \mathscr{G}_n\right) \leq \sum_{n=1}^{N} \delta^3(\mathscr{F}_n, \mathscr{G}_n) + \delta^3\left(\bigvee_{n=N+1}^{\infty} \mathscr{F}_n, \bigvee_{n=N+1}^{\infty} \mathscr{G}_n\right),$$

for any sequences $\langle \mathscr{F}_1, \mathscr{F}_2, \dots \rangle$ and $\langle \mathscr{G}_1, \mathscr{G}_2, \dots \rangle$ in $\mathfrak{F}$ and for any positive integer N. Theorem 1 and Zbâganu's result [(10)] imply that

$$\begin{aligned} \delta^3\left(\bigvee_{n=N+1}^{\infty} \mathscr{F}_n, \bigvee_{n=N+1}^{\infty} \mathscr{G}_n\right) &\leq 4\delta^1\left(\bigvee_{n=N+1}^{\infty} \mathscr{F}_n, \bigvee_{n=N+1}^{\infty} \mathscr{G}_n\right) \\ &\leq 64 \sum_{n=N+1}^{\infty} \delta^1(\mathscr{F}_n, \mathscr{G}_n). \end{aligned}$$

The limit of this last term as $N \to \infty$ is 0 if $\sum_{n=1}^{\infty} \delta^1(\mathscr{F}_n, \mathscr{G}_n)$ is finite, and so in the limit (15) becomes

$$\delta^3\left(\bigvee_{n=1}^{\infty} \mathscr{F}_n, \bigvee_{n=1}^{\infty} \mathscr{G}_n\right) \le \sum_{n=1}^{\infty} \delta^3(\mathscr{F}_n, \mathscr{G}_n).$$

Now apply Theorem 1 to complete the proof. □

Our second application is to static decision theory. Interpret $\omega \in \Omega$ as a state, $x \in X$ as an action, $f \in V(X)$ as a decision rule that specifies the action to be taken in each state, and $\mathscr{F} \in \mathfrak{F}$ as an information structure or an experiment. A decision rule is informationally feasible given $\mathscr{F}$ if and only if it is $\mathscr{F}$-measurable.

Let u: $V(X) \to \mathbb{R}$ be a uniformly continuous utility function that is bounded above. Note that u may have an additively separable representation such as

$$f \underset{u}{\mapsto} -\int_\Omega L(\omega, f(\omega))\, d\mu,$$

where L: $\Omega \times X \to \mathbb{R}$ is interpreted as a loss function, but that this is not necessary. Let

$$\hat{v}(\mathscr{F}) = \sup_{f \in \mathfrak{M}(\mathscr{F})} u(f).$$

Let $\mathscr{F}_0$ be the null information structure, that is, the complete sub-σ-field generated by $\{\Omega, \varnothing\}$. The value of $\mathscr{F} \in \mathfrak{F}$ is defined to be

$$v(\mathscr{F}) = \hat{v}(\mathscr{F}) - \hat{v}(\mathscr{F}_0);$$

it is the difference between the attainable utility given information $\mathscr{F}$ and the attainable utility given no information.

COROLLARY 3. *The value of information map* v: $\mathfrak{F} \to \mathbb{R}$ *is uniformly continuous with respect to the Hausdorff metric* δ^1 *on* $\mathfrak{F}$.

(The continuity of the value of information has been shown by Allen (1983) for u additively separable and X a compact, convex subset of $\mathbb{R}^n$, using the continuity of conditional expected utility.)

PROOF OF COROLLARY 3. Let $\varepsilon > 0$. Let δ be such that

$$\theta(f, g) < \delta \Rightarrow |u(f) - u(g)| < \varepsilon.$$

Suppose $\mathscr{F}, \mathscr{G} \in \mathfrak{F}$ are such that $\delta^1(\mathscr{F}, \mathscr{G}) < \delta/4$. By Theorem 1, for $f \in \mathfrak{M}(\mathscr{F})$ there is $g \in \mathfrak{M}(\mathscr{G})$ such that $\theta(f, g) < \delta$, and thus $u(f) - u(g) < \varepsilon$. The converse also holds. Thus, $|\hat{v}(\mathscr{F}) - \hat{v}(\mathscr{G})| < \varepsilon$. □

The two corollaries taken together indicate that the value of information is continuous when information is combined from finitely many sources.

REFERENCES

ALLEN, B. (1983). Neighboring information and distributions of agents' characteristics under uncertainty. *J. Math. Econom.* **12** 63–101.

BOYLAN, E. (1971). Equiconvergence of martingales. *Ann. Math. Statist.* **42** 552–559.

LANDERS, D. and ROGGE, L. (1986). An inequality for the Hausdorff-metric of σ-fields. *Ann. Probab.* **14** 724–730.

ROGGE, L. (1974). Uniform inequalities for conditional expectations. *Ann. Probab.* **2** 486–489.

ZBÂGANU, G. (1986). Some entropy-like indices and their connection with the metrization of the σ-algebras. *Stud. Cerc. Mat.* **38** 76–88.

DEPARTMENT OF ECONOMICS
PRINCETON UNIVERSITY
PRINCETON, NEW JERSEY 08544-1021