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A STRONG INVARIANCE PRINCIPLE CONCERNING THE J-UPPER ORDER STATISTICS FOR STATIONARY GAUSSIAN SEQUENCES

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It is shown that in the case of stationary Gaussian processes, the Jth ($J \geq 1$) record times $\{T_n, n \geq 1\}$ and the corresponding J-upper order statistics $\{X_{T_n - J+1, T_n}, \dots, X_{T_n, T_n}\}$ can almost surely be identified via a translation of the time index n to the corresponding elements defined on a sequence of independent and identically distributed random variables. A construction method for approximating sequences of record times and the corresponding upper order statistics introduced by Haiman (1987a, b) for the case $J = 1$ is extended and applied under weaker conditions concerning the covariance function, and also under different sets of new hypotheses.

1. Introduction. Let $\{X_n, -\infty < n < \infty\}$ be a stationary Gaussian sequence centered at 0 with covariance function

$$\Gamma(n) = E(X_i X_{i+n}), \qquad \Gamma(0) = 1, \qquad n = 0, \pm 1, \pm 2, \dots. \tag{1.1}$$

Let $X_{1,n} < X_{2,n} < \cdots < X_{n-1,n} < X_{n,n}$ denote the order statistics of $X_1, \dots, X_n$. Let $J \geq 1$ be a fixed integer and $\Theta_0 > 0$ a fixed real number. Denote

$$\begin{aligned} T_1 &= \inf\{n; n \geq J, X_{n-J+1,n} > \Theta_0\}, \\ \underline{\Theta}_1 &= \{X_{T_1-J+1,T_1}, \dots, X_{T_1,T_1}\} \end{aligned} \tag{1.2}$$

and, for any $k \geq 2$, set

$$\begin{aligned} T_k &= \inf\{n; n > T_{k-1}, X_{n-J+1,n} > X_{n-J,n-1}\}, \\ \underline{\Theta}_k &= \{X_{T_k-J+1,T_k}, \dots, X_{T_k,T_k}\} =: \{\Theta_k^J, \Theta_k^{J-1}, \dots, \Theta_k^1\}. \end{aligned} \tag{1.3}$$

We shall call Θ_k^J the kth Jth-*upper record*. Note that our definition is a slight variant of the usual definition of the Jth-upper record [see, e.g., Resnick (1987), page 243, and Deheuvels (1988)]. Note also that when $J > 1$, Θ_k^1 is not the kth-upper record. T_k's are the Jth record times, and $\{T_k, \underline{\Theta}_k\}_{k \geq 1}$ describe the history of the J largest values of the sample $X_1, \dots, X_n$ for $n \geq T_1$. In the sequel, we will call $\{T_k, \underline{\Theta}_k\}$ the Jth record sequence based on $\{X_n, n \geq 1\}$ and Θ_0.

DEFINITION. A sequence $\{\underline{r}_k = (r_k^J, \dots, r_k^1) \in \mathbb{R}^J, k \geq 1\}$ is called J-ordered if the following conditions hold: For any $k \geq 1$, $r_k^J < \cdots < r_k^1$ and, either

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$r_k^1 < r_{k+1}^1$ and then $r_{k+1}^J = r_k^{J-1}, r_{k+1}^{J-1} = r_k^{J-2}, \dots, r_{k+1}^2 = r_k^1$; or there exists j, $2 \le j \le J$, such that $r_k^j < r_{k+1}^j < r_k^{j-1}$, and then $r_{k+1}^i = r_k^{i-1}$ if $i > j$ and $r_{k+1}^i = r_k^i$ if $i \le j$.

Note that when $\{X_n,\ n \ge 1\}$ is a sequence of iid (independent and identically distributed) random variables with a continuous distribution, then for any increasing sequence of positive integers $\{t_k,\ k \ge 1\}$ and any J-ordered sequence $\{\underline{r}_k = (r_k^J, \dots, r_k^1),\ k \ge 1\}$, we have the following Markov behavior [see Deheuvels (1974)] of the sequence $\{T_k, \underline{\Theta}_k\}_{k \ge 1}$:

$$P\big[T_{n+1} = t_{n+1}, \Theta_{n+1}^j < r_{n+1}^j \big| T_1 = t_1, \underline{\Theta}_1 = \underline{r}_1, \dots, T_n = t_n, \underline{\Theta}_n = \underline{r}_n\big]$$

$$= \big[P\big(X_1 < r_n^J\big)\big]^{t_{n+1}-t_n-1} \cdot P\big[r_n^j < X_1 < r_{n+1}^j\big], \tag{1.4}$$

$$n \ge 1, j = 1, \dots, J.$$

Consider now Hypotheses H_1, H_2 and H_3.

HYPOTHESIS H_1.

$$\sum_{n=1}^{\infty} |\Gamma(n)| < 1/2. \tag{1.5}$$

(1.6) *There exists an* $\varepsilon > 0$ *such that* $\limsup_{n \to \infty} |\Gamma(n)| n^{4+\varepsilon} = 0.$

HYPOTHESIS H_2. $\{X_n, n \ge 1\}$ *has a spectral density* $f(\lambda) \ge 0$, $-\pi < \lambda < \pi$, *such that*

$$\int_{-\pi}^{\pi} [f(\lambda)]^{-1} \, d\lambda < \infty. \tag{1.7}$$

(1.8) *There exists an* $\varepsilon > 0$ *such that* $\limsup_{n \to \infty} |\Gamma(n)| n^{6+\varepsilon} = 0.$

To formulate Hypothesis H_3, we need the following preliminaries:

Let $\rho(\tau)$ be the coefficient of regularity of the sequence $\{X_n, -\infty < n < \infty\}$, defined by

$$\rho(\tau) = \max |E(\eta_1 \eta_2)|, \tag{1.9}$$

where the maximum is taken over all $\eta_1 \in H(-\infty, 0)$ and $\eta_2 \in H(\tau, \infty)$, $\tau \ge 1$, with $E(\eta_1^2) = E(\eta_2^2) = 1$, and where $H(r, s)$ denotes the Hilbert space of random variables generated by $\{X_n; r \le n \le s\}$.

HYPOTHESIS H_3. *There exists an* $\varepsilon > 0$ *such that*

$$\limsup_{n \to \infty} \rho(n) n^{6+\varepsilon} = 0. \tag{1.10}$$

Our main result is the following:

THEOREM 1.1. *Let Hypothesis* H_1, H_2 *or* H_3 *hold. Then, for any fixed* $J \geq 1$, *there exists a probability space which carries, in addition to the sequence* $\{X_n,\ n \in (-\infty, \infty)\}$, *an iid sequence* $\{X_n^*,\ n \geq 1\}$ *of random variables each having a* $\mathcal{N}(0,1)$ *distribution. Moreover, there exists a* Θ_0 (>0) *such that if* $\{T_k, \underline{\Theta}_k\}_{k \geq 1}$ *is the Jth record sequence based on* $\{X_n,\ n \in (-\infty, \infty)\}$ *and* Θ_0 *and* $\{S_k, \underline{R}_k\}_{k \geq 1}$ *is the Jth record sequence based on* $\{X_n^*,\ n \geq 1\}$ *and* Θ_0, *then there exist almost surely an* n_0 *and* q *such that for all* $n \geq n_0$, *we have*

$$S_n = T_{n-q} \quad \text{and} \quad \underline{R}_n = \underline{\Theta}_{n-q}. \tag{1.11}$$

Moreover, if $\{\underline{M}_n\}_{n \geq J} := \{(X_{n-J+1,n}, \ldots, X_{n,n})\}_{n \geq J}$ *and* $\{\underline{M}_n^*\}_{n \geq J} := \{(X_{n-J+1,n}^*, \ldots, X_{n,n}^*)\}_{n \geq J}$, *then there exists almost surely an* n_1 *and* q' *such that for any* $n \geq n_1$, *we have* $\underline{M}_n^* = \underline{M}_{n-q'}$.

The proof of this theorem is given in Section 2.

REMARK 1. For $J = 1$, Haiman (1987a) proved Theorem 1.1 under:

HYPOTHESIS H_1'. (i) $\sum_{n=1}^{\infty} |\Gamma(n)| < 1/2$ *and* (ii) *there exists an* $\varepsilon > 0$ *such that* $\limsup_{n \to \infty} |\Gamma(n)| n^{4((1+\delta)/(1-\delta))+\varepsilon} = 0$ *with* $\delta = \max_{n \geq 1} |\Gamma(n)| < 1$.

It may be observed that H_1'(ii) is more restrictive than (1.6) [since $((1+\delta)/(1-\delta)) > 1$].

REMARK 2. Note that under Hypotheses H_1, H_2 or H_3, f is continuous and $f(\lambda) = 1 + 2\sum_{n=1}^{\infty} \Gamma(n) \cos n\lambda > 0$. Consequently, (1.5) $\Rightarrow \inf_\lambda f(\lambda) > 0 \Rightarrow$ (1.7) and (1.8) $\Rightarrow$ (1.6).

REMARK 3. A weakly stationary sequence $\{Y_n,\ n \geq 1\}$ is called strong mixing (or α-mixing) if

$$\alpha(\tau) = \max \left| P(A \cap B) - P(A) \cdot P(B) \right| \to 0 \quad \text{as } \tau \to \infty, \tag{1.12}$$

where the maximum is over $A \in \sigma\{Y_j;\ j \leq t\}$ and $B \in \sigma\{Y_l;\ l \geq t + \tau\}$.

When $\{Y_n, n \geq 1\}$ is Gaussian, then for any $\tau > 0$ [see Ibragimov and Rozanov (1974), page 133],

$$\alpha(\tau) \leq \rho(\tau) \leq 2\pi\alpha(\tau). \tag{1.13}$$

Thus in (1.10), $\rho(n)$ may be replaced by the strong mixing coefficient $\alpha(n)$.

REMARK 4. It is easy to show [cf. Ibragimov and Rozanov (1974), page 174], that for any $n \geq 1$, $\Gamma(n) \leq \rho(n) \int_{-\pi}^{\pi} f(\lambda)\, d\lambda$. Thus (1.10) $\Rightarrow$ (1.8).

Moreover, for Gaussian sequences, complete regularity [i.e., $\lim_{n \to \infty} \rho(n) = 0$] implies regularity [see Ibragimov and Rozanov (1974), pages 130–136] which is equivalent to the well-known condition

$$\int_{-\pi}^{\pi} \log f(\lambda)\, d\lambda > -\infty. \tag{1.14}$$

Thus (1.10) $\Rightarrow$ (1.14).

It may be shown [see Grenander and Rosenblatt (1957), pages 65–69 and 82–83] that (1.7) $\Rightarrow$ (1.14), which gives a compensation phenomenon similar to that observed in Remark 2.

To summarize, we have the following relationship between the hypotheses:

$$\begin{array}{c} (1.5) \Rightarrow (1.7) \Rightarrow \int_{-\pi}^{\pi} \log f(\lambda)\, d\lambda > -\infty \\ \Uparrow \\ (1.6) \Leftarrow (1.8) \Leftarrow \text{Hypothesis } H_3. \end{array}$$

The following theorem gives the full characterization in terms of spectral density of the weakly stationary sequences (not necessarily Gaussian) satisfying a condition similar to (1.10).

THEOREM 1.2 [Ibragimov and Rozanov (1974), page 212]. *A necessary and sufficient condition for*

$$\rho(\tau) = O(\tau^{-r-\beta}), \qquad \tau \to \infty,\ r \in \mathbb{N},\ 0 < \beta < 1,$$

is that

$$f(\lambda) = |P(e^{i\lambda})|^2 \omega(\lambda),$$

where $P(Z)$ is a polynomial with zeros on $|Z| = 1$, $\omega(\lambda) \geq m > 0$, and the rth derivative of ω satisfies the Lipschitz condition of order β (for every λ_1, λ_2, $|\omega(\lambda_1) - \omega(\lambda_2)| < c|\lambda_1 - \lambda_2|^{\beta}$).

Theorem 1.1 has direct applications in extreme value theory by enabling one to reduce limiting theorems for dependent sequences to the iid case by means of a strong invariance principle. For the general classical results dealing with the extreme value theory for stationary processes, the reader is referred to Leadbetter and Rootzén (1988) and the references cited therein.

We shall prove Theorem 1.1 by first constructing on the probability space (eventually enlarged by products) on which $\{X_n, n \geq 1\}$ is defined, a sequence $\{S_n, \underline{R}_n\}_{n\geq 1}$ such that: (a) $\{S_k, \underline{R}_k\}_{k\geq 1}$ has the same distribution as the Jth record sequence based on an iid sequence of $\mathcal{N}(0,1)$ random variables and Θ_0 [i.e., satisfying (1.4) with T_k and $\underline{\Theta}_k$ replaced by S_k and $\underline{R}_k$, respectively]; (b) there exists almost surely an n_0 and a q such that for any $n \geq n_0$, we have $S_n = T_{n-q}$ and $\underline{R}_n = \underline{\Theta}_{n-q}$. The construction of the iid sequence $\{X_n^*, n \geq 1\}$ such that $\{S_k, \underline{R}_k\}_{k\geq 1}$ is the jth record sequence based on $\{X_n^*, n \geq 1\}$ and Θ_0 is then straightforward.

2. Proof of Theorem 1.1 for $J = 1$.

2.1. *Some preliminaries.* In this section, we shall first prove some basic lemmas which are needed to prove Theorem 1.1. To make the results self-contained and to provide proper motivation, we shall first prove Theorem 1.1 for $J = 1$.

The following lemma is a specialization of the classical comparison theorem of Berman (1964).

LEMMA 2.1. *Suppose $\xi_1, \dots, \xi_n$ are standard normal random variables with positive definite covariance matrix $\Lambda^1 = (\Lambda^1_{ij})$ and $\eta_1, \dots, \eta_n$ are standard normal random variables with positive definite covariance matrix $\Lambda^0 = (\Lambda^0_{ij})$. Further, let $a_1 < b_1, \dots, a_n < b_n$ be real numbers. Then*

$$|P\{a_1 < \xi_1 < b_1, \dots, a_n < \xi_n < b_n\} - P\{a_1 < \eta_1 < b_1, \dots, a_n < \eta_n < b_n\}|$$

$$\leq \frac{2}{\pi} \sum_{1 \leq i < j \leq n} \left|\Lambda^1_{ij} - \Lambda^0_{ij}\right| \left(1 - \delta^2_{ij}\right)^{-1/2} \exp\left(-\frac{c_i^2 + c_j^2}{2(1 + \delta_{ij})}\right), \tag{2.1}$$

where $c_i = \min(|a_i|, |b_i|)$, $i = 1, \dots, n$, and $\delta_{ij} = \max(|\Lambda^1_{ij}|, |\Lambda^0_{ij}|)$.

PROOF. Let f_1 be the joint density of $(\xi_1, \dots, \xi_n)$ and f_0 the joint density of $(\eta_1, \dots, \eta_n)$ (which exist because Λ^1 and Λ^0 are positive definite). Writing $\underline{a} = (a_1, \dots, a_n)$ and $\underline{b} = (b_1, \dots, b_n)$, we have

$$P(a_1 < \xi_1 < b_1, \dots, a_n < \xi_n < b_n) = \int_{\underline{a}}^{\underline{b}} \cdots \int f_1(y_1, \dots, y_n)\, d\underline{y}$$

and

$$P(a_1 < \eta_1 < b_1, \dots, a_n < \eta_n < b_n) = \int_{\underline{a}}^{\underline{b}} \cdots \int f_0(y_1, \dots, y_n)\, d\underline{y}.$$

For any $0 < h < 1$, put

$$\Lambda^h = (\Lambda_{ij}\colon 1 \leq i, j \leq n) = h\Lambda^1 + (1 - h)\Lambda^0, \tag{2.2}$$

which is positive definite.

Let f_h be the density of $\mathscr{N}(0, \Lambda^h)$ random variable and put

$$F(h) = \int_{\underline{a}}^{\underline{b}} \cdots \int f_h(y_1, \dots, y_n)\, d\underline{y}.$$

The difference in the first term in (2.1) equals $F(1) - F(0)$ and

$$F(1) - F(0) = \int_0^1 F'(h)\, dh = \int_0^1 \left[\int_{\underline{a}}^{\underline{b}} \cdots \int \frac{\partial f_h(y_1, \dots, y_n)}{\partial h}\, d\underline{y}\right] dh. \tag{2.3}$$

From (2.2), it follows that

$$\frac{\partial \Lambda_{ij}^h}{\partial h} = \Lambda_{ij}^1 - \Lambda_{ij}^0$$

and hence that

$$(2.4)\quad F'(h) = \sum_{i<j} \int \int_{\underline{a}}^{\underline{b}} \cdots \int \frac{\partial f_h}{\partial \Lambda_{ij}^h} \frac{\partial \Lambda_{ij}^h}{\partial h}\, d\underline{y} = \sum_{i<j} \left(\Lambda_{ij}^1 - \Lambda_{ij}^0\right) \int_{\underline{a}}^{\underline{b}} \cdots \int \frac{\partial f_h}{\partial \Lambda_{ij}^h}\, d\underline{y}.$$

But since f_h is Gaussian, we have

$$\frac{\partial f_h}{\partial \Lambda_{ij}^h} = \frac{\partial^2 f_h}{\partial y_i\, \partial y_j},$$

which combined with (2.4) implies

$$(2.5)\qquad F'(h) = \sum_{i<j} \left(\Lambda_{ij}^1 - \Lambda_{ij}^0\right) \int_{\underline{a}}^{\underline{b}} \cdots \int \frac{\partial^2 f_h}{\partial y_i\, \partial y_j}\, d\underline{y}.$$

By integrating first with respect to y_i and y_j in (2.4), we obtain

$$F'(h) = \sum_{i<j} \left(\Lambda_{ij}^1 - \Lambda_{ij}^0\right)$$

$$(2.6)\qquad \times \int_{\underline{a}'}^{\underline{b}'} \cdots \int \left[f_h(y_i = b_i, y_j = b_j) - f_h(y_i = b_i, y_j = a_j) \right.$$

$$\left. - f_h(y_i = a_i, y_j = b_j) + f_h(y_i = a_i, y_j = a_j) \right] d\underline{y}',$$

where $f_h(y_i = u_i, y_j = u_j)$ is the function of $n - 2$ variables obtained by fixing $y_i = u_i$ and $y_j = u_j$, and where $dy' = \prod_{k \neq i, k \neq j} dy_k$.

If $u_i = a_i$ or b_i and $u_j = a_j$ or b_j, $i < j$, then

$$\int_{\underline{a}}^{\underline{b}} \cdots \int f_h(y_i = u_i, y_j = u_j)\, d\underline{y}'$$

$$(2.7)\qquad \leq \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \cdots \int f_h(y_i = u_i, y_j = u_j)\, d\underline{y}'$$

$$= \frac{1}{2\pi\left(1 - \left(\Lambda_{ij}^h\right)^2\right)^{1/2}} \exp\left\{ - \frac{\left(u_i^2 - 2\Lambda_{ij}^h u_i u_j + u_j^2\right)}{2\left(1 - \left(\Lambda_{ij}^h\right)^2\right)} \right\}.$$

On the other hand,

$$(2.8)\qquad u_i^2 - 2\Lambda_{ij}^h u_i u_j + u_j^2 \geq \left(u_i^2 + u_j^2\right)\left(1 - \left|\Lambda_{ij}^h\right|\right)$$

and

$$|\Lambda_{ij}^h| \le \max\left(|\Lambda_{ij}^1|, |\Lambda_{ij}^0|\right) = \delta_{ij}. \tag{2.9}$$

By combining (2.3), (2.6), (2.7), (2.8) and (2.9), we deduce (2.1). □

Let $0 < K < 1$ be a constant and for any $r > 0$, define $\mu_K(r)$ by $P\{X_1 > \mu_K(r)\} = P\{X_1 > r\}^{1+K} = [1 - \Phi(r)]^{1+K}$, where $\Phi(\cdot)$ is the standard normal distribution function. Denote by $g(x_1, \dots, x_t)$ the joint density of $X_1, \dots, X_t$. For any integer $t \ge 1$ and any $0 < r < s$, put

$$\begin{aligned} Q(t, r, s) &= \int_D g(x_1, \dots, x_t)\, d\underline{x}_t, \\ D &= \{-\mu_K(r) < x_i < s,\ i = 1, \dots, t\}, \end{aligned} \tag{2.10}$$

and denote by $Q_I(t, r, s)$ the value of $Q(t, r, s)$ when the X_n are iid. [Though $Q(t, r, s)$ and $Q_I(t, r, s)$ depend on K, we have suppressed this fact for notational convenience.]

Lemma 2.2. *If $\sum_{n \ge 1} |\Gamma(n)| < \infty$, then there exist an $r_1 > 0$, two positive constants c_1 and c_2 and a $\tau_0 > 0$ such that for all $r \ge r_1$, $0 < \tau < \tau_0$ and $0 < K < 1$,*

$$\sup_{\substack{1 \le t \le (\tau/G(r)) \log(1/G(r)) \\ 0 < r \le s < \mu_K(r)}} \left| \frac{Q(t, r, s)}{Q_I(t, r, s)} - 1 \right| < c_1 [G(r)]^{c_2}, \tag{2.11}$$

where $G(r) = 1 - \Phi(r)$.

Proof. We first apply Lemma 2.1 with $\Lambda_{ij}^0 = 0$ and $\Lambda_{ij}^1 = \Gamma(i - j)$, $i \ne j$, for which clearly Λ^0 and Λ^1 are positive definite and $a_i = -\mu_K(r)$, $b_i = s$, $1 \le i \le n$. Then, for any $t \ge 1$ and $0 < r < s \le \mu_K(r)$, we have

$$\begin{aligned} &|Q(t, r, s) - Q_I(t, r, s)| \\ &\quad \le \frac{2}{\pi} \sum_{1 \le i < j \le t} |\Gamma(i - j)| \left(1 - (\Gamma(i - j))^2\right)^{-1/2} \\ &\qquad \times \exp\left\{ -\frac{s^2}{1 + |\Gamma(i - j)|} \right\} \\ &\quad \le (\text{const.}) t \sum_{k=1}^{t} |\Gamma(k)| \exp\left\{ \frac{-s^2}{1 + |\Gamma(k)|} \right\} \\ &\quad \le (\text{const.}) t \exp\left\{ \frac{-s^2}{1 + \delta} \right\} \\ &\quad \le (\text{const.}) t \exp\left\{ \frac{-r^2}{1 + \delta} \right\}, \end{aligned} \tag{2.12}$$

where $\delta = \max_{i \geq 1}|\Gamma(i)| < 1$ and (const.) is a generic constant (independent of K and τ) which may change from step to step. Note that $|Q(t, r, s) - Q_1(t, r, s)| \leq (\text{const.}) t \exp(-r^2/(1+\delta))$ when we have $0 < r < s$ only.

Now, if $0 < r < s < \mu_K(r)$ and $1 \leq t \leq (\tau/G(r))\log(1/G(r))$, we have, for $G(r) < 1/2$,

$$\begin{aligned} Q_I(t,r,s) &= \left(P\{-\mu_K(r) < X_1 < s\}\right)^t = \left(1 - [G(r)]^{1+K} - G(s)\right)^t \\ &\geq (1 - 2G(r))^{(\tau/G(r))\log(1/G(r))} \\ &\geq [G(r)]^{2\tau} \exp\left\{\frac{2\tau G(r)}{1 - 2G(r)} \log G(r)\right\}, \end{aligned}$$

by using $-\log(1-u) \leq u + u^2/(2(1-u))$ for $0 \leq u < 1$.

Thus, since $|G(r) \log G(r)| \leq 1/e$, there exists $r_1 > 0$ and $\tau_1 > 0$ such that for any $r \geq r_1$, $0 < \tau < \tau_1$ and $0 < r \leq s < \mu_K(r)$, we have

$$Q_I(t,r,s) > \frac{[G(r)]^{2\tau}}{2} \quad \text{for } 1 \leq t \leq \frac{\tau}{G(r)} \log \frac{1}{G(r)}. \tag{2.13}$$

Combining (2.12) and (2.13) and using the fact that $G(r) \sim (2\pi)^{-1/2}(1/r) e^{-r^2/2}$ as $r \to \infty$, we obtain for $r \geq r_1$, $0 < \tau < \tau_1$, $0 < r \leq s < \mu_K(r)$ and $1 \leq t \leq (\tau/G(r))\log(1/G(r))$,

$$\begin{aligned} \left|\frac{Q(t,r,s)}{Q_I(t,r,s)} - 1\right| &\leq (\text{const.})\left(\frac{\tau}{G(r)} \log \frac{1}{G(r)}\right)[G(r)]^{-2\tau} \exp\left\{\frac{-r^2}{1+\delta}\right\} \\ &\leq (\text{const.}) r^{3+2\tau} \exp\left\{r^2\left(\frac{-1}{1+\delta} + \frac{1+2\tau}{2}\right)\right\}. \end{aligned}$$

Now, noting that $-1/(1+\delta) + (1+2\tau)/2 < 0$ is equivalent to $2\tau < (1-\delta)/(1+\delta)$ and using the fact that for any $0 < c'' < c'$,

$$r^{3+2\tau} \exp\{-(c'/2)r^2\}/[G(r)]^{c''} \to 0 \quad \text{as } r \to \infty,$$

we readily obtain (2.11) by choosing τ and c_2 in such a way that

$$0 < \tau < \min\left(\tau_1, \frac{1-\delta}{2(1+\delta)}\right) \quad \text{and} \quad 0 < c_2 < \frac{1}{2}\left(\frac{1}{1+\delta} - \frac{1+2\tau}{2}\right). \qquad \square$$

For any integer t and any $0 < r < s$, put

$$P(t,r,s) = \int_{D'} g(x_1, \ldots, x_{t-1}, x_t = s) \prod_{i=1}^{t-1} dx_i \tag{2.14}$$

with

$$D' = \{-\mu_K(r) < x_i < r, i = 1, \ldots, t-1\},$$

and denote by $P_I(t, r, s)$ the probability in (2.14) when the X_n are iid. [Though

$P(t, r, s)$ and $P_I(t, r, s)$ depend on K, we have suppressed this fact for notational convenience.]

LEMMA 2.3. *If* $\sum_{n \geq 1} |\Gamma(n)| < \infty$ *and* $\int_{-\pi}^{\pi} \log f(\lambda)\, d\lambda > -\infty$, *then there exists an* $r_2 > 0$ *and two positive constants* c_1 *and* c_2, *a* $K_0 > 0$ *and a* $\tau_0 > 0$ *such that for all* $r \geq r_2$, $0 < \tau \leq \tau_0$ *and* $0 < K \leq K_0$,

$$\sup_{\substack{1 \leq t \leq (\tau/G(r))\log(1/G(r)) \\ 0 < r \leq s < \mu_K(r)}} \left| \frac{P(t, r, s)}{P_I(t, r, s)} - 1 \right| < c_1 [G(r)]^{c_2}. \tag{2.15}$$

[Note that the conditions of the lemma are satisfied if either one of Hypotheses H_1, H_2 or H_3 are satisfied (see Section 1, Remarks 3 and 4).]

PROOF. With the notation in Lemmas 2.1 and 2.2, let

$$\Lambda^0_{ij} = \Gamma(i - j), \qquad 1 \leq i, j \leq t - 1, \qquad \Lambda^0_{it} = 0 \quad \text{if } i \neq t \text{ and } \Lambda^0_{tt} = 1.$$

Set

$$\Lambda^1_{ij} = \Gamma(i - j), \qquad 1 \leq i, j \leq t, \tag{2.16}$$

and consider first with $Q(t, r, s)$ as defined in (2.10),

$$\begin{aligned} &P(t, r, s) - Q(t - 1, r, r)\varphi(s) \\ &=: F(1) - F(0) = \int_0^1 F'(h)\, dh, \qquad \varphi(s) = \frac{1}{\sqrt{2\pi}} e^{-s^2/2}. \end{aligned} \tag{2.17}$$

[Note that F in (2.17) is not the same as in (2.3).]

Let f_h be the density of an $\mathscr{N}(0, \Lambda^h)$ variable associated with $\Lambda^h = h\Lambda^1 + (1 - h)\Lambda^0$. Then, by proceeding as in (2.4) to (2.6), we have

$$\begin{aligned} F'(h) &= \sum_{i<j} \left(\Lambda^1_{ij} - \Lambda^0_{ij}\right) \\ &\quad \times \underbrace{\int_{-\mu_K(r)}^{r} \cdots \int_{-\mu_K(r)}^{r}}_{t-1 \text{ times}} \frac{\partial f_h}{\partial \Lambda^h_{ij}}(y_1, \ldots, y_{t-1}, y_t = s)\, dy_1 \cdots dy_{t-1} \\ &= \sum_{i=1}^{t-1} \Gamma(t - i) \int_{-\mu_K(r)}^{r} \cdots \int_{-\mu_K(r)}^{r} \frac{\partial^2 f_h}{\partial y_i\, \partial y_t}(y_1, \ldots, y_{t-1}, y_s)\, dy_1 \cdots dy_{t-1} \qquad (2.18) \\ &= \sum_{i=1}^{t-1} \Gamma(t - i) \underbrace{\int_{-\mu_K(r)}^{r} \cdots \int_{-\mu_K(r)}^{r}}_{t-2 \text{ times}} \left[\frac{\partial f_h}{\partial y_t}(y_1, \ldots; y_i = r, y_{i+1}, \ldots, y_t \geqq s) \right. \\ &\qquad \left. - \frac{\partial f_h}{\partial y_t}(y_1, \ldots, y_i = -\mu_K(r), y_{i+1}, \ldots, y_t = s) \right] \prod_{\substack{k=1 \\ k \neq i}}^{t-1} dy_k. \end{aligned}$$

Now, let for any $0 \le h \le 1$,

(2.19) $$X_j^h = X_j, \qquad j = 1, \dots, t-1, \quad \text{and} \quad X_t^h = hX_t + Y^h,$$

where Y^h is independent of $X_1, \dots, X_t$, and has $\mathscr{N}(0, 1-h^2)$ distribution. Let S^h be the lower triangular matrix such that the random variables

(2.20) $$Z_i^h = \sum_{j=1}^{i} S_{ij}^h X_j^h, \qquad i = 1, \dots, t,$$

form an orthonormal system [i.e., $E(Z_i^h Z_j^h) = 0$ if $i \neq j$ and $E(Z_i^h Z_j^h) = 1$ if $i = j$]. Then

(2.21) $$f_h(y_1, \dots, y_t) = \frac{|S^h|}{(2\pi)^{t/2}} \exp\left\{ -\frac{1}{2} \sum_{i=1}^{t} \left(\sum_{j=1}^{i} S_{ij}^h y_j \right)^2 \right\}.$$

From (2.19) and (2.20) it follows that for any $0 \le h \le 1$,

$$S_{ij}^h = S_{ij}^1 \quad \text{for } 1 \le i, j \le t-1.$$

Note that we can choose Z_t^h in (2.20) of the form

$$Z_t^h = a\left(\sum_{j=1}^{t-1} S_{tj}^1 X_j + \frac{S_{tt}^1}{h} X_t^h \right),$$

since for any $1 \le k \le t-1$,

$$E(Z_t^h Z_k^h) = 0,$$

and the condition $\|Z_t^h\|^2 := E((Z_t^h)^2) = 1$ is satisfied for

$$a^2 = \frac{h^2}{h^2 + (S_{tt}^1)^2 (1-h)^2} < 1.$$

Denote by $X_t^{X_r, \dots, X_s}$ the orthogonal projection of X_t on $H(r,s)$. Then we have

$$Z_t^1 = \left(X_t - X_t^{X_1, \dots, X_{t-1}} \right) \cdot \left\| X_t - X_t^{X_1, \dots, X_{t-1}} \right\|^{-1}$$

from which it follows that $S_{tt}^1 = \|X_t - X_t^{X_1, \dots, X_{t-1}}\|^{-1} > 1$ and [see Grenander and Rosenblatt (1957), page 69]

(2.22) $$|S_{tt}^1|^2 \le \frac{1}{\begin{pmatrix} \text{mean square error} \\ \text{of prediction} \\ \text{one step ahead} \end{pmatrix}} = \frac{1}{2\pi} \exp\left\{ -\frac{1}{2\pi} \int_{-\pi}^{\pi} \log f(\lambda)\, d\lambda \right\} =: L^2.$$

Thus, for $0 \le h \le 1$,

(2.23) $$|S_{ij}^h| \le |S_{ij}^1|, \qquad 1 \le i, j \le t.$$

Going back to (2.18), and using (2.21), (2.22) and (2.23) and the fact that for any $0 < \alpha < 1$, there exists $c(\alpha)$ such that

$$xe^{-x^2/2} \le c(\alpha)e^{-\alpha x^2/2}, \qquad x \ge 0,$$

we have, with $z_i^h = \Sigma_{j=1}^i S_{ij}^h y_j^h$, $i = 1, \dots, t$, $0 < \alpha < 1$,

$$\left|\frac{\partial f_h}{\partial y_t}(\underline{y}_t)\right| = \left|\sum_{j=1}^t S_{tj}^h y_j\right| |S_{tt}^h| f_h(\underline{y}_t) \le \frac{|S^h|}{(2\pi)^{t/2}} L |z_t^h| \exp\left\{-\frac{1}{2}\sum_{i=1}^t \left(z_i^h\right)^2\right\}$$

$$\text{(2.24)} \qquad \le \frac{|S^h|}{(2\pi)^{t/2}} Lc(\alpha) \exp\left\{-\frac{1}{2}\alpha \sum_{i=1}^t \left(z_i^h\right)^2\right\}$$

$$= \frac{|S^h|}{(2\pi)^{t/2}} Lc(\alpha) \exp\left\{-\frac{1}{2}\alpha \sum_{i=1}^t \left(\sum_{j=1}^i S_{ij}^h y_j\right)^2\right\}$$

$$= Lc(\alpha)\, f_h\left(\sqrt{\alpha}\, y_1, \dots, \sqrt{\alpha}\, y_t\right).$$

Next, if in (2.18) we write

$$\underbrace{\int_{-\mu_K(r)}^r \cdots \int_{-\mu_K(r)}^r}_{t-2 \text{ times}} \frac{\partial f_h}{\partial y_t}(y_1, \dots, y_i = r, y_{i+1}, \dots, y_t = s)\, d\underline{y}'$$

$$= \int_{-\mu_K(r)}^r \cdots \int_{-\mu_K(r)}^r \frac{\partial f_h}{\partial y_t}(y_i = r, y_t = s)\, d\underline{y}',$$

then by (2.24), for any $1 \le i \le t-1$, we have

$$\left|\int_{-\mu_K(r)}^r \cdots \int_{-\mu_K(r)}^r \frac{\partial f_h}{\partial y_t}(y_i = r, y_t = s)\, d\underline{y}'\right|$$

$$\le Lc(\alpha) \underbrace{\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty}}_{t-2 \text{ times}} f_h\left(\sqrt{\alpha}\, y_i = \sqrt{\alpha}\, r, \sqrt{\alpha}\, y_t = \sqrt{\alpha}\, s\right) d\underline{y}'$$

$$\text{(2.25a)} \qquad = \frac{Lc(\alpha)}{\alpha^{(t-2)/2} 2\pi\left(1 - h^2\Gamma^2(t-i)\right)}$$

$$\times \exp\left\{-\frac{1}{2}\alpha\left(\frac{r^2 - 2h\Gamma(t-i)rs + s^2}{1 - h^2\Gamma^2(t-i)}\right)\right\}$$

$$=: A_i(h),$$

and similarly for the term containing $\mu_K(r)$ in (2.18), for which we obtain

$$\left|\int_{-\mu_K(r)}^{r}\cdots\int_{-\mu_K(r)}^{r}\frac{\partial f_h}{\partial y_t}(y_i=\mu_K(r),y_t=s)\,d\underline{y}'\right|$$

(2.25b)
$$\leq \frac{L\cdot c(\alpha)}{\alpha^{(t-2)/2}2\pi(1-h^2\Gamma^2(t-i))}\times\exp\left\{-\frac{1}{2}\alpha\left(\frac{(\mu_K(r))^2-2h\Gamma(t-i)\mu_K(r)s+s^2}{1-h^2\Gamma^2(t-i)}\right)\right\}$$
$$=: A_i'(h).$$

By (2.17) and (2.18),

(2.26)
$$\begin{aligned}\left|\frac{P(t,r,s)-Q(t-1,r,r)\varphi(s)}{Q(t-1,r,r)\varphi(s)}\right| &\leq \int_0^1\frac{|F'(h)|}{Q(t-1,r,r)\varphi(s)}\,dh\\ &\leq \frac{\int_0^1\sum_{i=1}^{t-1}|\Gamma(t-i)|(A_i(h)+A_i'(h))\,dh}{Q(t-1,r,r)\varphi(s)}\\ &\leq \sum_{n\geq 1}|\Gamma(n)|\times\max_{1\leq i\leq t-1}\left[\frac{\sup_{0<h<1}(A_i(h)+A_i'(h))}{Q(t-1,r,r)\varphi(s)}\right].\end{aligned}$$

By Lemma 2.2 and (2.13), there exists an $r_0>0$ such that for any $0<\tau\leq\tau_0$, $0<K<1$ and $r_0\leq r$, we have

(2.27)
$$Q(t-1,r,r)>\frac{1}{2}Q_I(t-1,r,r)>\frac{1}{4}[G(r)]^{2\tau},$$
$$2\leq t\leq\frac{\tau}{G(r)}\log\frac{1}{G(r)}+1.$$

Next, if we put, for any $1\leq j$,

$$B_j=B_j(b,r,s,\alpha):=\exp\left\{-\frac{1}{2}\alpha\left(\frac{r^2-2h\Gamma(j)rs+s^2}{1-h^2\Gamma^2(j)}\right)+\frac{s^2}{2}\right\},$$
$$r\leq s\leq\mu_K(r),$$

we have, by (2.25) and (2.27),

(2.28)
$$\frac{A_i(h)}{Q(t-1,r,r)\varphi(s)}\leq\frac{Lc(\alpha)}{\alpha^{(t-2)/2}(2\pi)^{1/2}(1-h^2\Gamma^2(t-i))}B_{t-i}4[G(r)]^{-2\tau},$$
$$1\leq i\leq t,\ 2\leq t\leq\frac{\tau}{G(r)}\log\frac{1}{G(r)}+1,$$
$$0<\tau\leq\tau_0,\ 0<K<1,\ 0<r_0\leq r.$$

By using the fact that $\mu_K \sim \sqrt{1+K}\,r$ as $r \uparrow \infty$ and writing B_j in the form

$$B_j = \exp\left\{-\frac{1}{2}\frac{r^2 b_j}{(1-h^2\Gamma^2(j))}\right\}, \qquad j \ge 1,$$

it follows by routine computations that there exists $r_1 \ge r_0$ such that for any $r \ge r_1$,

$$(2.29) \quad b_j \ge \begin{cases} (1-\varepsilon)\left[1-2\sqrt{\varepsilon}(1+k)\right]-(1+k)^2\varepsilon, & \text{if } h^2\Gamma^2(j) \le \varepsilon, \\ (1-|\Gamma(j)|)^2 - 2(|\Gamma(j)|k + \varepsilon(1-|\Gamma(j)|(1+k))), & \\ & \text{if } h^2\Gamma^2(j) > \varepsilon, \end{cases}$$

where $k := \sqrt{1+K} - 1$ and $\alpha = 1 - \varepsilon$.

Notice that for k_0, $k_0 < (1-\delta)^2/4\delta$, we have for all $k \le k_0$,

$$\max_{j\ge 1}\left[(1-|\Gamma(j)|)^2 - 2k|\Gamma(j)|\right] \ge \frac{(1-\delta)^2}{2}, \qquad \delta := \max_{n\ge 1}|\Gamma(n)| < 1.$$

By (2.29), it is possible to choose $\varepsilon_0 > 0$ such that for any $0 < \varepsilon \le \varepsilon_0$ and $0 < k \le k_0$, we have $\inf_{j\ge 1} b_j \ge (1-\delta)^2/4$.

From there it follows from (2.28) that there exists an $r_1 > 0$ such that for any $2 \le t \le (\tau/G(r))\log(1/G(r)) + 1$, $0 < \varepsilon \le \varepsilon_0$, $0 < \tau \le \tau_0$, $0 < K \le K_0$ [where $K_0 = (1+k_0)^2 - 1$] and $r_1 \le r$, we have

$$(2.30) \quad \begin{aligned} &\frac{\max_{1\le i\le t-1}\sup_{0<h<1} A_i(h)}{Q(t-1,r,r)\varphi(s)} \\ &\quad \le \frac{2L}{(1-\delta^2)}\cdot\frac{c(1-\varepsilon)}{(1-\varepsilon)^{t/2}}\exp\left\{-\frac{1}{8}r^2(1-\delta^2)\right\}[G(r)]^{-2\tau}. \end{aligned}$$

Next, if we take $\varepsilon = \varepsilon(r) = G(r)/2$, then by the same arguments as in (2.13), there exists $0 < \tau_1 \le \tau_0$ such that for any $0 < \tau \le \tau_1$ and $r > 0$, we have

$$(1-\varepsilon(r))^{t/2} > \frac{[G(r)]^\tau}{2}, \qquad 2 \le t \le \frac{\tau}{G(r)}\log\frac{1}{G(r)} + 1.$$

If r_2 is such that $(G(r_2)/2) \le \varepsilon_0$, then by combining this inequality and (2.30), we obtain, for any $2 \le t \le (\tau/G(r))\log(1/G(r)) + 1$, $r \ge r_2 \ge r_1$, $0 < \tau \le \tau_1$ and $0 < K \le K_0$,

$$\begin{aligned} &\frac{\max_{1\le i\le t-1}\sup_{0<h<1} A_i(h)}{Q(t-1,r,r)\varphi(s)} \\ &\quad \le \frac{4L}{(1-\delta^2)}c\left(1-\frac{G(r)}{2}\right)\exp\left\{-\frac{1}{8}r^2(1-\delta^2)\right\}[G(r)]^{-3\tau}. \end{aligned}$$

Thus, since

$$\lim_{\alpha \to 1} c(\alpha) = 1 \quad \text{and} \quad G(r) \sim \frac{1}{\sqrt{2\pi}} \frac{1}{r} e^{-r^2/2} \quad \text{as } r \to \infty,$$

we deduce that there exist $\tau_2 < \tau_1$ and two constants $u_1 > 0$ and $u_2 > 0$ such that for any $r > 0$, $0 < \tau < \tau_2$ and $0 < K < K_0$, we have

$$\max_{1 \le t \le (\tau/G(r))\log(1/G(r))} \left(\frac{\max_{1 \le i \le t-1} \sup_{0 < h < 1} A_i(h)}{Q(t-1, r, r)\varphi(s)} \right) \le u_1 [G(r)]^{u_2}.$$

From (2.25b), we obtain in the same way and under the same conditions the analogous inequality for the term corresponding to $A'_i(h)$ in (2.26):

$$\max_{\substack{1 \le t \le (\tau/G(r))\log(1/G(r)) \\ 0 < r < s < \mu_K(r)}} \left(\frac{\max_{1 \le i \le t-1} \sup_{0 < h < 1} A'_i(h)}{Q(t-1, r, r)\varphi(s)} \right) \le u_1 [G(r)]^{u_2}.$$

Thus, we deduce that there exist $\tau_2 > 0$ and two constants $v_1 > 0$ and $v_2 > 0$ such that for any $0 < \tau \le \tau_2$ and $0 < K \le K_0$, we have

$$\max_{\substack{1 \le t \le (\tau/G(r))\log(1/G(r)) \\ 0 < r \le s < \mu_K(r)}} \left| \frac{P(t, r, s)}{Q(t-1, r, r)\varphi(s)} - 1 \right| < v_1 [G(r)]^{v_2}. \tag{2.31}$$

(2.15) follows by combining (2.31) and Lemma 2.2, since $P_I(t, r, s) = Q(t-1, r, r)\varphi(s)$. □

LEMMA 2.4. *Let Y be a random variable taking values in a measurable space $(\mathbb{E}, \mathscr{E})$. Let $\psi \in \mathscr{E}$ and let $\hat{P}$ be a probability measure on $(\mathbb{E}, \mathscr{E})$ such that $0 < \hat{P}(\psi) < 1$. Assume that on ψ, the probability measure endowed by Y has a Radon–Nikodym derivative $(dP_Y/d\hat{P})(y)$ with respect to $\hat{P}$ such that*

$$\sup_{y \in \psi} \left| \frac{dP_Y}{d\hat{P}}(y) - 1 \right| \left(1 - \hat{P}(\psi)\right)^{-1} =: q < 1. \tag{2.32}$$

Let Q be a Bernoulli random variable independent of Y such that $P(Q = 0) = q$. Then there exist two $\mathscr{E}$-measurable random variables Y' and $\overline{Y}$, taking values, respectively, in $\mathbb{E}$ and $\overline{\psi}$ (the complement of ψ), independent of Y and Q, and such that if we put

$$\hat{Y} = \begin{cases} Y, & \text{if } Q = 1 \text{ and } Y \in \psi, \\ \overline{Y}, & \text{if } Q = 1 \text{ and } Y \in \overline{\psi}, \\ Y', & \text{if } Q = 0, \end{cases} \tag{2.33}$$

then the probability measure endowed by $\hat{Y}$ is $\hat{P}$.

PROOF. See Haiman [(1987a), lemma, page 448]. □

In the construction of $\{(S_n, R_n),\ n \ge 1\}$ we shall utilize the recursive method used in Haiman (1987a). Let us recall this method by first constructing (S_1, R_1).

2.2. *Construction of* (S_1, R_1). We apply Lemma 2.4 with

$$\mathbb{E} := \mathbb{N} \times \{(-\infty, -\mu_K(\Theta_0)) \cup (\Theta_0, +\infty)\}, \qquad \Theta_0 > 0, \tag{2.34}$$

and Y defined by

$$\begin{aligned} &\{Y = (t,\rho), (t,\rho) \in \mathbb{E}\} \\ &\Leftrightarrow \{-\mu_K(\Theta_0) < X_i < \Theta_0, i = 1,\dots,t-1, X_t = \rho\}, \end{aligned} \tag{2.35}$$

$$\psi := \left\{(t,\rho) \in \mathbb{E}, 1 \le t \le \frac{\tau}{G(\Theta_0)} \log \frac{1}{G(\Theta_0)}, \Theta_0 < \rho \le \mu_K(\Theta_0)\right\}, \tag{2.36}$$

$\hat{P}$ being the probability distribution of Y when the X_n are iid $\mathscr{N}(0,1)$. Thus

$$dP_Y(t,\rho) = P(t,\rho,\Theta)\delta\rho, \qquad d\hat{P}_Y(t,\rho) = P_I(t,\rho,\Theta)\,\delta\rho,$$

with $P(\cdot,\cdot,\cdot)$ and $P_I(\cdot,\cdot,\cdot)$ as in (2.14). By Lemma 2.3, there exist $\tau_0 > 0$, $K_0 > 0$, $c_1 > 0$ and $c_2 > 0$ such that for any $0 < \tau \le \tau_0$ and $0 < K \le K_0$,

$$\sup_{y \in \psi} \left| \frac{dP_Y}{d\hat{P}}(y) - 1 \right| \le c_1[G(\Theta_0)]^{c_2}.$$

Next, by the same arguments as in (2.13) we see that, with $\bar{\psi}$ denoting the complement of ψ,

$$\begin{aligned} \hat{P}(\bar{\psi}) &\ge \left\{1 - G(\Theta_0) - [G(\Theta_0)]^{1+K}\right\}^{(\tau/G(\Theta))\log(1/G(\Theta_0))} \\ &> \frac{[G(\Theta_0)]^{2\tau}}{2} \quad \text{for any } 0 < \tau \le \tau_1. \end{aligned}$$

Combining these inequalities we deduce that for any $0 < \tau \le \inf(\tau_0, \tau_1, c_2/4)$ and $0 < K \le K_0$, we have

$$\sup_{y \in \psi} \left| \frac{dP_Y}{d\hat{P}}(y) - 1 \right| \left[\hat{P}(\bar{\psi})\right]^{-1} < \alpha[G(\Theta_0)]^{\beta} =: q \to 0 \quad \text{as } \Theta_0 \to \infty, \tag{2.37}$$

with $\alpha = 2c_1$ and $\beta = c_2/2$.

Thus, for Θ_0 large enough, we have $q < 1$ and we can apply Lemma 2.4. Let $\hat{Y}$ be the corresponding random vector given by (2.33) and put

$$(\Delta\hat{S}, \hat{\mathscr{R}}) := \hat{Y}. \tag{2.38}$$

Let $Y^+ = (\Delta S^+, \mathscr{R}^+)$ be a random vector independent of $\sigma\{X_n, n \ge 1\} \times \sigma\{Y, Q, Y', \bar{Y}\}$ and such that

$$f_{Y^+}(t,\rho) := \frac{d}{d\rho} P\{\Delta S^+ = t, \mathscr{R}^+ < \rho\} = \Phi(\Theta_0)^{t-1}\varphi(\rho), \qquad t \ge 1, 1 < \Theta_0 < \rho. \tag{2.39}$$

Put

$$S_1 = \Delta\hat{S}1_{\{\hat{\mathscr{R}} > 0\}} + (\Delta\hat{S} + \Delta S^+)1_{\{\hat{\mathscr{R}} < 0\}}$$

and

$$R_1 = \hat{\mathscr{R}} 1_{\{\hat{\mathscr{R}} > 0\}} + \mathscr{R}^+ 1_{\{\hat{\mathscr{R}} < 0\}}. \tag{2.40}$$

Then, it is obvious that for any $s \geq 1$ and $r > \Theta_0$,

$$P\{S_1 = s, R_1 > r\} = [\Phi(\Theta_0)]^{s-1}(1 - \Phi(r)). \tag{2.41}$$

We shall further make use of the preceding construction of (S_1, R_1) under the following more general form.

Let $r > \Theta_0$. Let $n \geq 1$ be fixed and define $Y_r(n)$ by the following relation:

$$Y_r(n) = (t, \rho) \text{ if and only if}$$

$$\{-\mu_K(r) < X_i < r, i = n+1, \dots, n+t-1; X_{n+t} = \rho\}, \tag{2.42}$$
$$(t, \rho) \in \mathbb{E}(r) := \mathbb{N} \times \{(-\infty, -\mu_K(r)) \cup (r, +\infty)\}.$$

Let $\tau > 0$ be given and let

$$\begin{aligned} \psi &= \psi(r) \\ &:= \{(t, \rho) \in \mathbb{E}(r); 1 \leq t \leq \tau[G(r)]^{-1}(-\log(G(r))), \\ &\qquad r < \rho < \mu_K(r)\}. \end{aligned} \tag{2.43}$$

Let η_{d-1} be a random vector defined on $(\Omega, \sigma\{X_n, n \geq 1\}, P)$ and taking values in $\mathscr{E}_d := (\mathbb{N}^+ \times \mathbb{R}^+)^d$, $d \geq 2$, and of the form $\eta_{d-1} = ((S_1, R_1), \dots, (S_{d-1}, R_{d-1}), (N_{d-1}, M_{d-1}))$ (these notations are motivated by the iterative construction of Proposition 2.1). For $d = 2$, S_1 and R_1 are the random variables constructed above. Moreover, if we denote by Y_1, Q_1, $\overline{Y}_1$, Y_1' and Y_1^+ the corresponding random variables and vectors used in the construction of S_1 and R_1, and if we put

$$\begin{aligned} Y_1 &:= ((Y_1)_1, (Y_1)_2), \qquad (Y_1)_1 \in \mathbb{N}, \\ (Y_1)_2 &\in (-\infty, -\mu_K(\Theta_0)) \cup (\Theta_0, +\infty), \end{aligned} \tag{2.44}$$

then

$$M_1 := \max\{R_1, Q_1 1_{\{|(Y_1)_2| > \mu_K(\Theta_0)\}} |(Y_1)_2|\} \tag{2.45}$$

and

$$N_1 := Q_1 1_{\{Y_1 \in \overline{\psi}(\Theta_0)\}} 1_{\{(Y_1)_1 - S_1 > 0\}}((Y_1)_1 - S_1). \tag{2.46}$$

PROPOSITION 2.1. *Assume that η_{d-1}, $d \geq 2$, defined above is such that for any integers $s_1 < \cdots < s_{d-1}$, $\nu_{d-1} \geq 0$ and any $0 < \rho_1 < \cdots < \rho_{d-1} \leq m_{d-1}$, the events*

$$\{\eta_{d-1}; (S_1 = s_1, R_1 < \rho_1), \dots, (S_{d-1} = s_{d-1}, R_{d-1} < \rho_{d-1}),$$
$$(N_{d-1} = \nu_{d-1}, M_{d-1} \leq m_{d-1})\}$$

are $\sigma\{X_n, n \leq s_{d-1} + z_{d-1}; z_{d-1} = [(G(m_{d-1}))^{A-1}] + \nu_{d-1}\} \times \sigma'$-measurable, where σ' is a σ-field independent of $\sigma\{X_n, n \geq 1\}$. Furthermore, put

$Z_{d-1} = [(G(M_{d-1}))^{A-1}\} + N_{d-1}$ *and assume that the conditional distribution of* $Y_d := Y_{R_{d-1}}(S_{d-1} + Z_{d-1})$ *given* $\eta_{d-1} = l_{d-1}$ *exists for a.a.* (*almost all with respect to the probability distribution of* η_{d-1}) l_{d-1} *and has a density function with respect to* $\hat{P}$ *denoted by*

$$\frac{dP_{Y_d}^{\eta_{d-1}=l_{d-1}}}{d\hat{P}}(y)$$

such that for a.a. $l_{d-1} \in \mathscr{E}_d$,

$$\sup_{y \in \psi(r_{d-1})} \left| \frac{dP_{Y_d}^{\eta_{d-1}=l_{d-1}}}{d\hat{P}}(y) - 1 \right| \hat{P}^{-1}\big(\bar{\psi}(r_{d-1})\big) =: q_d < 1. \tag{2.47}$$

Then, there exists a $\sigma\{X_n,\ n > s_{d-1} + z_{d-1}\} \times \sigma'$-*measurable random vector* $(\Delta S_d, \mathscr{R}_d)$, $d \geq 2$, *taking values in* $\mathbb{N} \times (\mathbb{R}^+, \mathscr{B}^+)$, *such that*

$$P\{\Delta S_d = t, \mathscr{R}_d > \rho | \eta_{d-1} = l_{d-1}\} = \big[\Phi(r_{d-1})\big]^{t-1}(1 - \Phi(\rho)), \qquad t \geq 1, \rho > r_{d-1}, \tag{2.48}$$

and

$$\begin{aligned} P\big\{\Delta S_d &= \min\{t \geq 1,\ X_{s_{d-1}+z_{d-1}+t} > r_{d-1}\},\\ \mathscr{R}_d &= X_{s_{d-1}+z_{d-1}+\Delta S_d} | \eta_{d-1} = l_{d-1}\big\}\\ &= 1 - \big(1 - \big[\Phi(r_{d-1})\big]^{z_d}\big)(1 - q_d)\\ &\quad \times P\big\{Y_{r_{d-1}}(s_{d-1} + z_{d-1}) \in \psi(r_{d-1}) | \eta_{d-1} = l_{d-1}\big\}. \end{aligned} \tag{2.49}$$

PROOF. We apply Lemma 2.4 by taking, for a.a. l_{d-1}, $d \geq 2$, the conditional probability distribution of Y_d, given $\eta_{d-1} = l_{d-1}$. Note that the probability distribution of η_{d-1} is concentrated on the set of l_{d-1} such that $l_{d-1} = ((s_1, r_1), \dots, (s_{d-1}, r_{d-1}), (\nu_{d-1}, m_{d-1}))$ with $1 \leq s_1 < \cdots < s_{d-1}, \nu_{d-1} \geq 0$, and $0 < \Theta_0 < r_1 < \cdots < r_{d-1} \leq m_{d-1}$. Thus, in the following, l_{d-1} will be assumed to belong to this set. Let $\mathbb{E} := \mathbb{E}(r_{d-1})$ as in (2.42). Writing $\psi := \psi(r_{d-1})$ as in (2.43) we note that (2.47) implies (2.32). Put

$$\big(\Delta\hat{S}_d, \hat{\mathscr{R}}_d\big) := \hat{Y}_d \tag{2.50}$$

with $\hat{Y}_d := \hat{Y}$ as in (2.33) (in which we also put, in order to keep homogeneity, $Q := Q_d$, $Y' := Y'_d$ and $\bar{Y} := \bar{Y}_d$), these random variables and vectors being chosen independent of $\sigma\{X_n,\ n \leq s_{d-1} + z_{d-1}\} \times \sigma'$.

Let $(\Delta S_d^+, \mathscr{R}_d^+)$ be a random vector independent of

$$\sigma\{X_n;\ n < s_{d-1} + z_{d-1}\} \times \sigma'$$

and such that

$$\begin{aligned} &\frac{d}{d\rho} P\{\Delta S_d^+ = t, \mathscr{R}_d^+ < \rho | \eta_{d-1} = l_{d-1}\}\\ &\quad = \big[\Phi(r_{d-1})\big]^{t-1}\varphi(\rho) =: f_{Y_{d+}}(t, \rho), \qquad t \geq 1, r_{d-1} < \rho. \end{aligned} \tag{2.51}$$

Let

$$\Delta\tilde{S}_d = \Delta\hat{S}_d \cdot 1_{\{\hat{\mathscr{R}}_d > 0\}} + \left(\Delta\hat{S}_d + \Delta S_d^{+}\right)1_{\{\hat{\mathscr{R}}_d < 0\}} \tag{2.52}$$

and

$$\tilde{R}_d = \hat{\mathscr{R}}_d \cdot 1_{\{\hat{\mathscr{R}}_d > 0\}} + \mathscr{R}_d^{+} 1_{\{\hat{\mathscr{R}}_d < 0\}}. \tag{2.53}$$

Let L_d be a Bernoulli random variable such that

$$P\{L_d = 1 | \eta_{d-1} = l_{d-1}\} = 1 - \left[\Phi(r_{d-1})\right]^{z_{d-1}} \tag{2.54}$$

and let $(\Delta \underline{S}_d, \underline{\mathscr{R}}_d)$ be a random vector such that

$$\begin{aligned} &\frac{d}{d\rho} P\{\Delta\underline{S}_d = t, \underline{\mathscr{R}}_d < \rho | \eta_{d-1} = l_{d-1}\} \\ &\quad = \left[\Phi(r_{d-1})\right]^{t-1}\varphi(\rho)\left[P(L_d = 1|\eta_{d-1} = l_{d-1})\right]^{-1}, \\ &\qquad\qquad t \in \{1, \dots, z_{d-1}\},\ r_{d-1} < \rho. \end{aligned} \tag{2.55}$$

L_d, $\Delta\underline{S}_d$ and $\underline{\mathscr{R}}_d$ are mutually independent, independent of $\sigma\{X_n, n \le s_{d-1} + z_{d-1}\} \times \sigma'$ and depend on the other previously defined random variables only through η_{d-1}. Put

$$\Delta S_d := L_d \Delta\underline{S}_d + (1 - L_d)\left(Z_{d-1} + \Delta\tilde{S}_d\right)$$

and

$$\mathscr{R}_d := L_d \underline{\mathscr{R}}_d + (1 - L_d)\tilde{\mathscr{R}}_d. \tag{2.56}$$

It may easily be checked that these random variables satisfy (2.49). □

2.3. *Construction of* (S_d, R_d) *for* $d \ge 2$. We apply Proposition 2.1 recursively to define

$$S_d := S_{d-1} + \Delta S_d, \qquad R_d := \mathscr{R}_d, \qquad d \ge 2, \tag{2.57}$$

$$M_d := \max\left\{R_d, M_{d-1}, (1 - L_d)Q_d 1_{\{|(Y_d)_2| > \mu_K(R_{d-1})\}}|(Y_d)_2|\right\} \tag{2.58}$$

and

$$\begin{aligned} N_d := {}& (1 - L_d)Q_d 1_{\{Y_d \in \bar{\psi}(R_{d-1})\}} 1_{\{\Delta S_d < Z_{d-1} + (Y_d)_1\}} \\ & \times \left(Z_{d-1} + (Y_d)_1 - \Delta S_d\right), \end{aligned} \tag{2.59}$$

where

$$\begin{aligned} &Y_d := ((Y_d)_1, (Y_d)_2), \qquad (Y_d)_1 \in \mathbb{N}, \\ &(Y_d)_2 \in \left(-\infty, -\mu_K(R_{d-1})\right) \cup (R_{d-1}, \infty), \end{aligned} \tag{2.60}$$

M_1 and N_1 are defined, respectively, in (2.45) and (2.46), and the other random variables are those defined in Proposition 2.1 and in its proof.

It is assumed that at any step of the construction, $d = 1, 2, \dots,$ the hypotheses of Proposition 2.1 are satisfied. Before proving this fact, it is an easy exercise to check that the process $\{S_k, R_k\}_{k\geq 1}$ constructed in this way has the same probability structure as the record sequence based on an iid $\mathcal{N}(0,1)$ sequence and Θ_0, that is, we have:

$$P\{S_{n+1} = s_{n+1}, R_{n+1} < \rho_{n+1} | S_1 = s_1, R_1 = r_1, \dots, S_n = s_n, R_n = r_n\}$$
$$= [\Phi(r_n)]^{s_{n+1}-s_n-1} \cdot P(r_n < X_1 < \rho_{n+1}),$$
$$n \geq 1, 1 \leq s_1 < \cdots < s_n < s_{n+1}, 0 < \Theta_0 < r_1 < \cdots < r_n < \rho_n.$$

We now prove that $\{\eta_d = (S_1, R_1, \dots, S_d, R_d, N_d, M_d),\ d \geq 1\}$ satisfies the hypotheses of Proposition 2.1.

Let $1 < s_1 < s_2 < \cdots < s_n$ and $\Theta_0 < r_1 < \cdots < r_n$ be given. Let $\mathcal{S}$ be the set of vectors ζ of the form

$$\zeta = \{(s_{i_1}, r_{i_1}), \dots, (s_{i_l}, r_{i_l})\}, \tag{2.61}$$

with

$$1 \leq i_1 < i_2 < \cdots < i_l \leq n,$$

or, if (s, m_n) is such that $s_{i_{k-1}} < s < s_{i_k}$ for some $1 \leq k < l$ and $m_n > r_n$, then

$$\zeta = \{(s_{i_1}, r_{i_1}), \dots, (s_{i_{k-1}}, r_{i_{k-1}}), (s, m_n), (s_{i_k}, r_{i_k}), \dots, (s_{i_l}, r_{i_l})\}. \tag{2.61$'$}$$

Let E be an event of the form

$$E = \{a_i < X_{t_i} < b_i, i = 1, \dots, n\} \tag{2.62}$$

and $\zeta \in \mathcal{S}$ such that

$$\{t_1, \dots, t_n\} \cap \{s_{i_1}, \dots, s_{i_l}\} = \varnothing \quad \text{if } \zeta \text{ is of the form (2.61)} \tag{2.63}$$

or

$$\{t_1, \dots, t_n\} \cap \{s_{i_1}, \dots, s_{i_{k-1}}, s, s_{i_k}, \dots, s_{i_l}\} = \varnothing \quad \text{if } \zeta \text{ is of the form (2.61$'$)}. \tag{2.63$'$}$$

Let

$$P(E|\zeta) := \begin{cases} P\{E | X_{s_{i_1}} = r_1, \dots, X_{s_{i_l}} = r_l\}, & \text{if } \zeta \text{ is of the form (2.61)}, \\ P\{E | X_{s_{i_1}} = r_1, \dots, X_{s_{i_{k-1}}} = r_{i_{k-1}}, \\ \quad X_s = m_n, X_{s_{i_k}} = r_{i_k}, \dots, X_{s_{i_l}} = r_{i_l}\}, & \text{if } \zeta \text{ is of the form (2.61$'$)}, \end{cases} \tag{2.64}$$

which exists for any $\zeta \in \mathcal{S}$, since the covariance function Γ of the process is positive definite. Put

$$P_\zeta(E) = P(E|\zeta) g(\zeta), \tag{2.65}$$

where $g(\zeta)$ is the Gaussian joint density of $X_{s_{i_1}}, \dots, X_{s_{i_l}}$ in which we take $X_{s_{i_j}} = r_{i_j}$, $j = 1, \dots, l$ $(X_s = m_n)$.

LEMMA 2.5. (i) *For any $d \geq 1$, the random vector η_d has a density function with respect to the Lebesgue measure given by*

$$
\begin{aligned}
f_{\eta_d}(s_1, r_1, \dots, s_d, r_d, \nu_d, m_d) &= f_{\eta_d}(l_d) \\
&= \sum_{\zeta \in \mathscr{S}(l_d)} \sum_{1 \leq j \leq J(\zeta, l_d)} A_j(\zeta, l_d) P_\zeta(E_j(l_d)) \\
&\quad + \sum_{1 \leq k \leq J'(l_d)} B_k(l_d) P(E'_k(l_d)) \\
&\quad + C(l_d) 1_{\{\nu_d = 0,\, m_d = r_d\}},
\end{aligned} \tag{2.66}
$$

where $\mathscr{S}(l_d)$ denotes the set of ζ defined in (2.61) *and* (2.61′) *associated with $(s_1, r_1), \dots, (s_d, r_d)$ and m_d, $J(l_d)$ and $J'(l_d)$ are finite integers and the E_j's and the E'_k's are of the form* (2.62) *with*

$$
\Big\{a_i < X_{t_i} < b_i,\ i = 1, \dots, n(j);\ n(j) < \tau d[G(r_d)]^{-1}(-\log G(r_d)),
$$
$$
t_{n(j)} < s_d + \nu_d, \max\{(|a_i|, |b_i|), 1 \leq i \leq n(j)\} \leq m_d\Big\}
$$

and the A_j's, the B_k's and C are > 0.

(ii) *For all $l_d \in \mathscr{E}_d$, the conditional probability distribution of Y_{d+1} given $\eta_d = l_d$ has a density function with respect to the Lebesgue measure, denoted by $f_{Y_{d+1}}(t, \rho | l_d)$, such that for any $t \geq 1$ and $r_d < \rho$,*

$$
\begin{aligned}
&f_{Y_{d+1}}(t, \rho | l_d) f_{\eta_d}(l_d) \\
&= \Bigg\{ \sum_{\zeta \in \mathscr{S}(l_d)} \sum_{1 \leq j \leq J(\zeta, l_d)} A_j(\zeta, l_d) \frac{\partial}{\partial \rho} P_\zeta\big(E_j(l_d) \cap F_{r_d, s_d + z_d}(t, \rho)\big) \\
&\quad + \sum_{1 \leq k \leq J'(l_d)} B_k(l_d) \frac{\partial}{\partial \rho} P\big(E'_k(l_d) \cap F_{r_d, s_d + z_d}(t, \rho)\big) \\
&\quad + C(l_d) 1_{\{\nu_d = 0,\, m_d = r_d\}} \frac{\partial}{\partial \rho} P\big(F_{r_d, 1}(t, \rho)\big) \Bigg\},
\end{aligned} \tag{2.67}
$$

with

$$
\begin{aligned}
F_{r_d, s}(t, \rho) = \{&-\mu_K(r_d) < X_i < r_d, \\
&i = s + 1, \dots, s + t - 1,\ r_d < X_{s+t} < \rho\}
\end{aligned} \tag{2.68}
$$

and

$$z_d = \left[(G(m))^{A-1} \right] + \nu_d.$$

PROOF. Let us recall the construction formulas of $(S_d, \mathscr{R}_d)$ for any $d \geq 1$. We have:

$$\left(\Delta \hat{S}_d, \hat{\mathscr{R}}_d \right) =: \hat{Y}_d = Q_d \left(Y_d 1_{\{Y_d \in \psi(R_{d-1})\}} + \overline{Y}_d 1_{\{Y_d \in \overline{\psi}(R_{d-1})\}} \right) + (1 - Q_d) Y_d' \qquad [\text{see } (2.33)],$$

$$\Delta \tilde{S}_d = \Delta \hat{S}_d \cdot 1_{\{\hat{\mathscr{R}}_d > 0\}} + \left(\Delta \hat{S}_d + \Delta S_d^+ \right) 1_{\{\hat{\mathscr{R}}_d < 0\}} \qquad [\text{see } (2.52)],$$

$$\tilde{\mathscr{R}}_d = \hat{\mathscr{R}}_d \cdot 1_{\{\hat{\mathscr{R}}_d > 0\}} + \mathscr{R}_d^+ 1_{\{\hat{\mathscr{R}}_d < 0\}} \qquad [\text{see } (2.53)].$$

If $d \geq 2$,

$$\Delta S_d = L_d \Delta \underline{S}_d + (1 - L_d) \left(Z_{d-1} + \Delta \tilde{S}_d \right) \qquad [\text{see } (2.56)],$$

$$\mathscr{R}_d = L_d \underline{\mathscr{R}}_d + (1 - L_d) \tilde{\mathscr{R}}_d \qquad [\text{see } (2.56)]$$

and

$$S_d := S_{d-1} + \Delta S_d, \qquad R_d := \mathscr{R}_d \qquad [\text{see } (2.57)].$$

If $d = 1$,

$$S_1 = \Delta \hat{S} 1_{\{\hat{\mathscr{R}} > 0\}} + (\Delta \hat{S} + \Delta S^+) 1_{\{\hat{\mathscr{R}} < 0\}}$$

$$R_1 = \tilde{\mathscr{R}} 1_{\{\hat{\mathscr{R}} > 0\}} + \mathscr{R}^+ 1_{\{\mathscr{R} < 0\}} \qquad [\text{see } (2.40)].$$

Let us first consider the terms of (2.66) when $d = 1$.

By the proof of Lemma 2.4 [see Haiman (1987a)], if the conditional probability distribution of $\hat{Y}_d$ in (2.33) given $R_{d-1} = r_{d-1}$ is absolutely continuous with respect to the Lebesgue measure of E_d and has a density function $f_{\hat{Y}_d}$, where

$$f_{\hat{Y}_1}(s_1, r_1) = [\Phi(\Theta_0)]^{s_1 - 1} \varphi(r_1), \qquad s_1 \geq 1, \Theta_0 < r_1,$$

then Y_d, $\overline{Y}_d$ and Y_d' are also absolutely continuous with respect to the Lebesgue measure, with density functions denoted by f_{Y_d}, $f_{\overline{Y}_d}$ and $f_{Y_d'}$, respectively.

In particular, for any $s_1 \geq 1$ and $r_1 \in (-\infty, -\mu_K(\theta_0)) \cup (\Theta_0, +\infty)$, we have, with the notation in (2.60)–(2.65), in which we take $\zeta := (s_1, r_1)$,

$$(2.69) \quad f_{Y_1}(s_1, r_1) = \frac{d}{dr_1} P\{ -\mu_K < X_i < \Theta_0, i = 1, \dots, s_1 - 1, X_{s_1} < r_1 \}.$$

Let $t_0 = \tau(G(\Theta_0))(-\log(G(\Theta_0)))$ and put

$$a_X(t, r_1) := \sum_{k=1}^{t} P\{-\mu_K(\Theta_0) < X_i < \Theta_0, i = 1, \dots, k-1, \mu_K(\Theta_0) < |X_k| < r_1\}, \quad t \le t_0, \tag{2.70}$$

$$\alpha'_X(k, r_1, m) := \frac{\partial}{\partial m} P\{-\mu_K(\Theta_0) < X_i < \Theta_0, i = 1, \dots, k-1, \mu_K(\Theta_0) < r_1 < |X_k| < m\}, \quad 1 \le k \le t_0, \mu_K(\Theta_0) < r_1 < m, \tag{2.71}$$

$$a'_X(t, r_1, m) := \sum_{k=1}^{t} \alpha'(k, r_1, m), \qquad 1 \le t \le t_0, \tag{2.72}$$

$$b_X := P\{-\mu_K(\Theta_0) < X_i < \Theta_0, i = 1, \dots, t_0\}, \tag{2.73}$$

$$a_{\bar{Y}}(t, r_1) := \sum_{j=1}^{\min(t, t_0)} P\{(\bar{Y}_1)_1 = j, (\bar{Y}_1)_2 < -\mu_K(\Theta_0)\} \times f_{Y_1^+}(t-j, r_1) + f_{\bar{Y}_1}(t, r_1) \tag{2.74}$$

with $f_{Y_1^+} = f_{Y^+}$ as in (2.39),

$$a_{Y'}(t, r_1) = \sum_{j=1}^{t} P\{(Y')_1 = j, (Y'_1)_2 < -\mu_K(\Theta_0)\} \times f_{Y_1^+}(t-j, r_1) + f_{Y'}(t, r_1). \tag{2.74'}$$

With this notation, it is easy to check that if $q_1 := q$ defined in (2.37) $=$ $P(Q_1 = 0)$, we have: If $\nu_1 = 0$ and $m_1 = r_1$,

$$f_{\eta_1}(l_1) = q_1 a_{Y'}(s_1, r_1) + (1 - q_1) \tag{2.75}$$

$$\times \begin{cases} (a_X(t_0, r_1) + b_X(t_0)) a_{\bar{Y}}(s_1, r_1), & \text{if } r_1 > \mu_K(\Theta_0),\ s_1 > t_0, & (1) \\ a_X(s_1, r_1) \times a_{\bar{Y}}(s_1, r_1), & \text{if } r_1 > \mu_K(\Theta_0),\ s_1 \le t_0, & (2) \\ b_X a_{\bar{Y}}(s_1, r_1), & \text{if } r_1 < \mu_K(\Theta_0),\ s_1 > t_0, & (3) \\ f_{Y_1}(s_1, r_1), & \text{if } r_1 < \mu_K(\Theta_0),\ s_1 \le t_0. & (4) \end{cases}$$

If $\nu_1 > 0$ or $m_1 > r_1$, we have the following situations:
If $\nu_1 > 0$ and $m_1 = r_1$, then

$$f_{\eta_1}(l_1) = (1 - q_1)$$

$$(2.75)\qquad \times \begin{vmatrix} a_X(s_1 + n_1, r_1) \cdot a_{\bar{Y}}(s_1, r_1), & \begin{matrix}\text{if } r_1 > \mu_K(\Theta_0), \\ s_1 + \nu_1 < t_0,\end{matrix} & (5) \\ b_X \cdot a_{\bar{Y}}(s_1, r_1), & \text{if } s_1 + \nu_1 = t_0. & (6) \end{vmatrix}$$

If $\nu_1 = 0$ and $r_1 < m_1$, then

$$f_{\eta_1}(l_1) = (1 - q_1)$$

$$(2.75)\qquad \times \begin{vmatrix} a'_X(t_0, r_1, m_1) a_{\bar{Y}}(s_1, r_1), & \begin{matrix}\text{if } r_1 > \mu_K(\Theta_0), \\ s_1 > t_0,\end{matrix} & (7) \\ a'_X(s_1, r_1, m_1) a_{\bar{Y}}(s_1, r_1), & \begin{matrix}\text{if } r_1 > \mu_K(\Theta_0), \\ s_1 \le t_0,\end{matrix} & (8) \\ a'_X(t_0, \mu_K(\Theta_0), m_1) a_{\bar{Y}}(s_1, r_1), & \begin{matrix}\text{if } r_1 < \mu_K(\Theta_0), \\ s_1 > t_0,\end{matrix} & (9) \\ a'_X(s_1, \mu_K(\Theta_0), m_1) a_{\bar{Y}}(s_1, r_1), & \begin{matrix}\text{if } r_1 < \mu_K(\Theta_0), \\ s_1 \le t_0.\end{matrix} & (10) \end{vmatrix}$$

If $\nu_1 > 0$ and $r_1 < m_1$, then

$$f_{\eta_1}(l_1) = (1 - q_1)$$

$$(2.75)\qquad \times \begin{vmatrix} a'_X(s_1 + n_1, r_1, m_1) a_{\bar{Y}}(s_1, r_1), & \begin{matrix}\text{if } r_1 > \mu_K(\Theta_0) \\ \text{and } s_1 + n_1 < t_0,\end{matrix} & (11) \\ a'_X(s_1 + n_1, \mu_K(\Theta_0), m_1) a_{\bar{Y}}(s_1, r_1), & \begin{matrix}\text{if } r_1 < \mu_K(\Theta_0) \\ \text{and } s_1 + n_1 < t_0.\end{matrix} & (12) \end{vmatrix}$$

Thus, the decomposition (2.66) corresponding to (2.75)-(1) is

$$C(l_1) = q_1 a_{Y'}(s_1, r_1),$$

$$E'_k = \{-\mu_K(\Theta_0) < X_i < \Theta, i = 1, \dots, k - 1, \mu_K(\Theta_0) < |X_k| < r_1\},$$

$$B_k = (1 - q_1) a_{\bar{Y}}(s_1, r_1), \qquad k = 1, \dots, t_0 =: J'(l_1) - 1;$$

$$E'_{J'} = \{-\mu_K(\theta_0) < X_i < \Theta, i = 1, \dots, t_0\};$$

$$B_{J'} = (1 - q_1) a_{\bar{Y}}(s_1, r_1).$$

There are no terms corresponding to $\Sigma\Sigma$ in (2.66). For the other terms in (2.75)-(2)–(12), one obtains similar identifications.

Thus, it is not difficult to see that the above decomposition may be extended to any $d \ge 1$. The term C corresponds to the event $\{[L_i = 1) \cup ((L_i = 0) \cap$

$(Q_i = 0))]$, $i = 1, \dots, d\}$ whereas the other terms (Σ and $\Sigma\Sigma$) correspond to the complement of this event.

The factors $P_\zeta(E_j(l_d))$ correspond to events $\{Y_t \in \psi(R_{t-1})\}$ or $\{Y_t \in \overline{\psi}(R_{t-1}), M_t > R_t\}$, $1 \le t \le d$, and the factors $P(E_k'(l_d))$ to events $\{Y_t \in \psi(R_{t-1}), M_t = R_t\}$, $1 \le t \le d$. The factors A_j and B_k are related to probabilities of events generated by $\{Q_t, L_t, \overline{Y}_t, Y_t', Y_t^+, t = 1, \dots, d\}$, which are independent of $\sigma\{X_i, i \ge 1\}$.

The decomposition (2.67) is a straightforward consequence of (2.66). □

Let us observe that the fact that for any $s_1 < \cdots < s_d$, $\nu_d \ge 0$ and $0 < \rho_1 < \cdots < \rho_d \le m_d$, the events

$$\{\eta_d; (S_1 = s_1, R_1 < \rho_1), \dots, (S_d = s_d, R_d < \rho_d), (N_d = \nu_d, M_d < m_d)\}, \qquad d \ge 1,$$

are $\sigma\{X_n, n \le s_d + z_d\} \times \sigma'$-measurable is a straightforward consequence of the method of construction of (S_n, R_n) [see (2.33)–(2.57)].

In order to prove (2.47), we shall utilize Lemma 2.5 together with the following:

LEMMA 2.6. *With the notations in* (2.68) *and* (2.69), *if one among Hypotheses* H_1, H_2 *or* H_3 *is satisfied, then there exist an* $A_0 > 0$ *and two constants* C_1 *and* $C_2 > 0$ *independent of* l_d, *such that for any* $0 < A < A_0$, *we have*

$$\sup_{\zeta \in \mathscr{S}(l_d)} \max_{1 \le j \le J(\zeta, l_d)} \left| \left\{ \frac{\partial}{\partial \rho} P_\zeta\big(E_j(l_d) \cap F_{r_d, s_d + z_d}(t, \rho)\big) \right\} \times \big\{P_\zeta(E_d(l_d)) \cdot P(t, r_d, \rho)\big\}^{-1} - 1 \right| < C_1[G(r_d)]^{C_2} \tag{2.76}$$

and

$$\max_{1 \le k \le J'(l_d)} \left| \left\{ \frac{\partial}{\partial \rho} P\big(E_k'(l_d) \cap F_{r_d, s_d + z_d}(t, \rho)\big) \right\} \times \big\{P(E_k'(l_d)) \cdot P(t, r_d, \rho)\big\}^{-1} - 1 \right| < C_1[G(r_d)]^{C_2}, \qquad t \ge 1, r_d < \rho, \tag{2.77}$$

with $P(\cdot, \cdot, \cdot)$ *as in* (2.14).

Before proving Lemma 2.6, let show that this lemma combined with Lemma 2.5 implies that at any step $d \ge 2$ of the construction, (2.47) is satisfied. By

(2.43) and the same arguments as in (2.13), we have

$$\hat{P}\big(\bar{\psi}(r_{d-1})\big) \geq \Big\{1 - G(r_{d-1}) - \big[G(r_{d-1})\big]^{1+K}\Big\}^{(\tau/G(r_{d-1}))\log(1/G(r_{d-1}))}$$

$$\geq \frac{\big[G(r_{d-1})\big]^{2\tau}}{2}. \tag{2.78}$$

Next, with the notation in (2.47) and $P_I(\cdot, \cdot, \cdot)$ defined in (2.14), we have

$$\frac{dP_{Y_d}^{\eta_{d-1}=l_{d-1}}}{d\hat{P}}(t,\rho) = \frac{f_Y(t,\rho|l_{d-1})}{P_I(t,r_{d-1},\rho)}, \qquad (t,\rho) \in \psi(r_{d-1}). \tag{2.79}$$

Observe that P and P_I satisfy (2.15) with $r := r_{d-1}$ and $s := \rho$. Thus $P(t, r_d, \rho)$ in (2.76) and (2.77) may be replaced by $P_I(t, r_d, \rho)$.

Notice that $(\partial/\partial\rho)P(F_{r_d,1}(t,\rho)) = P(t, r_d, \rho)$. Further, observe that if $p_i > 0$ and $\hat{p}_i > 0$, $i = 1,\dots,n$, are such that $\max_{1\leq i\leq n}|(p_i/\hat{p}_i) - 1| \leq q$, then for any $\alpha_i > 0$, $i = 1,\dots,n$, we have

$$\left|\frac{\Sigma\alpha_i p_i}{\Sigma\alpha_i \hat{p}_i} - 1\right| \leq q.$$

By using these remarks in (2.68) together with (2.76)–(2.79), we obtain (2.47) with

$$q_d \leq 2C_1\big[G(r_d)\big]^{C_2-2\tau}.$$

Thus, if we take $\tau < \inf(\tau_0, C_2/2)$ with τ_0 as in Lemma 2.3, we have

$$q_d \leq v_1\big[G(r_d)\big]^{v_2}, \qquad v_1 = 2C_1,\ v_2 = C_2 - 2\tau,\ d = 1,2,\dots \tag{2.80}$$

(so that for a sufficiently large Θ_0 we have $q_d < 1$ for any $d \geq 1$).

Let us now prove Lemma 2.6.

PROOF OF LEMMA 2.6. Let

$$\Big\{j_1 < \cdots < j_k;\ k \leq \tau d\big[G(r_d)\big]^{-1}(-\log(G(r_d)))$$

$$+d+1,\ j_k \leq s_d + \nu_d\Big\} \tag{2.81}$$

be the sequence formed by the t_i of (2.67) and the s_{i_j} (or s) of ζ in (2.60) and (2.61). Put

$$j_{k+1} = s_d + z_d + \nu_d, \dots, j_{k+t+1} = s_d + z_d + \nu_d + t,$$

$$1 \leq t \leq \tau\big[G(r_d)\big]^{-1}(-\log(G(r_d))). \tag{2.82}$$

Let $[R(u,v)]$ be the $(k+t+1)\times(k+t+1)$-matrix spanned by

$$R(u,v) = \Gamma(j_u - j_v) = E\big(X_0 X_{j_u - j_v}\big), \qquad u,v = 1,\dots,k+t+1, \tag{2.83}$$

and let

$$(2.84)\qquad R_I(u,v) = \begin{cases} R(u,v), & \text{if } 1 \le u \le k,\ 1 \le v \le k \\ & \text{or } k < u \le k+t+1, \\ & k < v \le k+t+1, \\ 0, & \text{otherwise.} \end{cases}$$

[Though k, $j_1, \dots, j_k$ and t depend on $l_d = ((s_1, r_1), \dots, (s_d, r_d), (\nu_d, m_d))$, we have suppressed this fact for notational convenience.] Since Γ is positive definite, R and R_I are positive definite. Thus, the inverses of these matrices exist; we denote them, respectively, by R^{-1} and R_I^{-1}.

We shall deduce Lemma 2.6 as a consequence of the following lemma.

LEMMA 2.7. *There exists a universal constant* [*denoted by* (const.)], *such that*: (i) *If Hypothesis* H_1 *is satisfied, then*

$$(2.85)\qquad \begin{aligned} &\max_{1 \le i, j \le k+t+1} \left| R^{-1}(i,j) - R_I^{-1}(i,j) \right| \\ &\quad \le (\text{const.}) k(t+1) \max_{j \ge [G(m_d)]^{A-1}} |\Gamma(j)| \end{aligned}$$

and

$$(2.86)\qquad \left|\operatorname{Det}\left(RR_I^{-1}\right) - 1\right| \le (\text{const.}) \left(k^2 \max_{j \ge [G(m_d)]^{A-1}} |\Gamma(j)| \right)^2.$$

(ii) *If Hypotheses* H_2 *or* H_3 *are satisfied, then*

$$(2.87)\qquad \begin{aligned} &\max_{1 \le i, j \le k+t+1} \left| R^{-1}(i,j) - R_I^{-1}(i,j) \right| \\ &\quad \le (\text{const.}) k^2 (t+1)^2 \max_{j \ge [G(m_d)]^{A-1}} |\Gamma(j)| \end{aligned}$$

and

$$(2.88)\qquad \left|\operatorname{Det}\left(RR_I^{-1}\right) - 1\right| \le (\text{const.}) \left((k(t+1))^3 \max_{j \ge (G(m_d))^{A-1}} |\Gamma(j)| \right)^2.$$

PROOF. See Appendix A. □

Put

$$\begin{aligned} F_{r_d, s}(t, (\rho, \rho + \delta\rho)) = \{ -\mu_K(r_d) < X_i < r_d, \\ i = s+1, \dots, s+t-1,\ \rho < X_{s+t} < \rho + \delta\rho \}, \\ t \ge 1,\ r_d < \rho < \rho + \delta\rho. \end{aligned}$$

We shall prove Lemma 2.6 with (2.76) and (2.77) replaced, respectively, by

$$\sup_{\zeta \in \mathscr{S}(l_d)} \max_{1 \le j \le J(\zeta, l_d)} \Big| P_\zeta\big(E_j(l_d) \cap F_{r_d, s_d+z_d}(t, \rho, \rho + \delta\rho)\big) \times \big\{P_\zeta(E_d(l_d)) \cdot P\big(F_{r_d,1}(t,(\rho,\rho+\delta\rho))\big)\big\}^{-1} - 1\Big| < C_1[G(r_d)]^{C_2} \tag{2.89}$$

and

$$\max_{1 \le k \le J'(l_d)} \Big| P\big(E'_k(l_d) \cap F_{r_d, s_d+z_d}(t,(\rho, \rho + \delta\rho))\big) \times \big\{P(E'_k(l_d)) P\big(F_{r_d,1}(t,(\rho,\rho+\delta\rho))\big)\big\}^{-1} - 1\Big| < C_1[G(r_d)]^{C_2}. \tag{2.90}$$

Clearly (2.89) and (2.90) are equivalent to (2.76) and (2.77), since

$$\frac{\partial}{\partial\rho} P_\zeta\big(E_j(l_d) \cap F_{r_d, s_d+z_d}(t,\rho)\big) = \lim_{\delta\rho \downarrow 0} \frac{1}{\delta\rho} P_\zeta\big(E_j(l_d) \cap F_{r_d, s_d+z_d}(t,(\rho,\rho+\delta\rho))\big),$$

$$\frac{\partial}{\partial\rho} P\big(E'_k(l_d) \cap F_{r_d, s_d+z_d}(t,\rho)\big) = \lim_{\delta\rho \downarrow 0} \frac{1}{\delta\rho} P\big(E'_k(l_d) \cap F_{r_d, s_d+z_d}(t,(\rho,\rho+\delta\rho))\big)$$

and

$$\frac{\partial}{\partial\rho} P\big(F_{r_d,1}(t,\rho)\big) = P(t, r_d, \rho) = \lim_{\delta\rho \downarrow 0} \frac{1}{\delta\rho} P\big(F_{r_d,1}(t,(\rho,\rho+\delta\rho))\big).$$

Put

$$\underline{x}' = (x_{j_1}, \dots, x_{j_{k+t+1}}). \tag{2.91}$$

Let $g(\underline{x}')$ be the Gaussian joint density of the random variables $X_{j_1}, \dots, X_{j_{k+t+1}}$ and let $g_\zeta(\underline{y}')$ be the function deduced from $g(\underline{x})$ by fixing $x_{s_{i_j}} = r_{i_j}$, $j = 1, \dots, l$ ($x_s = m$) (thus depending on the vector $\underline{y}'$ formed by the $k + t + 1 - l$

remaining variables). If we put $V = R^{-1} - R_I^{-1}$, then we have

$$g(\underline{x}') = (2\pi)^{-(k+t+1)/2}(\text{Det } R)^{-1/2}\exp\left\{-\frac{1}{2}\underline{x}'R^{-1}\underline{x}\right\}$$

(2.92)
$$= \left(g^I(\underline{x}) \times \left(\frac{\text{Det } R}{\text{Det } R_I}\right)^{-1/2} \exp\left\{-\frac{1}{2}\underline{x}'V\underline{x}\right\}\right),$$

where

$$g^I(\underline{x}) = (2\pi)^{-(k+t+1)/2}(\text{Det } R_I)^{-1/2}\exp\left\{-\frac{1}{2}\underline{x}'R_I^{-1}\underline{x}\right\}.$$

Next

$$P\left(E_k'(l_d) \cap F_{r_d, s_d+z_d}(t, (\rho, \rho + \delta\rho))\right)$$

(2.93)
$$= \int_{E_k'(l_d)\cap F_{r_d, s_d+z_d}(t, (\rho, \rho+\delta\rho))} g(\underline{x}')\, d\underline{x}'$$

[where we use the same notation for the domain of integration of $g(\underline{x}')$ and the corresponding event in (2.90)].

Observe that

$$\int_{E_k'(l_d)\cap F_{r_d, s_d+z_d}(t, (\rho, \rho+\delta\rho))} g^I(\underline{x}')\, d\underline{x}'$$

(2.94)
$$= P(E_d'(l_d))P\left(F_{r_d, 1}(t, (\rho, \rho + \delta\rho))\right)$$

and that for $\underline{x}'$ in the domain of integration in the left side above, we have

$$\max|\underline{x}'V\underline{x}| \le \max_{1\le i, j\le k+t+1} \left|R^{-1}(i,j) - R_I^{-1}(i,j)\right|$$

(2.95)
$$\times (k + t + 1)^2(m_d^2).$$

Furthermore, by (2.81), (2.82) (and again using the fact that $G(u) \sim (1/\sqrt{2\pi})(1/u)e^{-u^2/2}$ and $u^\alpha e^{-u} \to 0$ for any $\alpha > 0$ as $u \uparrow \infty$), for any $\eta > 0$ arbitrarily small, there exists a $C(\eta) > 0$ such that

$$(k + t + 1)^2 < C(\eta)[G(r_d)]^{-2-\eta},$$

(2.96)
$$k(t + 1) \le C(\eta)[G(r_d)]^{-2-\eta}.$$

Next,

(2.97)
$$\max_{j\ge[G(m_d)]^{A-1}} |\Gamma(j)| \le \begin{cases} [G(m_d)]^{(1-A)(4+\varepsilon)}, & \text{by } (1.6), \\ [G(m_d)]^{(1-A)(6+\varepsilon)}, & \text{by } (1.8) \text{ or } (1.10). \end{cases}$$

Combining (2.92), (2.93) and (2.94) we have for

$$D = E'_k(l_d) \cap F_{r_d, s_d+z_d}(t, (\rho, \rho + \delta\rho)),$$

$$(2.98)\quad \begin{aligned} &\Big| P\big(E'_k(l_d) \cap F_{r_d, s_d+z_d}(t, (\rho, \rho + \delta\rho))\big) \\ &\qquad \times \Big\{ P(E'_k(l_d)) \times P\big(F_{r_d, 1}(t, (\rho, \rho + \delta\rho))\big)\Big\}^{-1} - 1 \Big| \\ &= \left| \left(\int_D g(\underline{x}')\, d\underline{x}' \right) \left(\int_D g^I(\underline{x}')\, d\underline{x}' \right)^{-1} - 1 \right|, \\ &\le \max_{\underline{x}' \in D} \left| \frac{g(\underline{x}')}{g^I(\underline{x}')} - 1 \right| = \max_{\underline{x}' \in D} \left| \left(\frac{\text{Det } R}{\text{Det } R_I} \right)^{-1/2} \exp\left\{ -\frac{1}{2} \underline{x}' V \underline{x} \right\} - 1 \right| \\ &= O_1\left(\left(\frac{\text{Det } R}{\text{Det } R_I} \right) - 1 \right) + O_2 \max_{\underline{x}' \in D} ((\underline{x}' V \underline{x})), \end{aligned}$$

where $|O_i(u)| < \sigma|u|$, $i = 1, 2$, for some $\sigma > 0$ independent of l_d [we use the fact that $(1 + u)^{1/2} \sim 1 + (u/2)$ and $\exp v \sim 1 + v$ as u and $v \to 0$].

Now, combining (2.98), (2.85)–(2.88) and (2.93)–(2.97), it is easy to check that $O_1((|R|/|R_I|) - 1) = o(\max_{\underline{x}' \in D}(\underline{x}' V \underline{x}))$ as $G(r_d) \to 0$, so that the last term in (2.98) is dominated by

$$(2.99)\quad \begin{aligned} &2\sigma(m_d)^2 C(\eta)[G(r_d)]^{-2-\eta} \\ &\quad \times \begin{cases} (\text{const.}) C(\eta)[G(r_d)]^{-2-\eta}(G(m_d))^{(1-A)(4+\varepsilon)} \\ \quad \text{under } (1.6), \\ (\text{const.}) C^2(\eta)[G(r_d)]^{-4-2\eta}(G(m_d))^{(1-A)(6+\varepsilon)} \\ \quad \text{under } (1.8) \text{ or } (1.10). \end{cases} \end{aligned}$$

Finally (2.99) $\Rightarrow$ (2.90) since η is arbitrary, $(m_d)^2 \sim -2 \log G(m_d)$ as $m_d \to \infty$ and $G(m_d) \le G(r_d)$ for $r_d \le m_d$.

The above proof of (2.90) may readily be adapted to prove (2.89) by replacing (2.93) by

$$(2.100)\quad \begin{aligned} &P_\zeta\big(E_j(l_d) \cap F_{r_d, s_d+z_d}(t, (\rho, \rho + \delta\rho))\big) \\ &\qquad = \int_{E_j(l_d) \cap F_{r_d, s_d+z_d}(t, (\rho, \rho+\delta\rho))} g_\zeta(\underline{y}')\, d\underline{y}'. \end{aligned}$$

□

The above lemma completes the proof of the first part of Theorem 1.1 for $J = 1$.

We now prove (1.11), that is, that there exists almost surely an n_0 and q such that for all $n \ge n_0$ we have $S_n = T_{n-q}$ and $R_n = \Theta_{n-q}$. We need the following lemma.

LEMMA 2.8. *For any fixed $J \geq 1$, let $\{\Theta_n^J, n \geq 1\}$ be the sequence of Jth-upper records based on the sequence $\{X_n, n \geq 1\}$ of iid random variables with common continuous distribution function $F(x) = P(X_1 < x)$, $x \in \mathbb{R}$ and Θ_0. Then for any $0 < \alpha < 1$ we have*

$$P\{G(\Theta_n^J) > e^{-\alpha n/J} \ i.o.\} = 0 \tag{2.101}$$

and for any $\beta > 1$ we have

$$P\{G(\Theta_n^J) < e^{-\beta n/J} \ i.o.\} = 0. \tag{2.102}$$

PROOF. Observe first that $\{F(\Theta_n^J), n \geq 1\}$ is the sequence of Jth-upper records based on the sequence of independent random variables $V_n := F(X_n)$ uniformly distributed on $[0, 1]$ and $F(\Theta_0)$.

Consequently, for any $0 < \varepsilon < 1$, (2.101) and (2.102) are, respectively, equivalent to

$$P\{F(\Theta_n^J) < 1 - e^{-(n(1-\varepsilon)/J)} \text{ i.o.}\} = 0 \tag{2.103}$$

and

$$P\{F(\Theta_n^J) > 1 - e^{-(n(1+\varepsilon)/J)} \text{ i.o.}\} = 0. \tag{2.104}$$

Consider $Y_n = \log(1/(1 - V_n))$ which is exponentially distributed, and denote by Z_n^J $[= \log(1/(1 - F(\Theta_n^J)))]$ the nth Jth upper record based on this sequence. An application of Theorem 7 of Deheuvels (1984a) shows that $\{Z_n^J, n \geq 1\}$ forms a Markov chain with stationary transition probabilities such that for any $n \geq 1$, $Z_{n+1}^J - Z_n^J$ is independent of Z_n^J and

$$P(Z_{n+1}^J - Z_n^J > t - s | Z_n^J = s) = \exp\{-J(t - s)\}, \qquad 0 < s < t. \tag{2.105}$$

Thus Z_n^J is the sequence of arrival points of a Poisson process of intensity J and by the strong law of large numbers, we have

$$\lim_{n \to \infty} \frac{Z_n^J}{n} = E(Z_{n+1}^J - Z_n^J) = \frac{1}{J} \quad \text{a.s.} \tag{2.106}$$

Next,

$$P\{F(\Theta_n^J) > 1 - e^{-(n(1+\varepsilon)/J)}\} = P\left\{\frac{Z_n^J}{n} > \frac{1 + \varepsilon}{J}\right\}$$

and

$$P\{F(\Theta_n^J) < 1 - e^{-(n(1-\varepsilon)/J)}\} = P\left\{\frac{Z_n^J}{n} < \frac{1 - \varepsilon}{J}\right\},$$

which by (2.106) implies, respectively, (2.103) and (2.104). □

LEMMA 2.9. *With the notation in* (2.33)–(2.40) *we have*

$$P\{(\Delta S_d, \mathscr{R}_d) \in \bar{\psi}(R_{d-1})\ i.o.\} = P\{Q_d = 0\ i.o.\} = P\{L_d = 1\ i.o.\} = 0. \tag{2.107}$$

PROOF. Put

$$A_d = \{((\Delta S_d, \mathscr{R}_d) \in \bar{\psi}(R_{d-1})) \cap (G(R_d) < e^{-\alpha d})\}. \tag{2.108}$$

By (2.101), we have

$$P\{(\Delta S_d, \mathscr{R}_d) \in \bar{\psi}(R_{d-1})\ \text{i.o.}\} = 0 \Leftrightarrow P\{A_d\ \text{i.o.}\} = 0. \tag{2.109}$$

Now, with the notations in Proposition 2.1,

$$\begin{aligned} P\{(\Delta S, \mathscr{R}_d) &\in \bar{\psi}(r_{d-1}) | \eta_{d-1} = l_{d-1}\} \\ &= \hat{P}\{\bar{\psi}(r_{d-1})\} \\ &= 2[G(r_{d-1})]^K \\ &\quad + \left(1 - G(r_{d-1}) - [G(r_{d-1})]^{1+K}\right)^{\tau(1/G(r_{d-1}))(-\log G(r_{d-1}))} \\ &\le 2[G(r_{d-1})]^K + [G(r_{d-1})]^{\tau}. \end{aligned} \tag{2.110}$$

Thus, there exists a $d_0 > 0$ such that for any $d \ge d_0$, we have

$$\begin{aligned} P(A_d) &= \int P(A_d | \eta_{d-1} = l_{d-1})\, dP_{\eta_{d-1}}(l_{d-1}) \\ &\le \int_{G(r_{d-1}) < e^{-\alpha(d-1)}} \left(2[G(r_{d-1})]^K + [G(r_{d-1})]^{\tau}\right) dP_{R_{d-1}}(r_{d-1}) \\ &\le 2e^{-\alpha K(d-1)} + e^{-\alpha(d-1)\tau}, \qquad d \ge 1. \end{aligned} \tag{2.111}$$

Since the RHS of (2.111) is the general term of a convergent sum, the Borel–Cantelli lemma implies that $P\{A_d \text{ i.o.}\} = 0$. By (2.80),

$$P\{Q_d = 0 | \eta_{d-1} = l_{d-1}\} = q_d \le v_1[G(r_{d-1})]^{v_2}, \tag{2.112}$$

which by the same arguments as above implies

$$P\{Q_d = 0 \text{ i.o.}\} = 0. \tag{2.113}$$

Since $P\{A_d \text{ i.o.}\} = 0$ and $P\{Q_d = 0 \text{ i.o.}\} = 0$, we have

$$P\{Z_d \ne [G(R_d)]^{A-1} \text{ i.o.}\} = 0. \tag{2.114}$$

Thus, by (2.54) and using the same arguments as in the proof of (2.113),

$$\begin{aligned}P\Big\{(L_d = 1) \cap \Big(Z_d = [G(r_{d-1})]^{A-1}\Big) | \eta_{d-1} = l_{d-1}\Big\} &= 1 - [\Phi(r_{d-1})]^{[G(r_{d-1})]^{A-1}} \\ &= 1 - [1 - G(r_{d-1})]^{[G(r_{d-1})]^{A-1}} \\ &\leq 1 - \exp\left\{-G(r_{d-1})^A\left(1 + \frac{G(r_{d-1})}{2(1 - G(r_{d-1}))}\right)\right\} \\ &\leq G(r_{d-1})^A\left(1 + \frac{G(r_{d-1})}{2(1 - G(r_{d-1}))}\right),\end{aligned}$$

which by the preceding arguments implies

$$P\{L_d = 1 \text{ i.o.}\} = 0. \tag{2.115}$$
□

From (2.33)–(2.40) and (2.107) it follows that there exists a.s. $n_0 > 0$ such that for any $n \geq n_0$,

$$S_{n+1} = S_n + \min\Big\{t > [G(R_n)]^{A-1}; X_{S_n+t} > R_n\Big\} \tag{2.116}$$

and

$$R_{n+1} = X_{S_{n+1}}.$$

LEMMA 2.10. *For any fixed $J \geq 1$, let $\{(T_k, \underline{\Theta}_k),\ k \geq 1\}$ be the Jth record sequence based on $\{X_n,\ n \geq 1\}$ and Θ_0. Let $\{(T'_n, \underline{\Theta}'_n),\ n \geq 1\}$ be a sequence of random vectors such that there exists a.s. $p \geq 1$ and such that for any $n \geq p$, we have*

$$T'_{n+1} = \inf\{k;\ k \geq T'_n,\ X_{k-J+1,k} > X_{k-J,k-1}\}$$

and

$$\underline{\Theta}'_{n+1} = \Big\{X_{T'_{n+1}-J+1, T'_{n+1}}, \dots, X_{T'_{n+1}, T'_{n+1}}\Big\}. \tag{2.117}$$

Then, there exist a.s. two integers n_0 and q such that for any $n \geq n_0$, we have

$$T_n = T'_{n-q} \quad \text{and} \quad \underline{\Theta}_n = \underline{\Theta}'_{n-q}. \tag{2.118}$$

PROOF. See Section 2, Lemma 4 in Haiman (1987b). □

Put, for any $n \geq 1$,

$$C_n = \Big\{\max\Big(X_{S_n+t}: 1 \leq t \leq [G(R_{n-1})]^{(A-1)(1+K)}\Big) > R_n\Big\}. \tag{2.119}$$

LEMMA 2.11. *If one among Hypotheses H_1, H_2 or H_3 is satisfied, then there exists a $0 < A_0 < 1$ such that for any $0 < A \leq A_0$, there exists a $0 <$*

$K_0(A)$ *such that for all* $0 < K \le K_0(A)$,

$$P\{C_n \ i.o.\} = 0. \tag{2.120}$$

PROOF. See Appendix B. □

3. Proof of Theorem 1.1 for the general case ($J > 1$).

3.1. *Construction of* $\{S_n, \underline{R}_n\}_{n \ge 1}$. $(S_1, \underline{R}_1)$ is constructed independently of $\{X_n, n \ge 1\}$ (i.e., by using a sequence $\{X_n^*, n \ge 1\}$ independent of $\{X_n, n \ge 1\}$, of iid $\mathscr{N}(0,1)$ random variables, and by taking

$$\begin{aligned} S_1 &= \inf\{n;\, n \ge J, X^*_{n-J+1,n} > \Theta_0\}, \\ \underline{R}_1 &= \{X^*_{S_1-J+1,S_1}, \dots, X^*_{S_1,S_1}\}, \end{aligned} \tag{3.1}$$

where Θ_0 is a fixed real number). In order to construct $(S_n, \underline{R}_n)$ for $n \ge 2$ we shall adapt step by step the method of construction in the $J = 1$ case.

Observe [see (1.2) and (2.3)] that T_k, $k \ge 1$, are the times m at which $X_{m-j+1,m}$ changes. Moreover, for any $n \ge 2$, the J-ordered random sequence $\underline{\Theta}_2, \dots, \underline{\Theta}_n$ is uniquely determined by $\Theta_1^J, \dots, \Theta_1^1$ and the sequence of $n-1$ (not necessarily increasing) values corresponding to the consecutive changes, $\Theta_2, \Theta_3, \dots, \Theta_n, \Theta_1^J < \Theta_2, \Theta_n \le \Theta_n^1$, such that if $\underline{\Theta}_1 = \{\Theta_1^J, \Theta_1^{J-1}, \dots, \Theta_1^1\}$, then

$$\{\Theta_1^J, \Theta_1^{J-1}, \dots, \Theta_1^1\} \cup \{\Theta_2, \dots, \Theta_n\} = \{X_{T_n-(J+n-2),T_n}, \dots, X_{T_n,T_n}\}. \tag{3.2}$$

For any $r > \Theta_0$ and $n \ge 1$, let $Y_r(n)$ be given by (2.42) and $\psi(r)$ by (2.43).

Let $\boldsymbol{\eta}_{d-1}$, $d \ge 2$, be a random vector defined by

$$\boldsymbol{\eta}_{d-1} := ((S_1, \underline{R}_1), \dots, (S_{d-1}, \underline{R}_{d-1}), (N_{d-1}, M_{d-1})). \tag{3.3}$$

$\boldsymbol{\eta}_{d-1}$ takes values in the set of $((s_1, \underline{r}_1), \dots, (s_{d-1}, \underline{r}_{d-1}), (\nu_{d-1}, m_{d-1}))$ such that $J < s_1 < \cdots < s_{d-1}$, $\nu_{d-1} > 0$ and $\{r_k, 1 \le k \le d-1\}$ is a J-ordered sequence. For $d = 2$, S_1 and $\underline{R}_1$ are the random vectors given above, $M_1 = R_1^1$ (we have $\underline{R}_k := \{R_k^J, \dots, R_k^1\}$, $k \ge 1$) and $N_1 = 0$.

PROPOSITION 3.1. *Assume that* $\boldsymbol{\eta}_{d-1}$, $d \ge 2$, *is such that the events of probability* > 0 *of the form* $\{\boldsymbol{\eta}_{d-1};\ (S_1 = s_1, \underline{R}_1 \in b_1), \dots, (S_{d-1} = s_{d-1}, \underline{R}_{d-1} \in b_{d-1}), N_{d-1} = \nu_{d-1}, M_{d-1} \le m_{d-1}\}$, *where* $s_1 < \cdots < s_{d-1}$, $\nu_{d-1} \ge 0$, $b_1, \dots, b_{d-1}$ *are Borel subsets of* $(\mathbb{R}^+)^J$ *and* $m_{d-1} > 0$ *are* $\sigma\{X_n \le s_{d-1} + z_{d-1};\, z_{d-1} := [(G(m_{d-1}))^{A-1}] + \nu_{d-1}\} \times \sigma'$-*measurable* (*where* σ' *is independent of* $\sigma\{X_n, n \ge 1\}$). *Furthermore, set* $Z_{d-1} = [(G(M_{d-1}))^{A-1}] + N_{d-1}$ *and assume that the conditional distribution of* $Y_d := Y_{R_{d-1}^J}(S_{d-1} + Z_{d-1})$ *given* $\boldsymbol{\eta}_{d-1} = \boldsymbol{l}_{d-1}$ *exists for all* $\boldsymbol{l}_{d-1} \in \mathscr{E}_d$ *and has a density function with respect to* $\hat{P}$ *denoted by*

$$\frac{dP_{Y_d}^{\boldsymbol{\eta}_{d-1} = \boldsymbol{l}_{d-1}}}{d\hat{P}}(y)$$

such that for all $\boldsymbol{l}_{d-1} \in \mathscr{E}_d$,

$$(3.4) \qquad \max_{y \in \psi(r_{d-1}^J)} \left| \frac{dP_{Y_d}^{\boldsymbol{\eta}_{d-1}=\boldsymbol{l}_{d-1}}}{d\hat{P}}(y) - 1 \right| \hat{P}^{-1}\big(\bar{\psi}(r_{d-1}^J)\big) =: q_d < 1.$$

Then, there exists a $\sigma\{X_n, n > s_{d-1} + z_{d-1}\} \times \sigma'$*-measurable random variable* $(\Delta S_d, \mathscr{R}_d)$ *taking values in* $\mathbb{N} \times \mathbb{R}^+$ *such that* (2.49) *is satisfied with* η_{d-1} *and* l_{d-1} *replaced by* $\boldsymbol{\eta}_{d-1}$ *and* $\boldsymbol{l}_{d-1}$ *and* r_{d-1} *by* r_{d-1}^J.

PROOF. See the proof of Proposition 2.1 in which η_{d-1} and l_{d-1} are replaced by $\boldsymbol{\eta}_{d-1}$ and $\boldsymbol{l}_{d-1}$ and r_{d-1} by r_{d-1}^J. □

Construction of $(S_n, \underline{R}_n)$ *for* $n \geq 2$. We apply Proposition 3.1 and define

$$(3.5) \qquad S_d = S_{d-1} + \Delta S_d, \qquad \underline{R}_d = \Omega\big(R_{d-1}^{J-1}, \dots, R_1^1, \mathscr{R}_d\big), \qquad d \geq 2,$$

where $\Omega(x_1, \dots, x_j) = (x_{1j}, \dots, x_{jj})$ denotes the ordered sample $(x_1, \dots, x_j)$,

$$(3.6) \qquad M_d = \max\big[R_d^1, M_{d-1}, (1 - L_d)Q_d 1_{\{|(Y_d)_2| > \mu_K(R_{d-1}^J)\}} \times |(Y_d)_2|\big]$$

and

$$(3.7) \qquad \begin{aligned} N_d = {} & (1 - L_d)Q_d \times 1_{\{Y_d \in \bar{\psi}(R_{d-1}^J)\}} \\ & \times 1_{\{\Delta S_d < Z_{d-1} + (Y_d)_1\}}\big(Z_{d-1} + (Y_d)_1 - \Delta S_d\big), \end{aligned}$$

where L_d, Q_d and Y_d are the corresponding random vectors used in the construction of $(\Delta S_d, \mathscr{R}_d)$.

Let $r_2, r_3, \dots, r_n$, $r_1^J < r_2$, $r_n \leq r_n^1$, $n \geq 2$, be the sequence which together with $\underline{r}_1$ uniquely determines the J-ordered sequence $\underline{r}_1, \dots, \underline{r}_n$ [see remark on (3.2)].

Let $\mathscr{S}$ be the set of ζ defined in (2.60) with the above (not necessarily increasing) r_i, $i = 2, \dots, n$, $2 \leq i_1 < i_2 < \cdots < i_l \leq n$ (instead of $1 \leq i_1 < \cdots < i_l \leq n$) and $m_n > r_n^1$ [instead of $m_n > r_n$ in (2.61)].

The fact that at any step of the construction $d = 2, \dots$ the hypotheses of Proposition 3.1 are satisfied may be deduced by the same arguments as in the case $J = 1$ by the following substitutions in Lemmas 2.5 and 2.6.

$$(3.8) \qquad \begin{gathered} \eta_d := \boldsymbol{\eta}_d, \qquad l_d := \boldsymbol{l}_d \quad \text{in (2.66)–(2.68)}, \\ ((s_1, r_1), \dots, (s_d, r_d)) := ((s_2, r_2), \dots, (s_d, r_d)) \quad \text{in (2.66)}, \\ r_d := r_d^J \quad \text{in (2.67)–(2.69) and (2.76), (2.77)}. \end{gathered}$$

Let us now prove that the above construction of $\{(S_n, \underline{R}_n)\}_{n \geq 1}$ satisfies (1.11).

3.2. *Proof of Theorem* 1.1 ($J > 1$). With the notation in Proposition 3.1 (and its proof) we have the following version of Lemma 2.9:

LEMMA 3.1.

$$P\{(\Delta S_n, \mathscr{R}_d) \in \bar{\psi}(R^J_{d-1})\ i.o.\} = P\{Q_d = 0\ i.o.\} = 0, \tag{3.9}$$

$$P\{L_d = 1\ i.o.\} = 0. \tag{3.10}$$

PROOF. The proof of (3.9) is similar to the proof of Lemma 2.9 for $J = 1$. To prove (3.10) we need the following:

LEMMA 3.2. *There exists a sequence* $\{\eta_n\}_{n \ge 1}$, $\lim_{n\to\infty} \eta_n = 0$, *such that for any* $\varepsilon > 0$,

$$P\left\{G(R^1_n) \le \left(\frac{n}{J}\right)^{-(1+\varepsilon)} \exp\left(-\frac{n}{J}(1+\eta_n)\right)\ i.o.\right\} = 0. \tag{3.11}$$

PROOF. Observe that when $J > 1$, R^1_n has the same probability distribution as $\max(X_1, \dots, X_{S_n})$, where S_n is the nth Jth record time based on an independent identically $\mathscr{N}(0,1)$ distributed sequence of r.v.'s. Thus, the probability distribution of $G(R^1_n)$ is the same as that of $\min(U_1, \dots, U_{\nu_n}) = U_{1,\nu_n}$, where $\{U_n\}_n$ is a sequence of independent random variables uniformly distributed on $[0,1]$ and ν_n is the nth Jth record time based on this sequence.

By Deheuvels [(1986), page 134], for any $\varepsilon > 0$ there exists a.s. an $n_0 > 0$ such that for any $n \ge n_0$,

$$U_{1,n} > n^{-1}(\log n)^{-(1+\varepsilon)} \tag{3.12}$$

and by Deheuvels [(1984b), Theorem 7], there exist a.s. two sequences, $\eta_1(n) > 0$ and $\eta_2(n) > 0$, $\eta_i(n) \to 0$ as $n \to \infty$, $i = 1, 2$, such that

$$P\left\{\log \nu_n \ge \frac{n}{J}(1 + \eta_1(n))\ \text{i.o.}\right\} = 0 \tag{3.13}$$

and

$$P\left\{\log \nu_n \le \frac{n}{J}(1 - \eta_2(n))\ \text{i.o.}\right\} = 0. \tag{3.14}$$

By combining (3.12), (3.13) and (3.14), we get (3.11). □

By (3.9), it is equivalent to prove that

$$P\{L_d\ \text{i.o.}\} = 0 \quad \text{and} \quad P\{B_d\ \text{i.o.}\} = 0, \tag{3.15}$$

where

$$\begin{aligned} B_d = \Bigg(&\{L_d = 1\} \cap \{M_d = R^1_d\} \cap \{N_d = 0\} \\ &\cap \{G(R^J_{d-1}) < e^{-(\alpha(d-1)/J)}\} \\ &\cap \left\{G(R^1_{d-1}) > \left(\frac{d-1}{J}\right)^{-(1+\varepsilon)} \exp\left(-\frac{d-1}{J}(1+\eta_{d-1})\right)\right\}\Bigg), \end{aligned} \tag{3.16}$$

$$0 < \alpha < 1.$$

By (2.54),

$$(3.17)\qquad \begin{aligned} P\{B_d\} &= \int P(B_d | \boldsymbol{\eta}_{d-1} = \boldsymbol{l}_{d-1})\, dP_{\boldsymbol{\eta}_{d-1}}(\boldsymbol{l}_{d-1}) \\ &= \int_D \int \Big(1 - \big(\Phi(r^J_{d-1})\big)^{[G(r)]^{A-1}}\Big)\, dP_{(R^1_{d-1}, R^J_{d-1})}(r, r^J_{d-1}), \end{aligned}$$

with

$$(3.18)\qquad D = \begin{cases} G(r^J_{d-1}) \le e^{-(\alpha(d-1)/J)}, \\ G(r) \ge \left(\dfrac{d-1}{J}\right)^{-(1+\varepsilon)} \exp\left\{-\dfrac{d-1}{J}(1+\eta_{d-1})\right\}. \end{cases}$$

Next

$$(3.19)\qquad 1 - \big(\Phi(r^J_{d-1})\big)^{[G(r)]^{A-1}} = 1 - \exp\left\{[G(r)]^{A-1} \log\big(1 - G(r^J_{d-1})\big)\right\}$$

and $\log(1 - G(r^J_{d-1})) \sim -G(r^J_{d-1})$ as $d \to \infty$.

Hence, there exists a positive constant (independent of d) such that, on D,

$$(3.20)\qquad \begin{aligned} &-\left[\{G(r)\}^{A-1}\right] \log\big(1 - G(r^J_{d-1})\big) \\ &\quad \le (\text{const.}) e^{-(\alpha(d-1)/J)} \times \exp\left\{-\frac{(d-1)}{J}(1+\eta_{d-1})(A-1)\right\} \\ &\quad\quad \times \left(\frac{d-1}{J}\right)^{(1-A)(1+\varepsilon)} \\ &\quad = (\text{const.}) \left(\frac{d-1}{J}\right)^{(1-A)(1+\varepsilon)} \exp\left\{-\frac{d-1}{J}\big(\alpha + (1+\eta_{d-1})(A-1)\big)\right\} \\ &\quad \le \exp\left\{-\frac{d-1}{J}\frac{A}{2}\right\}, \qquad d \ge d_0, \text{ sufficiently large, } 0 < 1 - \alpha < \frac{A}{2}. \end{aligned}$$

Thus, since $e^x - 1 \sim x$ as $x \to 0$, by combining (3.17)–(3.20), for $d \ge d_0$, we have $P(B_d) \le \exp\{-((d-1)/J)(A/2)\}$, which implies (3.15) by Borel–Cantelli. □

From (2.33)–(2.40) for $J > 1$ [see (3.8), (3.9) and (3.10)] it follows that there exists a.s. $n_0 > 0$ such that for any $n \ge n_0$,

$$S_{n+1} = S_n + \min\left\{t > \left[\big(G(R^1_n)\big)^{A-1}\right], X_{S_n + t} > R^J_n\right\}$$

and

$$(3.21)\qquad \underline{R}_{n+1} = \Omega\big(R^{J-1}_n, \dots, R^1_n, X_{S_{n+1}}\big).$$

Thus, if we put $\mathscr{E}'_n = \{\max(X_{S_n+t};\ 1 \le t \le [(G(R^1_n))^{A-1}]) > R^J_n\}$ and if we prove that $P(\mathscr{E}'_n \text{ i.o.}) = 0$, then by (3.21) and Lemma 2.10, the result will be established.

Next, the fact that $P(\mathscr{C}'_n \text{ i.o.}\} = 0$ results as a straightforward consequence of the transcription of Lemma 2.11 with $R_{n-1} := R^J_{n-1}$ and $R_n := R^J_n$ in (2.119), the first equality in (3.9) and the following lemma.

LEMMA 3.3. *For any* $K > 0$,

$$P\Big\{G(R^1_n) < \big(G(R^J_{n-1})\big)^{1+K} \; i.o.\Big\} = 0. \tag{3.22}$$

PROOF. By (3.11), a.s. there exists an n_0 such that for $n > n_0$,

$$G(R^1_n) > \left(\frac{n}{J}\right)^{-(1+\varepsilon)} \exp\left(-\frac{n}{J}(1+\eta_n)\right), \qquad \eta_n \to 0, n \to \infty, \tag{3.23}$$

and by (2.99), for any $0 < \alpha < 1$ a.s. there exists an n_1 such that for $n > n_1$,

$$G(R^J_{n-1}) < \exp\left\{-\frac{(n-1)}{J}\alpha\right\}. \tag{3.24}$$

Thus, if $\alpha(1+K) > 1$,

$$\exp\left\{-\frac{n-1}{J}(1+K)\alpha\right\} \times \left[\left(\frac{n}{J}\right)^{-(1+\varepsilon)} \exp\left(-\frac{n}{J}(1+\eta_n)\right)\right]^{-1} \to 0$$

as $n \to \infty$, which implies (3.22). □

APPENDIX A

PROOF OF LEMMA 2.7

Proof under Hypothesis H_1. Put

$$U = R_I^{-1}(R - R_I). \tag{A1}$$

Then we have

$$R^{-1} - R_I^{-1} = (I+U)^{-1}R_I^{-1} - R_I^{-1} = (-U + U^2 - \cdots)R_I^{-1}. \tag{A2}$$

It may easily be checked [by using recursively the inequality $|AB(ij)| \le n \max_{1 \le k \le n} |A(ik)| \max_{1 \le l \le n} |B(lj)|$, $1 \le i, j \le n$] that $V^{2q+1} := U^{2q+1} \times R_I^{-1}$, $q \ge 0$, are of the form

$$V^{2q+1} = \begin{array}{|c|c|l} \hline 0 & V_h^{2q+1} & \}k \quad \text{with} \quad \max_{i,j} |V_h^{2q+1}(i,j)| \le \varrho, \\ \hline V_l^{2q+1} & 0 & \}t+1 \quad \text{with} \quad \max_{i,j} |V_l^{2q+1}(i,j)| \le \varrho, \\ \hline \end{array} \tag{A3}$$

where $\varrho = \rho(\rho\gamma k(t+1))^{2q+1}$

and where the maximum is taken over $1 \le i \le k$, $1 \le j \le t+1$ and

$$\text{(A4)} \qquad \begin{aligned} \rho &:= \max_{1 \le i,\, j \le k+t+1} \left| R_I^{-1}(i,j) \right|, \\ \gamma &:= \max_{1 \le i,\, j \le k+t+1} \left| (R - R_I^{-1})(i,j) \right|, \end{aligned}$$

and that $V^{2q} := U^{2q} \times R_I^{-1}$, $q \ge 1$, are of the form

$$\text{(A5)} \qquad V^{2q} = \begin{array}{|c|c|} \hline V_h^{2q} & 0 \\ \hline 0 & U_l^{2q} \\ \hline \end{array} \begin{array}{ll} \}k & \text{with} \quad \max_{i,j} \left| V_h^{2q}(i,j) \right| \le \varrho', \\ \}t+1 & \text{with} \quad \max_{i,j} \left| V_l^{2q}(i,j) \right| \le \varrho', \end{array}$$

where $\varrho' = \rho(\rho\gamma k(t+1))^{2q}$.

Thus, for any $n \ge 1$,

$$\text{(A6)} \qquad \max_{1 \le i,\, j \le k+t+1} \left| U^n R_I^{-1}(i,j) \right| \le \rho(\rho\gamma k(t+1))^n.$$

Let us now give bounds for ρ and γ defined in (A.4). We have

$$\text{(A7)} \qquad R_I^{-1} = (I + (R_I - I))^{-1} \sim I - S + S^2 \cdots$$

with $S := R_I - I$.

By (1.5) we have

$$\text{(A8)} \qquad \max_{1 \le v \le k+t+1} \sum_{u=1}^{k+t+1} |S(u,v)| < 2 \sum_{n=1}^{\infty} |\Gamma(n)| := \lambda < 1$$

and

$$\text{(A9)} \qquad \max_{1 \le u \le k+t+1} \sum_{v=1}^{k+t+1} |S(u,v)| < \lambda,$$

from which it may be easily seen that for any $n \ge 1$,

$$\text{(A10)} \qquad \max_{1 \le i,\, j \le k+t+1} |S^n(i,j)| < \lambda^n$$

and hence

$$\text{(A11)} \qquad \rho \le \sum_{n=1}^{\infty} \lambda^n = \frac{1}{1-\lambda}.$$

Next,

$$\text{(A12)} \qquad \gamma \le \max |\Gamma(j)|, \qquad j \ge (G(m_d))^{A-1}.$$

Thus, by combining (A2), (A6), (A11) and (A12) and since by (2.96) and (2.97),

$$k(t+1) \max_{j \ge (G(m_d))^{A-1}} |\Gamma(j)| \to 0 \quad \text{as } d \to \infty,$$

we deduce that

$$\text{(A13)}\qquad \max_{1\le i,\, j\le k+t+1} \left|\left((R^{-1}-R_I^{-1})+UR_I^{-1}\right)(i,j)\right| \times\left[k(t+1)\max_{j\ge (G(m_d))^{A-1}}|\Gamma(j)|\right]^{-1}\to 0$$

as $d\to\infty$, from which (2.85) follows.

Let us now prove (2.86). We have

$$\text{(A14)}\qquad \left|\operatorname{Det}(RR_I^{-1})-1\right| = \left|\operatorname{Det}(I+U)-1\right|$$

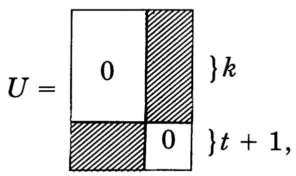
and

$$\operatorname{Det}(I+U)=\operatorname{Det}(e_1+U_{\cdot,1},\ldots,e_{k+t+1}+U_{\cdot,k+t+1}),$$

where e_n denotes the nth column of I and $U_{\cdot,n}$ is the nth column of U. By the multilinearity of the determinant we have

$$\text{(A15)}\qquad \operatorname{Det}(I+U)=1+\sum_{n=1}^{k+t+1}\begin{pmatrix}\text{determinants of all submatrices}\\ \text{of } U \text{ obtained by removing } n\\ \text{rows and columns of the same rank.}\end{pmatrix}.$$

Next, since U is of the form



among these determinants, only those belonging to the submatrices having the two zero blocks of equal size are not equal to 0.

Each such determinant of size $2p$, $1\le p\le \inf(k,t+1)$ (there are $C_k^p C_{t+1}^p$ of them) is bounded by $(p!)\times u^p$, where

$$\text{(A16)}\qquad u := \max_{1\le i,\, j\le k+t+1}|U(i,j)| \le \rho\gamma\max(k,t+1)=\rho\gamma k.$$

Thus

$$\text{(A17)}\qquad \begin{aligned}\left|\operatorname{Det}(I+U)-1\right| &\le \sum_{p=1}^{\inf(k,t+1)} C_k^p C_{t+1}^p (p!)^2 u^{2p} \le \sum_{p=1}^{\infty}\left(k(t+1)u^2\right)^p\\ &\le \sum_{p=1}^{\infty}\left(k^3(t+1)\rho^2\gamma^2\right)^p \le \sum_{p=1}^{\infty}\left(\rho^2k^4\gamma^2\right)^p,\end{aligned}$$

from which (2.86) follows since by (2.96) and (2.97),

$$k^2\gamma\le k^2\max_{j\ge (G(m_d))^{A-1}}|\Gamma(j)|\to 0\quad\text{as } d\to\infty.$$

Proof under Hypotheses H_2 *or* H_3. Let $R_1^{-1/2}$ be the lower triangular matrix of order k such that $R_1^{-1/2}(X_{j_1}, \dots, X_{j_k})'$ forms a complete orthonormal system of $H\{j_1, j_k\}$ and $R_2^{-1/2}$ the lower triangular matrix order $t+1$ such that $(R_2^{-1/2}(X_{j_{k+1}}, \dots, X_{j_{k+t+1}}))'$ forms a complete orthonormal system of $H\{j_{k+1}, j_{k+t+1}\}$.

Consider

$$R_I^{-1/2} = \begin{array}{|c|c|} \hline R_1^{-1/2} & 0 \\ \hline 0 & R_2^{-1/2} \\ \hline \end{array} \begin{array}{l} \}k \\ \}t+1 \end{array}$$

and observe that

(A18) $$(R_I^{-1/2})' R_I^{-1/2} = R_I^{-1}.$$

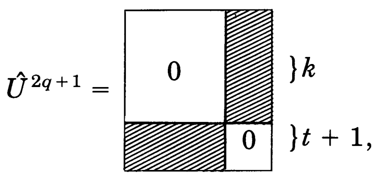
Put

(A19) $$\hat{U} = R_I^{-1/2}(R - R_I)(R_I^{-1/2})'.$$

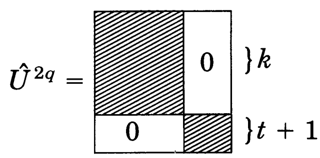
Then we have

(A20) $$\begin{aligned} R^{-1} - R_I^{-1} &= \left(R_I(I + R_I^{-1}(R - R_I))\right)^{-1} - R_I^{-1} \\ &= (R_I^{-1/2})'(-\hat{U} + \hat{U}^2 - \cdots)R_I^{-1/2}. \end{aligned}$$

$\hat{U}^{2q+1}$, $q \geq 0$, are of the form



$\hat{U}^{2q}$, $q \geq 1$, are of the form



and, as before, it may be easily checked that

(A21) $$\max_{1 \leq i, j \leq k+t+1} \left|\hat{U}^{2q+1}(i,j)\right| \leq \hat{u}^{2q+1} \times (k(t+1))^q, \qquad q \geq 0,$$

and

(A22) $$\max_{1 \leq i, j \leq k+t+1} \left|\hat{U}^{2q}(i,j)\right| \leq \hat{u}^{2q}(k(t+1))^q, \qquad q \geq 1,$$

with

(A23) $$\hat{u} := \max_{1 \leq i, j \leq k+t+1} \left|\hat{U}(i,j)\right| \leq \hat{\rho}^2 \gamma k \times (t+1),$$

where

$$\text{(A24)}\qquad \begin{aligned}\hat{\rho} &:= \max_{1\le i,\, j\le k+t+1} \left| R_I^{-1/2}(i,j)\right|,\\ \gamma &:= \max_{1\le i,\, j\le k+t+1} \left|(R-R_I)(i,j)\right|.\end{aligned}$$

Let us now give a bound for $\hat{\rho}$ in (A24).

Bound for $\hat{\rho}$ *under Hypothesis* H_2. For any $2 \le n \le k$, let $X_{j_n}^*$ be the projection of X_{j_n} on $H(j_1, j_{n-1})$ and put

$$\text{(A25)}\qquad X_{j_n}^* = \sum_{p=1}^{n-1} a_{n,p} X_{j_p}.$$

If we denote by σ_p the one-step prediction error [of the projection of X_0 on $H(-\infty, -1)$] and by σ_i the interpolation error of the projection of X_0 on $H(-\infty, -1) \cup H(1, +\infty)$, then, if (1.10) or (1.7) are satisfied (see Remark 4 in Section 1) we have

$$\text{(A26)}\qquad \sigma_p = \sqrt{2\pi}\, \exp\left\{ \frac{1}{4\pi} \int_{-\pi}^{\pi} \log f(\lambda)\, d\lambda \right\} > 0.$$

If (1.7) is satisfied, then we have

$$\text{(A27)}\qquad \sigma_i = 2\pi \left(\int_{-\pi}^{\pi} \frac{d\lambda}{f(\lambda)} \right)^{-1/2} > 0$$

and

$$\text{(A28)}\qquad \| X_{j_n} - X_{j_n}^* \| \ge \sigma_p \ge \sigma_i > 0.$$

Put

$$\begin{pmatrix} Y_1 \\ \vdots \\ Y_k \end{pmatrix} = R_1^{-1/2} \begin{pmatrix} X_{j_1} \\ \vdots \\ X_{j_k} \end{pmatrix},$$

which, by the definition of $R_1^{-1/2}$, is an orthonormal complete system of $H(X_{j_1}, \dots, X_{j_k})$ such that $Y_1 = X_{j_1}$ and for any $2 \le n \le k$,

$$\text{(A29)}\qquad Y_n = \frac{X_{j_n} - X_{j_n}^*}{\| X_{j_n} - X_{j_n}^* \|}.$$

Thus, for any $2 \le n \le k$,

$$\text{(A30)}\qquad R_1^{-1/2}(n,n) = \frac{1}{\| X_{j_n} - X_{j_n}^* \|} \le \frac{1}{\sigma_p} \le \frac{1}{\sigma_i}$$

and for any $1 \le p \le n-1$,

$$\text{(A31)}\qquad R_1^{-1/2}(n,p) = \frac{-a_{n,p}}{\| X_{j_n} - X_{j_n}^* \|}.$$

Let $1 \le p_0 \le n-1$ and $\hat{X}_{j_{p_0}}$ be the projection of $X_{j_{p_0}}$ on $H(X_{j_p}, p = 1, \dots, n-1; p \ne p_0)$ (the Hilbert space generated by X_{j_p}, $p = 1, \dots, n-1$; $p \ne p_0$).

Then, we have

(A32)
$$X_{j_n}^* = \sum_{\substack{p=1 \\ p \ne p_0}}^{n-1} a_{n,p} X_{j_p} + a_{n,p_0}\left(\eta_{j_{p_0}} + \hat{X}_{j_{p_0}}\right),$$

where $\eta_{j_{p_0}}$ is orthogonal to $H(X_{j_p}, p = 1, \dots, n-1; p \ne p_0)$. By the orthogonal projection theorem, we have

(A33)
$$|a_{n,p_0}| \times \|\eta_{j_{p_0}}\| \le \|X_{j_n}\| = 1$$

and since $\|\eta_{j_{p_0}}\| \ge \sigma_i$, we deduce that for any $1 \le p \le n-1$,

(A34)
$$|a_{n,p}| \le \frac{1}{\sigma_i} = \frac{\left(\int_{-\pi}^{\pi} \frac{d\lambda}{f(\lambda)}\right)^{1/2}}{2\pi} < \infty.$$

By combining (A30), (A31) and (A34), which are also valid for $R_2^{-1/2}$, we obtain that

(A35)
$$\hat{\rho} \le \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{d\lambda}{f(\lambda)}.$$

Bound for $\hat{\rho}$ *under Hypothesis* H_3. Let $-i_{n+1} < -i_n < -i_{n-1} < \cdots < -i_1 < 0$ and put (with the notation defined in the proof of Lemma 2.3)

(A36)
$$X_0^{X_{-i_n}, \dots, X_{-i_1}} := \hat{X}_{0/n} := a_{i_1}^{(n)} X_{-i_1} + \cdots + a_{i_n}^{(n)} X_{-i_n}.$$

Let

(A37)
$$\hat{X}_{0/n+1} := a_{i_1}^{(n+1)} X_{-i_1} + \cdots + a_{i_n}^{(n+1)} X_{-i_n} + a_{i_{n+1}}^{(n+1)} X_{-i_{n+1}}$$

and observe that

(A38)
$$\begin{aligned}\hat{X}_{0/n} &= a_{i_1}^{(n+1)} X_{-i_1} + \cdots + a_{i_n}^{(n+1)} X_{-i_n} \\ &\quad + a_{i_{n+1}}^{(n+1)}\left(b_{i_1}^{(n)} X_{-i_n} + \cdots + b_{i_n}^{(n)} X_{-i_n}\right),\end{aligned}$$

where

(A39)
$$X_{-i_{n+1}}^{X_{-i_n}, \dots, X_{-i_1}} := \sum_{k=1}^{n} b_{i_k}^{(n)} X_{-i_k}.$$

Thus, by combining (A36) and (A38) we obtain

(A40)
$$a_{i_k}^{(n+1)} = a_{i_k}^{(n)} - a_{i_{n+1}}^{(n+1)} b_{i_k}^{(n)}, \qquad k = 1, \dots, n.$$

Let us now suppose that there exists a sequence $u_n > 0$, $\sum_{n=1}^{\infty} u_n < \infty$ such that for any $n \ge 1$ and $-i_n < \cdots < -i_1 < 0$, we have

(A41)
$$\left|a_{i_n}^{(n)}\right| \le u_n.$$

Thus, if we put $M_n := \sup_{1 \le i \le n-1} |a_{i_{n-1}}^{(n)}|$ and since $|b_{i_1}^{(n)}| \le u_n$, we have by (A40),

$$M_{n+1} \le M_n(1 + u_{n+1}) \tag{A42}$$

and hence

$$\sup_{n \ge 1} |M_n| < M_1 \prod_{p=1}^{\infty} (1 + u_{1+p}) := M < \infty. \tag{A43}$$

Next, with the notation in (A25) and (A36), by making the identifications

$$a_{n,p} = a_{j_{n-p}}^{(n-1)}, \qquad p = 1, \dots, n-1, \tag{A44}$$

we deduce, by (A31) and (A43), that

$$\hat{\rho} \le \frac{M}{\sigma_p} = \frac{M}{\sqrt{2\pi}} \exp\left\{ \frac{-1}{4\pi} \int_{-\pi}^{\pi} \log f(\lambda)\, d\lambda \right\} < \infty. \tag{A45}$$

Let us now show that if Hypothesis H_3 is satisfied, then there exists a sequence $u_n > 0$, $\sum_{n=1}^{\infty} u_n < \infty$ such that (A41) holds.

Let

$$\begin{aligned} \eta_1 &= X_{-i_1}, \qquad \eta_2 = \left(X_{-i_2} - X_{-i_2}^{X_{-i_1}} \right) \Big/ \left\| X_{-i_2} - X_{-i_2}^{X_{-i_1}} \right\|, \dots, \\ \eta_n &= \left(X_{-i_n} - X_{-i_n}^{X_{-i_1}, \dots, X_{-i_{n-1}}} \right) \Big/ \left\| X_{-i_n} - X_{-i_n}^{X_{-i_1}, \dots, X_{-i_{n-1}}} \right\| \end{aligned} \tag{A46}$$

be an orthonormal complete system of the Hilbert space generated by $X_{-i_1}, \dots, X_{-i_n}$.

With the notations in (A36) we have

$$\hat{X}_{0/n} = \sum_{i=1}^{n} E(X_0 \cdot \eta_i) \eta_i, \tag{A47}$$

where, by expressing the η_i's as linear combinations of the X_{-i_k}, $k = 1, \dots, n$, only η_n contains X_{-i_n}.

Thus, by (A46),

$$a_{i_n}^{(n)} = E(X_0 \cdot \eta_n) \left\| X_{-i_n} - X_{-i_1}^{X_{-i_1}, \dots, X_{-i_n-1}} \right\|^{-1}. \tag{A48}$$

Next, by observing that $\eta_n \in H(-\infty, -i_n)$,

$$|E(X_0 \cdot \eta_n)| \le \rho(i_n) \le \rho(n) \le Cn^{-(6+\varepsilon)} \tag{A49}$$

[where $\rho(n)$ is as in (1.9)], we obtain finally

$$\begin{aligned} \left| a_{i_n}^{(n)} \right| &\le \frac{c}{\sigma_p} n^{-(6+\varepsilon)} \\ &= \frac{C}{\sqrt{2\pi}} \exp\left\{ \frac{-1}{2\pi} \int_{-\pi}^{\pi} \log f(\lambda)\, d\lambda \right\} n^{-(6+\varepsilon)} \\ &=: u_n, \end{aligned} \tag{A49'}$$

where C is a positive constant.

Let us now return to (A18)–(A24). We have by (A23) and (A12),

$$\hat{u} \le \hat{\rho}^2 k(t+1) \max_{j \ge (G(m_d))^{A-1}} |\Gamma(j)| \tag{A50}$$

and if we put

$$\hat{V} = \left(R_I^{-1/2}\right)' \hat{U} R_I^{-1/2}, \tag{A51}$$

it is easy to check that

$$\max_{1 \le i,\, j \le k+t+1} \left|\hat{V}(i,j)\right| \le \hat{\rho}^4 k^2 (t+1)^2 \max_{j \ge (G(m_d))^{A-1}} |\Gamma(j)|. \tag{A52}$$

Thus, from (A20) and (A50) we deduce, by routine calculations, that

$$\begin{aligned} &\max_{1 \le i,\, j \le k+t+1} \left|\left(R^{-1} - R_I^{-1} + \hat{V}\right)(i,j)\right| \\ &\quad \times \left[k^2(t+1)^2 \max_{j \ge (G(m_d))^{A-1}} |\Gamma(j)|\right]^{-1} \to 0 \quad \text{as } d \to \infty, \end{aligned} \tag{A53}$$

which implies (2.87).

In order to prove (2.88) we can use (A14)–(A17).

Thus we have

$$\left|\operatorname{Det}\left(R R_I^{-1}\right) - 1\right| \le \sum_{p=1}^{\infty} \left(k(t+1)u^2\right)^p \tag{A54}$$

with u as in (A16), from which (2.88) follows. □

APPENDIX B

PROOF OF LEMMA 2.11. Put

$$\begin{aligned} \Lambda(R_n) &= \left[\tau(G(R_n))^{-1}(-\log G(R_n))\right] \\ &\quad + \left[G(R_n)^{-(1+K)(1-A)}\right], \qquad n \ge 1, \end{aligned} \tag{B1}$$

and

$$\mathscr{G}_n = \left\{\max\left(\left|X_{S_n+Z_n+i}\right|; i = 1, \ldots, \Lambda(R_n)\right) > \mu_K(R_n)\right\}, \qquad n \ge 1. \tag{B2}$$

We shall make use of the following propositions.

PROPOSITION B.1. *If one among the Hypotheses* H_1, H_2 *or* H_3 *is satisfied, then there exist a* $\tau_0 > 0$ *and* $A_0 > 0$ *such that for any* $0 < \tau < \tau_0$ *and* $0 < A < A_0$, *there exists* $0 < K_0(A)$ *such that if* $0 < K < K_0(A)$, *then*

$$P\{\mathscr{G}_n \ i.o.\} = 0. \tag{B3}$$

PROOF. For any $r > 0$, $n \ge 1$ and $t \ge 1$, put

$$\mathscr{F}_{r,n}(t) = \{\max(|X_{n+k}|, k = 1, \ldots, t) < \mu_K(r)\} \tag{B4}$$

and observe that

$$\overline{\mathscr{G}_n} = \mathscr{F}_{R_n, S_n+Z_n}(\Lambda(R_n)), \tag{B5}$$

where, if $(1 + K)(1 - A) < 1$ [and since by (2.101) and Lemma 2.8, $G(R_n) \downarrow 0$ a.s. as $n \to \infty$], we have

$$\Lambda(R_n) = \tau(G(R_n))^{-1}(-\log G(R_n))(1 + o(1)) \quad \text{as } n \to \infty. \tag{B6}$$

The proof of Lemma 2.5 implies the following analogues of (2.67):

$$\begin{aligned} &P\{\mathscr{F}_{R_d, S_d+Z_d}(t) | \eta_d = l_d\} f_{\eta_d}(l_d) \\ &\quad = \sum_{\zeta \in \mathscr{S}(r_d)} \sum_{1 \le j \le J(\zeta, l_d)} A_j(\zeta, l_d) P_\zeta\big(E_j(l_d) \cap \mathscr{F}_{r_d, s_d+z_d}(t)\big) \\ &\qquad + \sum_{1 \le k \le J'(l_d)} B_k(l_d) P\big(E'_k(l_d) \cap \mathscr{F}_{r_d, s_d+z_d}(t)\big) \\ &\qquad + C(l_d) 1_{\{\nu_d = 0, m_d = r_d\}} P\big(\mathscr{F}_{r_d, 1}(t)\big) \\ &\qquad + C(l_d) 1_{\{\nu_d = 0, m_d = r_d\}} P\big(\mathscr{F}_{r_d, 1}(t)\big). \end{aligned} \tag{B7}$$

Similarly, by using the proof of Lemma 2.6, we obtain:

LEMMA 2.6′. *If one among Hypotheses* H_1, H_2 *or* H_3 *is satisfied, then there exist an* $0 < A_0 < 1$ *and two positive constants* C_1 *and* C_2 *independent of* l_d, *such that for any* $0 < A \le A_0$, *we have*

$$\begin{aligned} &\max_{\zeta \in \mathscr{S}(l_d)} \max_{1 \le j \le J(\zeta, l_d)} \Big| P_\zeta\big(E_j(l_d) \cap \mathscr{F}_{r_d, s_d+z_d}(t)\big) \\ &\qquad \cdot \big\{P_\zeta\big(E_j(l_d)\big) P\big(\mathscr{F}_{r_d, 1}(t)\big)\big\}^{-1} - 1 \Big| \\ &\quad \le C_1 [G(r_d)]^{C_2} \end{aligned} \tag{B8}$$

and

$$\begin{aligned} &\max_{1 \le k \le J'(l_d)} \Big| P\big(E'_k(l_d) \cap \mathscr{F}_{r_d, s_d+z_d}(t)\big) \cdot \big\{P\big(E'_k(l_d)\big) P\big(\mathscr{F}_{r_d, 1}(t)\big)\big\}^{-1} - 1 \Big| \\ &\quad \le C_1 [G(r_d)]^{C_2}. \end{aligned} \tag{B9}$$

By combining (B7), (B8) and (B9) and by again using the fact that

$$\begin{aligned} &\left(\max_{\substack{1 \le i \le n \\ p_i > 0,\, \hat{p}_i > 0,\, 1 \le i \le n}} \left| \frac{p_i}{\hat{p}_i} - 1 \right| \le q \right) \\ &\quad \Rightarrow \left| \frac{\Sigma \alpha_i p_i}{\Sigma \alpha_i \hat{p}_i} - 1 \right| \le q \quad \text{for any } \alpha_i > 0, 1 \le i \le n, \end{aligned} \tag{B10}$$

we obtain

$$\text{(B11)} \qquad \max_{t \ge 1} \left| \frac{P\{\mathscr{F}_{R_d, S_d + Z_d}(t) | \eta_d = l_d\}}{P\{\mathscr{F}_{r_d, 1}(t)\}} - 1 \right| < C_1 [G(r_d)]^{C_2}.$$

Next, if we apply Lemma 2.2 with $\mathscr{F}_{r_d,1}(t) := Q(t, r_d, r_d)$ and $(1 + K)(1 - A) < 1$ [in order to have (B6)], it follows that there exist two positive constants c_1 and c_2 and $\tau_0 > 0$ such that for any $0 < \tau \le \tau_0$,

$$\text{(B12)} \qquad \max_{1 \le t \le \Lambda(r_d)} \left| \frac{P\{\mathscr{F}_{r_d,1}(t)\}}{(1 - 2G(\mu_K(r_d)))^t} - 1 \right| < c_1 [G(r_d)]^{c_2},$$

since $Q_I(t, r_d, r_d) = (1 - 2G(\mu_K(r_d)))^t$. Furthermore,

$$\text{(B13)} \qquad \max_{1 \le t \le \Lambda(r_d)} \left| (1 - 2G(\mu_K(r_d)))^t - 1 \right| \le 3\tau [G(r_d)]^K (-\log G(r_d))$$

[by the classical inequality $(1 - at) < (1 - a)^t < 1$, $0 < a < 1$, $t > 0$.] Thus, by combining (B5), (B11), (B12) and (B13), we obtain that if $0 < K \le K_0(A) = A/(1 - A)$, $A < A_0$, then

$$\text{(B14)} \qquad \begin{aligned} &|P\{\mathscr{G}_d | \eta_d = l_d\}| \\ &\le C_1 [G(r_d)]^{C_2} + c_1 [G(r_d)]^{c_2} + 3\tau [G(r_d)]^K (-\log G(r_d)) \\ &\le a [G(r_d)]^b \quad \text{for some positive constants } a \text{ and } b. \end{aligned}$$

Finally, for any $0 < \alpha < 1$,

$$\text{(B15)} \qquad \begin{aligned} &P\{\mathscr{G}_d \cap (G(R_d) < e^{-\alpha d})\} \\ &\le \int_{\{G(r_d) < e^{-\alpha d}\}} a [G(r_d)]^b \, dP_{\eta_{d-1}}(l_{d-1}) \le a e^{-\alpha d b}, \end{aligned}$$

from which, by using the Borel–Cantelli lemma and (2.101), we deduce (B3). □

Now, for any $0 < r < \rho < \rho + \delta\rho \le \mu_K(r)$, $n \ge 1$, $t \ge 1$ and $t' \ge 1$, put

$$\text{(B16)} \qquad \begin{aligned} H_r^{n,t,t'}(\rho, \delta\rho) = &\{-\mu_K(r) < X_{n+s} < r; s = 0, \dots, t - 1\} \\ &\cap \{\rho < X_t < \rho + \delta\rho\} \\ &\cap \{-\mu_K(r) < X_{n+t+s'} < \rho + \delta\rho; s' = 1, \dots, t' - 1\} \\ &\cap \{\rho + \delta\rho < X_{n+t+t'} < \mu_K(r)\}. \end{aligned}$$

PROPOSITION B.2. *If one among Hypotheses* H_1, H_2 *or* H_3 *is satisfied, then there exists a* $0 < A_0 < 1$ *such that for any* $0 < A \le A_0$ *and* $K \le A/(1 - A)$

we have

$$\text{(B17)}\quad \max_{\substack{1\le t\le \tau(G(r_d))^{-1}(-\log G(r_d))\\ 1\le t'\le (G(r_d))^{-(1-A)(1+K)}}} dP\{H_{r_d}^{s_d+z_d,t,t'}(\rho)|\eta_d = l_d\} \le (\text{const.})\, dP\{H_{r_d}^{1,t,t'}(\rho)\},$$

where $dP\{H_{r_d}^{n,t,t'}(\rho)\} := P\{H_{r_d}^{n,t,t'}(\rho,\delta\rho)\}$ *and* (const.) *is a positive constant independent of* ρ *and* l_d.

PROOF. As in the proof of Proposition B.1, we may replace in (B7), (B8) and (B9), $\mathscr{F}_{r_d, s_d+z_d}(t)$ and $\mathscr{F}_{r_d,1}(t)$, respectively, by $H_{r_d}^{s_d+z_d,t,t'}(\rho)$ and $H_{r_d}^{1,t,t'}(\rho)$ in order to obtain, by (B10), that there exist c_1 and $c_2 > 0$ such that the inequality corresponding to (B11) [which clearly implies (B17)] holds. □

PROPOSITION B.3. *If one among Hypotheses* H_1, H_2 *or* H_3 *is satisfied, then there exist* $c > 0$ *and* $\varepsilon > 0$ *such that for any* $r > 0$, *we have*

$$\text{(B18)}\quad \sum_{(r)}\sum := \sum\sum_{\substack{1\le t\le \tau(G(r))^{-1}(-\log G(r))\\ 1\le t' < (G(r))^{-(1-A)(1+K)}}} \int_r^{\mu_K r} dP\{H_r^{1,t,t'}(\rho)\} \le c[G(r)]^{\varepsilon}.$$

PROOF. For any $0 < r < \rho < \rho + \delta\rho < \mu_K(r)$, $t \ge 1$ and $t' \ge 1$, we have

$$\begin{aligned}
dP\{H_r^{1,t,t'}(\rho)\} &\le P\{\rho < X_0 < \rho + \delta\rho,\ X_{t'} > \rho + \delta\rho\}\\
&\sim \frac{\delta\rho}{2\pi(1-\Gamma^2(t'))^{1/2}} \int_\rho^\infty \exp\left\{-\frac{\rho^2 2\Gamma(t')\rho x + x^2}{2(1-\Gamma(t'))}\right\} dx \qquad (\delta\rho \to \infty)\\
&\le \frac{\delta\rho}{2\pi(1-\Gamma^2(t'))^{1/2}} \exp\left\{-\frac{\rho^2}{2(1+|\Gamma(t')|)}\right\}\\
&\qquad \times \int_\rho^\infty \exp\left\{-\frac{x^2}{2(1+|\Gamma(t')|)}\right\} dx \qquad \text{[by (2.8)]}\\
&\le \frac{\delta\rho}{(1-\Gamma^2(t'))^{1/2}} \times \frac{d}{d\rho}\left\{(-G^2)'\left(\frac{\rho}{(1+|\Gamma(t')|)^{1/2}}\right)\right\}\\
&= \frac{1}{(1-|\Gamma(t')|)^{1/2}}\, d\left(-G^2\left(\frac{\rho}{(1+|\Gamma(t')|)^{1/2}}\right)\right).
\end{aligned}\tag{B19}$$

Thus, by splitting the summation with respect to t' in (B18) into

$$\text{(B20)}\quad \sum_{(r)}\sum = \sum_{t=1}^{\tau(G(r))^{-1}(-\log G(r))} \left(\sum_{t'=1}^{[T]} + \sum_{t'=T+1}^{[(G(r))]^{-(1-A)(1+K)}}\right),$$

where $1 \le T \le G(r)^{-(1-A)(1+K)}$, and by using (B19), we obtain

$$\sum_{(r)}\sum \le \frac{\tau}{(1-\delta)^{1/2}} \times \Bigg(G^2\left(\frac{r}{(1+\delta)^{1/2}}\right) \times (G(r))^{-1}(-\log G(r))T + G^2\left(\frac{r}{(1+\delta(T))^{1/2}}\right) \times (G(r))^{-1-(1-A)(1+K)}(-\log G(r))\Bigg) \tag{B21}$$

$$=: G_{1,n} + G_{2,n},$$

where $\delta = \max_{1 \le t}|\Gamma(t)|$ and $\delta(T) = \max_{T \le t}|\Gamma(t)|$. Next,

$$G^2\left(\frac{r}{(1+\delta)^{1/2}}\right) \times (G(r))^{-1} \sim \frac{(1+\delta)}{\sqrt{2\pi}} \cdot \frac{1}{r}\exp\left\{-r^2\left(\frac{1}{1+\delta} - \frac{1}{2}\right)\right\}, \qquad r \to \infty.$$

Observe that $(1/(1+\delta)) - (1/2) > 0$ since $0 < \delta < 1$ and take $T = T(r) = e^{\eta r^2}$ with

$$0 < \eta < \frac{1}{2}\inf\left(\left(\frac{1}{1+\delta} - \frac{1}{2}\right), (1-A)(1+K)\right).$$

Thus [since $-\log(G(r)) \sim (r^2/2)$ and for any $a, b > 0$, $r^a e^{-r^{2b}} \to 0$ as $r \to \infty$], there exist a $c > 0$ and $\varepsilon > 0$ such that $G_{1,n} \le (c/2)(G(r))^{\varepsilon}$. Likewise, if we consider $G_{2,n}$, we have

$$G^2\left(\frac{r}{(1+\delta(T))^{1/2}}\right)(G(r))^{-1-(1-A)(1+K)} \sim \frac{(1+\delta(T))}{\sqrt{2\pi}} r^{(1-A)(1+K)-1} \times \exp\left\{-r^2\left[\frac{A + A\delta(T) - 2\delta(T) + K(1-A)(1+\delta(T))}{2(1+\delta(T))}\right]\right\}, \tag{B22}$$

$$r \to \infty.$$

Thus, since by Hypotheses H_1, H_2 or H_3, $\lim_{T\to\infty}\delta(T) = 0$, for $r \ge r_0$ sufficiently large, the factor of $-r^2$ in (B22) is larger than

$$\frac{A}{2} + \frac{K(1-A)}{2} > 0.$$

This, by the same arguments as above, completes the proof of (B18). □

We can now prove Lemma 2.11. Put

$$D_n = \left\{ ((\Delta S_{n+1}, \mathscr{R}_{n+1}) \in \psi(R_n)) \cap \left(Z_n = \left[G(R_n)^{A-1} \right] \right) \cap (G(R_n) < e^{-\alpha n}) \right\}, \qquad n \geq 1. \tag{B23}$$

By Lemmas 2.8, 2.9 and 2.10 and Proposition B.1, it is equivalent to prove (2.120) and

$$P\{\mathscr{E}_n \text{ i.o.}\} = 0, \tag{B24}$$

where

$$\mathscr{E}_n = C_{n+1} \cap D_n \cap \overline{\mathscr{I}}_n, \qquad n \geq 2. \tag{B25}$$

We have, with the notation in (B18),

$$P(\mathscr{E}_n) = \int_{G(r_n) < e^{-\alpha n}} \left(\sum_t \sum_{t'} \int_{r_n}^{\mu_K(r_n)} dP\{H_{r_n}^{s_n + z_n, t, t'}(\rho) | \eta_n = e_n\} \right) dP_{\eta_n}(e_n), \tag{B26}$$

where $\sum\sum$ is over $1 < t < \tau(G(r_n))^{-1}(-\log G(r_n))$ and $1 \leq t' \leq (G(r_n))^{1+A(1+K)}$. Using (B17) and (B18) in (B26), we obtain

$$P(\mathscr{E}_n) \leq (\text{const.}) \cdot c \cdot e^{-\alpha \varepsilon n} \tag{B27}$$

from which (B24) follows by the Borel–Cantelli lemma. □

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