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CHARACTERISTIC EXPONENTS FOR TWO-DIMENSIONAL BOOTSTRAP PERCOLATION

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Bootstrap percolation is a model in which an element of $\mathbf{Z}^2$ becomes occupied in one time unit if two appropriately chosen neighbors are occupied. Schonmann [4] proved that starting from a Bernoulli product measure of positive density, the distribution of the time needed to occupy the origin decays exponentially. We show that for $\alpha > 1$, the exponent can be taken as $\delta p^{2\alpha}$ for some $\delta > 0$, thus showing that the associated characteristic exponent is at most two. Another characteristic exponent associated to this model is shown to be equal to one.

1. Introduction. Bootstrap percolation in dimension 2 is a discrete time process on $X = \{0,1\}^{\mathbf{Z}^2}$, where $\mathbf{Z}^2$ is the planar integer lattice. Elements of $\mathbf{Z}^2$ are called sites and when the process is at state $\eta \in X$, we say that x is occupied or vacant according to whether $\eta(x) = 1$ or $\eta(x) = 0$, respectively. For $n = 0, 1, 2 \dots$, η_n will represent the state of the process at time n. Denoting by $\{e_1, e_2\}$ the canonical basis of $\mathbf{R}^2$, we describe the evolution of the process by means of the following rule: A necessary and sufficient condition for $\eta_{n+1}(x) = 0$ is that either

$$\eta_n(x - e_1) = \eta_n(x) = \eta_n(x + e_1) = 0$$

or

$$\eta_n(x - e_2) = \eta_n(x) = \eta_n(x + e_2) = 0.$$

This means that an occupied site remains occupied forever and that a vacant site is occupied in one time unit if at least one of its horizontal neighbors and one of its vertical neighbors are occupied. This evolution is deterministic but the initial state η_0 may be random. In this paper we are interested in the behavior of the process for initial states η_0 satisfying the following condition: The random variables $\eta_0(x)$ $(x \in \mathbf{Z}^2)$ are i.i.d. Bernoulli with parameter $p \in [0,1]$. For each value of p, the probability measure generated by this initial distribution on the space of trajectories of the process will be denoted by P_p.

This model has been recently studied by Schonmann [3] and a slightly modified version of it is treated in van Enter [4] and Aizenman and Lebowitz [1]. The interested reader will find in these papers some older references on the

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subject. To describe some of the known results, let

$$T = \inf\{n \geq 0: \eta_n(0) = 1\},$$

that is, the time we have to wait for the origin to be occupied. Schonmann [3] adapted a proof of van Enter [4] to show that for all $p > 0$, $T < \infty$, P_p-a.s., and using this result he also proved that for $p > 0$, there exist C and γ in $(0, \infty)$ such that

$$P_p(T \geq n) \leq Ce^{-\gamma n} \qquad \forall\, n \geq 0. \tag{1.1}$$

Then he introduced the quantity

$$\gamma(p) = \sup\{\gamma > 0: \exists\, C \text{ such that } P_p(T \geq n) \leq Ce^{-\gamma n}\ \forall\, n\},$$

and raised the following question: How does $\gamma(p)$ behave as $p \downarrow 0$? Does the characteristic exponent

$$\nu = \lim_{p \downarrow 0} \frac{\log(\gamma(p))}{\log p}$$

exist? What value does it take? He also gave a simple argument showing that

$$\liminf_{p \downarrow 0} \frac{\log(\gamma(p))}{\log p} \geq 1, \tag{1.2}$$

which we present for the sake of completeness: Note that

$$P_p(T \geq n) \geq P_p(\eta_0(ie_1) = 0,\ -n \leq i \leq n) = (1-p)^{2n+1},$$

hence $\gamma(p) \leq -2\ln(1-p)$, therefore

$$\limsup_{p \downarrow 0} \frac{\gamma(p)}{p} \leq 2,$$

which implies (1.2).

It is harder to give upper bounds for ν (assuming it exists). Our main result shows that it cannot be bigger than two:

THEOREM 1.3.

$$\limsup_{p \downarrow 0} \frac{\log \gamma(p)}{\log(p)} \leq 2.$$

Another characteristic exponent can be defined in the following way: For $\Delta \subset \mathbf{Z}^2$ and $\eta \in X$, let

$$\eta^{\Delta}(x) = \begin{cases} 1, & \text{if } x \in \Delta \text{ and } \eta(x) = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Then, say that Δ is internally spanned by η if for all $x \in \Delta$, $\lim_{n \to \infty}(\eta^{\Delta})_n(x) = 1$. This concept was introduced by Aizenman and Lebowitz [1] and was later used by Schonmann [3]. Connected to it is the following probability:

$$N(L,p) = P_p([0,L] \times [0,L] \text{ is internally spanned}).$$

In both of the above mentioned papers it is shown that $N(L,p)$ converges exponentially to 1 as L goes to infinity. In view of this we let

$$\bar{\gamma}(p) = \sup\{\gamma > 0 \colon \exists\, C \text{ such that } 1 - N(L,p) \le Ce^{-\gamma L}\ \forall\, L \ge 0\}$$

and we define a new critical exponent:

$$\bar{\nu} = \lim_{p \downarrow 0} \frac{\log \bar{\gamma}(p)}{\log(p)} \quad \text{if the limit exists.}$$

The techniques developed to prove Theorem 1.3 give us better information about $\bar{\nu}$:

THEOREM 1.4. *The critical exponent $\bar{\nu}$ exists and is equal to one.*

The paper is organised as follows. In Section 2 we compare oriented bond percolation to oriented site percolation. This will enable us to use, in the context of site percolation, known results concerning bond percolation. In Section 3 we give the proofs of Theorems 1.3 and 1.4. Throughout this paper the following notation will be in force: $e_1 = (1,0)$ and $e_2 = (0,1)$ are the elements of the canonical basis in $\mathbf{R}^2$; in that same space $\langle\ \rangle$ and $\|\ \|$ will denote the Euclidean inner product and Euclidean norm, respectively. For $A \subset \mathbf{Z}^2$, $|A|$ will denote its cardinality. The natural logarithm of x will be written as $\log x$ and, finally, we define

$$\mathbf{Z}_0 = \{n \in \mathbf{Z} \colon n \ge 0\}$$

and

$$\mathbf{Z}_{-1} = \{n \in \mathbf{Z} \colon n \ge -1\}.$$

2. A comparison between oriented bond percolation and oriented site percolation. Consider the following oriented bond percolation model on $\mathbf{Z}_0 \times \mathbf{Z}_{-1}$. From each element $x \in \mathbf{Z}_0 \times \mathbf{Z}_{-1}$, start two oriented bonds whose respective endpoints are $x + e_2$ and $x + e_1 + e_2$. These oriented bonds will be open, independently from each other, with probability ρ. We will say that the event $x \to_b y$ occurs if $x = y$ or there exists a finite sequence $x_0 = x, x_1, \dots, x_n = y$ of elements in $\mathbf{Z}_0 \times \mathbf{Z}_{-1}$ such that for all $i = 0, 1, \dots, n-1$, there is an open bond starting at x_i and having x_{i+1} as its endpoint. It is easily seen that this model is equivalent to the model studied by Durrett [2]. From this last paper we will need the following results. Let $\rho = 9/10$ and for $A \subset \mathbf{Z}_0 \times \mathbf{Z}_{-1}$, define

$$C_A = \{y \in \mathbf{Z}_0 \times \mathbf{Z}_{-1} \colon x \to_b y \text{ for some } x \in A\}.$$

Then there exist positive constants C_1 and γ_1 satisfying

$$(2.1) \qquad P(|C_A| < \infty) \le C_1 e^{-\gamma_1 |A|} \quad \forall\, A \subset \mathbf{Z}_0 \times \{-1\}.$$

Now, for $A \subset \mathbf{Z}_0 \times \{-1\}$ and $n \in \mathbf{Z}_{-1}$, define

$$\Gamma_n^A = \{y \in C_A: \langle y, e_2 \rangle = n\}.$$

Elementary arguments show that given $\beta > 0$,

$$\begin{aligned} P(|C_A| < \infty) &\geq P(\text{for some } n, |\Gamma_n^A| < \beta|A|)\left(1 - \tfrac{9}{10}\right)^{2\beta|A|} \\ &= P(\text{for some } n, |\Gamma_n^A| < \beta|A|)e^{-\log(10)2\beta|A|}. \end{aligned}$$

This and (2.1) are easily seen to imply that for some constants $\gamma_2 > 0$ and $\beta > 0$, we have

$$(2.2) \qquad P(|\Gamma_n^A| < \beta|A| \text{ for some } n) \leq C_1 e^{-\gamma_2|A|} \qquad \forall A \subset \mathbf{Z}_0 \times \{-1\}.$$

Although (2.1) and (2.2) have been shown for $\rho = \frac{9}{10}$, standard coupling techniques prove that the same inequalities (with the same constants C_1, β, γ_1 and γ_2) hold for all $\rho \geq \frac{9}{10}$.

The oriented site percolation model we will consider is defined in the following way: elements $x \in (\mathbf{Z}_0)^2$ are called sites. Sites will be open independently with probability $\bar{\rho}$. We say that the event $\{x \to_s y\}$ occurs if there exists a finite sequence of open sites $x_0 = x, x_1, \dots, x_n = y$ such that

$$x_{i+1} - x_i = e_1 \quad \text{or} \quad x_{i-1} - x_i = e_2, \qquad i = 0, 1, \dots, n-1,$$

$$\langle x_{i+2} - x_i, e_2 \rangle \geq 1, \qquad i = 0, 1, \dots, n-2.$$

This means that among two consecutive steps at least one is upwards. As in the previous model, we define for $A \subset (\mathbf{Z}_0)^2$,

$$\overline{C}_A = \left\{y \in (\mathbf{Z}_0)^2: x \to_s y \text{ for some } x \in A\right\}$$

and

$$\overline{\Gamma}_n^A = \left\{y \in \overline{C}_A: \langle y, e_2 \rangle = n\right\}.$$

The following proposition will allow us to compare these two percolation models.

PROPOSITION 2.3. *Let $\rho \in (0,1)$. Then there exists a probability space Ω supporting two sequences of iid Bernoulli random variables $(X_{(i/2)})_{i \geq -1}$ and $(Z_i)_{i \geq 0}$, satisfying* (a) $P(X_{(i/2)} = 0) = 1 - \rho$, (b) $P(Z_i = 0) = \rho(1-\rho)^3$, (c) $P(Z_i \geq \max\{X_{i-(1/2)}, X_i, X_{i+(1/2)}\}) = 1\ \forall\ i \geq 0$.

PROOF. Consider a probability space Ω supporting two independent sequences of iid random variables $(X_{(i/2)})_{i \geq -1}$ and $(U_i)_{i \geq 0}$ with the following properties: each $X_{i/2}$ has a Bernoulli distribution with parameter ρ and each $(U_i)_{i \geq 0}$ has a uniform distribution in the interval $[0,1]$. In this space Ω we will construct an i.i.d. sequence $(Z_i)_{i \geq 0}$ satisfying (b) and (c). First, let

$$Y_i = \max\{X_{i-(1/2)}, X_i, X_{i+(1/2)}\}.$$

Then for any sequence $a_0, \ldots, a_k$ of zeros and ones, define

$$c(a_0, \ldots, a_k) = P(Y_{k+1} = 0 | Y_i = a_i, 0 \le i \le k).$$

The coefficients $c(a_0, \ldots, a_k)$ satisfy the inequalities

$$\begin{aligned} &c(a_0, \ldots, a_{k-1}, 0) \\ &\quad = \frac{P(Y_i = a_i, 0 \le i \le k-1, X_{k-(1/2)} = X_k = 0, Y_{k+1} = 0)}{P(Y_i = a_i, 0 \le i \le k-1, Y_k = 0)} \\ &\quad = \frac{P(Y_i = a_i, 0 \le i \le k-1, X_{k-(1/2)} = X_k = 0) P(Y_{k+1} = 0)}{P(Y_i = a_i, 0 \le i \le k-1, Y_k = 0)} \\ &\quad \ge P(Y_{k+1} = 0) = (1-\rho)^3 \end{aligned}$$

and

$$\begin{aligned} &c(a_0, \ldots, a_{k-1}, 1) \\ &\quad \ge \frac{P(X_{k+(1/2)} = X_{k+1} = X_{k+(3/2)} = 0, X_k = 1, Y_i = a_i, 0 \le i \le k-1)}{P(Y_{k+1} = 0, Y_k = 1, Y_i = a_i, 0 \le i \le k-1)} \\ &\quad = \frac{\rho(1-\rho)^3 P(Y_i = a_i, 0 \le i \le k-1)}{P(Y_{k+1} = 0, Y_k = 1, Y_i = a_i, 0 \le i \le k-1)} \ge \rho(1-\rho)^3. \end{aligned}$$

Hence $c(a_0, \ldots, a_k) \ge \rho(1-\rho)^3 \ \forall (a_0, \ldots, a_k) \in \{0,1\}^{k+1}$. We now proceed to construct the random variables Z_i:

$$Z_0 = \begin{cases} 0, & \text{if } Y_0 = 0 \text{ and } U_0 < \rho, \\ 1, & \text{otherwise,} \end{cases}$$

$$Z_{i+1} = \begin{cases} 0, & \text{if } Y_{i+1} = 0 \text{ and } U_{i+1} \le \dfrac{\rho(1-\rho)^3}{c(Y_0, \ldots, Y_i)}, \\ 1, & \text{otherwise.} \end{cases}$$

Since condition (c) is clearly satisfied, we only have to show that the Z_i's are i.i.d. Bernoulli with parameter $1 - \rho(1-\rho)^3$. To do so, note that the random variables $U_0, \ldots, U_k$ are independent of the random variables $Y_0, \ldots, Y_k, Z_{k+1}$. This implies that for $u_i \in [0,1]$,

$$\begin{aligned} &P(Z_{k+1} = 0 | (Y_0, \ldots, Y_k) = (a_0, \ldots, a_k), U_i \le u_i, 0 \le i \le k) \\ &\quad = P\left(Y_{k+1} = 0, U_{k+1} \le \frac{\rho(1-\rho)^3}{c(a_0, \ldots, a_k)} \middle| (Y_0, \ldots, Y_k) = (a_0, \ldots, a_k)\right) \\ &\quad = P\left(U_{k+1} \le \frac{\rho(1-\rho)^3}{c(a_0, \ldots, a_k)}\right) P(Y_{k+1} = 0 | (Y_0, \ldots, Y_k) = (a_0, \ldots, a_k)) \\ &\quad = \rho(1-\rho)^3. \end{aligned}$$

This shows that each Z_k has the desired distribution and that Z_{k+1} is independent of the random variables $(Y_0, \dots, Y_k, U_0, \dots, U_k)$. Since Z_k is a function of $U_k, Y_0, \dots, Y_k$, it follows that Z_{k+1} is independent of $Z_1, \dots, Z_k$. □

COROLLARY 2.4. *Suppose that* ρ *and* $\bar{\rho} \in [0,1]$ *satisfy* $\bar{\rho} \ge 1 - \rho(1-\rho)^3$. *Then we can construct on the same probability space an oriented bond percolation model with parameter* ρ *and an oriented site percolation model with parameter* $\bar{\rho}$ *in such a way that* $\forall\, x, y \in (\mathbf{Z}_0)^2$, $x - e_2 \neq y$, *we have*

$$\{x - e_2 \to_b y\} \subset \{x \to_s y\} \quad a.s.$$

PROOF. By standard coupling techniques it suffices to prove the result for $\bar{\rho} = 1 - \rho(1-\rho)^3$. It follows easily from Proposition 2.3 that there exists a probability space containing random variables $(X_{(i/2),j}, X_{-(1/2),j}, Z_{i,j})_{i,j\ge 0}$ such that all the X's are i.i.d. Bernoulli with parameter ρ, all the Z's are i.i.d. Bernoulli with parameter $1 - \rho(1-\rho)^3$ and

$$P\big(Z_{i,j} \ge \max\{X_{i-(1/2),j}, X_{i,j}, X_{i+(1/2),j}\}\big) = 1 \qquad \forall\, i,j \in \mathbf{Z}_0.$$

Now let the site $(i,j) \in (\mathbf{Z}_0)^2$ be open if $Z_{i,j} = 1$, let the bond connecting $(i, j-1)$ to (i,j) be open if $X_{i,j} = 1$ and let the bond connecting $(i, j-1)$ to $(i+1, j)$ be open if $X_{i+(1/2),j} = 1$. It is now easy to see that these two percolation models satisfy the desired properties. □

COROLLARY 2.5. *There exist constants* C, γ *and* $\beta \in (0,\infty)$ *such that for* $\bar{\rho} \ge 1 - \frac{9}{10}(\frac{1}{10})^3$, *we have*

$$P\big(|\bar{\Gamma}_n^A| \ge \beta|A| \ \forall\, n \ge 1\big) \ge 1 - Ce^{-\gamma|A|} \qquad \forall\, A \subset \mathbf{Z}_0 \times \{0\}.$$

PROOF. This is an immediate consequence of (2.2) and the previous corollary. □

3. Proofs of Theorems 1.3 and 1.4. The first important tool in the proofs of both theorems is renormalization. Instead of considering single points in $\mathbf{Z}^2$ as sites, we define renormalized sites of size N. These are squares of the form

$$[kN, (k+1)N - 1] \times [lN, (l+1)N - 1] \qquad k, l \in \mathbf{Z}.$$

The above renormalized site will be denoted by (k,l). This establishes a natural correspodance between renormalized sites and $\mathbf{Z}^2$. Renormalization techniques were used by Schonmann [3] to prove (1.1). However, his renormalized sites are much larger than the ones involved in our proofs. Here we fix $\alpha > 1$ and our renormalized sites are of size $N = [1/p^\alpha]$, where [ ] denotes integer part of. In the sequel, to simplify the notation the symbol [ ] will be omitted, since we believe that this will not confuse the reader.

The second important tool in our proofs is the following definition: A renormalized site is *easily covered* if at time zero each line and each column of that renormalized site have at least one occupied site. The reasons for introducing this appear in the following lemmas.

LEMMA 3.1. *If a renormalized site is easily covered and one of its neighbors is entirely occupied at time n, then it will be entirely occupied at time $n + (1/p^\alpha)^2$.*

The proof is obvious and we omit it.

LEMMA 3.2. *The probability that a renormalized site is easily covered converges to one as $p \downarrow 0$.*

PROOF. The probability that a renormalized site is not easily covered is bounded above by

$$2\frac{1}{p^\alpha}(1-p)^{1/p^\alpha} = 2\frac{1}{p^\alpha}\left[(1-p)^{1/p}\right]^{(1/p^\alpha)p}.$$

For p small enough, this is less than

$$2\frac{1}{p^\alpha}\left(\frac{1}{2}\right)^{(1/p^\alpha)p},$$

which goes to zero as p goes to zero. □

From now on we use the following abbreviations: e.c.r.s. will mean easily covered renormalized site and i.s.r.s. will mean internally spanned renormalized site. Of course, each i.s.r.s. is an e.c.r.s. but the converse is false. However, for any $p > 0$, e.c.r.s.'s are i.s.r.s.'s independently, with probability $f(p) > 0$.

We need now to introduce four different types of oriented paths of renormalized sites. All these paths are formed by a finite or infinite sequence of renormalized sites $(k_0, l_0), (k_1, l_1), \dots, (k_n, l_n), \dots$ such that $\|(k_{i+1}, l_{i+1}) - (k_i, l_i)\| = 1$, $\forall i \geq 0$. A path will be said of type 1 if $k_{i+2} - k_i \geq 1$ and $l_{i+1} - l_i \leq 0 \ \forall\, i \geq 0$, type 2 if $k_{i+1} - k_i \geq 0$ and $l_{i+2} - l_i \geq 1 \ \forall\, i \geq 0$, type 3 if $k_{i+1} - k_i \leq 0$ and $l_{i+2} - l_i \geq 1 \ \forall\, i \geq 0$, type 4 if $k_{i+1} - k_i \geq 0$ and $l_{i+1} - l_i \leq 0 \ \forall\, i \geq 0$.

REMARK. In Section 2 we considered paths of type 2.

For $k > 0$, we introduce the following events:

$$A_k = \{\exists \text{ an infinite path of type 3 formed by e.c.r.s.'s and such that } (k_0, l_0) \in [3k, 4k-1] \times \{0\}\},$$

$$B_k = \{\exists \text{ an infinite path of type 1 formed by e.c.r.s.'s and such that } (k_0, l_0) \in \{0\} \times [3k, 4k-1]\}.$$

Note that although Corollary 2.5 was written for paths of type 2, considering the appropriate symmetries we see that it also holds for paths of types 1 and 3. In view of this and of Lemma 3.2, we conclude that there exist strictly positive constants p_0, C and γ such that

$$P_p(A_k) \geq 1 - Ce^{-\gamma k} \qquad \forall\, k \geq 1 \qquad \forall\, p \in (0, p_0),$$

and

$$P_p(B_k) \geq 1 - Ce^{-\gamma k} \qquad \forall\, k \geq 1 \qquad \forall\, p \in (0, p_0).$$

Of course, p_0 has been chosen in such a way that the probability that a renormalized site is an e.c.r.s. is, for all $p \in (0, p_0)$, large enough to allow us to use Corollary 2.5 for e.c.r.s.'s. Since we are interested in limits as $p \downarrow 0$, we need only consider values of p in $(0, p_0)$. In the exponential estimates below, the constants will not depend on p as long as it belongs to that interval. In particular, γ will remain fixed in the sequel. Now let $C_k = \{\exists$ a path of type 4 formed by e.c.r.s.'s and such that $(k_0, l_0) \in \{0\} \times [3k, 4k - 1]$ and for some n, $(k_n, l_n) \in [3k, 4k - 1] \times \{0\}\}$. Obviously, $C_k \supset A_k \cap B_k$, hence

$$P_p(C_k) \geq 1 - 2Ce^{-\gamma k}.$$

We state now a lemma that will be needed later. Its proof is easy and we omit it.

LEMMA 3.3. *Consider a path as in the definition of C_k. If it is entirely covered at time n, then the origin will be occupied at time $n + 8k(1/p^{\alpha})$.*

Let $D_k = \{|\bar{\Gamma}_n^{[0,k]}| \geq \beta k,\ \forall\ n \geq 0\}$. Here we have applied the definitions of Section 2 to the renormalized lattice and we have considered a renormalized site as open if it is easily covered. Besides this, the constant β is as in Corollary 2.5. It now follows from the same corollary that for $p \in (0, p_0)$, we have

$$P_p(D_k) \geq 1 - Ce^{-\gamma k}.$$

Therefore,

$$P_p(C_k \cap D_k) \geq 1 - 3Ce^{-\gamma k}. \tag{3.4}$$

PROOF OF THEOREM 1.3. Consider a path of type 4 joining an element of $\{0\} \times [3k, 4k - 1]$ to an element of $[3k, 4k - 1] \times \{0\}$ and a path of type 2 contained in the set $[0, 6k] \times [0, 6k]$ (see Figure 1). Note that the sum of the lengths of these paths is at most $8k + 12k = 20k$. Suppose now that both paths are formed by e.c.r.s.'s and that at least one of these is an i.s.r.s. Suppose also that these paths intersect. It now follows from Lemma 3.1 that both paths will be entirely covered by time $20k(1/p^{\alpha})^2$. Hence, by Lemma 3.3, the origin will be occupied at time $20k(1/p^{\alpha})^2 + 8k(1/p^{\alpha}) \leq 28k(1/p^{\alpha})^2$. Suppose now that C_k occurs; then there exists a path of type 4 as above. In fact, there may be many such paths and we simply choose one of them in an arbitrary manner. Supposing that D_k also occurs, the chosen path is connected by paths of type 2 formed by e.c.r.s.'s to a large number of e.c.r.s.'s contained in the region $[0, 6k] \times [4k, 5k]$. This number is bounded below by

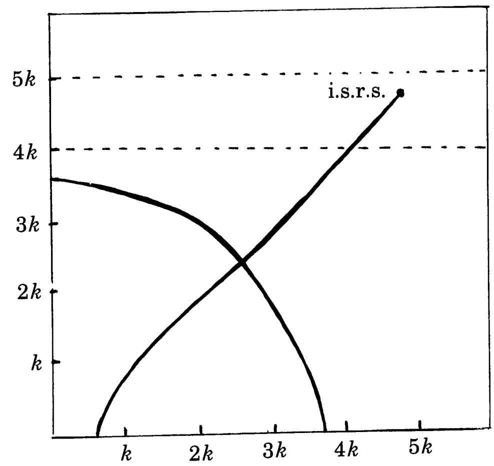

FIG. 1.

βk^2. Hence, the probability that at least one of these e.c.r.s.'s is an i.s.r.s. is bounded below by

$$1 - (1 - f(p))^{\beta k^2} = 1 - e^{[\log(1 - f(p))\beta k]k},$$

which for $k \geq \gamma[-\log(1 - f(p))\beta]^{-1}$ is at least $1 - e^{-\gamma k}$. This shows that for those k's, we have

$$P_p\left(T > 28k\left(\frac{1}{p^\alpha}\right)^2 \middle| C_k \cap D_k\right) \leq e^{-\gamma k}.$$

Recalling (3.4), we conclude that

$$P_p\left(T > 28k\left(\frac{1}{p^\alpha}\right)^2\right) \leq 3Ce^{-\gamma k} + e^{-\gamma k}$$

for $k \geq \gamma[-\log(1 - f(p))\beta]^{-1}$. Therefore, for n large enough (depending on p) there exists $\delta > 0$ and $K > 0$ depending on α but not on p such that

$$P_p(T > n) \leq Ke^{-\delta p^{2\alpha} n}.$$

This implies that

$$\limsup_{p \downarrow 0} \frac{\log \gamma(p)}{\log p} \leq 2\alpha.$$

Since this holds for any $\alpha > 1$, the theorem is proved. □

PROOF OF THEOREM 1.4. The proof of Theorem 1.3 shows that starting from $\eta^{[0, 6k(1/p^\alpha)] \times [0, 6k(1/p^\alpha)]}$, where η has as distribution a Bernoulli product measure with positive density p, all the sites of $[0, 3k(1/p^{\alpha})] \times \{0\} \cup \{0\} \times [0, 3k(1/p^{\alpha})]$ will be eventually be occupied, with probability at least

$$1 - 3Ce^{-\gamma k} - e^{-\gamma k}.$$

Using the symmetries of the model we see that all the sites of the boundary of $[0, 6k(1/p^{\alpha})] \times [0, 6k(1/p^{\alpha})]$ will eventually be occupied with probability at least

$$1 - 12Ce^{-\gamma k} - 4e^{-\gamma k}.$$

The inequality

$$\limsup_{p \downarrow 0} \frac{\log \bar{\gamma}(p)}{\log(p)} \le \alpha$$

now follows from the fact that once the boundary of a square is occupied, the whole square will eventually be occupied. To complete the proof of the theorem, observe that a square cannot be internally spanned if on a given column, no site is occupied at time zero. Hence

$$1 - N(L, p) \ge (1 - p)^L \quad \forall L \ge 1,$$

therefore $\bar{\gamma}(p) \le -\log(1 - p)$ and this implies that

$$\liminf_{p \downarrow 0} \frac{\log \bar{\gamma}(p)}{\log p} \ge 1. \qquad \square$$

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