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REFINEMENTS IN ASYMPTOTIC EXPANSIONS FOR SUMS OF WEAKLY DEPENDENT RANDOM VECTORS

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Let S_n denote the nth normalized partial sum of a sequence of mean zero, weakly dependent random vectors. This paper gives asymptotic expansions for $Ef(S_n)$ under weaker moment conditions than those of Götze and Hipp (1983). It is also shown that an expansion for $Ef(S_n)$ with an error term $o(n^{-(s-2)/2})$ is valid without any Cramér-type condition, if f has partial derivatives of order $(s-1)$ only. This settles a conjecture of Götze and Hipp in their 1983 paper.

1. Introduction. Let $X_1, X_2, \dots$ be a sequence of $\mathbb{R}^k$-valued random vectors with $EX_j = 0$ for all $j \geq 1$. Write $S_n = n^{-1/2}\sum_{i=1}^{n} X_i$. Edgeworth expansions for S_n have been derived by several authors under various sets of conditions on the sequence $\{X_n\}$. When $X_1, X_2, \dots$ are independent and identically distributed (iid), a more or less complete theory on the Edgeworth expansions for S_n is known; see Bhattacharya and Ranga Rao (1986) (hereafter referred to as BR). For an iid sequence $\{X_n\}$ satisfying (i) $E\|X_1\|^s < \infty$, $s \geq 3$, and (ii) the Cramér condition [cf. Condition 1 of Section 2], Theorem 20.1 of BR entails

$$(1.1) \qquad \left| Ef(S_n) - \int f \, d\Psi_{n,s} \right| = o(n^{-(s-2)/2}) \quad \text{as } n \to \infty,$$

for all $f \in \mathscr{F}_1$, where $\mathscr{F}_1$ is a large class of Borel measurable functions from $\mathbb{R}^k \to \mathbb{R}$ and $\Psi_{n,s}$ is a signed measure. In this case, $E\|X_1\|^s < \infty$ is a necessary condition for (1.1) to hold. In general, a similar expansion is not valid in the weak dependence setup. Depending on the dependence structure, one needs to impose additional conditions for the asymptotic normality of S_n, let alone Edgeworth expansion.

Under stronger conditions, expansions of the form (1.1) are proved by Statulevicius (1969, 1970) for finite order Markov chains and by Hipp (1985), Malinovskii (1987) and Jensen (1989) for strongly mixing Harris recurrent Markov chains. For general strongly mixing random vectors the best available result [Götze and Hipp (1983), hereafter referred to as GH2] assumes $E\|X_1\|^{s+1} < \infty$ to prove (1.1).

One of the major objectives of this paper is to obtain expansions for $Ef(S_n)$ under reduced moment conditions. Assuming that a conditional Cramér condition holds and $\{X_n\}$'s are approximately strongly mixed at an exponential rate, Theorem 2.1 gives an expansion for $Ef(S_n)$ under the weaker moment condi-

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tion $\sup\{E\|X_j\|^s(\log(1+\|X_j\|))^{\beta}\colon j \geq 1\} < \infty$ for some constant $\beta \equiv \beta(s) > 0$. Compared to the earlier results, this reduces the moment condition to the optimal one except for the logarithmic term. Indeed, if the $\{X_n\}$'s are stationary m-dependent, then the logarithmic factor can be removed. In this case, Theorem 2.2 yields an expansion under the minimal condition $E\|X_1\|^s < \infty$.

Next consider expansions for $Ef(S_n)$ when f is smooth. It is well known that in such situations, expansions for $Ef(S_n)$ remain valid without the Cramér condition on $\{X_n\}$. Indeed, Theorem 3.6 of Götze and Hipp (1978) (hereafter referred to as GH1) shows that in the iid case, an expansion of the form (1.1) holds if $E\|X_1\|^s < \infty$ and f has $(s-2)$ continuous partial derivatives. In the weak dependence situation, GH2 proves a similar result under the stronger conditions: (i) $\sup\{E\|X_j\|^{s+1}\colon j \geq 1\} < \infty$, and (ii) f is infinitely differentiable. In fact, from their proof it follows that an expansion holds if f has partial derivatives of order $[(s-2)/2\varepsilon] + 1$ for some $\varepsilon > 0$ small (cf. first line, page 218, GH2). Here, for any real number x, $[x]$ denotes its integer part. In view of the preceding results, GH2 conjectured (first paragraph, page 218, GH2) that in the weak dependence case, (1.1) holds if f has only $(s-1)$ continuous partial derivatives. Theorem 2.3 settles this conjecture. Furthermore, it proves an analogous result for m-dependent $\{X_n\}$'s under the minimal conditions: (i) $E\|X_1\|^s < \infty$, and (ii) f is $(s-2)$ times continuously differentiable.

The layout of the paper is as follows: Section 2 contains the main results of the paper and the proofs of all the results are given in Section 3.

2. Main results. Let $X_1, X_2, \ldots$ be a sequence of mean zero random k, vectors defined on a common probability space $(\Omega, \mathscr{A}, P)$. Unless otherwise stated, the X_n's are not assumed to be stationary. Set $S_n = (X_1 + \cdots + X_n)/\sqrt{n}$. Let $\mathscr{D}_0, \mathscr{D}_{\pm 1}, \ldots$ be a sequence of sub-σ-fields of $\mathscr{A}$. Write $\mathscr{D}_p^q$ for the σ-field generated by $\mathscr{D}_j\colon p \leq j \leq q$. The following conditions will be used in the sequel for proving the results.

Condition 1. (i) $EX_j = 0$ for all $j \geq 1$ and $\Sigma = \lim_{n\to\infty} \operatorname{Disp}(S_n)$ exists and is nonsingular.

(ii) There exists $d > 0$ such that $\inf\{t' \operatorname{Disp}(X_{j+1} + \cdots + X_{j+m})t\colon \|t\| = 1\} > dm$ for all $j > d^{-1}$, $m > d^{-1}$.

Condition 2. For some integer $s \geq 3$ and some real number $\beta(s) > s^2$,

$$\beta \equiv \sup\left\{E\|X_j\|^s\left(\log\left(1+\|X_j\|\right)\right)^{\beta(s)}\colon j \geq 1\right\} < \infty.$$

Condition 3. There exists a positive constant d such that for $n, m = 1, 2, \ldots$ with $m > d^{-1}$, there exists a $\mathscr{D}_{n-m}^{n+m}$-measurable random k-vector $\bar{Y}_{n,m}$ for which

$$E\|X_n - \bar{Y}_{n,m}\| \leq d^{-1}\exp(-dm).$$

CONDITION 4. There exists $d > 0$ such that for all $m, n = 1, 2, \dots,$ $A \in \mathscr{D}_{-\infty}^{n}$, $B \in \mathscr{D}_{n+m}^{\infty}$,

$$|P(A \cap B) - P(A)P(B)| \le d^{-1}e^{-dm}.$$

CONDITION 5. There exists $d > 0$ such that for all $m, n = 1, 2, \dots,$ $d^{-1} < m < n$ and all $t \in \mathbb{R}^k$ with $\|t\| \ge d$,

$$E|E(\exp(it'(X_{n-m} + \cdots + X_{n+m}))|\mathscr{D}_j: j \ne n)| \le e^{-d}.$$

CONDITION 6. There exists $d > 0$ such that for all $m, n, p = 1, 2, \dots$ and $A \in \mathscr{D}_{n-p}^{n+p}$,

$$E|P(A|\mathscr{D}_j: j \ne n) - P(A|\mathscr{D}_j: 0 < |n - j| \le m + p)| \le d^{-1}e^{-dm}.$$

All the conditions except Conditions 1(ii) and 2 have been used by GH2. Here, Condition 1(ii) is used for obtaining bounds on the derivatives of $E\exp(it'S_n)$ for moderately large values of t. In view of Condition 1(i), this is satisfied if the $\{X_n\}$'s are second order stationary. Condition 2 relaxes the moment condition $\sup\{E\|X_j\|^{s+1}: j \ge 1\} < \infty$ used by GH2.

Next define the Edgeworth polynomials $P_{r,n}(t)$ by the identity (in $\tau \in \mathbb{R}$)

$$\exp\left(\sum_{r=3}^{s} (r!)^{-1}\tau^{r-2}n^{(r-2)/2} \cdot \chi_{r,n}(t)\right) = 1 + \sum_{r=1}^{\infty} \tau^r P_{r,n}(t),$$

where $\chi_{r,n}(t)$ is the rth cumulant of $t'S_n$, $2 \le r \le s$ and for any matrix A, A' denotes its transpose. Let $\Psi_{n,s}$ denote the signed measure with Fourier transform $\hat{\Psi}_{n,s}(t) = \exp(-\chi_{2,n}(t)/2)(1 + \sum_{r=1}^{s-2} n^{-r/2}P_{r,n}(it))$, $t \in \mathbb{R}^k$.

Let $s_0 = 2[s/2]$. For a Borel measurable function f: $\mathbb{R}^k \to \mathbb{R}$ and $\varepsilon > 0$, write $\omega(f; \varepsilon) = \int \sup\{|f(x + y) - f(x)|: \|y\| \le \varepsilon\}\Phi_\Sigma(dx)$, where Φ_Σ is the normal distribution on $\mathbb{R}^k$ with mean zero and dispersion matrix Σ. Let $1(B)$ denote the indicator of a set B. Set $\mathbb{Z}^+ = \{0, 1, \dots\}$. For $\alpha = (\alpha_1, \dots, \alpha_k)' \in (Z^+)^k$, define the differential operator D^α by $D^\alpha = \partial^{|\alpha|}/\partial t_1^{\alpha_1} \cdots \partial t_k^{\alpha_k}$.

THEOREM 2.1. *Assume that Conditions* 1–6 *hold. Let* $f: \mathbb{R}^k \to \mathbb{R}$ *be a Borel measurable function satisfying* $\sup\{(\|x\|^{s_0} + 1)^{-1}|f(x)|: x \in \mathbb{R}^k\} \le M$ *for some constant* $M > 0$. *Then, for any real number* $a > 0$,

$$\left|Ef(S_n) - \int f\,d\Psi_{n,s}\right| \le C\omega(f; n^{-a}) + o(n^{-(s-2)/2}). \tag{2.1}$$

Theorem 2.1 improves Theorem 2.8 of GH2 requiring the weaker moment condition, Condition 2. The next theorem further reduces the requirement for m-dependent random vectors.

THEOREM 2.2. *Assume that Conditions* 1 *and* 5 *hold with* $\mathscr{D}_j = \sigma(X_j)$, $j \ge 1$ *and* $\mathscr{D}_j$*'s are m-dependent. Let* $\delta_{1n} = \max\{E\|X_j\|^s 1(\{X_j\|^4 > n\})$:

$1 \le j \le n\} \to 0$ *as* $n \to \infty$. *Then for any bounded, measurable function* f *and any* $a > 0$,

$$(2.2)\qquad \left| Ef(S_n) - \int f \, d\Psi_{n,s} \right| \le C \cdot \omega(f; n^{-a}) + o(n^{-(s-2)/2}).$$

Next consider expansions of $Ef(S_n)$ where f is smooth but the $\{X_n\}$'s do not necessarily satisfy the Cramér condition. In this case, GH2 conjectured the validity of expansion (1.1) for f's having only $(s-1)$ continuous partial derivatives on $\mathbb{R}^k$. The following theorem establishes the conjecture with a further reduction in the moment condition.

THEOREM 2.3. (a) *Assume that Conditions* 1–4 *and* 6 *hold. Let* $f: \mathbb{R}^k \to \mathbb{R}$ *be a function satisfying* (i) $f \in C^{s-1}(\mathbb{R}^k)$ *and* (ii) *for* $\alpha \in (\mathbb{Z}^+)^k$ *with* $0 \le |\alpha| \le s-1$, $\sup\{(1 + \|x\|^{p(\alpha)})^{-1} |D^\alpha f(x)|: x \in \mathbb{R}^k\} < M_\alpha$ *for some* $M_\alpha > 0$, $p(\alpha) \in \mathbb{Z}^+$ *with* $p(0) = s_0$. *Then*

$$(2.3)\qquad \left| Ef(S_n) - \int f \, d\Psi_{n,s} \right| = o(n^{-(s-2)/2}),$$

where the RHS depends on f *only through the* M_α*'s.*

(b) *Let* $\{\mathscr{D}_j = \sigma(X_j): j \ge 1\}$ *be m-dependent, Condition* 1 *hold and* δ_{1n} *of Theorem* 2.2 *tend to zero as* $n \to \infty$. *Then,* (2.3) *holds for every bounded* $f \in C^{s-2}(\mathbb{R}^k)$ *satisfying* (ii) *of part* (a).

A consequence of this theorem is the following moderate deviation bound.

THEOREM 2.4. *Let* λ_1 *denote the largest eigenvalue of* Σ, *and let* $\lambda > \lambda_1$.

(a) *Assume that Conditions* 1–4 *and* 6 *hold. Then*

$$E(1 + \|S_n\|^{s_0}) 1\left(\|S_n\| > ((s-2)\lambda \log n)^{1/2}\right) = o(n^{-(s-2)/2}).$$

(b) *Under the conditions of Theorem* 2.3(b),

$$P\left(\|S_n\| > ((s-2)\lambda \log n)^{1/2}\right) = o(n^{-(s-2)/2}).$$

3. Proofs. In the following, we will mostly use the notation of GH2. Let $\psi: [0, \infty) \to [0, \infty)$ be an infinitely differentiable function satisfying $\psi(x) = x$ for $0 \le x \le 1$, ψ is increasing and $\psi(x) = 2$ for $x \ge 2$. Define the truncation function $T: \mathbb{R}^k \to \mathbb{R}^k$ by $T(x) = \|x\|^{-1} c_n x \psi(c_n^{-1}\|x\|)$, $x \in \mathbb{R}^k$, where $c_n = \sqrt{n}/(\log n)^2$. For $j = 1, \dots, n$, let $Y_j = T(X_j)$, $\mathbb{Z}_j = Y_j - EY_j$. Write $S_n^* = n^{-1/2}(\mathbb{Z}_1 + \cdots + \mathbb{Z}_n)$ and $H_n(t) = E\exp(it'S_n^*)$. For any $\alpha = (\alpha_1, \dots, \alpha_k)' \in (\mathbb{Z}^+)^k$, set $|\alpha| = \alpha_1 + \cdots + \alpha_k$ and $\alpha! = \alpha_1! \cdots \alpha_k!$. Let $C(\cdot), C$ denote generic constants depending only on their arguments, if any. For any integrable U, write $E_t U = H_n^{-1}(t) EU \exp(it'S_n^*)$. For a set $B \subseteq \mathbb{R}$, let $|B|$, $\Lambda(B)$ and $\lambda(B)$, respectively, denote the cardinality, the supremum and the infimum of B.

Write $S(I) = \sum_{j \in I} it'\mathbb{Z}_j$, $I \subseteq \{1, \dots, n\}$. Fix $s^2 < \alpha(s) < \beta(s)$ and let

$$\delta_n = \sup\Big\{E\|X_j\|^s \log\big(1 + \|X_j\|\big)^{\alpha(s)} 1\big(\|X_j\| > c_n\big): 1 \le j \le n\Big\}, \quad n \ge 1. \tag{3.1}$$

Next define the semiinvariants (of order p) in the variables $V_1, \dots, V_p$ by

$$K_t(V_1, \dots, V_p) = \frac{\partial}{\partial \varepsilon_1} \cdots \frac{\partial}{\partial \varepsilon_p}\bigg|_{\varepsilon_1 = \cdots = \varepsilon_p = 0} \log E \exp\big(it'S_n^* + \varepsilon_1 V_1 + \cdots + \varepsilon_p V_p\big). \tag{3.2}$$

Write $K_t(V_1^p, V_2^q) = K_t(V_1, \dots, V_1, V_2, \dots, V_2)$ where, in the RHS, V_1 appears p times and V_2 appears q times. Then, by Taylor's expansion, one has

$$\log H_n(t) = \sum_{r=2}^{s} K_0(t'S_n^{*r}) + R(t), \tag{3.3}$$

where

$$R(t) = \left[\int_0^1 (1 - \eta)^s K_{\eta t}\big(t'S_n^{*(s+1)}\big)\, d\eta\right] \Big/ s!.$$

For $I \subseteq \{1, \dots, n\}$ and $a_{pj} \in \mathbb{R}^k$ with $\|a_{pj}\| \le 1$, define $\mathbb{Z}(I) = \prod_{j \in I} \prod_{p=1}^{r_j} a'_{pj} \mathbb{Z}_j$. To prove Lemmas 3.1–3.5, we will assume that Conditions 1–4 and 6 hold.

LEMMA 3.1. *Let $I_i \subseteq \{1, \dots, n\}$, $i = 1, 2$, with $\lambda(I_2) - \Lambda(I_1) \ge l$. Then, given any integer m, $1 \le m \le l$ and a real number ζ, $0 < \zeta < 1$, there exists a constant $C_1 = C(d, \zeta, \beta)$ such that for all $\|t\| < C_1 \cdot (n/m)^{1/2}$ and $1 \le K \le [l/m]$,*

$$\begin{aligned} &|H_n(t)|^2 |E_t \mathbb{Z}(I_1)\mathbb{Z}(I_2) - E_t \mathbb{Z}(I_1) \cdot E_t \mathbb{Z}(I_2)| \\ &\quad \le C \cdot n^{\gamma}\big[K 2^K \exp(-dm/3) + \zeta^K\big], \end{aligned}$$

where $2\gamma = |I_1| + |I_2| + 2$.

PROOF. Let $\{W_i, 1 \le i \le n\}$ denote an independent copy of $\{\mathbb{Z}_i, 1 \le i \le n\}$ and let $T_n = n^{-1/2}(W_1 + \cdots + W_n)$. Then

$$|H_n(t)|^2 |E_t \mathbb{Z}(I_1)\mathbb{Z}(I_2) - E_t \mathbb{Z}(I_1) E_t \mathbb{Z}(I_2)| = |EU_1 U_2 U_3|,$$

where

$$U_1 = \mathbb{Z}(I_1) - W(I_1),$$

$$U_2 = \mathbb{Z}(I_2) \cdot \prod_{j = \lambda(I_2)}^{n} \exp\Big(it'(\mathbb{Z}_j + W_j)/\sqrt{n}\Big)$$

and

$$U_3 = \prod_{j < \lambda(I_2)} \exp\Big(it'(\mathbb{Z}_j + W_j)/\sqrt{n}\Big).$$

Define

$$S(r) = \sum_{j<\lambda(I_2)-mr} (\mathbb{Z}_j + W_j)/\sqrt{n}\,,$$

$$R_r = \exp(it'S(r))$$

and

$$\Delta_r = -1 + \exp(it'(S(r-1) - S(r))), \qquad 0 \le r \le K$$

(setting $\Delta_0 = 0$). Using the iterative method of Tikhomirov (1980), one can show that

$$\begin{aligned} EU_1U_2U_3 &= EU_1U_2R_1 + EU_1U_2\Delta_1R_1 \\ &\;\;\vdots \\ &= \sum_{r=1}^{K} EU_1U_2\left(\prod_{j=1}^{r-1}\Delta_j\right)R_r + EU_1U_2\left(\prod_{j=1}^{K}\Delta_j\right)R_K. \end{aligned}$$

Hence, by Conditions 3 and 4, one has

$$\left|EU_1U_2\left(\prod_{j=1}^{r-1}\Delta_j\right)R_r - (EU_1R_r)\left(EU_2\left(\prod_{j=1}^{r-1}\Delta_j\right)\right)\right| \le C\cdot n^{\gamma}2^K\exp(-dm/3).$$

Since Δ_j and Δ_{j+2} are weak dependent and by Lemma 3.3, $|\Delta_j| \le \min\{2, C(d,\beta)(\|t\|^2 m/n)^{1/2}\}$, it follows that

$$\left|EU_1U_2\prod_{j=1}^{K}\Delta_jR_K\right| \le C\cdot n^{\gamma}\Big(\Big[C(d,\beta)\|t\|\cdot(m/n)^{1/2}\Big]^{K/2} + K2^K\exp(-dm/3)\Big).$$

Since $EU_1R_r = 0$ for all $1 \le r \le K$, the lemma is proved. □

LEMMA 3.2. *Given a real number ζ, $0 < \zeta < 1$, and an integer $l \ge 0$, there exists a sequence of real numbers $\{a_{1n}\}$ (depending only on ζ, d and $s + l$) with $a_{1n}^{-1} = o(1)$, such that*

$$\left|\frac{\partial^l}{\partial\varepsilon^l}R(t + a\varepsilon)\right|_{\varepsilon=0} \le C(s,l,\beta,d,\zeta)\big(1 + \theta_n(t)^p\big)(1 + \|t\|^p)\cdot o(n^{-(s-2)/2})$$

for any $a \in \mathbb{R}^k$, $\|a\| < 1$ and for $t \in \{u \in \mathbb{R}^k\colon \|u\| < C(\zeta,\beta)n^{(1-\zeta)/2}, \theta_n(u) < \infty\}$, where $|H_n(t)|\theta_n(t) = \sup\{|E\exp(S_I^{(i)})|\colon 1 \le i \le n^{\zeta/4}, |I| \le p\} + n^{-a_{1n}}$, $p = s + 1 + l$ and $S_I^{(i)}$ is as defined on page 227 of GH2.

PROOF. By (3.2) and (3.3), it is enough to estimate

$$\sum_{g=0}^{n-1}\sum^{(g)}\left|K_{\eta t}(V_{j_1},\dots,V_{j_r})\right| \tag{3.4}$$

for $s+1 \le r \le p$ and $0 \le \eta \le 1$, where $V_j = it'\mathbb{Z}_j/\sqrt{n}$ and the summation $\Sigma^{(g)}$ extends over all $1 \le j_1 \le \cdots \le j_r \le n$ with maximal gap g. Using Lemma 3.16 of GH2 with $m = [n^{\zeta/2}]$, $K = [\sqrt{m}\,]$, it can be shown that for all $\|t\| < C(\zeta, d, \beta)n^{(1-\zeta)/2}$ and all $\{j_1 \cdots j_r\} \subseteq \{1 \cdots n\}$, $r \le p$,

$$|K_t(V_{j_1}, \dots, V_{j_r})| \le C(l, \beta, \zeta, d)\beta_{rn}(1 + \|t\|^p)(1 + |\theta_n(t)|^p), \tag{3.5}$$

where

$$\beta_{rn} = n^{-r/2}c_n^{r-s-1} \cdot \Lambda\Big\{E\|X_j\|^{s+1}1(\|X_j\| < c_n): j \ge 1\Big\}.$$

To estimate (3.4), divide the range of g into three disjoint sets J_1, J_2, J_3, where $J_1 = \{i: 0 \le i \le a_n\}$, $J_2 = \{i: a_n < i \le n^{\zeta/2}\}$ and $J_3 = \{i: n^{\zeta/2} < i < n\}$ with [$\alpha(s)$ of (3.1)] $a_n = (\log n)^{2+[(\alpha(s)-2s)/2p]}$. Denote the summation over the set J_i by Σ_i, $1 \le i \le 3$. Hence, by Condition 2 it follows that for all $0 \le \eta \le 1$, $s+1 \le r \le p$ and $\|t\| < C(\zeta, \beta, l) \cdot n^{(1-\zeta)/2}$,

$$\sum_1 \sum^{(g)} |K_{\eta t}(V_{j_1}, \dots, V_{j_r})|$$
$$\le C(\zeta, \beta, d, p) \cdot n^{-(s-2)/2}a_n^r(\log n)^{-\alpha(s)-2(r-s)}(1 + \theta_n(t)^p)(1 + \|t\|^p).$$

Next applying Lemma 3.1 with $m = [\sqrt{g}\,]$, $K = [d\sqrt{g}/(3+d)]$ to the terms under $\Sigma_2\Sigma^{(g)}$ and with $m = [n^{\zeta/2}]$, $K = [\sqrt{m}\,]$ to those under $\Sigma_3\Sigma^{(g)}$, and using (3.14) of GH2, one can complete the proof of Lemma 3.2. □

LEMMA 3.3. *For any* $a_1, \dots, a_r \in \mathbb{R}^k$, $\|a_1\| = \cdots = \|a_r\| = 1$ *and* $2 \le r \le s$,

(i) $|K_0(a_1'S_n^*, \dots, a_r'S_n^*) - K_0(a_1'S_n, \dots, a_r'S_n)| \le C(s, \beta, d) \cdot o(n^{-(s-2)/2})$.
(ii) $|K_0(a_1'S_n^*, \dots, a_r'S_n^*)| \le C(s, \beta, d)n^{-(r-2)/2}$.

PROOF. Note that for proving part (i), it is enough to consider the sum

$$\Big(\sum_1 + \sum_2\Big)\sum^{(g)} |K_0(V_{j_1} \cdots V_{j_r}) - K_0(W_{j_1} \cdots W_{j_r})|$$

where $V_{j_i} = a_i'\mathbb{Z}_{j_i}$, $W_{j_i} = a_i'X_{j_i}$, Σ_i stands for the summation over $g \in J_i$, $i = 1, 2$, $J_1 = \{i: 0 \le i \le a_n\}$ and $J_2 = \{i: a_n < i < n\}$ with $a_n = \log n/(\delta_n)^{1/\alpha(s)}$, and $\Sigma^{(g)}$ is as in (3.4).

Considering the cases $2 \le r < s$ and $r = s$ separately and using a further truncation argument [see Lahiri (1990)], one can show that

$$|K_0(a_1'S_n^*, \dots, a_r'S_n^*) - K_0(a_1'S_n, \dots, a_r'S_n)|$$
$$\le C(\beta, s, d) \cdot n^{-(s-2)/2} \cdot \delta^{(\alpha(s)-s^2)/(s\alpha(s))}, \quad \text{if } 2 \le r < s,$$
$$\le C(s, \beta, d)n^{-(s-2)/2}(\log n)^{s-\alpha(s)/s}, \quad \text{for } r = s.$$

Hence, the first part of the lemma follows. Part (ii) can be proved similarly. □

LEMMA 3.4. *Let* $f: \mathbb{R}^k \to \mathbb{R}$ *be a Borel measurable function with* $\sup\{|f(x)| \cdot (1 + \|x\|^{s_0})^{-1}: x \in \mathbb{R}^k\} = M < \infty$. *Then, for any* $a > 0$,

$$\left|Ef(S_n) - \int f\, d\Psi_{n,s}\right| \le \omega(g; n^{-a}) + o(n^{-(s-2)/2}) + C(\beta, d, a)$$
$$\times M \sup_{|\alpha| \le k+1+s_0} \int \left|D^\alpha\left[\left(H_n(t) - \hat{\Psi}_{n,s}(t)\right)\hat{K}(n^{-a}t)\right]\right| dt,$$

where $g(x) = f(x) \cdot (1 + \|x\|^{s_0})^{-1}$, $x \in \mathbb{R}^k$, *and* $\hat{K}$ *is a characteristic function which vanishes outside a compact set.*

PROOF. Let $S_{1n} = n^{-1/2}\sum_{i=1}^n Y_i$, $n \ge 1$. Define the sets $A_1 = \{\|S_n\| \le \log n\}$ and $B_1 = \{\|S_{1n}\| \le \log n\}$. Then, using Lemmas 3.2 and 3.3 and the arguments in the proof of Lemma 3.3 of GH2 (with A and B of GH2 respectively replaced by A_1 and B_1), one can prove Lemma 3.4. See Lahiri (1990) for details. □

LEMMA 3.5. *Let* $0 < \zeta < 1$ *be given. Then, for any nonnegative integral vector* α,

$$|D^\alpha H_n(t)| \le C(\alpha, d, \zeta) \cdot n^{|\alpha|+3}\left(\exp(-dm/3) + \exp\left(-d\|t\|^2/8\right)\right)$$

for all $\|t\| < C(\beta, d)n^{(1-\zeta)/2}$, *where* $m = [n^{\zeta/2}]$.

PROOF. Let $|\alpha| = r$. Fix $j_1, \dots, j_r \in \{1, \dots, n\}$ and $a_i \in \mathbb{R}^k$, $\|a_i\| = 1$, $1 \le i \le r$. Then, as in the proof of Lemma 3.43 of GH2, one can show that for $\|t\| < C \cdot n^{(1-\zeta)/2}$ and $m = [n^{\zeta/2}]$,

$$\left|Ea_1'\mathbb{Z}_{j_1}, \dots, a_r\mathbb{Z}_{j_r} \exp(it'S_n^*)\right|$$
$$\le C(d, r) \cdot n^{r/2}\left(n^3 \exp(-dm/3) + \prod_{p=1}^{l} E\left|E\left(A_p|\mathscr{D}_j: j \ne i_p\right)\right|\right), \tag{3.6}$$

where $A_p = \exp(S(I_p)/\sqrt{n})$ for some $I_p \subseteq \{1, \dots, n\}$, $1 \le p \le l$ and $l = O(n/m)$.

Next note that for any σ-field $\mathscr{C} \subseteq \mathscr{A}$,

$$\left|E\left(\exp(S(I)/\sqrt{n})|\mathscr{C}\right)\right|^2 \le 1 - n^{-1}E\left(S(I)^2|\mathscr{C}\right) + 2n^{-3/2}E\left(|S(I)|^3|\mathscr{C}\right) \quad \text{a.s. } (P).$$

Hence, by Condition 1, it follows that for all j, $p > d^{-1}$, $I = \{J + 1, \dots, j + p\}$ and $\|t\| < dn^{1/2}/(4(1 + \beta)|I|^2)$,

$$\left\{E|E\left(\exp(S(I)/\sqrt{n})|\mathscr{C}\right)|\right\}^2 \le \exp\left(-I|I|d\|t\|^2/2n\right).$$

Now using (3.6) and the definition of A_p, one can complete the proof of Lemma 3.5. See Lahiri (1990) for details. □

PROOF OF THEOREM 2.1. Following the proof of Lemma 3.33 of GH2 with n^{ε} replaced by $a_{1n}^{1/4}(\log n)^{1/2}$ (where a_{1n} is as in Lemma 3.2) and using Lemmas 3.2, 3.3 and 3.5, it can be shown that for any $\alpha \in (\mathbb{Z}^{+})^{k}$, $|\alpha| \le k + s_0 + 1$,

$$\int_{\|t\| < a_{2n}} \left| D^{\alpha}\left(H_n(t) - \hat{\psi}_n(t)\right)\right| dt = o\left(n^{-(s-2)/2}\right),$$

where $a_{2n} = C(\alpha, d, \zeta) \cdot n^{(1-\zeta)/2}$. Now, the proof of Theorem 1 can be completed using Lemma 3.4 above and Lemma 3.43 of GH2. □

PROOF OF THEOREM 2.2. This is similar to the proof of Theorem 2.1. □

PROOF OF THEOREM 2.3. Part (a) follows from Lemmas 3.2, 3.4, 3.5 and the arguments on page 80 of GH1. See Lahiri (1990) for further details.

Part (b) can be proved along the lines of the proof of Theorem 3.6 of GH1 since, in this case, Lemma 3.5 remains valid over $\|t\| < C(d, m)\sqrt{n}$, even under the reduced moment condition. □

PROOF OF THEOREM 2.4. This follows from Lemmas 3.2, 3.4 and 3.5, as in the proof of Theorem 2.11 of GH2. □

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