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PATHWISE NONLINEAR FILTERING ON ABSTRACT WIENER SPACES

BY OGNIAN ENCHEV

Boston University

The nonlinear filtering problem is studied for models where the samples of the signal and the noise are elements of some general abstract Wiener space. The signal is allowed to depend on the noise and the optimal filter is expressed as an explicit functional of the observed sample (trajectory). It is shown that this functional satisfies the Zakai equation. As a necessary technical tool, a class of shift transformations on the Wiener space is studied and an analog of Cameron–Martin–Girsanov's theorem is obtained.

1. Introduction. We are concerned with communication systems of the type "observation = signal + noise," that is, systems described as

$$y = h + \omega, \tag{1.1}$$

where h is a sample (or realization) of the signal, ω is a sample of the noise and y is the observed sample (received information). We assume that h runs through some general separable Hilbert space H and ω runs through some Banach space E. H is assumed to be included densely into E via the embedding $H \hookrightarrow_{\iota} E$. An important assumption to be made is that the triplet (ι, H, E) forms an abstract Wiener space (AWS). The latter generates a probability measure μ on the Borel sets in E, and we will assume that the noise-sample ω is distributed in E according to the law μ. The sample of the signal $h \in H$ is a function of $\omega \in E$ and also of some parameter u which is distributed in some general set U according to a known probability law Π. Usually, u is interpreted as a "message," which upon transmission produces the signal $h \equiv h(u, \omega)$. Thus, we write (1.1) as

$$\begin{aligned} &y[u, \omega] = h[u, \omega] + \omega, \\ &\qquad u \in U,\ \omega \in E,\ h[u, \omega] \in H,\ y[u, \omega] \in E. \end{aligned} \tag{1.2}$$

If (ι, H, E) is taken to be the classical AWS, in which case $E \equiv C_0[0, T]$ is the space of continuous functions on $[0, T]$ vanishing at 0, and $H \equiv \mathscr{C}'[0, T]$ is the space of all functions of the form $f(t) = \int_0^t \dot{f}(s)\, ds$, $0 \le t \le T$, $\dot{f} \in L^2[0, T]$, then (1.2) takes the form

$$\begin{aligned} &y_t(u, \omega) = \int_0^t \phi_s(u, \omega)\, ds + \omega(t), \quad 0 \le t \le T, \\ &\qquad \omega \in C_0[0, t],\ \phi_{\cdot}(u, \omega) \in L^2[0, T]. \end{aligned} \tag{1.3}$$

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In this case the canonical Gaussian measure μ on $C_0[0,T]$, induced by $\mathscr{E}'[0,T]$, is exactly the standard Wiener measure.

We will investigate the model (1.2), assuming that the signal $h[u,\omega]$ has the form of an integral w.r.t. some H-valued measure on the interval $[0,T]$. Our main objective is, for a given (in general nonlinear) $(\Pi \otimes \mu)$-integrable function $f\colon U \times E \mapsto \mathbb{R}$, to compute

$$\mathbb{E}\{f|\mathscr{O}\},$$

where $\mathscr{O}$ is the σ-field in $U \times E$ generated by the mapping $y\colon U \times E \mapsto E$. $\mathscr{O}$ is usually referred to as the observation σ-field. It is customary to consider f as a function of some process (x_t), called the system process. The latter describes the evolution of a system, which cannot be observed directly. Therefore, one is interested in estimating $f(x_t)$ via the information provided by the observation y.

It is natural to assume that the observed sample $y \in E$ evolves in time, in which case the continuously received information is described as a nondecreasing family of σ-fields $\mathscr{O}_t$, $t \geq 0$. Thus, the estimator of $f(x_t)$ also evolves in time, and, for each $t \geq 0$, is given by

$$\mathbb{E}\{f(x_t)|\mathscr{O}_t\}.$$

The last quantity is known as the *optimal filter*. It can be described by various means. In the works [11, 12, 17 and 18], models somewhat similar to (1.3) are studied. The observation process (y_t) and the system process (x_t) are taken to be diffusion processes, and, under suitable assumptions, the optimal filter is expressed as

$$\mathbb{E}\{f(x_t)|\mathscr{O}_t\} = \frac{\int_{-\infty}^{+\infty} f(x)p_t(x)\,dx}{\int_{-\infty}^{+\infty} p_t(x)\,dx}.$$

Here $p_t(x)$ is the so-called unnormalized conditional density and it is described as a solution of a certain stochastic PDE. In yet another setup, Fujisaki, Kallianpur and Kunita derived a SDE for the optimal filter (cf. [15]). This equation involves the so-called innovation process (cf. [7] and Chapter 8 in [6] for more details). From a different point of view the filtering problem was studied by Kushner and Zakai (cf. [13] and [19]). They derived a measure-valued SDE for the conditional distribution $\mathbb{E}\{f(x_t) \in \cdot|\mathscr{O}_t\}$.

The strategy adopted here is the following. For a given $f\colon E \times U \mapsto \mathbb{R}$, we aim to construct explicitly a functional $\Phi^f\colon E \mapsto \mathbb{R}$, so that

$$\mathbb{E}\{f|\mathscr{O}\} = \Phi^f(y[u,\omega]). \tag{1.4}$$

Although theoretically such a functional always exists, its explicit form is far from obvious. This approach is motivated by the fact that, from a practical point of view, the only available information is the observed sample $y \in E$. The principal idea is due to Kallianpur and Striebel (cf. [9]) who studied the case of independent signal and noise. In [10] the same authors derived an equation for the optimal filter. Such an equation may be viewed as a recursive expression for the current value of the filter, as a function of the previous (already calculated) values. In 1984, Ocone [16] derived a similar equation, but under weaker assumptions. The problem of nonlinear filtering was treated "pathwise" also by Clark [3], Davis [4], Kushner [14] and some other authors. They were concerned mainly with the case where signal and noise are independent. The present paper aims to extend these methods by allowing certain dependence between signal and noise.

In [7] and [8], Kallianpur and Karandikar reformulated the model, investigated by Kallianpur and Striebel, in terms of the so-called finitely-additive white noise. Their approach is based on the assumption that signal and noise are uncorrelated and all samples (noise, signal and observation) are vectors in a fixed separable Hilbert space H. This leads to the following communication system:

$$(1.5)\quad y[u,\omega] = h[u] + \omega, \qquad u \in U,\ \omega \in H,\ h[u] \in H,\ y[u,\omega] \in H.$$

A principal assumption made in [7] and [8], is that the distribution of the noise sample is given by the canonical cylinder Gaussian measure on H. This assumption, as explained in [7], is motivated mainly by practical problems arising in electrical engineering and some other areas. Since the canonical cylinder Gaussian measure on H is finitely additive (f.a.), but not countably additive (c.a.), this approach involves a special probabilistic technique.

Here we adopt the more conventional countably additive approach, that is, as explained previously, the noise sample is considered to be distributed in the Banach space E and is not restricted to $H \subset E$. It should be noted, however, that the countably additive model (1.2) cannot be substituted for the finitely additive one in (1.5). This is because the canonical f.a. cylinder Gaussian measure on H is defined for a much larger class of sets than the c.a. Gaussian measure μ on E, generated by the AWS (ι, H, E). Another principal difference between the model (1.2) adopted here and Kallianpur and Karandikar's model (1.5) is that in (1.2) signal and noise are allowed to be correlated.

In the last section we derive a stochastic equation for the optimal filter. This equation generalizes those studied by Kallianpur and Striebel in [10] and Ocone in [16], which treat the case of classical AWS and uncorrelated signal and noise. In these works the independence between signal and noise is essential, so that our derivation will follow a different plan.

2. Preliminaries. We fix once and for all an abstract Wiener space (ι, H, E) and set $\langle \ell | \omega \rangle \equiv \ell(\omega)$, $\ell \in E^*$, $\omega \in E$. We have

$$E^* \hookrightarrow_{\iota^*} H^* \equiv H \hookrightarrow_{\iota} E,$$

where the embeddings ι^* and ι are both dense and continuous. With no ambiguity we will treat H as a proper dense subset of E and E^* as a proper dense subset of H, that is, we will not distinguish between $h \in H$ and its image $\iota h \in E$, nor between $\ell \in E^*$ and its image $\iota^* \ell \in H^* \equiv H$. Thus, for every $\ell \in E^*$ and $h \in H \subset E$, $\langle \ell | h \rangle = (\ell | h)_H$.

We also fix a compact interval $[0, T]$ and a vector-valued measure $Z(\cdot)$, defined on $\mathscr{B}_T \equiv$ the Borel σ-field in $[0, T]$, which assumes values in H and obeys the following assumption.

ASSUMPTION 2.1. (a) $Z(\cdot)$ is a nonatomic H-valued measure;
(b) $Z(A) \perp Z(B)$, for $A, B \in \mathscr{B}_T$, $A \cap B = \varnothing$;
(c) For every $t \in [0, T]$, $Z_t \equiv Z([0, t]) \in E^* \subset H$;
(d) the family $\{Z_t \equiv Z([0, t]): 0 \leq t \leq T\}$ spans E^*, in that all linear combinations of functionals from this family are dense in E^*, relative to the uniform norm.

For $t \in [0, T]$ define $\mathscr{L}_t \subset E^*$ to be the vector space of all finite linear combinations

$$\sum \alpha_i Z_{s_i}, \qquad \alpha_i \in \mathbb{R}, 0 \leq s_i \leq t.$$

It follows from Assumption 2.1(d), that $\{Z_t \equiv Z([0, t]): 0 \leq t \leq T\}$ is a separating family for E, in that $\omega \in E$ and $\langle Z_t | \omega \rangle = 0$, for all $t \in [0, T]$, implies $\omega = 0$. This yields that $\{Z_t\colon 0 \leq t \leq T\}^\perp = \{0\}$. Notice that the uniform norm on E^* is stronger than the Hilbert norm which E^* inherits from H. Every element $\omega \in E$ may be regarded as a function on $\mathscr{L}_T$,

$$\mathscr{L}_T \ni \ell \mapsto \omega(\ell) \equiv \langle \ell | \omega \rangle$$

(for the classical AWS this is the usual interpretation of the elements of $C_0[0, T]$ as functions of $t \in [0, T]$, cf. Example 2.1). Since $\{Z_t \equiv Z([0, t])$: $0 \leq t \leq T\}$ is a separating family for E, two elements of E that are indistinguishable as functions on $\mathscr{L}_T$ are automatically identical as elements of E. The fact that $\omega, \omega' \in E$ coincide on $\mathscr{L}_t \subset \mathscr{L}_T$, that is, $\langle \ell | \omega \rangle = \langle \ell | \omega' \rangle$, for all $\ell \in \mathscr{L}_t$, we will denote as $\omega \restriction \mathscr{L}_t \equiv \omega' \restriction \mathscr{L}_t$.

Denote by $\mathscr{S}$ the family of all simple real Borel functions on $[0, T]$ and for $f \in \mathscr{S}$,

$$f(\tau) = \sum \alpha_i I_{\Delta_i}(\tau), \qquad 0 \leq \tau \leq T, \alpha_i \in \mathbb{R}_1,$$

define the integral

$$Z[f] \equiv \int_0^T f(\tau) Z(d\tau) = \sum \alpha_i Z(\Delta_i).$$

Obviously

$$\| Z[f] \|_H^2 = \int_0^T | f(\tau) |^2 \nu(d\tau), \qquad f \in \mathscr{S},$$

where ν is defined on $\mathscr{B}_T$ by $\nu(A) = \|Z(A)\|_H^2$, $A \in \mathscr{B}_T$. Thus $Z[\cdot]$ extends to a unitary equivalence between $L^2(\nu; \mathbb{R})$ and H. For $f \in L^2(\nu; \mathbb{R})$ and $0 \leq s \leq t \leq T$, $\int_s^t f(\tau) Z(d\tau)$ is simply another notation for $Z[I_{[s, t]} \times f]$. For every $h \in H$, there exists a unique function (in fact, a class of ν-equivalent functions)

$\hat{h} \in L^2(\nu)$, such that $Z[\hat{h}] = h$, and

$$(h_1|h_2)_H = \int_0^T \hat{h}_1(\tau)\hat{h}_2(\tau)\nu(d\tau), \qquad h_1, h_2 \in H.$$

EXAMPLE 2.1. Take the classical AWS $(\iota, \mathscr{C}'[0,T], C_0[0,T])$ and for $t \in [0,T]$ set

$$\zeta_t(\tau) = \begin{cases} \tau, & \text{if } 0 \le \tau \le t, \\ t, & \text{if } \tau > t. \end{cases}$$

The canonical embedding $\mathscr{M} \hookrightarrow_{\iota^*} \mathscr{C}'[0,T]$, where $\mathscr{M}$ is the set of all finite signed Borel measures on $[0,T]$, identifies each ζ_t with Dirac's measure at the point $\{t\}$: $\zeta_t \equiv \iota^*(\delta_t)$ (cf. n° 6.7 in [1]). It is easy to see that there is exactly one measure $Z(\cdot)$ on $\mathscr{B}_T$ with values in $\mathscr{C}'[0,T]$, which satisfies Assumption 2.1, and is such that $\zeta_t(\cdot) = Z([0,t])$, for all $t \in [0,T]$. The corresponding scalar measure $\nu(A) = \|Z(A)\|^2_{\mathscr{C}'[0,T]}$, $A \in \mathscr{B}_T$, is the usual Lebesgue measure, and, for $f \in L^2[0,T]$, the integral of f relative to the vector-valued measure $Z(\cdot)$ is given by

$$Z[f] \equiv \int_0^T f(\tau) Z(d\tau) = \int_0^{(\cdot)} f(\tau)\, d\tau \in \mathscr{C}'[0,T].$$

We remark that for $\omega \in C_0[0,T]$ and for $0 \le t \le T$ $\langle Z_t|\omega\rangle = \omega(t)$, that is, $\omega \restriction \mathscr{L}_t \equiv \omega' \restriction \mathscr{L}_t$ simply means that ω and ω', as functions from $C_0[0,T]$, coincide on the interval $[0,t]$.

Denote by $\mathscr{E}$ the Borel σ-field in the Banach space E and let μ be the canonical Gaussian measure on $\mathscr{E}$, induced by the AWS (ι, H, E). For $\ell \in E^*$, $E \ni \omega \mapsto \langle \ell|\omega\rangle$ is a zero-mean Gaussian r.v. on $(E, \mathscr{E}, \mu)$, and

$$\int_E \langle \ell|\omega\rangle\langle \ell'|\omega\rangle \mu(d\omega) = (\ell|\ell'), \qquad \ell, \ell' \in E^*.$$

Note that all Gaussian r.v.'s considered in this article have vanishing mean, and in this paper "Gaussian distribution" actually means "Gaussian distribution with vanishing mean." We remark that the Borel σ-field in E is generated by the functionals from its dual E^* and we define the following filtration in E:

$$\mathscr{E}_t = \sigma\{\langle Z_s|\cdot\rangle : 0 \le s \le t\}, \qquad 0 \le t \le T.$$

CLAIM 2.1. Let, for some $0 \le t \le T$, $\xi\colon E \mapsto \mathbb{R}$ be an $\mathscr{E}_t$-measurable mapping. Then $\omega, \omega' \in E$ and $\omega \restriction \mathscr{L}_t \equiv \omega' \restriction \mathscr{L}_t$ implies that $\xi(\omega) = \xi(\omega')$.

For $t \in [0,T]$, $\gamma_t(\omega) \equiv \langle Z_t|\omega\rangle$, $\omega \in E$ is an *everywhere defined* Gaussian r.v. on $(E, \mathscr{E}, \mu)$.

CLAIM 2.2. $\{\gamma_t\colon 0 \le t \le T\}$ and $\{\gamma_t^2 - \nu[0,t]\colon 0 \le t \le T\}$ are continuous martingales on $(E, \mathscr{E}, \mu)$, relative to $\{\mathscr{E}_t\colon 0 \le t \le T\}$.

We also fix an arbitrary probability space $(U, \mathscr{A}, \Pi)$ with probability law Π independent of μ. All r.v.'s under consideration will be defined on the product

$$(U \times E, \mathscr{A} \otimes \mathscr{E}, \Pi \otimes \mu).$$

With no ambiguity, r.v's defined on $(E, \mathscr{E}, \mu)$, or on $(U, \mathscr{A}, \Pi)$, will also be regarded as r.v.'s on the above product-probability space. On $U \times E$ define the filtration $\mathscr{F}_t \equiv \mathscr{A} \otimes \mathscr{E}_t$, $0 \le t \le T$. In what follows we will be dealing with stochastic integrals relative to the process (γ_t), regarded as a continuous Gaussian martingale on $U \times E$, with respect to $\{\mathscr{F}_t\colon 0 \le t \le T\}$. For that purpose, introduce the σ-field $\tilde{\mathscr{P}}$ of all predictable sets in $[0, T] \times U \times E$ relative to $(\mathscr{F}_t)$, and the σ-field $\mathscr{P}$ of all predictable sets in $[0, T] \times E$ relative to $(\mathscr{E}_t)$. It is easy to see that $\tilde{\mathscr{P}} = \mathscr{A} \otimes \mathscr{P}$. By $\mathfrak{L}^2_{\mathrm{loc}}(\tilde{\mathscr{P}})$ we denote the family of all $\tilde{\mathscr{P}}$-measurable functions $\phi_t(u, \omega)$, $t \in [0, T]$, $u \in U$, $\omega \in E$, with

$$\int_0^T |\phi_t(u, \omega)|^2 \nu(dt) < \infty, \quad \text{for } \mu\text{-a.e. } \omega \in E, \text{ for } \Pi\text{-a.e. } u \in U.$$

For any $\phi \in \mathfrak{L}^2_{\mathrm{loc}}(\tilde{\mathscr{P}})$, the following stochastic integral is well defined as an equivalence class on $(U \times E, \mathscr{A} \otimes \mathscr{E}, \Pi \otimes \mu)$:

$$\mathscr{I}^{\phi} = \int_0^T \phi_t(u, \omega)\, d\gamma_t(\omega).$$

On the other hand, for every fixed $u \in U$, $\phi_{(\cdot)}(u, \cdot\,)$ is a $\mathscr{P}$-measurable function on $[0, T] \times E$, and so, for Π-a.e. (*fixed*) $u \in U$, the following stochastic integral is well defined, as an equivalence class on $(E, \mathscr{E}, \mu)$:

$$\mathscr{J}_u^{\phi} = \int_0^T \phi_t(u, \omega)\, d\gamma_t(\omega).$$

CLAIM 2.3. The two integrals $\mathscr{I}^{\phi}$ and $\mathscr{J}_u^{\phi}$ coincide a.e. in the following sense. If $\tilde{\mathscr{I}}^{\phi}(u, \omega)$ is a representative of the class $\mathscr{I}^{\phi}$, then for Π-a.e. $u \in U$, $\tilde{\mathscr{I}}^{\phi}(u, \cdot\,)$, as a function on E, is a representative of the class $\mathscr{J}_u^{\phi}$.

3. Shift transformations on the Wiener space. Our concern in this section is a class of nonlinear transformations of the Banach space E, which has the form

$$E \ni \omega \mapsto \omega + h[u, \omega] \in E, \qquad h[u, \omega] \in H.$$

More specifically, we are interested in the case where the h term above has the form of an integral relative to the H-valued measure $Z(\cdot)$,

$$h[u, \omega] = Z\big[\phi_{(\cdot)}(u, \omega)\big] \equiv \int_0^T \phi_t(u, \omega) Z(dt).$$

Here $\phi_t(u, \omega)$ is a function from $\mathfrak{L}^2_{\mathrm{loc}}(\tilde{\mathscr{P}})$. This implies that, for $(\Pi \otimes \mu)$-a.e. $(u, \omega) \in U \times E$, the function $\phi_{(\cdot)}(u, \omega)$ is from $L^2(d\nu)$, and therefore, by the construction from Section 2, the integral above is well defined a.e. in $U \times E$.

DEFINITION 3.1. For every $\phi \in \mathfrak{L}^2_{\mathrm{loc}}(\tilde{\mathscr{P}})$, the mapping $\mathscr{T}^{\phi}\colon U \times E \mapsto E$, defined $(\Pi \otimes \mu)$-a.e. by

$$\mathscr{T}^{\phi}(u,\omega) = \omega + \int_0^T \phi_t(u,\omega) Z(dt),$$

will be referred to as the shift transformation with kernel ϕ. For $0 \le t \le T$, we define also the shift

$$\mathscr{T}_t^{\phi}(u,\omega) = \omega + \int_0^t \phi_s(u,\omega) Z(ds).$$

Every shift is a measurable mapping from its domain in $U \times E$ into E. Indeed, for every $\ell \in E^*$, $(u,\omega) \mapsto \langle \ell | \mathscr{T}^{\phi}(u,\omega)\rangle$ is measurable, for

$$\langle \ell | \mathscr{T}^{\phi}(u,\omega)\rangle = \langle \ell | \omega \rangle + \int_0^T \hat{\ell}(t) \phi_t(u,\omega) \nu(dt).$$

The main goal of this section is to investigate conditions for $\phi \in \mathfrak{L}^2_{\mathrm{loc}}(\tilde{\mathscr{P}})$, under which the measure $\mathscr{T}^{\phi} \circ (\Pi \otimes \mu)$ is absolutely continuous relative to μ. For the classical AWS this problem was studied by Cameron and Martin [2] and by Girsanov [5]. Here we modify their result in the context of the AWS and the shifts $\mathscr{T}^{\phi}$. Our proof, however, is based on a different idea and does not involve martingale methods or Itô calculus.

First, we introduce the family $\mathfrak{S} \subset \mathfrak{L}^2_{\mathrm{loc}}(\tilde{\mathscr{P}})$ of all simple processes of the form

$$\psi_t(u,\omega) = \sum_{j=1}^{n} \xi_j(u,\omega) I_{(t_{j-1},t_j]}(t), \qquad 0 \le t \le T, \tag{3.1}$$

where $0 = t_0 < t_1 < \cdots < t_n = 1$, and each ξ_j is a $\mathscr{F}_{t_{j-1}}$-measurable r.v., with $\mathbb{E}_{(\Pi\otimes\mu)}\{\xi_j^2\} < \infty$. Let us fix a simple process as in (3.1). Then

$$b_j \equiv \langle Z_{t_j} - Z_{t_{j-1}} | \omega \rangle, \qquad j = 1,\dots,n$$

are all independent Gaussian r.v.'s, each having variance $\Delta_j = \|Z_{t_j} - Z_{t_{j-1}}\|_H^2 \equiv \nu((t_{j-1},t_j])$. Further, the r.v.'s b_j and the r.v.'s ξ_j from (3.1) have the following property.

LEMMA 3.1. *For $1 \le j \le n$, and for any Borel function $\alpha\colon \mathbb{R}^{n+j-1} \mapsto \mathbb{C}$, the following identity holds in the sense that the existence of either side implies the existence of the other one and the equality*:

$$\begin{aligned}\mathbb{E}_{(\Pi\otimes\mu)}\Big\{\alpha(\xi_1,b_1;\dots;\xi_{j-1},b_{j-1};b_j+\xi_j\Delta_j;b_{j+1};\dots;b_n)\\ \times\exp\big[-\xi_j b_j - \tfrac{1}{2}\xi_j^2\Delta_j\big]\Big\}\\ = \mathbb{E}_{(\Pi\otimes\mu)}\{\alpha(\xi_1,b_1;\dots;\xi_{j-1},b_{j-1};b_j;b_{j+1};\dots;b_n)\}\end{aligned} \tag{3.2}$$

(*the shift term $\xi_j\Delta_j$ disappears in the right side*).

PROOF. It is enough to consider the case where α is positive. In this case both sides of (3.2) are well defined and the Fubini theorem can be applied. Let

$$M(dy_1, dx_1; \dots; dy_{j-1}, dx_{j-1}; dy_j)$$

be the probability distribution in $\mathbb{R}^{2j-1}$ of the random vector

$$(\xi_1, b_1; \dots; \xi_{j-1}, b_{j-1}; \xi_j).$$

The latter is independent of the vector $(b_j; \dots; b_n)$, so that the distribution of

$$(\xi_1, b_1; \dots; \xi_{j-1}, b_{j-1}; \xi_j, b_j; b_{j+1}; \dots; b_n)$$

is

$$M(dy_1, dx_1; \dots; dy_{j-1}, dx_{j-1}; dy_j) \times N_j(dx_j) \times \cdots \times N_n(dx_n),$$

$$N_j(dx) = \frac{1}{\sqrt{2\pi\Delta_j}} e^{-x^2/2\Delta_j}\, dx, \qquad 0 \le j \le n.$$

The left side of (3.2) equals

$$\begin{aligned}
&\frac{1}{\sqrt{2\pi\Delta_j}} \int_{\mathbb{R}^{n-j}} N_{j+1}(dx_{j+1}) \times \cdots \times N_n(dx_n)\\
&\quad \times \int_{\mathbb{R}^{2j-1}} M(dy_1, dx_1; \dots; dy_{j-1}, dx_{j-1}; dy_j)\\
&\quad \times \int_{\mathbb{R}} \alpha(y_1, x_1; \dots; y_{j-1}, x_{j-1}; x_j + y_j\Delta_j; x_{j+1}; \dots; x_n)\\
&\quad \times e^{-y_j x_j - (1/2) y_j^2 \Delta_j} \times e^{-x_j^2/2\Delta_j}\, dx_j.
\end{aligned}$$

Changing the variable in the last integral to $z = x_j + y_j\Delta_j$ leads to the following expression, which is exactly the right side of (3.2) (note that the integrand below does not depend on y_j):

$$\begin{aligned}
&\int_{\mathbb{R}^{n+1-j}} N_j(dz) N_{j+1}(dx_{j+1}) \times \cdots \times N_n(dx_n)\\
&\quad \times \int_{\mathbb{R}^{2j-1}} \alpha(y_1, x_1; \dots; y_{j-1}, x_{j-1}; z; x_{j+1}; \dots; x_n)\\
&\quad \times M(dy_1, dx_1; \dots; dy_{j-1}, dx_{j-1}; dy_j). \qquad \square
\end{aligned}$$

For $\phi \in \mathscr{L}^2_{\text{loc}}(\tilde{\mathscr{P}})$, define

$$R^{\phi} = \exp\left[-\int_0^T \phi_t\, d\gamma_t - \tfrac{1}{2}\int_0^T |\phi_t|^2 \nu(dt) \right].$$

LEMMA 3.2. *Let $\psi \in \mathfrak{S}$. Then $\mathbb{E}_{(\Pi\otimes\mu)}\{R^{\psi}\} = 1$ and for every $\ell \in E^*$,*

$$\mathbb{E}_{(\Pi\otimes\mu)}\left\{ e^{i\langle \ell | \mathscr{T}^{\psi}(u,\omega)\rangle} R^{\psi}(u,\omega) \right\} = e^{-1/2\|\ell\|_H^2}. \tag{3.3}$$

PROOF. Since $\mathscr{L}_T$ is dense in E^*, and since the Hilbert norm $\|\cdot\|_H$ is a continuous function on E^* (relative to the uniform norm), it is enough to show that (3.3) holds for every $\ell \in E$ of the form

$$\ell = \sum_{j=1}^{n} \alpha_i \left(Z_{t_j} - Z_{t_{j-1}}\right),$$

for some choice of $0 = t_0 < t_1 < \cdots < t_n = 1$ and $\alpha_i \in \mathbb{R}$. With no loss of generality we may and do assume that $\psi \in \mathfrak{S}$ is written as

$$\psi_t(u, \omega) = \sum_{i=1}^{n} \xi_j(u, \omega) I_{(t_{j-1}, t_j]}(t),$$

where each ξ_j is measurable with respect to $\mathscr{F}_{t_{j-1}}$. Then

$$\begin{aligned} &\mathbb{E}_{(\Pi\otimes\mu)}\left\{e^{i\langle \ell | \mathscr{T}^{\psi}(u,\omega)\rangle} R^{\psi}(u,\omega)\right\} \\ &\quad = \mathbb{E}_{(\Pi\otimes\mu)}\left\{\exp\left[i \sum_{j=1}^{n} \left(\alpha_j b_j + \alpha_j \xi_j \Delta_j\right)\right] \prod_{j=1}^{n} \exp\left[-\xi_j b_j - \tfrac{1}{2}\xi_j^2 \Delta_j\right]\right\}. \end{aligned}$$

Applying Lemma 3.1 consecutively for $j = n, n-1, \dots, 1$, we get

$$\begin{aligned} \mathbb{E}_{(\Pi\otimes\mu)}\left\{e^{i\langle \ell | \mathscr{T}^{\psi}(u,\omega)\rangle} R^{\psi}(u,\omega)\right\} &= \mathbb{E}_{(\Pi\otimes\mu)}\left\{\exp\left[i \sum_{j=1}^{n} \alpha_j b_j\right]\right\} \\ &= \mathbb{E}_{(\Pi\otimes\mu)}\left\{\exp\left[-\tfrac{1}{2}\|\ell\|_H^2\right]\right\}. \end{aligned}$$

In particular, for $\ell = 0$ we get $\mathbb{E}_{(\Pi\otimes\mu)}\{R^{\psi}\} = 1$. □

What follows is our version of the theorem of Cameron–Martin–Girsanov.

THEOREM 3.1. *Let $\phi \in \mathfrak{L}^2_{loc}(\tilde{\mathscr{P}})$ and let $\mathbb{E}_{(\Pi\otimes\mu)}\{R^{\phi}\} = 1$. Then, for every $\ell \in E^*$,*

$$\mathbb{E}_{(\Pi\otimes\mu)}\left\{e^{i\langle \ell | \mathscr{T}^{\phi}(u,\omega)\rangle} R^{\phi}(u,\omega)\right\} = e^{(-1/2)\|\ell\|_H^2}.$$

PROOF. Let $\phi \in \mathfrak{L}^2_{\mathrm{loc}}(\tilde{\mathscr{P}})$ and $\mathbb{E}_{(\Pi\otimes\mu)}\{R^{\phi}\} = 1$. Choose a sequence $\{\psi^n \in \mathfrak{S}: n \geq 1\}$ with the following two properties:

(i) $\displaystyle\lim_{n\to\infty} \int_0^T \left|\phi_t(u,\omega) - \psi_t^n(u,\omega)\right|^2 \nu(dt) = 0, \qquad (\Pi\otimes\mu)\text{-a.s.};$

(ii) $\displaystyle\lim_{n\to\infty} \int_0^T \psi_t^n(u,\omega)\, d\gamma_t(u,\omega) = \int_0^T \phi_t(u,\omega)\, d\gamma_t(u,\omega), \qquad (\Pi\otimes\mu)\text{-a.s.}$

To see why such a sequence exists, notice first that

$$\lim_{n\to\infty} \int_0^T \left|\phi_t(u,\omega) - n \wedge \left(\phi_t(u,\omega)\right)\right|^2 \nu(dt) = 0, \qquad (\Pi\otimes\mu)\text{-a.s.}$$

On the other hand, as is well known, $\mathfrak{S}$ is dense in $L^2(U \times E, \tilde{\mathscr{P}}, \Pi\otimes\mu)$, so that for each $n \geq 1$ there exists $\psi^n \in \mathfrak{S}$ with

$$\mathbb{E}_{(\Pi \otimes \mu)} \left\{ \int_0^T \left| \psi_t^n(u, \omega) - n \wedge (\phi_t(u, \omega)) \right|^2 \nu(dt) \right\} \leq 2^{-n}.$$

It is now clear that some subsequence of $\{\psi^n: n \geq 1\}$ must satisfy (i) and we remark that if (i) holds, then the convergence in (ii) takes place in probability and therefore a.s. for some subsequence.

Next, notice that (i) and (ii) yield

$$\lim_{n \to \infty} R^{\psi^n}(u, \omega) = R^{\phi}(u, \omega), \qquad (\Pi \otimes \mu)\text{-a.s.}$$

and

$$\lim_{n \to \infty} \left\| \mathscr{T}^{\phi}(u, \omega) - \mathscr{T}^{\psi^n}(u, \omega) \right\|_E^2 = 0, \qquad (\Pi \otimes \mu)\text{-a.s.}$$

From the last relation, for every $\ell \in E^*$

$$\lim_{n \to \infty} e^{i\langle \ell | \mathscr{T}^{\psi^n}(u, \omega)\rangle} = e^{i\langle \ell | \mathscr{T}^{\phi}(u, \omega)\rangle}, \qquad (\Pi \otimes \mu)\text{-a.s.}$$

By Lemma 3.2 we have $\mathbb{E}_{(\Pi \otimes \mu)}\{R^{\psi^n}(u, \omega)\} = 1$, $n \geq 1$. Thus, the assumption $\mathbb{E}_{(\Pi \otimes \mu)}\{R^{\phi}\} = 1$ implies the uniform integrability of the family $\{R^{\psi^n}(u, \omega): n \geq 1\}$. But then, for every $\ell \in E^*$ the family $\{e^{i\langle \ell | \mathscr{T}^{\psi^n}(u, \omega)\rangle} R^{\psi^n}(u, \omega): n \geq 1\}$ is also uniformly integrable and since

$$\lim_{n \to \infty} e^{i\langle \ell | \mathscr{T}^{\psi^n}(u, \omega)\rangle} R^{\psi^n}(u, \omega) = e^{i\langle \ell | \mathscr{T}^{\phi}(u, \omega)\rangle} R^{\phi}(u, \omega), \qquad (\Pi \otimes \mu)\text{-a.s.},$$

we conclude that

$$\begin{aligned} \mathbb{E}_{(\Pi \otimes \mu)}\left\{ e^{i\langle \ell | \mathscr{T}^{\phi}(u, \omega)\rangle} R^{\phi}(u, \omega) \right\} &= \lim_{n \to \infty} \mathbb{E}_{(\Pi \otimes \mu)}\left\{ e^{i\langle \ell | \mathscr{T}^{\psi^n}(u, \omega)\rangle} R^{\psi^n}(u, \omega) \right\} \\ &= e^{(-1/2)\|\ell\|_H^2}. \end{aligned}$$

□

For $\phi \in \mathfrak{L}^2_{\mathrm{loc}}(\tilde{\mathscr{P}})$, with $\mathbb{E}_{(\Pi \otimes \mu)}\{R^{\phi}\} = 1$, consider the following probability law on $U \times E$:

$$d\mathbb{P}^{\phi} = R^{\phi} d(\Pi \otimes \mu).$$

In fact, the last theorem claims that the distribution of $\mathscr{T}^{\phi}(u, \omega)$ in E, relative to the law $\mathbb{P}^{\phi}$, coincides with the canonical Gaussian measure μ. We remark that in our considerations Π is an arbitrary probability law on $(U, \mathscr{A})$ and the last result obviously applies for $\Pi \equiv \delta_{u_0}$ (δ_{u_0} is Dirac's measure concentrated at $u_0 \in U$). In the latter case, if $\int_0^T |\phi_t(u_0, \omega)|^2 \nu(dt) < \infty$, for μ-a.e. $\omega \in E$, and $\mathbb{E}_{\mu}\{R^{\phi}(u_0, \cdot\,)\} = 1$ we conclude (cf. Claim 2.3) that relative to the measure $R^{\phi}(u_0, \cdot\,)\, d\mu$, the element $\mathscr{T}^{\phi}(u_0, \omega)$ is distributed in E according to the law μ. Thus, the following result is established.

Corollary 3.1. *Let $\phi \in \mathfrak{L}^2_{loc}(\tilde{\mathscr{P}})$ and let $\mathbb{E}_{(\Pi \otimes \mu)}\{R^{\phi}\} = 1$. Then, for every Borel function f: $E \mapsto \mathbb{R}$, the following identity holds, in the sense that the* *existence of either side implies the existence of the other one and the equality*:

$$\mathbb{E}_{(\Pi\otimes\mu)}\{f(\mathscr{T}^{\phi}(u,\omega))R^{\phi}(u,\omega)\} = \mathbb{E}_{\mu}\{f(\omega)\}.$$

If $\mathbb{E}_{\mu}\{R^{\phi}(u,\omega)\} = 1$, *for some* $u \in U$, *then*

$$\mathbb{E}_{\mu}\{f(\mathscr{T}^{\phi}(u,\omega))R^{\phi}(u,\omega)\} = \mathbb{E}_{\mu}\{f(\omega)\}.$$

As is immediate from the proof of Theorem 3.1, $\mathbb{E}_{(\Pi\otimes\mu)}\{R^{\phi}\} = 1$ is equivalent to the uniform integrability of $\{R^{\psi^n}(u,\omega)\colon n \geq 1\}$, for some (i.e., for any) approximating sequence $\psi^n \in \mathfrak{S}$, $n \geq 1$, as in Theorem 3.1. The following implication is known from Lemma 7.1.2 in [6]. (In [6] this fact is proved for γ_t defined as the Wiener process, but the same reasoning applies in our case also. An independent proof, using uniform integrability of exponents of simple processes, is also possible.)

$$\text{(3.4)}\qquad \begin{aligned} &u \in U \quad\text{and}\quad \int_0^T |\phi(u,\omega)|^2 \nu(dt) < C, \quad \text{for } \mu\text{-a.e. } \omega \in E,\\ &\Rightarrow \quad \mathbb{E}_{\mu}\{R^{\phi}(u,\omega)\} = 1. \end{aligned}$$

We will show next that $\mathbb{E}_{\mu}\{R^{\phi}\} = 1$, $u \in U$, always holds, if the shift $\mathscr{T}^{\phi}(u,\cdot)$ is invertible over a set with probability 1, the inverse being also a shift defined a.e. We will illustrate with examples that such a situation naturally arises when strong solutions of stochastic equations are considered. First we introduce the following assumption for the kernel $\phi \in \mathfrak{L}^2_{\text{loc}}(\tilde{\mathscr{P}})$.

ASSUMPTION 3.1. For Π-a.e. $u \in U$ there exist sets S^u, $\hat{S}^u \in \mathscr{E}$ with the following properties:

(a) $\mu(S^u) = \mu(\hat{S}^u) = 1$;
(b) for every $\omega \in S^u$,

$$\int_0^T |\phi_t(u,\omega)|^2 \nu(dt) < \infty;$$

(c) the mapping $\mathscr{T}^{\phi}(u,\cdot)$ provides a one-to-one correspondence between S^u and $\hat{S}^u$ and the inverse $[\mathscr{T}^{\phi}(u,\cdot)]^{-1}\colon \hat{S}^u \mapsto S^u$ is measurable;
(d) the following implication holds for every $t \in [0,T]$:

$$\omega,\omega' \in S^u \quad\text{and}\quad \mathscr{T}^{\phi}(u,\omega)\restriction\mathscr{L}_t \equiv \mathscr{T}^{\phi}(u,\omega')\restriction\mathscr{L}_t \quad\Rightarrow\quad \omega\restriction\mathscr{L}_t \equiv \omega'\restriction\mathscr{L}_t.$$

THEOREM 3.2. (a) *Let* $\phi \in \mathfrak{L}^2_{loc}(\tilde{\mathscr{P}})$. *Then* $\mathbb{E}_{\mu}\{R^{\phi}(u,\omega)\} \leq 1$, *for* Π-*a.e.* $u \in U$.

(b) *Let* $\phi \in \mathfrak{L}^2_{loc}(\tilde{\mathscr{P}})$ *obey Assumption* 3.1. *Then* $\mathbb{E}_{\mu}\{R^{\phi}(u,\omega)\} = 1$, *for* Π-*a.e.* $u \in U$.

PROOF. For every $n \geq 1$, define the set

$$A_n = \left\{(u,\omega) \in U \times E\colon \int_0^T |\phi_s(u,\omega)|^2 \nu(ds) \leq n\right\}$$

and let

$$\tau_n(u,\omega) = \begin{cases} \inf\left\{t \in [0,T]\colon \int_0^t |\phi_s(u,\omega)|^2 \nu(ds) > n\right\}, & \text{if } (u,\omega) \notin A_n, \\ T, & \text{if } (u,\omega) \in A_n. \end{cases}$$

Then define $\phi_t^n(u,\omega) = I_{[0,\tau_n]}(t)\phi_t(u,\omega)$, $n \geq 1$. Clearly, $\phi_t^n(u,\omega) = \phi_t(u,\omega)$ for $(u,\omega) \in A_n$ and for $t \in [0,T]$. Thus, $\mathscr{T}^{\phi^n} \restriction A_n = \mathscr{T}^{\phi} \restriction \mathrm{A}_n$ and $I_{A_n}(u,\omega)R^{\phi}(u,\omega) = I_{A_n}(u,\omega)R^{\phi^n}(u,\omega)$, $(\Pi \otimes \mu)$-a.e. By (3.4), for every $u \in U$,

$$\mathbb{E}_\mu\{R^{\phi^n}(u,\omega)\} = 1, \qquad n \geq 1.$$

PROOF OF (a). For Π-a.e. $u \in U$,

$$\mathbb{E}_\mu\left\{I_{A_n}(u,\omega)R^{\phi}(u,\omega)\right\} \equiv \mathbb{E}_\mu\left\{I_{A_n}(u,\omega)R^{\phi^n}(u,\omega)\right\} \leq \mathbb{E}_\mu\{R^{\phi^n}(u,\omega)\} = 1.$$

Passing to the limit as $n \to \infty$, we get $\mathbb{E}_\mu\{R^{\phi}(u,\omega)\} \leq 1$, for Π-a.e. $u \in U$.

PROOF OF (b). First we will show that for every $n \geq 1$ and $u \in U$ the following assertion holds: If $\omega, \omega' \in S^u$ and $\mathscr{T}^{\phi^n}(u,\omega') = \mathscr{T}^{\phi^n}(u,\omega)$, then $\omega = \omega'$.

Let $u \in U$, and let $\omega, \omega' \in S^u$ be such that $\mathscr{T}^{\phi^n}(u,\omega') = \mathscr{T}^{\phi^n}(u,\omega)$. With no loss of generality we assume that $\tau_n(u,\omega) \leq \tau_n(u,\omega')$ and set $t^* = \tau_n(u,\omega)$. If $t^* = T$, then $\mathscr{T}^{\phi}(u,\omega) = \mathscr{T}^{\phi}(u,\omega')$ and therefore $\omega \restriction \mathscr{L}_T \equiv \omega' \restriction \mathscr{L}_T$, that is, $\omega = \omega'$. So, let $t^* < T$. Then

$$\mathscr{T}^{\phi}(u,\omega) \restriction \mathscr{L}_{t^*} \equiv \mathscr{T}^{\phi^n}(u,\omega) \restriction \mathscr{L}_{t^*} = \mathscr{T}^{\phi^n}(u,\omega') \restriction \mathscr{L}_{t^*} \equiv \mathscr{T}^{\phi}(u,\omega') \restriction \mathscr{L}_{t^*}.$$

Due to (d) in Assumption 3.1, this yields $\omega \restriction \mathscr{L}_{t^*} \equiv \omega' \restriction \mathscr{L}_{t^*}$. Since $\phi_s(u,\cdot)$, $0 \leq s \leq t^*$, are all $\mathscr{E}_{t^*}$-measurable functions on E, we have $\phi_s(u,\omega') = \phi_s(u,\omega)$, $0 \leq s \leq t^*$. Thus

$$\int_0^{t^*} |\phi_s(u,\omega')|^2 \nu(dt) = \int_0^{t^*} |\phi_s(u,\omega)|^2 \nu(dt) = n,$$

which shows that

$$\int_{t^*}^{\tau_n(u,\omega')} |\phi_s(u,\omega')|^2 \nu(dt) = 0.$$

Hence

$$\int_{t^*}^{\tau_n(u,\omega')} \phi_s(u,\omega') Z(dt) = 0$$

and

$$\begin{aligned} \int_0^{t^*} \phi_s(u,\omega') Z(dt) + \omega' &= \mathscr{T}^{\phi^n}(u,\omega') \\ &= \mathscr{T}^{\phi^n}(u,\omega) = \int_0^{t^*} \phi_s(u,\omega) Z(dt) + \omega. \end{aligned} \tag{3.5}$$

Since $\omega \restriction \mathscr{L}_{t^*} \equiv \omega' \restriction \mathscr{L}_{t^*}$, $\langle Z_t | \omega \rangle = \langle Z_t | \omega' \rangle$, $t \in [0,t^*]$. For $t \in [t^*,T]$, by (3.5), we get

$$\langle Z_t - Z_{t*}|\omega'\rangle = \langle Z_t - Z_{t*}|\mathscr{T}^{\phi^n}(u,\omega')\rangle = \langle Z_t - Z_{t*}|\mathscr{T}^{\phi^n}(u,\omega)\rangle = \langle Z_t - Z_{t*}|\omega\rangle.$$

Hence, $\langle Z_t|\omega\rangle = \langle Z_t|\omega'\rangle$, for all $t \in [0,T]$, which implies that $\omega = \omega'$.

Let, for $u \in U$, A_n^u be the u-slice of the set $A_n \subset U \times E$ and let

$$B_n^u = \mathscr{T}^{\phi}(u,[A_n^u \cap S^u]) \equiv \mathscr{T}^{\phi^n}(u,[A_n^u \cap S^u]), \qquad u \in U, n \geq 1.$$

For Π-a.e. $u \in U$, we have $B_n^u \in \mathscr{E}$ (because $[\mathscr{T}^{\phi}(u,\cdot)]^{-1}$ is measurable on $\hat{S}^u$), and, by Corollary 3.1,

$$\mathbb{E}_{\mu}\{I_{B_n^u}(\omega)\} = \mathbb{E}_{\mu}\{I_{B_n^u}(\mathscr{T}^{\phi^n}(u,\omega))R^{\phi^n}(u,\omega)\}, \qquad n \geq 1.$$

Because of the invertibility of $\mathscr{T}^{\phi^n}(u,\cdot)$ over S^u established above, we have

$$I_{S^u}(\omega)I_{A_n^u}(\omega) = I_{S^u}(\omega)I_{B_n^u}(\mathscr{T}^{\phi^n}(u,\omega)), \qquad \omega \in E.$$

Hence, for Π-a.e. $u \in U$,

$$\begin{aligned}\mathbb{E}_{\mu}\{I_{A_n}(u,\omega)R^{\phi}(u,\omega)\} &\equiv \mathbb{E}_{\mu}\{I_{A_n}(u,\omega)R^{\phi^n}(u,\omega)\}\\ &= \mathbb{E}_{\mu}\{I_{B_n^u}(\mathscr{T}^{\phi^n}(u,\omega))R^{\phi^n}(u,\omega)\} = \mathbb{E}_{\mu}\{I_{B_n^u}\}.\end{aligned} \tag{3.6}$$

But $S^u \subset \bigcup_n A_n^u$, and so,

$$\mu\Big(\bigcup_n B_n^u\Big) = \mu(\mathscr{T}^{\phi}(u,[S^u])) = \mu(\hat{S}^u) = 1, \qquad \Pi\text{-a.s.}$$

Passing to the limit in (3.6), we get $\mathbb{E}_{\mu}\{R^{\phi}(u,\omega)\} = 1$, for Π-a.e. $u \in U$. □

Consider next the following modification of Assumption 3.1.

ASSUMPTION 3.2. Same as Assumption 3.1, with (d) replaced by
(d′): there exists a kernel $\hat{\phi} \in \mathfrak{L}^2_{\mathrm{loc}}(\tilde{\mathscr{P}})$, such that

$$\int_0^T |\hat{\phi}_t(u,\omega)|^2 \nu(dt) < \infty, \qquad \omega \in \hat{S}^u$$

and $[\mathscr{T}^{\phi}(u,\cdot)]^{-1} \upharpoonright \hat{S}^u \equiv \mathscr{T}^{\hat{\phi}}(u,\cdot) \upharpoonright \hat{S}^u$.

When it exists, $\hat{\phi}$ will be called the inverse of ϕ.

It is easy to see that (d′) implies (d) in Assumption 3.1. Indeed, let $u \in U$ and let $y = \mathscr{T}^{\phi}(u,\omega)$, $y' = \mathscr{T}^{\phi}(u,\omega')$, for some $\omega, \omega' \in S^u$. If ϕ satisfies Assumption 3.2, then

$$\omega = \int_0^T \hat{\phi}_t(u,y)Z(dt) + y,$$

$$\omega' = \int_0^T \hat{\phi}_t(u,y')Z(dt) + y',$$

from which relation it follows that if $y \upharpoonright \mathscr{L}_t \equiv y' \upharpoonright \mathscr{L}_t$, then $\omega \upharpoonright \mathscr{L}_t \equiv \omega' \upharpoonright \mathscr{L}_t$.

Clearly, the inverse $\hat{\phi}$, if it exists, satisfies the same Assumption 3.2, with $\hat{S}^u$ playing the role of S^u and with $\hat{\hat{\phi}} = \phi$. Hence, if ϕ satisfies Assumption 3.2, and $\hat{\phi}$ is its inverse, then by Theorem 3.2 $\mathbb{E}_\mu\{R^\phi(u,\omega)\} = \mathbb{E}_\mu\{R^{\hat{\phi}}(u,\omega)\} = 1$, for Π-a.e. $u \in U$, and according to Corollary 3.1, for every measurable mapping $f\colon U \times E \mapsto \mathbb{R}$,

$$\begin{aligned}\mathbb{E}_\mu\{f(u,\mathscr{T}^\phi(u,\omega))\} &= \mathbb{E}_\mu\{f(u,\mathscr{T}^\phi(u,\mathscr{T}^{\hat{\phi}}(u,\omega)))R^{\hat{\phi}}(u,\omega)\} \\ &= \mathbb{E}_\mu\{f(u,\omega)R^{\hat{\phi}}(u,\omega)\},\end{aligned} \tag{3.7}$$

for Π-a.e. $u \in U$. Since $R^{\hat{\phi}} > 0$, $(\Pi \otimes \mu)$-a.e., taking $f(u,\omega) = I_A(\omega)$ above, for $A \in \mathscr{E}$, we get

$$\mathbb{E}_{(\Pi\otimes\mu)}\{I_A(\mathscr{T}^\phi(u,\omega))\} = 0 \quad \Leftrightarrow \quad \mu(A) = 0,$$

that is, $\mu^\phi \equiv (\Pi \otimes \mu)\circ(\mathscr{T}^\phi)^{-1}$ and μ are mutually absolutely continuous.

4. Kallianpur–Striebel formula for signal correlated with noise. In this section we investigate the model (1.2), under the assumption that the signal $h[u,\omega]$ has the form of an integral w.r.t some H-valued measure $Z(\cdot)$, which satisfies Assumption 2.1. More specifically, we study the following model:

$$y = \mathscr{T}^\phi(u,\omega) \equiv \int_0^T \phi_t(u,\omega)Z(dt) + \omega, \qquad u \in U, \omega \in E. \tag{4.1}$$

We assume once and for all that the kernel $\phi \in \mathfrak{L}^2_{\mathrm{loc}}(\tilde{\mathscr{P}})$ obeys Assumption 3.2 and $\hat{\phi} \in \mathfrak{L}^2_{\mathrm{loc}}(\tilde{\mathscr{P}})$ will denote the inverse of ϕ. As was illustrated in Example 2.1, our considerations include the classical model (1.3) as a particular case.

Consider now the observation σ-field $\mathscr{O} \equiv (\mathscr{T}^\phi)^{-1}[\mathscr{E}] \subset \mathscr{A} \otimes \mathscr{E}$, related to the model (4.1). The main goal of this section is to express $\mathbb{E}_{(\Pi\otimes\mu)}\{\cdot|\mathscr{O}\}$ as an explicit function of the observation $y \equiv \mathscr{T}^\phi(u,\omega)$. For any $(\Pi \otimes \mu)$-integrable function $f\colon U \times E \mapsto \mathbb{R}$, define the following two functionals on E:

$$\sigma(f;\omega) = \int_U f(u,\mathscr{T}^{\hat{\phi}}(u,\omega))R^{\hat{\phi}}(u,\omega)\Pi(du), \quad \text{for } \mu\text{-a.e. } \omega \in E,$$

$$\Phi(f;\omega) = \frac{\sigma(f;\omega)}{\sigma(1;\omega)}, \quad \text{for } \mu\text{-a.e. } \omega \in E.$$

The key step in our approach is the following simple result.

THEOREM 4.1. *Let $f\colon U \times E \mapsto \mathbb{R}$ be a $(\Pi \otimes \mu)$-integrable function. Then, for every bounded Borel function $g\colon E \mapsto \mathbb{R}$,*

$$\begin{aligned}&\mathbb{E}_{(\Pi\otimes\mu)}\{g(\mathscr{T}^\phi(u,\omega))f(u,\omega)\} \\ &\qquad = \mathbb{E}_{(\Pi\otimes\mu)}\{g(\mathscr{T}^\phi(u,\omega))\Phi(f;\mathscr{T}^\phi(u,\omega))\}.\end{aligned} \tag{4.2}$$

PROOF. By Corollary 3.1 we have

$$\begin{aligned}
&\mathbb{E}_{(\Pi\otimes\mu)}\{g(\mathscr{T}^{\phi}(u,\omega))f(u,\omega)\} \\
&\quad\equiv \mathbb{E}_{(\Pi\otimes\mu)}\{g(\mathscr{T}^{\phi}(u,\mathscr{T}^{\hat{\phi}}(u,\omega)))f(u,\mathscr{T}^{\hat{\phi}}(u,\omega))R^{\hat{\phi}}(u,\omega)\} \\
&\quad= \mathbb{E}_{\mu}\{g(\omega)\mathbb{E}_{\Pi}\{f(u,\mathscr{T}^{\hat{\phi}}(u,\omega))R^{\hat{\phi}}(u,\omega)\}\} \qquad (4.3)\\
&\quad= \mathbb{E}_{\mu}\{g(\omega)\Phi(f;\omega)\sigma(1;\omega)\} \\
&\quad= \mathbb{E}_{\mu}\{g(\omega)\Phi(f;\omega)\mathbb{E}_{\Pi}\{R^{\hat{\phi}}(u,\omega)\}\}.
\end{aligned}$$

On the other hand, (3.7) implies that

$$\begin{aligned}
&\mathbb{E}_{(\Pi\otimes\mu)}\{g(\mathscr{T}^{\phi}(u,\omega))\Phi(f;\mathscr{T}^{\phi}(u,\omega))\} \\
&\quad= \mathbb{E}_{\Pi}\{\mathbb{E}_{\mu}\{g(\omega)\Phi(f;\omega)R^{\hat{\phi}}(u,\omega)\}\} \\
&\quad= \mathbb{E}_{\mu}\{g(\omega)\Phi(f;\omega)\mathbb{E}_{\Pi}\{R^{\hat{\phi}}(u,\omega)\}\},
\end{aligned}$$

which completes the proof. □

As (4.2) shows,

$$\begin{aligned}
\mathbb{E}_{(\Pi\otimes\mu)}\{f(u,\omega)|\mathscr{O}\} &= \Phi(f;\mathscr{T}^{\phi}(u,\omega)) \\
&\equiv \frac{\sigma(f;\mathscr{T}^{\phi}(u,\omega))}{\sigma(1;\mathscr{T}^{\phi}(u,\omega))}, \qquad (\Pi\otimes\mu)\text{-a.e.} \qquad (4.4)
\end{aligned}$$

REMARK. Let $\mathring{\mathbb{E}}\{\cdot|\mathscr{O}\}$ denote the conditional expectation relative to the law $d\mathbb{P}^{\phi} = R^{\phi}d(\Pi\otimes\mu)$. It is easy to show that

$$\mathbb{E}_{(\Pi\otimes\mu)}\{f|\mathscr{O}\} = \frac{\mathring{\mathbb{E}}\{f\times(R^{\phi})^{-1}|\mathscr{O}\}}{\mathring{\mathbb{E}}\{(R^{\phi})^{-1}|\mathscr{O}\}}, \qquad (\Pi\otimes\mu)\text{-a.e.} \tag{4.5}$$

Many authors refer to the last relation as the Kallianpur–Striebel formula, whereas the term seems more appropriate to the expression in (4.4). Indeed, the original idea of Kallianpur and Striebel [9] was to express $\mathbb{E}_{(\Pi\otimes\mu)}\{f|\mathscr{O}\}$ as a functional of the observed sample path, and they constructed explicitly this functional for the classical model (1.3), with kernel ϕ, which does not depend upon ω.

For some fixed $t\in[0,T]$, consider the following observation scheme:

$$y_t = \mathscr{T}_t^{\phi}(u,\omega) \equiv \int_0^t \phi_s(u,\omega)Z(ds) + \omega, \qquad u\in U,\ \omega\in E, \tag{4.6}$$

and let $\mathscr{O}_t = (\mathscr{T}_t^{\phi})^{-1}[\mathscr{E}] \subset \mathscr{A} \otimes \mathscr{E}$ be the corresponding observation σ-field. This, of course, is a particular case of the model (4.1), with ϕ replaced by $I_{[0,t]}\phi$. It is easy to see that $I_{[0,t]}\phi$ also satisfies Assumptions 3.2 with $(I_{[0,t]}\phi)^{\wedge} = I_{[0,t]}\hat{\phi}$. Thus, for any $(\Pi \otimes \mu)$-integrable function $f: U \times E \mapsto \mathbb{R}$,

$$\begin{aligned} \mathbb{E}_{(\Pi\otimes\mu)}\{f(u,\omega)|\mathscr{O}_t\} &= \Phi_t(f; \mathscr{T}_t^{\phi}(u,\omega)) \\ &\equiv \frac{\sigma_t(f; \mathscr{T}_t^{\phi}(u,\omega))}{\sigma_t(1; \mathscr{T}_t^{\phi}(u,\omega))}, \qquad (\Pi \otimes \mu)\text{-a.e.,} \end{aligned} \tag{4.7}$$

where

$$\sigma_t(f;\omega) = \int_U f\big(u, \mathscr{T}_t^{\hat{\phi}}(u,\omega)\big) R_t^{\hat{\phi}}(u,\omega)\Pi(du),$$

$$R_t^{\hat{\phi}}(u,\omega) = \exp\left[-\int_0^t \hat{\phi}_s(u,\omega)\,d\gamma_s(\omega) - \tfrac{1}{2}\int_0^t \big|\hat{\phi}_s(u,\omega)\big|^2 \nu(ds)\right].$$

It is trivial to show that

$$\int_0^T \big|\phi_s\big(u, \mathscr{T}_s^{\hat{\phi}}(u,\omega)\big) + \hat{\phi}_s(u,\omega)\big|^2 \nu(ds) = 0, \quad \text{for } \mu\text{-a.e. } \omega, \text{ for } \Pi\text{-a.e. } u$$

and so, the exponent $R_t^{\hat{\phi}}$ can be written in the following equivalent form:

$$\begin{aligned} R_t^{\hat{\phi}}(u,\omega) = \exp\bigg[&\int_0^t \phi_s\big(u, \mathscr{T}_s^{\hat{\phi}}(u,\omega)\big)\,d\gamma_s(\omega) \\ &-\tfrac{1}{2}\int_0^t \big|\phi_s\big(u, \mathscr{T}_s^{\hat{\phi}}(u,\omega)\big)\big|^2 \nu(ds)\bigg]. \end{aligned} \tag{4.8}$$

Note that by definition $\Phi_0(f;\omega) = \sigma_0(f;\omega) = \mathbb{E}_{\Pi}\{f(u,\omega)\}$.

Consider now the following filtration: $\tilde{\mathscr{O}}_t \equiv (\mathscr{T}^{\phi})^{-1}[\mathscr{E}_t]$, $0 \le t \le T$. Clearly, $\tilde{\mathscr{O}}_t \subset \mathscr{O}_t$, $0 \le t \le T$. We can now compute $\mathbb{E}_{(\Pi\otimes\mu)}\{f|\tilde{\mathscr{O}}_t\}$, for $f: U \times E \mapsto \mathbb{R}$, which is $(\mathscr{A} \otimes \mathscr{E}_t)$-measurable and satisfies $\mathbb{E}_{(\Pi\otimes\mu)}\{|f|\} < \infty$. By (4.4),

$$\mathbb{E}_{(\Pi\otimes\mu)}\{f|\mathscr{O}_t\} = \Phi_t(f; \mathscr{T}_t^{\phi}(u,\omega))$$

and since $[\mathscr{T}_t^{\hat{\phi}}]^{-1}[\mathscr{E}_t] \subset \mathscr{A} \otimes \mathscr{E}_t$, $f(u, \mathscr{T}_t^{\hat{\phi}}(u,\omega))$ is $(\mathscr{A} \otimes \mathscr{E}_t)$-measurable. But then

$$\Phi_t(f;\cdot) \equiv \frac{\int_U f\big(u, \mathscr{T}_t^{\hat{\phi}}(u,\cdot)\big) R_t^{\hat{\phi}}(u,\cdot)\Pi(du)}{\int_U R_t^{\hat{\phi}}(u,\cdot)\Pi(du)} : E \mapsto \mathbb{R}$$

is an $\mathscr{E}_t$-measurable mapping, and therefore $\Phi_t(f; \mathscr{T}_t^{\phi}(u,\omega))$ is $\tilde{\mathscr{O}}_t$ measurable. Hence

$$\mathbb{E}_{(\Pi\otimes\mu)}\{f|\tilde{\mathscr{O}}_t\} = \mathbb{E}_{(\Pi\otimes\mu)}\{f|\mathscr{O}_t\} \equiv \Phi_t(f; \mathscr{T}_t^{\phi}(u,\omega)). \tag{4.9}$$

As the above reasoning indicates, the role of the inverse kernel $\hat{\phi}$ is essential for the explicit construction of the filter. The next theorem provides a method for computing $\hat{\phi}$.

THEOREM 4.2. *Let $\phi \in \mathfrak{L}^2_{loc}(\tilde{\mathscr{P}})$ satisfy the following assumptions*:

(i) *For every $u \in U$ and $\omega \in E$*,

$$A_{u,\omega} \equiv \sup\{|\phi_t(u,\omega)|: 0 \le t \le T\} < \infty.$$

(ii) *For every $u \in U$ and $\omega \in E$*,

$$B_{u,\omega} \equiv \sup\left\{\frac{1}{\|h\|_H}|\phi_t(u,\omega+h) - \phi_t(u,\omega)|: 0 \le t \le T,\, h \in H,\, h \ne 0\right\} < \infty.$$

Then there exists another kernel $\hat{\phi} \in \mathfrak{L}^2_{loc}(\tilde{\mathscr{P}})$, which also satisfies (i) *and* (ii) *and is such that*

$$\begin{aligned}&\mathscr{T}^{\phi}(u, \mathscr{T}^{\hat{\phi}}(u,\omega))\\ &\quad = \mathscr{T}^{\hat{\phi}}(u, \mathscr{T}^{\phi}(u,\omega)) = \omega, \quad \textit{for every } u \in U \textit{ and every } \omega \in E.\end{aligned}$$

PROOF. Everything here is a suitable modification of the standard argument involving Picard's method. For $0 \le t \le T$ denote by P_t the orthogonal projector in H with Range(P_t) defined as the closure in H of $\mathscr{L}_t$ and observe that $\mathscr{T}^{\phi}(u,\omega) = \mathscr{T}^{\phi}(u,\omega')$ is equivalent to

$$\omega' - \omega = h \equiv \int_0^T [\phi_s(u,\omega') - \phi_s(u,\omega)] Z(ds).$$

This implies that

$$\|P_t h\|_H^2 \le B_{u,\omega}^2 \int_0^t \|P_t h\|_H^2 \nu(dt), \qquad 0 \le t \le T$$

and therefore that $\omega' - \omega \equiv h = 0$. To complete the proof we have to show that for any given $u \in U$ and $\omega \in E$ there exists (uniquely, according to the above) $\omega' \in E$ such that $\mathscr{T}^{\phi}(u,\omega') = \omega$. We also have to show that $\omega' \equiv \mathscr{T}^{\hat{\phi}}(u,\omega)$ for some universal $\hat{\phi} \in \mathfrak{L}^2_{\rm loc}(\tilde{\mathscr{P}})$ which satisfies (i) and (ii). To accomplish this program, define the transformations $\mathscr{S}_n$: $U \times E \mapsto E$, $n \ge 0$, as

$$\begin{aligned}\mathscr{S}_n(u,\omega) &= \mathscr{T}^{\psi^n}(u,\omega), \qquad n \ge 0,\\ \psi_t^0(u,\omega) &\equiv 0,\\ \psi_t^{n+1}(u,\omega) &= \phi_t(u, \mathscr{S}_n(u,\omega)), \qquad n \ge 1\end{aligned}$$

and set

$$\rho_t^n(u,\omega) = \psi_t^n(u,\omega) - \psi_t^{n-1}(u,\omega), \qquad n \ge 1.$$

Then observe that

$$|\rho_t^{n+1}(u,\omega)|^2 \le B_{u,\omega}^2 \int_0^t |\rho_s^n(u,\omega)|^2 \nu(ds)$$

and that

$$|\rho_t^1(u,\omega)| \le A_{u,\omega}, \qquad 0 \le t \le T.$$

Iterating, we get

$$\left|\rho_t^{n+1}(u,\omega)\right|^2 \le A_{u,\omega}^2\left(B_{u,\omega}^2\right)^n \int_0^t \nu(ds_1)\int_0^{s_1}\nu(ds_2)\cdots\int_0^{s_{n-1}}\nu(ds_n)$$

$$\equiv \frac{1}{n!}A_{u,\omega}^2\left(B_{u,\omega}^2\nu([0,t])\right)^n$$

and so, we can legitimately define

$$\hat{\phi}_t(u,\omega) = \sum_{n=1}^{\infty}\rho_t^n(u,\omega).$$

It is now easy to check that

$$\sup_{t\in[0,T]}\left|\hat{\phi}_t(u,\omega)\right| \le A_{u,\omega}\sum_{n=0}^{\infty}\frac{\left(B_{u,\omega}\sqrt{\nu([0,T])}\right)^n}{\sqrt{n!}} < \infty,$$

$$\left|\hat{\phi}_t(u,\omega+h) - \hat{\phi}_t(u,\omega)\right|^2 \le \|h\|_H^2 \exp\left[2B_{u,\omega}^2\nu([0,T])\right]$$

and

$$\mathscr{T}^{\phi}\left(u, \mathscr{T}^{\hat{\phi}}(u,\omega)\right) = \mathscr{T}^{\hat{\phi}}\left(u, \mathscr{T}^{\phi}(u,\omega)\right) = \omega. \qquad \square$$

As follows from the last result, conditions (i) and (ii) imply Assumption 3.2. The next example clarifies the role of the inverse kernel $\hat{\phi}$ and the invertibility discussed above.

EXAMPLE 4.1. Let $\varphi_t(x)$, $t \in [0,T]$, $x \in C_0[0,T]$, be a jointly measurable, causal functional on $C_0[0,T]$, such that, for every $x \in C_0[0,T]$, the following two conditions hold:

(i) $$\sup\{|\varphi_t(x)| : 0 \le t \le T\} < \infty.$$

(ii) $$\sup\left\{\frac{1}{\|h\|_H}|\varphi_t(x+h) - \varphi_t(x)| : 0 \le t \le T,\ h \in \mathscr{C}'[0,T],\ h \ne 0\right\} < \infty.$$

According to the last theorem, these assumptions guarantee existence and uniqueness of a strong solution (y_t) to the following stochastic equation:

$$y_t(\omega) = -\int_0^t \varphi_s(y_\cdot(\omega))\,ds + W_t(\omega), \qquad 0 \le t \le T. \tag{4.10}$$

Here the probability space is taken to be $C_0[0,T]$, provided with the standard Wiener measure, and W_t is the coordinate Brownian motion $W_t(\omega) = \omega(t)$, $0 \le t \le T$, $\omega \in C_0[0,T]$. Equation (4.10) may be regarded as a functional relation between the Brownian path $W_\cdot(\omega) \equiv \omega$ and the solution's sample path $y_\cdot(\omega)$. As was explained in Example 2.1, the function $t \mapsto \int_0^t \varphi_s\,ds$, as an element of $\mathscr{C}'[0,T]$, coincides with $\int_0^T \varphi_s Z(ds)$, where $Z(\cdot)$ is the $\mathscr{C}'[0,T]$-

valued measure described there. Therefore (4.10) is equivalent to

$$\omega(\cdot) = \mathscr{T}^{\varphi}(y_\cdot) \equiv y_\cdot + \int_0^{(\cdot)} \varphi(y_\cdot)\, ds, \qquad y, \omega \in C_0[0,T].$$

By Theorem 4.2, $\mathscr{T}^{\varphi}\colon C_0[0,T] \mapsto C_0[0,T]$ is one-to-one and invertible with inverse $(\mathscr{T}^{\varphi})^{-1} \equiv \mathscr{T}^{\hat{\varphi}}\colon C_0[0,T] \mapsto C_0[0,T]$, where $\hat{\varphi}_t(x)$ is another jointly measurable causal functional, which also satisfies (i) and (ii). So, the solution (y_t) can be expressed in terms of the inverse kernel $\hat{\varphi}$ as

$$y_t(\omega) = \int_0^t \hat{\varphi}_s(\omega)\, ds + \omega(t), \qquad 0 \le t \le T.$$

Note that this solves (4.10) "ω-wise."

EXAMPLE 4.2. Consider the following modification of the model (4.1):

$$y \equiv y(u,\omega) = \int_0^T \varphi_t(u,y) Z(dt) + \omega, \qquad y, \omega \in E, \tag{4.11}$$

where $\varphi \in \mathfrak{L}^2_{\text{loc}}(\tilde{\mathscr{P}})$. We regard this relation as an equation for the observation $y \equiv y(u,\omega)$, that is, we assume that for Π-a.e. $u \in U$ and for μ-a.e. $\omega \in E$ there exists $y(u,\omega) \in E$, such that:

(i) $\varphi_t(u, y(u,\omega)) \in \mathfrak{L}^2_{\text{loc}}(\tilde{\mathscr{P}})$ and (4.11) is satisfied $(\Pi \otimes \mu)$-a.e.
(ii) $\mu\{y(u,\omega)\colon \omega \in E\} = 1$, for Π-a.e. $u \in U$.

Clearly, this is satisfied if, for example, φ obeys (i) and (ii) in Theorem 4.2.
Setting $\phi_t(u,\omega) = \varphi_t(u, y(u,\omega))$, (4.11) takes the form

$$y \equiv y(u,\omega) = \mathscr{T}^{\phi}(u,\omega), \qquad y, \omega \in E.$$

It is easy to see that ϕ obeys Assumption 3.2, with $\hat{\phi} = -\varphi$. Thus, for the model (4.11), the nonlinear filter $\Phi_t(f; \cdot)$ [cf. (4.9)] has the following form:

$$\Phi_t(f;\omega) = \frac{\int_U f(u, \mathscr{T}_t^{-\varphi}(u,\omega)) \exp\left[\int_0^t \varphi_t(u,\omega)\, d\gamma_t(\omega) - \frac{1}{2}\int_0^t |\varphi_t(u,\omega)|^2 \nu(dt)\right] \Pi(du)}{\int_U \exp\left[\int_0^t \varphi_t(u,\omega)\, d\gamma_t(\omega) - \frac{1}{2}\int_0^t |\varphi_t(u,\omega)|^2 \nu(dt)\right] \Pi(du)},$$

$$\omega \in E.$$

In particular, if $f(u,\omega) \equiv f(u)$ the last relation is easily seen to coincide with the one obtained by Kallianpur and Striebel [9].

5. The finite energy condition and the Zakai equation. Let x_t, $0 \le t \le T$, be a stochastic process on $(U, \mathscr{A}, \Pi)$, which is a strong solution of the following SDE:

$$dx_t = a_t(x_\cdot)\, dt + b_t(x_\cdot)\, d\beta_t, \qquad 0 \le t \le T. \tag{5.1}$$

Here (β_t) is a Brownian motion process on $(U, \mathscr{A}, \Pi)$ [thereby (β_t) is independent of the white noise $\omega \in E$], and a, b: $[0, T] \times C_0[0, T] \mapsto \mathbb{R}$ are appropriate nonanticipative functionals with

$$\int_0^T |a_t(x.)|\, dt < \infty \quad \text{and} \quad \int_0^T |b_t(x.)|^2\, dt < \infty \qquad \Pi\text{-a.e.}$$

We regard (x_t) as a system process, that is, x_t is interpreted as the state of some system at moment t. In general, (x_t) can be taken to be an $\mathbb{R}^d$-valued diffusion process. Here we assume that (x_t) is one dimensional only for the sake of simplicity and all results in this section can be easily modified for $x_t \in \mathbb{R}^d$.

In what follows, we will employ the results of Sections 3 and 4 in the case where $(C[0, T], \mathscr{B}_{C[0,T]}, \pi)$, $\pi \equiv$ the law of the system process (x_t) is substituted for $(U, \mathscr{A}, \Pi)$. All notation will be adjusted to this case in an obvious way. Therefore, the role of the parameter u will be played by the trajectory $x.$ of the process (x_t). A principal assumption for the system process is that it cannot be observed directly, and information for (x_t) is provided only by observing $y \in E$, given by

$$y = \mathscr{T}^{\phi}(x., \omega) \equiv \int_0^T \phi_t(x, \omega) Z(dt) + \omega, \qquad \omega \in E. \tag{5.2}$$

Here ϕ: $[0, T] \times C[0, T] \times E \mapsto \mathbb{R}$ is a kernel from the class $\mathfrak{L}^2_{\mathrm{loc}}(\tilde{\mathscr{P}})$. Everywhere below we will assume that $\phi_t(x., \omega)$ obeys Assumption 3.2, that is, for π-a.e. $x. \in C[0, T]$, the mapping

$$\mathscr{T}^{\phi}(x., \cdot): E \mapsto E$$

is invertible over a set $S^{x.} \in \mathscr{E}$, with $\mu(S^{x.}) = \mu(\mathscr{T}^{\phi}(x., [S^{x.}]) = 1$, and the inverse has the form $[\mathscr{T}^{\phi}(x., \cdot\,)]^{-1} = \mathscr{T}^{\hat{\phi}}(x., \cdot\,) \equiv \int_0^T \hat{\phi}_t(x., \cdot\,) Z(dt) + (\cdot)$. This always holds if, for example, ϕ obeys (i) and (ii) in Theorem 4.2. The model discussed in Example 4.2, with the sample path $x.$ substituted for the parameter u, presents an important particular case of the observation scheme (5.2).In the latter case the model already provides the inverse kernel $\hat{\phi} \equiv -\varphi$.

Following our considerations in Section 4, for a $(\pi \otimes \mu)$-integrable function $f(x., \omega)$ and for $0 \le t \le T$, we set

$$\Phi_t(f; \omega) = \frac{\sigma_t(f; \omega)}{\sigma_t(1; \omega)},$$

$$\sigma_t(f; \omega) = \int_{C[0,T]} f\big(x., \mathscr{T}_t^{\hat{\phi}}(x., \omega)\big) R_t^{\hat{\phi}}(x., \omega) \pi(dx.),$$

$$R_t^{\hat{\phi}}(x., \omega) = \exp\bigg[\int_0^t \phi_s\big(x., \mathscr{T}_s^{\hat{\phi}}(x., \omega)\big)\, d\gamma_s(\omega) - \tfrac{1}{2}\int_0^t \big|\phi_s\big(x., \mathscr{T}_s^{\hat{\phi}}(x., \omega)\big)\big|^2 \nu(ds)\bigg].$$

The observation y in the model (5.2) generates the filtration $\tilde{\mathscr{O}}_t \equiv (\mathscr{T}^{\phi})^{-1}[\mathscr{E}_t]$,

$0 \le t \le T$, and for every Borel function $f: \mathbb{R} \mapsto \mathbb{R}$, with $\mathbb{E}_\pi\{|f(x_t)|\} < \infty$, $0 \le t \le T$, we have by (4.9) that $\mathbb{E}_{(\pi\otimes\mu)}\{f(x_t)|\tilde{\mathscr{O}}_t\} = \Phi_t(f(x_t); \mathscr{T}^\phi(x_\cdot, \omega))$. As processes on $(E, \mathscr{E}, \mu)$, $\Phi_t \equiv \Phi_t(f(x_t); \omega)$, $0 \le t \le T$, and $\sigma_t \equiv \sigma_t(f(x_t); \omega)$, $0 \le t \le T$, are adapted to the filtration $\mathscr{E}_t$, $0 \le t \le T$. Our main objective in this section is the computation of Ito differentials $d\Phi_t$ and $d\sigma_t$. That would allow the current values of Φ_t and σ_t to be computed recursively. Notice that as a r.v. on $(E, \mathscr{E}, \mu)$, $\Phi_t(f(x_t); \omega)$, *does not* represent $\mathbb{E}_{(\pi\otimes\mu)}\{f(x_t)|\tilde{\mathscr{O}}_t\} \equiv \Phi_t(f(x_t); \mathscr{T}^\phi(x_\cdot, \omega))$. However, since the measure $\mu^\phi = \mathscr{T}^\phi \circ (\pi \otimes \mu)$, induced on E by the observation $y = \mathscr{T}^\phi(x_\cdot, \omega)$, and the measure μ, are mutually absolutely continuous, any equation which holds μ-a.e. in E, will hold $(\pi \otimes \mu)$-a.e. in $C[0,T] \times E$, if $\omega \in E$ is replaced by $y = \mathscr{T}^\phi(x_\cdot, \omega)$. Therefore, we are free to choose different modifications of the processes (Φ_t) and (σ_t), which are adapted to the augmented filtration $\mathscr{E}_t^\mu$, $0 \le t \le T$ ($\mathscr{E}_t^\mu$ is obtained by augmenting $\mathscr{E}_t$ with all μ-null sets). One can show that $(\mathscr{E}_t^\mu)$ is right continuous and this property is important for what follows. We fix once and for all a continuous modification of the process $R_t^{\hat{\phi}}(x_\cdot, \omega)$, $0 \le t \le T$, and everywhere below we will operate only with this modification. It follows trivially from Fubini's theorem that $\mathbb{E}_\pi\{R_t^{\hat{\phi}}(x_\cdot, \omega)\}$, $0 \le t \le T$, is an $(\mathscr{E}_t^\mu)$ martingale, relative to the measure μ and therefore it admits a right continuous modification. By Doob's inequality, for any such modification,

$$\mu\left\{\omega \in E: \sup_{0 \le t \le T} \left|\mathbb{E}_\pi\{R_t^{\hat{\phi}}(x_\cdot, \omega)\}\right| < \infty\right\} = 1.$$

Lemma 5.1. *Let $\phi \in \mathfrak{L}^2_{loc}(\tilde{\mathscr{P}})$ be such that $\int_0^T \mathbb{E}_{(\pi\otimes\mu)}\{|\phi_t(x_\cdot, \omega)|^2\}\nu(dt) < \infty$. Then the following identity holds μ-a.e. in E*:

$$\mathbb{E}_\pi\left\{\int_0^T \phi_t(x_\cdot, \omega)\, d\gamma_t(\omega)\right\} = \int_0^T \mathbb{E}_\pi\{\phi_t(x_\cdot, \omega)\}\, d\gamma_t(\omega).$$

Proof. If $\phi \in \mathfrak{S}$, there is nothing to prove. In the general case, choose a sequence $\{\psi^n \in \mathfrak{S}: n \ge 1\}$ so that

$$\lim_{n\to\infty} \int_0^T \mathbb{E}_{(\pi\otimes\mu)}\left\{|\phi_t(x_\cdot, \omega) - \psi_t^n(x_\cdot, \omega)|^2\right\}\nu(dt) = 0$$

and notice that, since

$$\int_0^T \mathbb{E}_\mu\left\{|\mathbb{E}_\pi\{\phi_t(x_\cdot, \omega)\} - \mathbb{E}_\pi\{\psi_t^n(x_\cdot, \omega)\}|^2\right\}\nu(dt)$$

$$\le \int_0^T \mathbb{E}_{(\pi\otimes\mu)}\left\{|\phi_t(x_\cdot, \omega) - \psi_t^n(x_\cdot, \omega)|^2\right\}\nu(dt) \to_{n\to\infty} 0,$$

one has

$$\int_0^T \mathbb{E}_\pi\{\psi_t^n(x_\cdot, \omega)\}\, d\gamma_t(\omega) \to_{n\to\infty} \int_0^T \mathbb{E}_\pi\{\phi_t(x_\cdot, \omega)\}\, d\gamma_t(\omega)$$

in $L^2(\mu)$. On the other hand,

$$\int_0^T \psi_t^n(x_\cdot, \omega)\, d\gamma_t(\omega) \to_{n\to\infty} \int_0^T \phi_t(x_\cdot, \omega)\, d\gamma_t(\omega) \quad \text{in } L^2(\pi \otimes \mu),$$

so that μ-a.e.,

$$\mathbb{E}_\pi\left\{\int_0^T \phi_t(x_\cdot, \omega)\, d\gamma_t(\omega)\right\} = \lim_{k\to\infty} \mathbb{E}_\pi\left\{\int_0^T \psi_t^{n_k}(x_\cdot, \omega)\, d\gamma_t(\omega)\right\}$$

for some appropriate subsequence. □

The following theorem presents the main result in this section. The equation we derive for $\sigma_t(f;\omega)$, may be regarded as a "pathwise version" of the Zakai equation.

THEOREM 5.1. *Let* $f\colon \mathbb{R} \mapsto \mathbb{R}$ *be a twice-differentiable function, with* $|f(x)| \le C$, $|f'(x)| \le C$, $x \in \mathbb{R}$. *Assume that*:

(i) $\mathbb{E}_{(\pi\otimes\mu)}\{|\phi_t(x_\cdot, \omega)|\} < \infty$, *for every* $t \in [0, T]$.
(ii) $\int_0^T \mathbb{E}_{(\pi\otimes\mu)}\{|\phi_t(x_\cdot, \omega)|^2\}\nu(dt) < \infty$.
(iii) $\int_0^T \mathbb{E}_\pi\{|L_t f(x_\cdot)|\}\, dt < \infty$, *where*

$$L_t f(x_\cdot) = a_t(x_\cdot) f'(x_t) + \tfrac{1}{2} b_t(x_\cdot) f''(x_t).$$

Then, for every $t \in [0, T]$, *the following identity holds* μ-*a.e. in* E:

$$\sigma_t(f(x_t);\omega) = \mathbb{E}_\pi\{f(x_0)\} + \int_0^t \sigma_s(L_s f(x_\cdot);\omega)\, ds + \int_0^t \sigma_s(f(x_s)\phi_s;\omega)\, d\gamma_s(\omega). \tag{5.3}$$

REMARK. Condition (ii) above is known as the finite energy condition. Its role in deriving the Zakai equation was studied by Ocone [16] in the case of uncorrelated signal and noise.

PROOF. We will show first that both integrals in the right side of (5.3) are meaningful. For the first integral we have

$$\mathbb{E}_\mu\left\{\int_0^T |\sigma_s(L_s f(x_\cdot);\omega)|\, ds\right\} \le \int_0^T \mathbb{E}_\pi\left\{|L_s f(x_\cdot)|\, \mathbb{E}_\mu\{R_s^{\hat\phi}(x_\cdot, \omega)\}\right\} ds$$

$$= \int_0^T \mathbb{E}_\pi\{|L_s f(x_\cdot)|\}\, ds < \infty.$$

To show that the stochastic integral in (5.3) exists, it is enough to verify that for μ-a.e. $\omega \in E$,

$$A(\omega) \equiv \int_0^T \left| \sigma_s \equiv \mathbb{E}_\pi\left\{R_s^{\hat\phi}(x.,\omega) f(x_s)\phi_s\left(x., \mathscr{T}_s^{\hat\phi}(x.;\omega)\right)\right\}\right|^2 \nu(ds)$$

$$\le C^2 \int_0^T \mathbb{E}_\pi\left\{R_s^{\hat\phi}(x.,\omega)\right\}\mathbb{E}_\pi\left\{R_s^{\hat\phi}(x.,\omega)\left|\phi_s\left(x., \mathscr{T}_s^{\hat\phi}(x.;\omega)\right)\right|^2\right\}\nu(ds) < \infty.$$

In fact, we only need to show that the last relation holds for some modification of the process $\mathbb{E}_\pi\{R_s^{\hat\phi}(x.,\omega)\}$, $0 \le s \le T$. Taking a right-continuous modification, we get

$$A(\omega) \le C^2\left(\sup_{0\le s\le T} \mathbb{E}_\pi\left\{R_s^{\hat\phi}(x.,\omega)\right\}\right)$$

$$\times \int_0^T \mathbb{E}_\pi\left\{R_s^{\hat\phi}(x.,\omega)\left|\phi_s\left(x., \mathscr{T}_s^{\hat\phi}(x.;\omega)\right)\right|^2\right\}\nu(ds).$$

The last integral is finite for μ-a.e. $\omega \in E$, because by Corollary 3.1,

$$\mathbb{E}_\mu\left\{\int_0^T \mathbb{E}_\pi\left\{R_s^{\hat\phi}(x.,\omega)\left|\phi_s\left(x., \mathscr{T}_s^{\hat\phi}(x.;\omega)\right)\right|^2\right\}\nu(ds)\right\}$$

$$= \mathbb{E}_\pi\left\{\int_0^T \mathbb{E}_\mu\left\{R_s^{\hat\phi}(x.,\omega)\left|\phi_s\left(x., \mathscr{T}_s^{\hat\phi}(x.;\omega)\right)\right|^2\right\}\nu(ds)\right\}$$

$$= \mathbb{E}_\pi\left\{\int_0^T \mathbb{E}_\mu\left\{\left|\phi_s(x.,\omega)\right|^2\right\}\nu(ds)\right\} < \infty.$$

Thus, the right side of (5.3) is correctly defined.

Now we will show the identity in (5.3). Define the stopping times

$$\sigma_n'(u) = \begin{cases} \inf\left\{t : t \in [0,T],\ \int_0^t |a_s(x.)|\,ds > n \text{ or } \int_0^t |b_s(x.)|^2\,ds > n\right\}, \\ T, \qquad \text{if the above set is empty,} \end{cases}$$

$$\sigma_n''(x.,\omega) = \begin{cases} \inf\left\{t : t \in [0,T],\ \int_0^t \left|\phi_s\left(x., \mathscr{T}_s^{\hat\phi}(x.,\omega)\right)\right|^2 \nu(ds) > n \right. \\ \qquad \left. \text{or } \int_0^t \left| R_s^{\hat\phi}(x.,\omega)\phi_s\left(x., \mathscr{T}_s^{\hat\phi}(x.,\omega)\right)\right|^2 \nu(ds) > n\right\}, \\ T, \qquad \text{if the above set is empty.} \end{cases}$$

Set $\tau_n = \min\{\sigma_n', \sigma_n''\}$ and $\chi_s^n = I_{[0,\tau_n]}(s) \equiv I_{[0,\sigma_n']}(s) I_{[0,\sigma_n'']}(s)$, $0 \le s \le T$. Then, for every $t \in [0, T]$, the following equalities hold a.e.:

$$x_{t \wedge \tau_n}(u) = x_0 + \int_0^t \chi_s^n a_s(x_.)\, ds + \int_0^t \chi_s^n b_s(x_.)\, d\beta_s(u),$$

$$R^{\hat{\phi}}_{t \wedge \tau_n}(x_., \omega) = \exp\left[\int_0^t \chi_s^n \phi_s\big(x_., \mathscr{T}_s^{\hat{\phi}}(x_., \omega)\big)\, d\gamma_s(\omega) - \tfrac{1}{2}\int_0^t \chi_s^n \big|\phi_s\big(x_., \mathscr{T}_s^{\hat{\phi}}(x_., \omega)\big)\big|^2 \nu(ds)\right].$$

Itô's differential rule now yields

$$\begin{aligned} f(x_{t\wedge\tau_n}) R^{\hat{\phi}}_{t\wedge\tau_n} =_{(\text{a.e.})}\ & f(x_0) + \int_0^t \chi_s^n R_s^{\hat{\phi}}(x_., \omega)(L_s f)(x_s)\, ds \\ & + \int_0^t \chi_s^n R_s^{\hat{\phi}}(x_., \omega) f'(x_s) b_s(x_.)\, d\beta_s(u) \\ & + \int_0^t \chi_s^n R_s^{\hat{\phi}}(x_., \omega) f(x_s) \phi_s\big(x_., \mathscr{T}_s^{\hat{\phi}}(x_., \omega)\big)\, d\gamma_s(\omega) \\ \equiv\ & f(x_0) + \mathscr{I}_1 + \mathscr{I}_2 + \mathscr{I}_3. \end{aligned} \tag{5.4}$$

Next we will take the expectation $\mathbb{E}_\pi$ on both sides of (5.4), but first we need to check the validity of this operation. We have, for every $t \in [0, T]$,

$$\begin{aligned} &\mathbb{E}_\mu\Big\{\big|R^{\hat{\phi}}_{t\wedge\sigma_n''}(x_., \omega)\big|^2\Big\} \\ &\quad = \mathbb{E}_\mu\left\{R^{2\hat{\phi}}_{t\wedge\sigma_n''}(x_., \omega) \times \exp\left[\int_0^{t\wedge\sigma_n''} \big|\phi_s\big(x_., \mathscr{T}_s^{\hat{\phi}}(x_., \omega)\big)\big|^2 \nu(ds)\right]\right\} \\ &\quad \le e^n \mathbb{E}_\mu\Big\{R^{2\hat{\phi}}_{t\wedge\sigma_n''}(x_., \omega)\Big\} = e^n. \end{aligned}$$

Therefore

$$\begin{aligned} &\mathbb{E}_{(\pi\otimes\mu)}\left\{\int_0^t \big|\chi_s^n R_s^{\hat{\phi}}(x_., \omega) f'(x_s) b_s(x_.)\big|^2 ds\right\} \\ &\quad \le C^2 \mathbb{E}_\pi\left\{\int_0^{\sigma_n'} |b_s(x_.)|^2 \mathbb{E}_\mu\Big\{I_{[0,\sigma_n'']}(s)\big|R_s^{\hat{\phi}}(x_., \omega)\big|^2\Big\}\, ds\right\} \\ &\quad \le C^2 e^n \mathbb{E}_\pi\left\{\int_0^{\sigma_n'} |b_s(x_.)|^2\, ds\right\} \\ &\quad \le C^2 n e^n. \end{aligned}$$

The last relation shows that, for μ-a.e. $\omega \in E$, $\mathscr{I}_2(\cdot, \omega)$ is π-square-integrable with $\mathbb{E}_\pi\{\mathscr{I}_2(u,\omega)\} = 0$, μ-a.e. On the other hand,

$$\mathbb{E}_{(\pi\otimes\mu)}\left\{\int_0^t \left|\chi_s^n R_s^{\hat\phi}(x.,\omega) f(x_s)\phi_s\big(x.,\mathscr{T}_s^{\hat\phi}(x.,\omega)\big)\right|^2 \nu(ds)\right\}$$
$$\le C^2 \mathbb{E}_{(\pi\otimes\mu)}\left\{\int_0^{\sigma_n''} \left|R_s^{\hat\phi}(x.,\omega)\phi_s\big(x.,\mathscr{T}_s^{\hat\phi}(x.,\omega)\big)\right|^2 \nu(ds)\right\}$$
$$\le C^2 n,$$

and so, by Lemma 5.1,

$$\mathbb{E}_\pi\{\mathscr{I}_3\} = \int_0^t \mathbb{E}_\pi\left\{\chi_s^n R_s^{\hat\phi}(x.,\omega) f(x_s)\phi_s\big(x.,\mathscr{T}_s^{\hat\phi}(x.,\omega)\big)\right\} d\gamma_s(\omega), \qquad \mu\text{-a.e.}$$

We can now apply $\mathbb{E}_\pi$ to both sides of (5.4) to get

$$\begin{aligned}\mathbb{E}_\pi&\left\{f(x_{t\wedge\tau_n})R_{t\wedge\tau_n}^{\hat\phi}\right\}\\ &=_{(\text{a.e.})} \mathbb{E}_\pi\{f(x_0)\} + \int_0^t \mathbb{E}_\pi\left\{\chi_s^n R_s^{\hat\phi}(x.,\omega)(L_s f)(x_s)\right\} ds\\ &\quad + \int_0^t \mathbb{E}_\pi\left\{\chi_s^n R_s^{\hat\phi}(x.,\omega) f(x_s)\phi_s\big(x.,\mathscr{T}_s^{\hat\phi}(x.,\omega)\big)\right\} d\gamma_s(\omega).\end{aligned} \tag{5.5}$$

We have $R_{t\wedge\tau_n}^{\hat\phi} \to_{(n)} R_t^{\hat\phi}$, $(\pi\otimes\mu)$-a.e., and

$$1 = \mathbb{E}_{(\pi\otimes\mu)}\left\{R_{t\wedge\tau_n}^{\hat\phi}\right\} \to_{(n)} \mathbb{E}_{(\pi\otimes\mu)}\left\{R_t^{\hat\phi}\right\} = 1.$$

Thus, the family $\{f(x_{t\wedge\tau_n})R_{t\wedge\tau_n}^{\hat\phi}: n \ge 1\}$ is uniformly integrable (f is bounded), so

$$\lim_{n\to\infty} \mathbb{E}_\mu\mathbb{E}_\pi\left\{\left|f(x_{t\wedge\tau_n})R_{t\wedge\tau_n}^{\hat\phi} - f(x_t)R_t^{\hat\phi}\right|\right\} = 0.$$

Hence, for some sequence (k_n),

$$\lim_{n\to\infty} \mathbb{E}_\pi\left\{f(x_{t\wedge\tau_{k_n}})R_{t\wedge\tau_{k_n}}^{\hat\phi}\right\} = \mathbb{E}_\pi\left\{f(x_t)R_t^{\hat\phi}\right\}, \qquad \mu\text{-a.e.}$$

The dominated convergence theorem yields

$$\int_0^t \mathbb{E}_\pi\left\{\left|(1-\chi_s^n)R_s^{\hat\phi}(x.,\omega)(L_s f)(x_s)\,ds\right|\right\} ds \to_{\mu\text{-a.s.}} 0,$$

$$\int_0^t \mathbb{E}_\pi\left\{\left|(1-\chi_s^n)R_s^{\hat\phi}(x.,\omega) f(x_s)\phi_s\big(x.,\mathscr{T}_s^{\hat\phi}(x.,\omega)\big)\right|^2\right\}\nu(ds) \to_{\mu\text{-a.s.}} 0.$$

By the last relation, we have in μ-probability,

$$\lim_{n\to\infty} \int_0^t \mathbb{E}_\pi\left\{\chi_s^{k_n} R_s^{\hat\phi}(x.,\omega) f(x_s)\phi_s\big(x.,\mathscr{T}_s^{\hat\phi}(x.,\omega)\big)\right\} d\gamma_s(\omega)$$
$$= \int_0^t \mathbb{E}_\pi\left\{R_s^{\hat\phi}(x.,\omega) f(x_s)\phi_s\big(x.,\mathscr{T}_s^{\hat\phi}(x.,\omega)\big)\right\} d\gamma_s(\omega).$$

We now replace (k_n) with an appropriate subsequence, so that the last relation holds μ-a.s. then, substituting k_n for n in (5.5), and passing to the limit as $n \to \infty$, we get the result. □

Knowing the Itô differentials $d\sigma_t(f(x_t); \omega)$ and $d\sigma_t(1; \omega)$, by the Itô formula one can calculate directly the Itô differential of the process

$$\Phi_t(f(x_t); \omega) = \frac{\sigma_t(f(x_t); \omega)}{\sigma_t(1; \omega)}, \qquad 0 \le t \le T.$$

The result is the following.

COROLLARY 5.1. *Let all assumptions of Theorem* 5.1 *be met. Then the following equation holds* μ*-a.e. in* E:

$$\begin{aligned}\Phi_t(f(x_t); \omega) &= \mathbb{E}_\pi\{f(x_0)\} + \int_0^t \Phi_s(L_s f(x_\cdot); \omega)\, ds \\ &\quad + \int_0^t [\Phi_s(f(x_s)\phi_s; \omega) - \Phi_s(f(x_s); \omega)\Phi_s(\phi_s; \omega)] \\ &\quad \times [d\gamma_s(\omega) - \Phi_s(\phi_s; \omega)\nu(ds)].\end{aligned} \tag{5.6}$$

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DEPARTMENT OF MATHEMATICS
BOSTON UNIVERSITY
BOSTON, MASSACHUSETTS 02215