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MODERATELY LARGE DEVIATIONS AND EXPANSIONS OF LARGE DEVIATIONS FOR SOME FUNCTIONALS OF WEIGHTED EMPIRICAL PROCESSES[1]

BY TADEUSZ INGLOT AND TERESA LEDWINA

Technical University, Wrocław

Let α_n be the classical empirical process. Assume T, defined on $D[0,1]$, satisfies the Lipschitz condition with respect to a weighted sup-norm in $D[0,1]$. Explicit bounds for $P(T(\alpha_n) \geq x_n\sqrt{n})$ are obtained for every $n \geq n_0$ and all $x_n \in (0, \sigma]$, where n_0 and σ are also explicitly given. These bounds lead to moderately large deviations and expansions of the asymptotic large deviations for $T(\alpha_n)$. The present theory closely relates large and moderately large deviations to tails of the asymptotic distributions of considered statistics. It unifies and generalizes some earlier results. In particular, some results of Groeneboom and Shorack are easily derived.

1. Introduction and summary. Let $U_1, U_2, \dots$ be a sequence of independent uniform $(0, 1)$ random variables, and for each $n \geq 1$, let F_n denote the empirical distribution function based on $U_1, \dots, U_n$. Let

$$\alpha_n(t) = n^{1/2}(F_n(t) - t), \qquad 0 \leq t \leq 1,$$

denote the uniform empirical process.

This paper deals with some large and moderately large deviation results about some functionals T of α_n. So, some asymptotics of $P(T(\alpha_n) \geq t_n)$, $t_n \to \infty$, are studied. The case $t_n = O(n^{1/2})$ corresponds to large deviations and the case $t_n = o(n^{1/2})$ to moderately large deviations. Since, by the Komlós, Major and Tusnády (KMT) (1975) inequality, for every n, α_n can be well approximated by some Brownian bridge B and since probabilities $P(T(B) \geq t_n)$ are well investigated for a large class of T's, it is very useful to know under which conditions on T the difference between $P(T(\alpha_n) \geq t_n)$ and $P(T(B) \geq t_n)$ is negligible. Inglot and Ledwina (1990) have shown that if T is Lipschitz with respect to the uniform norm on $D[0,1]$ and for a positive constant a,

$$\log P(T(B) \geq y) = -\frac{a}{2}y^2(1 + o(1)) \quad \text{as } y \to \infty, \tag{1.1}$$

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then for $t_n = o(n^{1/2})$,

$$\lim_{n\to\infty} t_n^{-2} \log P(T(\alpha_n) \geq t_n) \text{ exists and equals}$$

$$\lim_{n\to\infty} t_n^{-2} \log P(T(B) \geq t_n) = -\frac{a}{2}. \tag{1.2}$$

So, this application of the KMT inequality gives the desired negligibility and in this way unifies and clarifies isolated earlier results. For more details and some examples see Inglot and Ledwina (1990). Further applications can be found in Koning (1991). Moreover, in Inglot and Ledwina (1990) it has been shown that using the Lipschitz property and KMT inequality in the case $t_n = O(n^{1/2})$ leads to different lower and upper bounds for $(1/n)\log P(T(\alpha_n) \geq t_n)$ (cf. also Theorem 4.1 and Proposition 5.1 of the present paper). So, the approach does not ensure that

$$I(x) = \lim_{n\to\infty} n^{-1} \log P\big(T(\alpha_n) \geq x\sqrt{n}\,\big) \text{ exists.} \tag{1.3}$$

Existence of $I(x)$ is guaranteed in a very general case by Sanov-type theorems proved by, among others, Groeneboom, Oosterhoff and Ruymgaart (1979) and Groeneboom and Shorack (1981). Moreover, Sanov-type theorems state that $I(x)$ is the infimum of the Kullback–Leibler information over an appropriate set of functions. However, for practical applications one needs to know a more explicit form of $I(x)$, at least for small x's. Usually, some expansions of $I(x)$ are sufficient. In general, to find such expansions is a nontrivial problem. For several special cases many different methods have been applied. For example, Nikitin (1980) and Groeneboom and Shorack (1981) related the minimization of the Kullback–Leibler information to associated differential equations. Kremer (1979, 1981) analysed some integral equations defining $I(x)$ for rank statistics. Ledwina (1987) used Hoeffding's result to provide simple solution of problems considered by Kremer. The first general result on expansion of $I(x)$ is due to Kallenberg and Ledwina (1987). Their approach was, however, restricted to asymptotically normal statistics. Recently, Jeurning and Kallenberg (1990) extended this approach to quadratic statistics. Inglot and Ledwina (1990) developed a new approach, showing that in the case $x \to 0$ for T satisfying (1.1), (1.3) and the Lipschitz condition, the KMT Brownian bridge approximation is again sufficient and results in the following:

$$\begin{aligned} \lim_{x\to 0+} x^{-2} I(x) &= \lim_{x\to 0+} x^{-2} \lim_{n\to\infty} n^{-1} \log P\big(T(B) \geq x\sqrt{n}\,\big) \\ &= \lim_{y\to\infty} y^{-2} \log P(T(B) \geq y). \end{aligned} \tag{1.4}$$

So, this application of the KMT inequality allows us to replace the analytical problem concerning the left-hand side of (1.4) by a probabilistic one concerning the right-hand side of (1.4), whose solution is already known in many cases.

In the present paper we extend (1.2) and (1.4) to the case of T satisfying the following weighted Lipschitz condition:

$$(1.5)\quad |T(x) - T(y)| \leq \sup_{0\leq t\leq 1} \{|x(t) - y(t)|/w(t)\}, \quad \text{for all } x, y \in D[0,1],$$

where w satisfies some mild conditions precisely stated in Section 2. For example, the Anderson–Darling statistic defined by

$$T_{\mathrm{AD}}(x) = \left\{\int_0^1 \frac{x^2(t)}{t(1-t)}\,dt\right\}^{1/2}$$

and the statistic $T_{\mathrm{GS}}(\alpha_n)$, where

$$T_{\mathrm{GS}}(x) = \sup_{0\leq t\leq 1}\left|x(t)\log\frac{1}{t(1-t)}\right|,$$

introduced and investigated by Groeneboom and Shorack (1981), satisfy our assumptions. Counterparts of (1.2) and (1.4) under (1.5), that is, Propositions 2.1–2.5, are formulated in Section 2. Similarly as in Inglot and Ledwina (1990), these results are derived from some explicit lower and upper bounds for $P(T(\alpha_n) \geq t_n)$ given in Theorem 4.1 of Section 4. To derive the bounds, in addition to exploiting (1.5) and the KMT inequality, some delicate analysis of $\sup_{0\leq t\leq b}\{\alpha_n(t)/w(t)\}$ for b close to 0 is necessary. To this end a useful inequality is stated and proved in Section 3. This section also contains a version of a known result on the behaviour of $\sup_{0\leq t\leq b}\{B(t)/w(t)\}$ that we need to get the bounds. Section 5 contains proofs of Propositions 2.1–2.5.

2. Expansions of large deviations and moderately large deviations. In this section we shall collect and comment on some counterparts of (1.2) and (1.4) for T satisfying the weighted Lipschitz condition (1.5).

We begin with a precise description and some discussion of the class of weights appearing in (1.5). Set $w = h(t(1-t))$, $t \in [0,1]$, and assume h satisfies the following conditions:

CONDITION 1. $h(0) = 0$, h is continuous and increasing on $[0, 1/4]$.

CONDITION 2. $t/h^2(t)$ is nondecreasing on $(0, \nu)$, $0 \leq \nu \leq 1/4$.

CONDITION 3. $\int_0^{1/4} dt/h^2(t) = H < \infty$.

CONDITION 4. There exists a constant A_0, $A_0 \geq 1$, such that for every $A \geq A_0$ there exist $\varepsilon = \varepsilon(A)$ and η, η not depending on A, $0 \leq \varepsilon < \eta \leq 1/4$, such that the function $h(t)\log\{At/h(t)\}$ is nonincreasing on $(0, \varepsilon]$ and nondecreasing on (ε, η).

REMARK 2.1. Conditions (1) and (2) are standard assumptions to get exponential bounds for the supremum of a weighted Brownian bridge. We shall use them together with the unrestrictive Condition 3 to get a variant of the Itô–McKean inequality (see Proposition 3.1). Condition 4 is crucial to get useful bounds for the supremum of weighted empirical processes (Proposition 3.2). The essence of Condition 4 consists in the following property: There exists η, $\eta \in (0, 1/4]$, such that for large A the function $h(t)\{\log At/h(t)\}$ is u-shaped for $t \in (0, \eta]$. After the present paper was submitted for publication, Einmahl kindly informed us that he imposed a similar assumption to get a LIL for weighted tail empirical processes [cf. Einmahl (1992)].

REMARK 2.2. The functions $h_\delta(t) = \log^{-\delta}(1/t)$, $\delta \ge 1$, $h_\gamma(t) = t^\gamma$, $\gamma < 1/2$, $h_k(t) = 1/\{\log(1/t)\}\{l_k(1/t)\}$, where $l_k(x)$ is defined inductively as $l_1(x) = \log(\max(x, e))$ and $l_k(x) = \log(\max(e, l_{k-1}(x))$, satisfy Conditions 1 to 4. Some elementary calculations connected with checking Condition 4 for h_δ and h_k are contained in Section 5.

REMARK 2.3. Both T_{AD} and T_{GS} satisfy (1.5) with $w(t) = ch_1(t(1-t))$, where $h_1(t) = \log^{-1}(1/t)$ and c is suitably chosen. The function h_1 satisfies Conditions 1–4 with $\varepsilon = 0$, $A_0 = e$, $\eta = 1/4$ and $\nu = e^{-2}$. Note that h_1 plays a fundamental role in large deviations theory for weighted Kolmogorov–Smirnov statistics. As shown by Groeneboom and Shorack (1981), the heaviest weight $w(t) = h(t(1-t))$ under which $\sup_{0 \le t \le 1}\{|\alpha_n(t)|/w(t)\}$ has nonnull large deviations probabilities has to satisfy

$$\liminf_{t \to 0+} h(t)\log(1/t) = \theta \in (0, +\infty]. \tag{2.1}$$

The function h_1 is also a key function in our considerations (cf. Remark 5.1).

Throughout this section we shall assume that T satisfies (1.1), that is, for a Brownian bridge B and some positive constant a,

$$\log P(T(B) \ge y) = -\frac{a}{2}y^2(1 + o(1)), \quad \text{as } y \to \infty.$$

REMARK 2.4. Note that (1.1) is not a restrictive assumption. In the statistical literature, since Bahadur (1960), it is one of the standard assumptions on $T(\alpha_n)$. In particular, (1.1) holds for T_{AD} with $a = 2$ [cf. Gregory (1980), pages 119 and 127] and for T_{GS} with $a = \{\sup_{0 \le t \le 1} \mathrm{Var}(B(t) \cdot \log(1/t(1-t))\}^{-1} = e^2/4$ [see Marcus and Shepp (1972)]. A general result of Borell [(1975), page 214] is also applicable for both statistics. The result specific to our situation is as follows. Suppose T: $D[0,1] \to (-\infty, \infty]$ is sublinear, that is, $T(x + y) \le T(x) + T(y)$, $T(cx) = cT(x)$ for all $c \ge 0$ and $x \in D[0,1]$. Moreover, assume T is finite B-a.s. Set $\|T\|_B = \sup_{x \in O_k} T(x)$, where O_k is the unit ball in the reproducing kernel Hilbert space of B. Then (1.1) holds with $a = \|T\|_B^{-2}$. For related general results see also Kallianpur and Oodaira (1978).

The first two results presented in this section will concern a functional T satisfying (1.5), with h obeying (2.1).

PROPOSITION 2.1. *Assume T satisfies* (1.1) *and* (1.5) *with h fulfilling Conditions* 1–4 *and* (2.1). *Then*

$$\lim_{x\to 0+} x^{-2} \liminf_{n\to\infty} n^{-1} \log P\big(T(\alpha_n) \geq x\sqrt{n}\big)$$
$$= \lim_{x\to 0+} x^{-2} \limsup_{n\to\infty} n^{-1} \log P\big(T(\alpha_n) \geq x\sqrt{n}\big) = -\frac{a}{2},$$

where a is defined by (1.1). *In particular, if* (1.3) *holds then*

$$\lim_{x\to 0+} x^{-2} I(x) = -\frac{a}{2}. \tag{2.2}$$

REMARK 2.5. Groeneboom and Shorack (1981) got (2.2) for T_{AD} and T_{GS} by rather complicated analytical considerations on the infima of the Kullback–Leibler information over some special sets. Earlier Nikitin (1980) had derived (2.2) for T_{AD} via solving a related variational problem. Recently, Jeurnink and Kallenberg (1990) proposed an alternative way to obtain (2.2) for T_{AD}.

PROPOSITION 2.2. *Assume T satisfies* (1.1) *and* (1.5) *with h obeying Conditions* 1–4 *and* (2.1). *Then, for arbitrary $x_n \to 0$ such that $nx_n^2 \to \infty$, we have*

$$\lim_{n\to\infty} (1/nx_n^2) \log P\big(T(\alpha_n) \geq x_n\sqrt{n}\big) = -\frac{a}{2}. \tag{2.3}$$

It is intuitively clear that even if (2.1) does not hold, some moderately large deviations can exist. Propositions 2.3, 2.4 and 2.5 describe important functions h and corresponding sequences of x_n's for which (2.3) still holds.

PROPOSITION 2.3. *For T satisfying* (1.1) *and* (1.5) *with* $h_\delta(t) = \log^{-\delta}(1/t)$, $\delta > 1$, (2.3) *holds for* $x_n = O(n^{-\rho})$ *with* $\rho > (\delta - 1)/(2\delta - 1)$.

PROPOSITION 2.4. *For T satisfying* (1.1) *and* (1.5) *with* $h_k(t) = (\log(1/t) l_k(1/t))^{-1}$, $k \geq 2$, (2.3) *holds for* $x_n = O((l_{k-1}(n))^{-\rho})$ *with an arbitrary* $\rho > 2$.

PROPOSITION 2.5. *For T satisfying* (1.1) *and* (1.5) *with* $h_\gamma(t) = t^\gamma$, $\gamma < 1/2$, (2.3) *holds for* $x_n \leq (An^{-1}\log n)^{1/2}$, *where* $A < (1 - 2\gamma)^2/\gamma(1 - \gamma)a$. *In particular,* (2.3) *holds for* $x_n = o((n^{-1}\log n)^{1/2})$.

Proofs of Propositions 2.1–2.5 are given in Section 5.

REMARK 2.6. As we have said before, Propositions 2.1–2.5 result from our basic bounds given in Theorem 4.1. Observe that Proposition 2.4 exemplifies the accuracy of our bounds, in the sense that for weights almost equal to $h_1(t) = \log^{-1}(1/t)$ the moderately large deviations we get hold for almost constant sequences $\{x_n\}$. Note also that some further moderately large deviations can be derived from Theorem 4.1 (cf., e.g., Proposition 5.2).

REMARK 2.7. Results like those given in Propositions 2.1–2.5 are important in mathematical statistics for comparing tests and estimators. Some recent applications of Proposition 2.3 to bound remainders in a decomposition of some L- and U-statistics can be found in Inglot, Kallenberg and Ledwina (1992) and Inglot and Ledwina (1993), respectively.

3. Inequalities for the weighted Brownian bridge and empirical processes. First we state a version of a known result on the weighted Brownian bridge B.

PROPOSITION 3.1. *Suppose h satisfies Conditions 1–4 and $w(t) = h(t(1-t))$. Then*

$$P\left(\sup_{0\le t\le b}\frac{|B(t)|}{w(t)} \ge \lambda\right) \le \frac{2H}{\lambda^2\sqrt{\pi}}\exp\left\{\frac{-\lambda^2h^2(b)}{3b}\right\} \tag{3.1}$$

provided that $0 < b \le \min(\nu, 1/8)$ and $\lambda h(b) \ge 6\sqrt{b}$.

PROOF OF PROPOSITION 3.1. By the formula (1.28) of Shorack and Wellner (1982), for example, one easily gets

$$P\left(\sup_{0\le t\le b}\frac{|B(t)|}{w(t)} \ge \lambda\right) \le \frac{2}{\sqrt{\pi}}\int_0^b \frac{1}{\lambda^2h^2(u)}\left(\frac{\lambda h(u)}{\sqrt{u}}\right)^3 e^{-(\lambda^2h^2(u)/2u)}\,du.$$

By Condition 2 the function $\lambda h(u)/\sqrt{u}$ is nonincreasing on $(0,\nu)$. The function $f(x) = x^3\exp(-x^2/2)$ decreases for $x > \sqrt{3}$ and satisfies $f(x) \le \exp(-x^2/3)$ for $x \ge 6$. Thus, by the choice of b and λ and Condition 3, the inequality (3.1) follows. □

The next inequality links the property of h described via Condition 4 (see Remark 2.1, also) with the behaviour of the distribution of the weighted empirical process near zero.

PROPOSITION 3.2. *Suppose h obeys Conditions 1, 2 and 4, and $w(t) = h(t(1-t))$. Define $b_0 = 2\nu/[1 + (1-4\nu)^{1/2}]$. Then for arbitrary $n \ge 1$, $1 \le m \le n$, $0 < b \le b_0$, and positive z such that $\max\{2b/w(b), 2m/nh(\eta)\} \le z \le 2e/(1-b)A_0$,*

$$\begin{aligned} &P\left(\sup_{0\le t\le b}\frac{|F_n(t)-t|}{w(t)} \ge z\right) \\ &\quad\le m\frac{enh^{-1}(2/nz)}{1-b} + m\left(\frac{enh^{-1}(2m/nz)}{m(1-b)}\right)^m + \left(\frac{ebn}{m}\right)^m. \end{aligned} \tag{3.2}$$

PROOF. By Condition 2 and the choice of z we have $t/w(t) \le z/2$ provided that $0 \le t \le b$. Hence

$$P\left(\sup_{0\le t\le b} \frac{|F_n(t)-t|}{w(t)} \ge z\right) \le P\left(\sup_{0\le t\le b} \frac{F_n(t)}{w(t)} \ge \frac{z}{2}\right). \tag{3.3}$$

Let now $U_{(1)} \le \cdots \le U_{(n)}$ denote the order statistics of uniform $(0,1)$ rv's $U_1,\dots,U_n$. Define also $U_{(n+1)} \equiv 1$. The probability (3.3) can be majorized as follows:

$$\begin{aligned}
P\left(\sup_{0\le t\le b} \frac{F_n(t)}{w(t)} \ge \frac{z}{2}\right) &\le \sum_{k=1}^{n} P\left(\max_{1\le i\le k} \frac{i}{nw(U_{(i)})} \ge \frac{z}{2}, U_{(k)} \le b < U_{(k+1)}\right)\\
&\le \sum_{k=1}^{m}\sum_{i=1}^{k} P\left(\frac{i}{nh\big(U_{(i)}(1-U_{(i)})\big)} \ge \frac{z}{2}, U_{(k)} \le b < U_{(k+1)}\right)\\
&\quad + \sum_{k=m}^{n} P\big(U_{(k)} \le b < U_{(k+1)}\big)\\
&\le \sum_{i=1}^{m}\sum_{k=i}^{m} P\left(\frac{i}{nh\big((1-b)U_{(i)}\big)} \ge \frac{z}{2}, U_{(k)} \le b \le U_{(k+1)}\right) \qquad (3.4)\\
&\quad + P\big(U_{(m)} \le b\big)\\
&= \sum_{i=1}^{m} P\left(h\big((1-b)U_{(i)}\big) \le \frac{2i}{nz}, U_{(i)} \le b < U_{(m)}\right)\\
&\quad + P\big(U_{(m)} \le b\big)\\
&\le \sum_{i=1}^{m} P\left(U_{(i)} \le \frac{h^{-1}(2i/nz)}{1-b}\right) + P\big(U_{(m)} \le b\big).
\end{aligned}$$

Since $P(U_{(i)} \le u) \le \binom{n}{i}u^i$, $1 \le i \le n$, $u \in (0,1)$, and $k! \ge k^k e^{-k}$, $k \ge 1$, then (3.4) is majorized by

$$\begin{aligned}
&\sum_{i=1}^{m}\left(\frac{neh^{-1}(2i/nz)}{i(1-b)}\right)^i + \left(\frac{ebn}{m}\right)^m\\
&\quad = \sum_{i=1}^{m} \exp\left\{\frac{nz}{2}\cdot\frac{2i}{nz}\cdot\log\left(\frac{h^{-1}(2i/nz)}{2i/nz}\cdot\frac{2e}{(1-b)z}\right)\right\} + \left(\frac{ebn}{m}\right)^m.
\end{aligned} \tag{3.5}$$

Defining now t_i via $h(t_i) = 2i/nz$, by Conditions 1 and 4 and the assumption $2m/nz \le h(\eta)$, we get $0 < t_1 \le \cdots \le t_m \le \eta$. Note that the exponent in the first term of (3.5) has the form

$$\frac{nz}{2}h(t_i)\log\left(\frac{At_i}{h(t_i)}\right), \qquad A = \frac{2e}{(1-b)z}.$$

By the assumption on z, $A \ge A_0$ and by Condition 4 the sum in (3.5) can be majorized by the first component plus the last component times m. This leads to (3.2). □

REMARK 3.1. The estimate (3.5) can be written in an integral form. For this purpose put $m = [nd]$, $i = [nx]$, where $[v]$ stands for the integral part of v. Then, under Conditions 1, 2 and 4, we get

$$P\left(\sup_{0\le t\le b}\frac{|F_n(t)-t|}{w(t)}\ge z\right)\le n\int_0^d\left(\frac{eh^{-1}(2x/z)}{(1-b)x}\right)^{nx}dx+\left(\frac{eb}{d-2/n}\right)^{[nd]-1}$$

for all $n\ge 1$, $0<b\le b_0$, $2/n<d\le 1$ and z satisfying

$$\max\{2b/w(b),2d/h(\eta)\}\le z\le 2e/(1-b)A_0.$$

4. Explicit bounds for $P(T(\alpha_n)\ge t_n)$. Let $\mathscr{T}_h$ denote the class of mappings T: $D[0,1]\to(-\infty,\infty]$ such that (1.5) holds with h satisfying Conditions 1–4 and such that $P(T(B)>y)>0$ for all $y>0$. Obviously, we have $\mathscr{T}_h\subset\mathscr{T}_{h^*}$ provided that $h^*<h$. For $T\in\mathscr{T}_h$ we shall state in this section explicit bounds for $P(T(\alpha_n)\ge x_n\sqrt{n})$ valid for all $x_n\in(0,\sigma]$ and $n\ge n_0$, where σ and n_0 are also explicitly given. To formulate the bounds some further notation is introduced.

For $T\in\mathscr{T}_h$ and a positive a define the function $g(y)$ via

$$\log P(T(B)\ge y)=-(a/2)y^2(1+g(y)),\qquad y\in R. \tag{4.1}$$

Since $P(T(B)>y)>0$ for all y, such a function g always exists. We do not require that (1.1) hold, that is, $g(y)\to 0$ as $y\to\infty$. However, to derive from Theorem 4.1 some asymptotic results like Propositions 2.1–2.5, the condition (1.1) obviously has to be imposed, in addition. Note also that using some more detailed information on g, the inequalities we shall state can be applied also to get some explicit bounds for the exact distribution of $T(\alpha_n)$. For an illustration see Section 3 of Inglot and Ledwina (1990).

For h satisfying Conditions 1–4 and arbitrary $D>0$ define

$$\mathscr{H}=\sup\{t\le\nu: t/h(t)\le 2e/DA_0\}$$

and set

$$x_0=\min\{(4e)^{-1},\eta,\mathscr{H}\}.$$

Finally, recall that Komlós, Major and Tusnády (1975) have constructed a sequence of independent uniform $(0,1)$ random variables $U_1,U_2,\dots$ and a sequence of Brownian bridges $B_1,B_2,\dots$ sitting on the same probability space such that for universal positive constants C, L and l,

$$P\left(\sup_{0\le t\le 1}n^{1/2}|\alpha_n(t)-B_n(t)|>C\log n+x\right)\le Le^{-lx},$$

for all $-\infty<x<\infty$ and all n. Recently Bretagnolle and Massart (1989) have shown that one can take $C=12$, $L=2$ and $l=1/6$.

THEOREM 4.1. *Let $T \in \mathscr{T}_h$ and set $K = a/l$. Then for arbitrary $1 < p \le 2$, $n \ge 1$ and $x_n > 0$ such that $nx_n^p \ge \max\{1, (2eK)^{-1}\}$ and $Kx_n^p \le x_0$,*

$$P\big(T(\alpha_n) \ge x_n\sqrt{n}\big) \le (1 + R_n)\exp\left\{-\frac{a}{2}nx_n^2(1 - s_n)^2(1 + r_n^-)\right\} \tag{4.2}$$

and

$$P\big(T(\alpha_n) \ge x_n\sqrt{n}\big) \ge (1 + L_n)\exp\left\{-\frac{a}{2}nx_n^2(1 + s_n)^2(1 + r_n^+)\right\}, \tag{4.3}$$

where

$$s_n = KDx_n^{p-1}/h\big(Kx_n^p(1 - Kx_n^p)\big), \qquad D \ge 16e,$$
$$r_n^- = g\big(x_n\sqrt{n}(1 - s_n)\big), \qquad r_n^+ = g\big(x_n\sqrt{n}(1 + s_n)\big),$$

while R_n and L_n are explicitly given by (4.14), (4.15) *and* (4.13).

PROOF. Define $z_n = x_n s_n$ and take B_n and the version of α_n defined in the KMT (1975) inequality. Then (1.5) yields

$$\begin{aligned} P\big(T(\alpha_n) \ge x_n\sqrt{n}\big) & \\ \le P\big(T(B_n) \ge x_n(1 - s_n)\sqrt{n}\big) & \\ + P\left(\sup_{0\le t\le 1}\frac{|\alpha_n(t) - B_n(t)|}{w(t)} \ge z_n\sqrt{n}\right) & \end{aligned} \tag{4.4}$$

and

$$\begin{aligned} P\big(T(\alpha_n) \ge x_n\sqrt{n}\big) & \\ \ge P\big(T(B_n) \ge x_n(1 + s_n)\sqrt{n}\big) & \\ - P\left(\sup_{0\le t\le 1}\frac{|\alpha_n(t) - B_n(t)|}{w(t)} \ge z_n\sqrt{n}\right). & \end{aligned} \tag{4.5}$$

The first components of (4.4) and (4.5) shall later on be written in exponential form resulting from (4.1). The rest of the proof consists of sufficiently precise estimation of the second term in (4.4) and (4.5). To get such an estimate, first of all observe that for arbitrary $0 < b_n \le 1/2$,

$$P\left(\sup_{0\le t\le 1}\frac{|\alpha_n(t) - B_n(t)|}{w(t)} \ge z_n\sqrt{n}\right)$$

$$\le 2P\left(\sup_{0\le t\le b_n}\frac{|B_n(t)|}{w(t)} \ge \frac{z_n\sqrt{n}}{2}\right) \tag{4.6}$$

$$+ 2P\left(\sup_{0\le t\le b_n}\frac{|\alpha_n(t)|}{w(t)} \ge \frac{z_n\sqrt{n}}{2}\right) \tag{4.7}$$

$$+ P\left(\sup_{0\le t\le 1}|\alpha_n(t) - B_n(t)| \ge w(b_n)z_n\sqrt{n}\right). \tag{4.8}$$

We shall estimate (4.6) and (4.7) by means of Propositions 3.1 and 3.2, respectively, and majorize (4.8) by using the KMT inequality. The involved parameters s_n, b_n and m shall be chosen in a way ensuring that the resulting estimates, for a wide class of h's, behave like $\Psi_i \exp\{-\psi_i n x_n^{\varphi}\}$, $1 < \varphi \le 2$, for large n and some positive constants Ψ_i, ψ_i, $i = 1,2,3$ [cf. (4.9), (4.11), (4.12) and Remark 4.1].

Set $b_n = Kx_n^p$. Since $Kx_n^p \le x_0 \le 1/4e$ then $b_n \le \min(\nu, 1/8)$. Besides, by $nx_n^p \ge 1/2eK$ and $D \ge 16e$ it easily follows that $\lambda = z_n\sqrt{n}/2 \ge 6\sqrt{b_n}/h(b_n)$. So, Proposition 3.1 yields the following bound for (4.6):

$$\frac{8H}{nz_n^2\sqrt{\pi}} \exp\left\{-\frac{nz_n^2h^2(b_n)}{12b_n}\right\} \le \frac{8H}{D^2\sqrt{\pi}} \exp\left\{-\frac{D^2Knx_n^p}{12} + 2\log\frac{h(Kx_n^p)}{K\sqrt{n}\,x_n^p}\right\},$$

where the last inequality follows by the definitions of z_n, b_n and the monotonicity of h. By $Kx_n^p \le 1/4e$, $nx_n^p \ge 1/2eK$ and Condition 1, one finally gets

$$P\left(\sup_{0\le t\le b_n} \frac{|B_n(t)|}{w(t)} \ge \frac{z_n\sqrt{n}}{2}\right) \le C_1 \exp\left\{-\frac{KD^2nx_n^p}{12} + \log n\right\}, \tag{4.9}$$

where $C_1 = 32He^2h^2(1/4e)/D^2\sqrt{\pi}$.

To majorize (4.7), Proposition 3.2 will be applied. To this end put $m = [2eKnx_n^p + 1]$. Observe that by Condition 1 and $Kx_n^p \le \mathscr{H}$ one has $z_n(1-b_n) \le 2e/A_0$. By $D \ge 16e$ one gets $2b_n/w(b_n) \le z_n/2$. Finally, by Condition 1, $Kx_n^p \le \eta$, $nx_n^p \ge 1/2eK$, $D \ge 16e$ and by the choice of m, we have that $4m/nz_n \le h(\eta)$. Hence

$$\begin{aligned} &P\left(\sup_{0\le t\le b_n} \frac{|\alpha_n(t)|}{w(t)} \ge \frac{z_n\sqrt{n}}{2}\right) \\ &\quad\le m\frac{enh^{-1}(4/nz_n)}{1-b_n} + m\left(\frac{enh^{-1}(4m/nz_n)}{m(1-b_n)}\right)^m + \left(\frac{enb_n}{m}\right)^m. \end{aligned} \tag{4.10}$$

By $nx_n^p \ge 1/2eK$ one has $m \le 4eKnx_n^p$. Observe that the choice of m implies that the second term in (4.10) simplifies considerably and behaves similarly to the third component of (4.10). Besides, for such m the first term of (4.10) will usually dominate. Using Condition 1 and the assumption $D \ge 16e$, the right-hand side of (4.10) can be majorized by

$$5e^2Kn^2x_n^ph^{-1}\left(\frac{4h(Kx_n^p)}{DKnx_n^p}\right) + (4eKnx_n^p + 1)2^{-2eKnx_n^p}.$$

By $nx_n^p \ge 1$ one finally gets

$$\begin{aligned} &P\left(\sup_{0\le t\le b_n} \frac{|\alpha_n(t)|}{w(t)} \ge \frac{z_n\sqrt{n}}{2}\right) \\ &\quad\le C_2n^2x_n^ph^{-1}\left(\frac{4h(Kx_n^p)}{DKnx_n^p}\right) + C_3nx_n^p\exp\{-2eK(\log 2)nx_n^p\}, \end{aligned} \tag{4.11}$$

where $C_2 = 5e^2K$ while $C_3 = 4eK + 1$.

By KMT (1975) the component (4.8) is majorized as follows:

$$P\Big(\sup_{0\le t\le 1}|\alpha_n(t)-B_n(t)|\ge w(b_n)z_n\sqrt{n}\Big) \le L\exp\{-lDKnx_n^p+lC\log n\}. \tag{4.12}$$

Put

$$M=\min\{e\log 4, Dl\},\qquad \mu=\max\{lC,1\}.$$

By $Kx_n^p\le 1/4e$ and $D\ge 16e$, (4.9), (4.11) and (4.12) yield

$$\begin{aligned}&P\left(\sup_{0\le t\le 1}\frac{|\alpha_n(t)-B_n(t)|}{w(t)}\ge z_n\sqrt{n}\right)\\ &\quad\le S_n=\left(2C_1+\frac{C_3}{4Ke+L}\right)\exp\{-KMnx_n^p+\mu\log n\}\\ &\qquad+2C_2n^2x_n^ph^{-1}\left(\frac{4h(Kx_n^p)}{DKnx_n^p}\right).\end{aligned}\tag{4.13}$$

Hence (4.2) and (4.3) follow with

$$R_n=S_n\exp\big\{(a/2)nx_n^2(1-s_n)^2(1+r_n^-)\big\} \tag{4.14}$$

and

$$L_n=S_n\exp\big\{(a/2)nx_n^2(1+s_n)^2(1+r_n^+)\big\}. \tag{4.15}$$

This concludes the proof of Theorem 4.1. □

REMARK 4.1. If $x_n\to 0$ and $nx_n^2\to\infty$, then for $p\in(1,2)$, $(nx_n^p)^{-1}\log n\to 0$ as $n\to\infty$. Moreover, if (2.1) is satisfied, then

$$h^{-1}\left(\frac{h(Kx_n^p)}{nx_n^p}\right)\le\Psi_4\exp\{-\psi_4nx_n^\varphi\} \tag{4.16}$$

holds with $\varphi=p$ and some positive constants Ψ_4, ψ_4. Hence $S_n\le\Psi\exp\{-\psi nx_n^p\}$ for some positive Ψ, ψ. Moreover, Propositions 2.3–2.5 provide examples of h's defining heavy weights [(2.1) does not hold] and related sequences $\{x_n\}$ for which (4.16) still holds, at least with $\varphi=2$ and ψ_4 sufficiently large to ensure that R_n and L_n tend to 0 (cf. Section 5).

5. Proofs of Propositions 2.1–2.5. First we shall state a little more general result than Proposition 2.1.

Setting in Theorem 4.1 $p=2$, $x_n=x$ and defining for arbitrary $0<\beta<1$

$$x^*=x^*(\beta)=\sup\{x\in(0,1/4]:1-s_n>\beta\}$$

and

$$x^{**}=\begin{cases}\{h^{-1}(\theta D/8l)/K\}^{1/2}, & \text{if } \theta D/8l\le h(K/16),\\ 1/4, & \text{otherwise},\end{cases}$$

one easily gets the following proposition.

PROPOSITION 5.1. *Assume $T \in \mathscr{T}_h$ with h satisfying* (2.1) *and suppose* (1.1) *holds. Then for every* $0 < x < \min\{x^*, x^{**}, (x_0/K)^{1/2}\}$,

$$\begin{aligned} -\frac{a}{2}x^2\left\{1 + \frac{DKx}{h(Kx^2(1-Kx^2))}\right\}^2 &\le \liminf_{n\to\infty} n^{-1}\log P\big(T(\alpha_n) \ge x\sqrt{n}\big) \\ &\le \limsup_{n\to\infty} n^{-1}\log P\big(T(\alpha_n) \ge x\sqrt{n}\big) \qquad (5.1)\\ &\le -\frac{a}{2}x^2\left\{1 - \frac{DKx}{h(Kx^2(1-Kx^2))}\right\}^2. \end{aligned}$$

PROOF OF PROPOSITION 2.1. Proposition 2.1 is an immediate consequence of (5.1). □

REMARK 5.1. Observe that every continuous function h satisfying (2.1) satisfies $h(t) \ge c_1 h_1(t)$ for some positive c_1 and all $t \in (0, 1/4]$. So, omitting the assumption $T \in \mathscr{T}_h$, one can get (5.1) with h replaced by $c_1 h_1$ and suitably changed x_0, x^* and x^{**}. Hence, obviously Proposition 2.1 follows, also. However, in this way one loses some precision in (5.1).

PROOF OF PROPOSITION 2.2. By Remark 5.1, it suffices to carry out the calculations for functionals T satisfying (1.5) with w defined via h_1 only. Taking arbitrary $1 < p < 2$, an elementary analysis of (4.2) and (4.3) yields Proposition 2.2 (cf. also Remark 4.1). □

When deriving asymptotic results under $x_n \to 0$ from Theorem 4.1, the main point is to prove that L_n and R_n tend to 0 as $n \to \infty$. The following proposition gives sufficient conditions for the convergence. In particular, proofs of Propositions 2.3 and 2.4 will be based on this proposition.

PROPOSITION 5.2. *Assume $T \in \mathscr{T}_h$ and* $\liminf_{t\to 0+} h(t)\log(1/t) = 0$. *Moreover, suppose*

$$p_0 = \inf\left\{p \in (1,2] : \lim_{t\to 0+} t^{p-1}/h(t^p) = 0\right\} < 2. \qquad (5.2)$$

Then, for every $x_n \to 0$ such that $nx_n^2 \to \infty$ and

$$\limsup_{n\to\infty}\big(\log h^{-1}(4h(Kx_n^p)/DKnx_n^p)\big)/nx_n^q < 0 \qquad (5.3)$$

for some p and q satisfying $p_0 < p < q < 2$, equality (2.3) *holds.*

PROOF. Take $p \in (p_0, 2)$. Then (5.2) implies $t^{p-1}/h(t^p) \to 0$, as $t \to 0+$. So, $s_n \to 0$ and consequently $r_n^+ \to 0$ and $r_n^- \to 0$. Thus, to get (2.3) from (4.2) and (4.3) it suffices to show $L_n \to 0$ and $R_n \to 0$. By the definition of S_n, R_n

is of the form $c_2 e^{w_1} + c_3 e^{w_2}$, where c_2 and c_3 are some constants while

$$w_1 = -nx_n^p \left\{ KM - \frac{a}{2} x_n^{2-p} (1-s_n)^2 (1+r_n^-) - \frac{(\mu \log n)}{(nx_n^2)^{p/2} n^{1-p/2}} \right\},$$

$$w_2 = nx_n^q \left\{ \frac{a}{2} x_n^{2-q} (1-s_n)^2 (1+r_n^-) + (nx_n^q)^{-1} \log h^{-1}\left(\frac{4h(Kx_n^p)}{DKnx_n^p} \right) + \left(\frac{p}{qnx_n^q} \right) \log nx_n^q + \frac{(2q-p)(\log n)}{qnx_n^q} \right\}.$$

Since $x_n \to 0$, $nx_n^2 \to \infty$ and $p < 2$, then $w_1 \to -\infty$. By (5.3) and $q < 2$, $w_2 \to -\infty$ also. L_n is completely analogous to R_n. □

Proof of Proposition 2.3. First we shall check Condition 4 for h_δ. Put $G_\delta(t) = h_\delta(t) \log\{At/h_\delta(t)\}$, where $h_\delta(t) = \log^{-\delta}(1/t)$, $\delta \geq 1$. Setting $u = \log(1/t)$ it is enough to investigate the function $f(u) = (\log A - u + \delta \log u)/u^\delta$, $u > 0$. We have $f'(u) = \delta^2[(\delta - 1)\delta^{-2} u - \delta^{-1} \log(A/e) - \log u]/u^{\delta+1}$. Consider $\delta > 1$ and set $A_0 = e$. For every $A \geq A_0$ define $u_0 = u_0(A, \delta)$ to be the largest root of the equation $(\delta - 1)\delta^{-2} u - \delta^{-1}\log(A/e) = \log u$. So, on $(0, e^{-u_0})$ the function $G_\delta(t)$ is decreasing. Moreover, for $A \geq e$, $f'(u) < 0$ on $(1, u_0)$. Thus, one can take $\eta = e^{-1}$, since $G_\delta(t)$ is increasing on (e^{-u_0}, e^{-1}). For $\delta = 1$ one gets $A_0 = e$, $\varepsilon = 0$, $\eta = 1/4$.

Now, it is enough to check (5.3) of Proposition 5.2. By $\log h_\delta^{-1}(u) = -u^{-1/\delta}$ one has

$$\begin{aligned}(\log h_\delta^{-1}(4h_\delta(Kx_n^p)/DKnx_n^p))/nx_n^q \\ = -\left(K^{q/p}(D/4)^{1/\delta} \log(1/b_n)\right)/\left(b_n n^{p(\delta-1)/(q\delta-p)}\right)^{(q\delta-p)/p\delta},\end{aligned}$$

where $b_n = Kx_n^p$. As $p_0 = 1$, one can choose p close to 1 and q close to 2 such that $\rho = (\delta - 1)/(q\delta - p)$. Then $b_n n^{p(\delta-1)/(q\delta-p)}$ is bounded from above and (5.3) follows. □

Proof of Proposition 2.4. We begin with checking Condition 4 for h_k. Define

$$G_k(t) = (\log A - \log(1/t) + l_2(1/t) + l_{k+1}(1/t))/(\log(1/t)) l_k(1/t),$$

set $u = \log(1/t)$ and denote $f(u) = G_k(e^{-u})$. Then $f'(u) = N_k(u)/u^2 l_{k-1}^2(u)$, where

$$\begin{aligned}N_k(u) = &\{u - \log A(1 + (l_{k-1}(u)) \cdot \cdots \cdot \log u)\}(l_{k-2}(u) \cdot \cdots \cdot \log u)^{-1} \\ &+ l_{k-1}(u) + ((l_{k-2}(u)) \cdot \cdots \cdot \log u)^{-1} - (\log u) l_{k-1}(u) \\ &-(l_k(u)) l_{k-1}(u) - ((l_{k-2}(u)) \cdot \cdots \cdot l_2(u))^{-1} \\ &-(l_k(u))((l_{k-2}(u)) \cdot \cdots \cdot \log u)^{-1},\end{aligned}$$

where $l_0(u) \equiv 1$. For large u the first term of $N_k(u)$ definitely dominates and for $A \geq A_0 = e$ there exists $u_0 = u_0(A, k)$ such that $f'(u) > 0$ for $u > u_0$. On the other hand, there exists u_1, not depending on A, such that $f'(u) < 0$ for $u \in (u_1, u_0)$. Hence Condition 4 follows.

Now, to check (5.3) define $G(u) = 1/ul_{k-1}(u)$. Thus $h_k(t) = G(\log(1/t))$ and hence $\log h_k^{-1}(u) = -G^{-1}(u)$. So

$$(5.4)\quad \begin{aligned}(nx_n^q)^{-1}\log h_k^{-1}\left(\frac{4h_k(Kx_n^p)}{DKnx_n^p}\right) &= -\frac{G^{-1}(4h_k(Kx_n^p)/DKnx_n^p)}{G^{-1}(G(nx_n^q))}\\ &= -\frac{G^{-1}\left(4\{DKnx_n^p\log(1/Kx_n^p)l_k(1/Kx_n^p)\}^{-1}\right)}{G^{-1}\left(\{nx_n^q l_{k-1}(nx_n^q)\}^{-1}\right)}.\end{aligned}$$

Now observe that G^{-1} has the following property. If $a_n \to 0$ and $a_n^* \to 0$ in such a way that $a_n/a_n^* < M/l_{k-1}(1/Ma_n^*)$ for some M, then $G^{-1}(a_n)/G^{-1}(a_n^*) > M^{-1}$ for n sufficiently large. Indeed, this statement easily follows by $G(u) \le u^{-1}$, $G^{-1}(u) \le u^{-1}$ and the monotonicity of $G^{-1}(u)$.

To verify that (5.4) is negative for large n set

$$a_n = 4/DKnx_n^p\log(1/Kx_n^p)l_k(1/Kx_n^p),$$

$$a_n^* = 1/nx_n^q l_{k-1}(nx_n^q), \qquad W_n = a_n(a_n^*)^{-1}l_{k-1}(1/a_n^*).$$

We shall show that $W_n < 1$ for large n. We have $x_n^{q-p} = O(l_{k-1}(n))^{-\rho(q-p)}$, where $\rho > 0$ is arbitrary. Choose $p > 1$, p close to 1, and q close to 2 such that $\xi = \rho(q-p) > 2$. Consequently,

$$(5.5)\quad W_n \le c_4\left\{(l_{k-1}(nx_n^q))^2 + (l_{2k-2}(n))^2\right\}\Big/(l_{k-1}(n))^{\xi}(l_k(1/Kx_n^p))$$

for some constant c_4. The right-hand side of (5.5) goes to 0 as $n \to \infty$. This concludes the proof. □

As (5.3) does not hold for h_γ, we shall derive Proposition 3.5 directly from Theorem 4.1.

PROOF OF PROPOSITION 3.5. For $h_\gamma(t) = t^\gamma$, (5.2) holds with $p_0 = 1/(1-\gamma)$. For $p_0 < p < 2$ we have $s_n \to 0$. Consequently, $r_n^+ \to 0$ and $r_n^- \to 0$. As in the proof of Proposition 5.2, R_n has the structure $c_2e^{w_1} + c_3e^{w_2}$ and $w_1 \to -\infty$. So, we shall consider only

$$c_3e^{w_2} = c_3n^2x_n^ph_\gamma^{-1}\left(\frac{4h_\gamma(Kx_n^p)}{DKnx_n^p}\right)\exp\left\{\frac{a}{2}nx_n^2(1-s_n)^2(1+r_n^-)\right\}.$$

Because $h_\gamma^{-1}(t) = t^{1/\gamma}$, for sufficiently large n we have

$$(5.6)\quad c_3e^{w_2} \le x_n^{p(2-1/\gamma)}n^{2-1/\gamma}\exp\left(anx_n^2/2\right).$$

Hence, by $x_n^2 \le An^{-1}\log n$ and $A < (1-2\gamma)^2/\gamma(1-\gamma)a$, the right-hand side of (5.6) goes to 0 as $n \to \infty$. The same reasoning applies to L_n. □

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INSTITUTE OF MATHEMATICS
TECHNICAL UNIVERSITY OF WROCŁAW
50-370 WROCŁAW
WYBRZEŻE WYSPIAŃSKIEGO 27
POLAND