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HILBERT SPACE REPRESENTATIONS OF m-DEPENDENT PROCESSES

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A representation of one-dependent processes is given in terms of Hilbert spaces, vectors and bounded linear operators on Hilbert spaces. This generalizes a construction of one-dependent processes that are not two-block-factors. We show that all one-dependent processes admit a representation. We prove that if there is in the Hilbert space a closed convex cone that is invariant under certain operators and that is spanned by a finite number of linearly independent vectors, then the corresponding process is a two-block-factor of an independent process.

Apparently the difference between two-block-factors and non–two-block-factors is determined by the geometry of invariant cones. The dimension of the smallest Hilbert space that represents a process is a measure for the complexity of the structure of the process.

For two-valued one-dependent processes, if there is a cylinder with measure equal to zero, then this process can be represented by a Hilbert space with dimension smaller than or equal to the length of this cylinder. In the two-valued case a cylinder (with measure equal to zero) whose length is minimal and less than or equal to 7 is symmetric.

We generalize the concept of Hilbert space representation to m-dependent processes and it turns out that all m-dependent processes admit a representation. Several theorems can be generalized to m-dependent processes.

1. Introduction. In this paper we consider *one-dependent processes*, which are discrete time stationary stochastic processes $(X_N)_{N \in \mathbb{Z}}$ with the property that for any given time t the past $(X_N)_{N<t}$ is independent of the future $(X_N)_{N>t}$.

Just like Markov processes, one-dependent processes are a weakening of independence, but in contrast to these we assume no knowledge about the present value X_t. Although Markov processes have been investigated thoroughly for a long time, the theory of one-dependence is still young but growing.

This paper is the first that uses Hilbert space techniques to investigate one-dependent processes. The concept of Hilbert space representations was initiated by Mike Keane. One-dependent processes arise in renormalization theory as limits of rescaling operations (see [21]). In statistical physics many models have rescaling properties for critical values (e.g., critical temperature) of their parameters (as is conjectured by physicists). This means that the

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model is invariant under rescaling operations (such as, e.g., fractals). Such random fields should therefore typically be one-dependent. The notion of one-dependence can be generalized to *m-dependence* ($m \in \mathbb{N}$); which means that for any given time t, $(X_N)_{N<t}$ and $(X_N)_{N \geq t+m}$ are independent.

Examples of m-dependent processes are $m + 1$-*block-factors*; let $(Y_N)_{N \in \mathbb{Z}}$ be an i.i.d. sequence and f a function of $m + 1$ variables. If we define

$$X_N := f(Y_N, \dots, Y_{N+m}),$$

then the $m + 1$-block-factor $(X_N)_{N \in \mathbb{Z}}$ is an m-dependent process, as follows immediately from the definition. It is easily checked that for $m + 1$-block-factors it is no restriction to assume that the underlying sequence $(Y_N)_{N \in \mathbb{Z}}$ is identically distributed with the uniform distribution over the unit interval.

Although for quite a time probabilists conjectured ([3], [9], [15], [16], [17] and [22]) that all m-dependent processes are $m + 1$-block-factors, in [2] a two-parameter family of counterexamples is shown of one-dependent processes (assuming only two values) that are not two-block-factors. Later Jon Aaronson, David Gilat and Mike Keane found an example of a five-state one-dependent Markov chain that is not a two-block-factor ([1]). Recently Burton, Goulet and Meester found a counterexample of a four-state one-dependent process that is not an m-block-factor for any $m \in \mathbb{N}$ ([4]). Several authors [3], [10], [13], [17] and [22]) used this conjecture as hypothesis and therefore some of their results on m-dependence are only valid for $m + 1$-block-factors. In [25] the authors prove that every process $(X_n)_{n \in \mathbb{N}}$ has an a.s. nonlinear regression representation $X_n = f_n(X_1, \dots, X_{n-1}, U_n)$ a.s. for some $(f_n)_{n \in \mathbb{N}}$ and some i.i.d. sequence $(U_n)_{n \in \mathbb{N}}$, where U_n is independent of $(X_1, \dots, X_{n-1})$. This implies the existence of a representation $X_n = g_n(U_1, \dots, U_n)$ a.s. for some $(g_n)_{n \in \mathbb{N}}$. For so-called monotone $m + 1$-block-factors this supplies a constructive method to obtain the $m + 1$-block-factor representation $X_n = g_n(U_{n-m}, \dots, U_n)$. In this article we generalize the construction of the counterexamples from [2] by representing one-dependent processes in terms of Hilbert spaces, vectors and bounded linear operators on Hilbert spaces. A crucial difference between the operators in Hilbert space representations (HSR) and the operators in quantum probability is that the HSR operators are defined on the whole space and are in general not self-adjoint and not even normal, while the quantum probability operators are defined on a subspace and are self-adjoint. These Hilbert space representations can supply new tools to investigate the structure of one-dependent processes and especially the essential difference between two-block-factors and non–two-block-factors. The dimension of the smallest Hilbert space that represents a process is a measure for the complexity of the structure of the process. One-dependent processes, represented by a one-dimensional Hilbert space, are i.i.d. sequences. One-dependent processes, represented by a two-dimensional Hilbert space, are two-block-factors. The counterexamples from [2] fit with a three-dimensional Hilbert space. The plan of this article is as follows.

In Section 2 we describe the Hilbert space representation and we show that it actually represents a consistent probability measure that is one-dependent

(Theorem 2.1). In Section 3 we show that each one-dependent process (Theorem 3.2) admits a Hilbert space representation. We give some examples. In Section 4 we introduce closed convex cones that are invariant under certain operators. We prove that if there is an invariant cone that is spanned by a finite number of linearly independent vectors, then the one-dependent process is a two-block-factor (Theorem 4.4). This implies that one-dependent processes with a two-dimensional Hilbert space representation are two-block-factors (Theorem 4.3). It seems that the difference between two-block-factors and non-two-block-factors is determined by the geometry of invariant cones. In Section 5 we make some remarks on minimal zero-cylinders and minimal dimension of the Hilbert space and we generalize the concept of Hilbert space representation to m-dependent processes. Several theorems on one-dependent processes can be generalized to m-dependent processes. In Section 6 we give a contribution to the perpetuation of mathematics by a list of conjectures and open problems.

2. The representation. In this section we describe the Hilbert space representation and we show that it actually gives rise to a consistent probability measure that is one-dependent. Let H be a real Hilbert space, and let $K \geq 2$ be an integer, let $A_1, \dots, A_K$: $H \to H$ be linear, continuous operators, and let $x, y \in H$ be two fixed vectors with $\langle x; y \rangle = 1$. We call $(H, x, y, A_1, \dots, A_K)$ a *weak Hilbert space representation* (weak HSR) of a one-dependent process $(X_N)_{N \in \mathbb{Z}}$ if

$$P[X_1 = i_1, \dots, X_N = i_N] = \langle A_{i_1} \cdots A_{i_N} y; x \rangle \quad \text{for all } n \in \mathbb{N} \text{ and all } i_1, \dots, i_N \in \{1, \dots, K\}. \tag{1}$$

We call $(H, x, y, A_1, \dots, A_K)$ a *strong* HSR (or just an HSR) of a one-dependent process $(X_N)_{N \in \mathbb{Z}}$ if in addition to (1) it satisfies the condition

$$(A_1 + \cdots + A_K) h = \langle h; x \rangle y \quad \text{for all } h \in H. \tag{2}$$

We will show that every one-dependent process admits an HSR. First we prove the converse, that every family $(H, x, y, A_1, \dots, A_K)$ that satisfies (2) and a nonnegativity condition (that is obviously necessary, because probabilities are nonnegative), represents a one-dependent process.

THEOREM 2.1. *Let H be a real Hilbert space, $A_1, \dots, A_K$: $H \to H$ linear continuous operators, $x, y \in H$, $\langle x, y \rangle = 1$. Assume that*

$$(A_1 + \cdots + A_K) h = \langle h; x \rangle y \quad \textit{for all } h \in H$$

and

$$\langle A_{i_1} \cdots A_{i_N} y; x \rangle \geq 0 \quad \textit{for all } N \in \mathbb{N} \textit{ and all } i_1, \dots, i_N \in \{1, \dots, K\}.$$

Then there exists a one-dependent process $(X_N)_{N \in \mathbb{Z}}$ with state space $\{1, \dots, K\}$ such that $(H, x, y, A_1, \dots, A_K)$ is a strong HSR of $(X_N)_{N \in \mathbb{Z}}$.

PROOF. We have to check that

$$P[X_1 = i_1, \dots, X_N = i_N] := \langle A_{i_1} \cdots A_{i_N} y; x \rangle \quad N \in \mathbb{N}, i_1, \dots, i_N \in \{1, \dots, K\}$$

defines consistently a probability measure on $\{1, \dots, K\}^{\mathbb{Z}}$ that gives rise to a one-dependent process. We have (using the definitions)

$$\begin{aligned}
\sum_{i_N=1}^{K} & P[X_1 = i_1, \dots, X_N = i_N] \\
&= \sum_{i_N=1}^{K} \langle A_{i_1} \cdots A_{i_N} y; x \rangle \\
&= \langle A_{i_1} \cdots A_{i_{N-1}} (A_1 + \cdots + A_K) y; x \rangle \\
&= \langle A_{i_1} \cdots A_{i_{N-1}} \langle y; x \rangle y; x \rangle \\
&= \langle A_{i_1} \cdots A_{i_{N-1}} y; x \rangle = P[X_1 = i_1, \dots, X_{N-1} = i_{N-1}]
\end{aligned}$$

and

$$\begin{aligned}
\sum_{i_1=1}^{K} & P[X_1 = i_1, \dots, X_N = i_N] \\
&= \sum_{i_1=1}^{K} \langle A_{i_1} \cdots A_{i_N} y; x \rangle \\
&= \langle (A_1 + \cdots + A_K) A_{i_2} \cdots A_{i_N} y; x \rangle \\
&= \langle \langle A_{i_2} \cdots A_{i_N} y; x \rangle y; x \rangle \\
&= \langle A_{i_2} \cdots A_{i_N} y; x \rangle = P[X_2 = i_2, \dots, X_N = i_N].
\end{aligned}$$

We see that

$$\begin{aligned}
\sum_{i=1}^{K} P[X_1 = i] &= \langle (A_1 + \cdots + A_K) y; x \rangle \\
&= \langle \langle y; x \rangle y; x \rangle = \langle y; x \rangle \langle y; x \rangle = 1
\end{aligned}$$

and we conclude that the inner product (which was required to be nonnegative) consistently defines a probability measure.

From

$$\begin{aligned}
\sum_{i=1}^{K} & P[X_1 = i_1, \dots, X_{N-1} = i_{N-1}, X_N = i, X_{N+1} = i_{N+1}, \dots, X_{N+M} = i_{N+M}] \\
&= \langle A_{i_1} \cdots A_{i_{N-1}} (A_1 + \cdots + A_K) A_{i_{N+1}} \cdots A_{i_{N+M}} y; x \rangle \\
&= \langle A_{i_1} \cdots A_{i_{N-1}} \langle A_{i_{N+1}} \cdots A_{i_{N+M}} y; x \rangle y; x \rangle \\
&= \langle A_{i_1} \cdots A_{i_{N-1}} y; x \rangle \langle A_{i_{N+1}} \cdots A_{i_{N+M}} y; x \rangle \\
&= P[X_1 = i_1, \dots, X_{N-1} = i_{N-1}] \cdot P[X_{N+1} = i_{N+1}, \dots, X_{N+M} = i_{N+M}],
\end{aligned}$$

we conclude that $(X_N)_{N \in \mathbb{Z}}$ is a one-dependent process. □

3. Examples of Hilbert space representations. In this section we show that every one-dependent process admits a Hilbert space representation and we give some examples of representations. First we need a technical theorem.

THEOREM 3.1. *Let $(X_N)_{N \in \mathbb{Z}}$ be a one-dependent process over $\{1, \dots, K\}^{\mathbb{Z}}$. Let $(H_0, x, x, A_1, \dots, A_K)$ be a weak HSR of $(X_N)_{N \in \mathbb{Z}}$.*

Then there exists a closed, separable subspace $H \subset H_0$ with $x \in H$, such that $(H, x, x, PA_1, \dots, PA_K)$ is a strong HSR of $(X_N)_{N \in \mathbb{Z}}$, where P: $H_0 \to H$ is the orthogonal projection from H_0 on H.

PROOF. We define the collection $\mathscr{H}$ of those closed subspaces H of H_0 with the property that for the orthogonal projection P: $H_0 \to H$, $(H, x, x, PA_1, \dots, PA_K)$ is a weak HSR of $(X_N)_{N \in \mathbb{Z}}$. We define a partial ordering on $\mathscr{H}$ by

$$H_1 \leq H_2 \quad \text{if } H_1 \supset H_2.$$

Note that $\mathscr{H} \neq \emptyset$ because $H_0 \in \mathscr{H}$.

CLAIM 1. We claim that every totally ordered subset of $\mathscr{H}$ has an upper bound.

PROOF OF CLAIM 1. Let $\mathscr{H}_1 = \{H_\theta \colon \theta \in \Theta\}$ be a totally ordered subset of $\mathscr{H}$. Define $H_1 := \bigcap_{\theta \in \Theta} H_\theta$. We will show that H_1 is an upper bound of $\mathscr{H}_1$. First we prove the following claim.

CLAIM 2. $H_1 \in \mathscr{H}$.

PROOF OF CLAIM 2. Because $H_1 \subset H_\theta$ for all θ, we have $H_1^\perp \supset H_\theta^\perp$ for all θ. So $H_1^\perp \supset \bigcup_\theta (H_\theta^\perp)$, and $H_1^\perp \supset \overline{\bigcup_\theta (H_\theta^\perp)}$. Assume that there exists a $h \in H_1^\perp$ such that $h \in (\bigcup_\theta (H_\theta^\perp))^\perp$. Then $h \in (H_\theta^\perp)^\perp = H_\theta$ for all θ, so $h \in \bigcap_\theta H_\theta = H_1$. But $h \in H_1^\perp$ and $h \in H_1$ implies $h = 0$. We conclude that $H_1^\perp = \overline{\bigcup_\theta (H_\theta^\perp)}$.

Let P_1: $H_0 \to H_1$ and P_θ: $H_0 \to H_\theta$ $(\theta \in \Theta)$ be the orthogonal projections.

Let $z \in H_1^\perp$. For any $\varepsilon > 0$ we can approximate z by a vector $h \in \bigcup_\theta (H_\theta^\perp)$ such that $\|z - h\| < \varepsilon$. So $h \in H_{\theta_0}^\perp$ for some θ_0. For $H_\theta \geq H_{\theta_0}$ we have $P_\theta h \in H_\theta \cap H_{\theta_0}^\perp \subset H_\theta \cap H_\theta^\perp = \{0\}$. Therefore,

$$\|P_\theta z\| = \|P_\theta z - P_\theta h\| \leq \|P_\theta\| \cdot \|z - h\| < \varepsilon$$

if $H_\theta \geq H_{\theta_0}$.

Now let $y \in H_0$. Take $z \in H_1^{\perp}$ and $w \in H_1$ such that $y = z + w$. Let $\varepsilon > 0$ be given. Take θ_0 as above. We have for $H_\theta \geq H_{\theta_0}$ that

$$\|(P_\theta - P_1)y\| = \|P_\theta(z + w) - P_1(z + w)\|$$
$$= \|P_\theta z + w - 0 - w\| = \|P_\theta z\| < \varepsilon.$$

We conclude that $P_\theta y \to_\theta P_1 y$ for all $y \in H_0$.

This implies that (for all i_1)

$$P[X_1 = i_1] = \langle P_\theta A_{i_1} x; x\rangle \to_\theta \langle P_1 A_{i_1} x; x\rangle.$$

Because

$$\|P_\theta A_{i_1} P_\theta A_{i_2} x - P_1 A_{i_1} P_1 A_{i_2} x\|$$
$$= \left\|P_\theta A_{i_1}\left(P_\theta A_{i_2} x - P_1 A_{i_2} x\right) + (P_\theta - P_1)\left(A_{i_1} P_1 A_{i_2} x\right)\right\|$$
$$\leq \|P_\theta\| \cdot \|A_{i_1}\| \cdot \left\|(P_\theta - P_1)\left(A_{i_2} x\right)\right\| + \left\|(P_\theta - P_1)\left(A_{i_1} P_1 A_{i_2} x\right)\right\|$$

and $\|P_\theta\| = 1$, we derive that (for all i_1, i_2)

$$P[X_1 = i_1, X_2 = i_2] = \langle P_\theta A_{i_1} P_\theta A_{i_2} x; x\rangle \to_\theta \langle P_1 A_{i_1} P_1 A_{i_2} x; x\rangle.$$

By induction (on N) we derive that

$$P[X_1 = i_1, \dots, X_N = i_N] = \langle P_\theta A_{i_1} \dots P_\theta A_{i_N} x; x\rangle \to_\theta \langle P_1 A_{i_1} \dots P_1 A_{i_N} x; x\rangle$$

(for all $N \in \mathbb{N}$ and all $i_1, \dots, i_N \in \{1, \dots, K\}$).

Because $x \in H_\theta$ for all $\theta \in \Theta$, we have $x \in H_1$. We conclude that $(H_1, x, x, P_1 A_1, \dots, P_1 A_K)$ is a weak HSR of $(X_N)_{N \in \mathbb{Z}}$. Thus $H_1 \in \mathscr{H}$. This proves Claim 2. Because $H_1 \subset H_\theta$ for all θ, H_1 is an upper bound of $\mathscr{H}_1$. This proves Claim 1. □

Now that we have proved (Claim 1) that every totally ordered subset of $\mathscr{H}$ has an upper bound, we can apply Zorn's lemma that implies the existence of a maximal element. Let H be a maximal element in $\mathscr{H}$. Let P: $H_0 \to H$ be the orthogonal projection on H.

Claim 3. We claim that $(H, x, x, PA_1, \dots, PA_K)$ is a strong HSR of $(X_N)_{N \in \mathbb{Z}}$.

Proof of Claim 3. Consider the restricted operators $PA_1 |_H, \dots, PA_K |_H$ from H to H. Let B_i: $H \to H$ be the adjoints of these restricted operators

($i = 1, \dots, K$). We define the separable space

$$H_B := \overline{sp}\{B_{j_1} \cdots B_{j_m}x \colon m \geq 0,\, j_1, \dots, j_m \in \{1, \dots, K\}\}.$$

To prove Claim 3 we first have to prove the following claim.

CLAIM 4. We claim that $H = H_B$.

Assume that $H \supsetneqq H_B$. Apparently $B_i H_B \subset H_B$ for all i. Consider the restricted operators $B_i|_{H_B}$ ($i = 1, \dots, K$).

Let $C_i \colon H_B \to H_B$ be the adjoints of these restricted operators ($i = 1, \dots, K$). Now we will show that $H_B \in \mathscr{H}$ and that $H_B > H$, which contradicts the maximality of H.

Let $P_B \colon H_0 \to H_B$ be the orthogonal projection, and let $y, z \in H_B$. Then

$$\begin{aligned}\langle P_B A_i y; z\rangle &= \langle P_B P A_i y; z\rangle \\ &= \langle PA_i y; P_B^* z\rangle = \langle (PA_i|_H)y; P_B z\rangle \\ &= \langle (PA_i|_H)y; z\rangle = \langle y; B_i z\rangle \\ &= \langle y; (B_i|_{H_B})z\rangle = \langle C_i y; z\rangle.\end{aligned}$$

This implies that $P_B A_i = C_i$ for all $i = 1, \dots, K$.

Further we have (for all N and for all $i_1, \dots, i_N$)

$$\begin{aligned}P[X_1 = i_1, \dots, X_N = i_N] &= \langle PA_{i_1} \cdots PA_{i_N}x; x\rangle \\ &= \langle (PA_{i_1}|_H) \cdots (PA_{i_N}|_H)x; x\rangle = \langle x; B_{i_N} \cdots B_{i_1}x\rangle \\ &= \langle x; (B_{i_N}|_{H_B}) \cdots (B_{i_1}|_{H_B})x\rangle = \langle C_{i_1} \cdots C_{i_N}x; x\rangle.\end{aligned}$$

Together with $x \in H_B$ (by definition of H_B) this implies that $H_B \in \mathscr{H}$. Because we assumed $H_B \subsetneqq H$, we have $H_B > H$, which contradicts the maximality of H. We conclude that $H = H_B$. This proves Claim 4. To prove Claim 3 we have to show that

$$(PA_1 + \cdots + PA_K)h = \langle h; x\rangle x$$

for all $h \in H$.

This is equivalent to

$$(*) \qquad \langle (PA_1 + \cdots + PA_K)h; g\rangle = \langle h; x\rangle\langle x; g\rangle$$

for all $g, h \in H$.

Because

$$H = \overline{sp}\{B_{j_1} \cdots B_{j_m}x \colon m \geq 0,\, j_1, \dots, j_m \in \{1, \dots, K\}\}$$

and

$$H = \overline{sp}\{PA_{i_1} \cdots PA_{i_N}x \colon N \geq 0,\, i_1, \dots, i_N \in \{1, \dots, K\}\}$$

(if the right-hand side is a proper subspace of H, then this would contradict the maximality of H) and because $(*)$ is a linear equation in h and g, it is sufficient to check $(*)$ for $h = PA_{i_1} \cdots PA_{i_N}x$ and $g = B_{j_1} \cdots B_{j_m}x$ (for all $N, m \in \mathbb{N}, i_1, \dots, i_N, j_1, \dots, j_m \in \{1, \dots, K\}$). For this h and g we have

$$\begin{aligned}
&\langle (PA_1 + \cdots + PA_K)h; g\rangle \\
&\quad = \langle (PA_1 + \cdots + PA_K)PA_{i_1} \cdots PA_{i_N}x; B_{j_1} \cdots B_{j_m}x\rangle \\
&\quad = \langle PA_{j_m} \cdots PA_{j_1}(PA_1 + \cdots + PA_K)PA_{i_1} \cdots PA_{i_N}x; x\rangle \\
&\quad = \sum_{i=1}^{K} P[X_{-m} = j_m, \dots, X_{-1} = j_1, X_0 = i, X_1 = i_1, \dots, X_N = i_N] \\
&\quad = P[X_{-m} = j_m, \dots, X_{-1} = j_1] \cdot P[X_1 = i_1, \dots, X_N = i_N] \\
&\quad = \langle PA_{j_m} \cdots PA_{j_1}x; x\rangle\langle PA_{i_1} \cdots PA_{i_N}x; x\rangle \\
&\quad = \langle x; B_{j_1} \cdots B_{j_m}x\rangle\langle PA_{i_1} \cdots PA_{i_N}x; x\rangle \\
&\quad = \langle x; g\rangle\langle h; x\rangle.
\end{aligned}$$

This proves $(*)$ and the proof of Claim 3 is finished. Claim 3 implies the theorem. □

REMARK. We restricted ourselves in Theorem 3.1 to $H \subset H_0$ because in general $(*)$ does not hold for all $h, g \in H_0$ (as is easy to see in the proof of Theorem 3.2, where we apply Theorem 3.1). Now we can prove the main theorem of this section.

THEOREM 3.2. *Let $(X_N)_{N \in \mathbb{Z}}$ be a K-valued (for some $K \in \mathbb{N}$) one-dependent process. Then there exists an HSR of $(X_N)_{N \in \mathbb{Z}}$.*

PROOF. Let $(X_N)_{N \in \mathbb{Z}}$ be a one-dependent process over $\{1, \dots, K\}^{\mathbb{Z}}$. $(X_N)_{N \in \mathbb{Z}}$ induces a probability measure P on $\{1, \dots, K\}^{\mathbb{N}}$. We define the Hilbert space $H_0 := L^2(P)$. Let $I \in H_0$ be the function that is identically one. We have $\langle I; I\rangle = 1$.

We define the operators $A_1, \dots, A_K$: $H_0 \to H_0$ by $(A_i h)(w_1, w_2, w_3, \dots) := I_i(w_1)h(w_2, w_3, \dots)$ for $h \in H_0$, where

$$I_i(w) := \begin{cases} 1, & \text{if } w = i, \\ 0, & \text{if } w \neq i. \end{cases}$$

Apparently $A_1, \dots, A_K$ are linear and continuous and they satisfy the equation

$$\begin{aligned}
\langle A_{i_1} \cdots A_{i_N}I; I\rangle &= \int I_{i_i}(w_1)I_{i_2}(w_2) \cdots I_{i_N}(w_N)\, dP(w) \\
&= P[X_1 = i_1, \dots, X_N = i_N]
\end{aligned}$$

for all $N \in \mathbb{N}$ and all $i_1, \dots, i_N \in \{1, \dots, K\}$. Thus $(H_0, I, I, A_1, \dots, A_K)$ is a weak HSR of $(X_N)_{N \in \mathbb{Z}}$.

Theorem 3.1 now implies the existence of a HSR of $(X_N)_{N \in \mathbb{Z}}$. □

The Hilbert space representation of a one-dependent process is not unique. We give some examples of HSR's.

EXAMPLE. Let $(X_N)_{N\in\mathbb{Z}}$ be a one-dependent process over $\{0,1\}^{\mathbb{Z}}$. We show a natural HSR of $(X_N)_{N\in\mathbb{Z}}$ with Hilbert space l^2.

In [2] (Theorem 1) it is proved that the distribution of a $0-1$ valued one-dependent process is uniquely determined by its values

$$[1^N] := P[X_1 = \cdots = X_N = 1] \qquad N \in \mathbb{N}.$$

Let

$$H := l^2, \qquad y := \begin{pmatrix} 1 \\ [1] \\ [11] \\ \vdots \\ [1^N] \\ \vdots \end{pmatrix}, \qquad x := \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ \vdots \end{pmatrix},$$

$$A_0 = \begin{pmatrix} 1 & -1 & 0 & 0 & \cdots & & \\ [1] & 0 & -1 & 0 & \cdots & & \\ [11] & 0 & 0 & -1 & \vdots & & \\ \vdots & & & & & & \\ [1^N] & 0 & 0 & \cdots & \cdots & -1 & \cdots \\ \vdots & & & & & & \end{pmatrix},$$

$$A_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & \\ 0 & 0 & 1 & 0 & \cdots & \\ 0 & 0 & 0 & 1 & & \\ \vdots & & & & & \\ 0 & 0 & 0 & 0 & \cdots & 1 \cdots \\ \vdots & & & & & \end{pmatrix}.$$

Because

$$\begin{aligned}[1^{N+M+1}] &\le P[X_1 = \cdots = X_N = 1, X_{N+2} = \cdots = X_{N+M+1} = 1] \\ &= P[X_1 = \cdots = X_N = 1]P[X_{N+2} = \cdots = X_{N+M+1} = 1] \\ &= [1^N]\cdot[1^M],\end{aligned}$$

it is easy to see that actually $x, y \in l^2$ and that A_0 and A_1 are continuous operators on l^2.

It is trivial that $(A_0 + A_1)h = \langle h; x\rangle y$ holds for all $h \in l^2$ and that $\langle x; y\rangle = 1$. From

$$\langle A_1^N y; x\rangle = \left\langle \begin{pmatrix} [1^N] \\ [1^{N+1}] \\ \vdots \end{pmatrix}; \begin{pmatrix} 1 \\ 0 \\ \vdots \end{pmatrix} \right\rangle = [1^N] \qquad \forall N \in \mathbb{N}$$

and Theorem 1 of [2], we conclude that (l^2, x, y, A_0, A_1) is an HSR of $(X_N)_{N\in\mathbb{Z}}$.

REMARK. The "special" processes in [2] are represented by $H = \mathbb{R}^3$,

$$y = \begin{pmatrix} 1 \\ \alpha \\ \beta \end{pmatrix}, \quad x = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad A_0 = \begin{pmatrix} 1 & -1 & 0 \\ \alpha & 0 & -1 \\ \beta & 0 & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

The two-parameter family of counterexamples of one-dependent processes that are not two-block-factors corresponds with HSR's of this type.

EXAMPLE 3.3. Let $(X_N)_{N\in\mathbb{Z}}$ be a K-valued (for some $K \in \mathbb{N}$) two-block-factor of an i.i.d. sequence. We show a natural HSR of $(X_N)_{N\in\mathbb{Z}}$ with Hilbert space $L^2[0,1]$.

Let $X_N = f(Y_N, Y_{N+1})$ for some function f and some i.i.d. sequence $(Y_N)_{N\in\mathbb{Z}}$ of random variables that are uniformly distributed over the unit interval.

We define the sets V_i $(i = 1, \dots, K)$ in the unit square

$$V_i := \{(t,s): f(t,s) = i\}.$$

Let $H = L^2[0,1]$, let the operators A_i be defined by

$$(A_i g)(t) := \int_0^1 I_{V_i}(t,s) g(s)\, ds \qquad i = 1, \dots, K,$$

where I_{V_i} is the indicator function of V_i. Let $\mathbb{1} \in H$ be the function that is identically one.

It is an easy exercise to prove that $(H, \mathbb{1}, \mathbb{1}, A_1, \dots, A_K)$ is a HSR of $(X_N)_{N\in\mathbb{Z}}$.

The construction in Example 3.3 can be generalized to K-valued one-dependent m-block-factors of i.i.d. sequences $(m \in \mathbb{N})$. Applying Theorem 3.1 leads to an HSR with a subspace of $L^2([0,1]^{m-1})$ as Hilbert space. Generally an m-block-factor is $(m-1)$-dependent, but for special choices of the function f the m-block-factor $X_N = f(Y_N, \dots, Y_{N+m-1})$ can be one-dependent. It is an open problem whether there exist one-dependent m-block-factors $(m \geq 3)$ that can not be written as a two-block-factor.

The reversed process of a one-dependent process is also one-dependent. The following theorem gives an HSR.

THEOREM 3.4. *Let* $(H, x, y, A_1, \dots, A_K)$ *be an HSR of a one-dependent process* $(X_N)_{N \in \mathbb{Z}}$. *Let* $(Y_N)_{N \in \mathbb{Z}}$ *be the reversed process; that is,* $Y_N := X_{-N}$, $N \in \mathbb{Z}$. *Then* $(H, y, x, A_1^*, \dots, A_K^*)$ *is an HSR of* $(Y_N)_{N \in \mathbb{Z}}$.

PROOF.

$$\langle (A_1^* + \cdots + A_K^*)h; g\rangle = \langle h; (A_1 + \cdots + A_K)g\rangle = \langle h; \langle g; x\rangle y\rangle$$
$$= \langle h; y\rangle\langle g; x\rangle = \langle\langle h; y\rangle x; g\rangle$$

for all $h, g \in H$. This implies that

$$(A_1^* + \cdots + A_K^*)h = \langle h; y\rangle x \qquad \forall\, h \in H.$$

Further,

$$\langle A_{i_1}^* \cdots A_{i_N}^* x; y\rangle = \langle x; A_{i_N} \cdots A_{i_1} y\rangle = P[X_1 = i_N, \dots, X_N = i_1]$$
$$= P[Y_1 = i_1, \dots, Y_N = i_N]. \qquad \square$$

4. Finite dimension and invariant cones. In this section we prove that an HSR with two-dimensional Hilbert space corresponds to a two-block-factor. Further we show that if there is an invariant (under $A_1, \dots, A_K$) cone spanned by a finite number of linearly independent vectors, then the HSR corresponds with a two-block-factor. The first theorem is just a special case of the other one. We need a technical theorem to show that it is no restriction to assume that the vectors x and y are equal.

THEOREM 4.1. *Let* $(H, x, y, A_1, \dots, A_K)$ *be an HSR of a one-dependent process* $(X_N)_{N \in \mathbb{Z}}$. *Then there exists a vector* $x_0 \in H$ *and there exist operators* $B_1, \dots, B_K \colon H \to H$ *such that* $(H, x_0, x_0, B_1, \dots, B_K)$ *is an HSR of* $(X_N)_{N \in \mathbb{Z}}$.

PROOF.

CASE 1. If x and y are linearly dependent, then it is easy to see that $(H, x/\|x\|, x/\|x\|, A_1, \dots, A_K)$ is an HSR of $(X_N)_{N \in \mathbb{Z}}$.

CASE 2. If x and y are linearly independent, then we consider the two-dimensional subspace H_0 that is spanned by x and y,

$$H_0 := sp\{x, y\}$$

and its orthogonal complement $H_0^\perp$,

$$H_0^\perp := \{h \in H \colon \langle h; x\rangle = \langle h; y\rangle = 0\}.$$

Take some orthonormal basis of H_0, and assume that $x = \binom{x_1}{x_2}$, $y = \binom{y_1}{y_2}$ with respect to this basis. We have $1 = \langle x; y\rangle = x_1 y_1 + x_2 y_2$.

Let $\lambda \in \mathbb{R}$, $\lambda \neq 0$. We define the linear operator $V: H \to H$ by

$$V|_{H_0} = \begin{pmatrix} y_1 & -\lambda x_2 \\ y_2 & \lambda x_1 \end{pmatrix}$$

and $V|_{H_0^\perp} =$ identity.

It is easy to see that V is invertible and

$$V^{-1}|_{H_0} = \frac{1}{\lambda}\begin{pmatrix} \lambda x_1 & \lambda x_2 \\ -y_2 & y_1 \end{pmatrix}$$

and $V^{-1}|_{H_0^\perp} =$ identity.

We claim that

$$\left(H, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, V^{-1}A_1V, \dots, V^{-1}A_KV\right)$$

is an HSR of $(X_N)_{N \in \mathbb{Z}}$. It is clear that $\left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle = 1$. Further, let $h \in H$. We have

$$\begin{aligned}(V^{-1}A_1V + \cdots + V^{-1}A_KV)h &= V^{-1}(A_1 + \cdots + A_K)Vh \\ &= V^{-1}\langle Vh; x\rangle y = \langle Vh; x\rangle V^{-1}y = \langle Vh; x\rangle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \langle h; V^*x\rangle \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \left\langle h; \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle \begin{pmatrix} 1 \\ 0 \end{pmatrix},\end{aligned}$$

and

$$\begin{aligned}&\left\langle V^{-1}A_{i_1}V \cdots V^{-1}A_{i_N}V\begin{pmatrix} 1 \\ 0 \end{pmatrix}; \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle \\ &\quad = \left\langle V^{-1}A_{i_1} \cdots A_{i_N}y; \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle = \left\langle A_{i_1} \cdots A_{i_N}y; (V^{-1})^*\begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle \\ &\quad = \langle A_{i_1} \cdots A_{i_N}y; x\rangle,\end{aligned}$$

which proves Theorem 4.1. □

REMARK. The fact that any orthonormal basis of H_0 and any $\lambda \neq 0$ can be chosen in the proof of Theorem 4.1 shows the non-uniqueness of the Hilbert space representations. In Theorem 4.3 we need the following lemma.

LEMMA 4.2. *Let $(H, x, x, A_1, \dots, A_K)$ be an HSR of a one-dependent process $(X_N)_{N \in \mathbb{Z}}$. Let $T := \overline{co}\{\alpha A_{i_1} \cdots A_{i_N}x\colon \alpha \geq 0,\ N \in \mathbb{N},\ i_1, \dots, i_N \in \{1, \dots K\}\}$. If $\exists\ v \in T$, $v \neq 0$ with $\langle v; x\rangle = 0$, then $(X_N)_{N \in \mathbb{Z}}$ has an HSR with Hilbert space*

$$H_0 = \{v \in T : \langle v; x\rangle = 0\}^\perp \subsetneq H.$$

PROOF. Let $V := \overline{sp}\{v \in T: \langle v; x\rangle = 0\}$, then $H_0 = V^{\perp}$. Note that $x \in H_0$. Let P be the orthogonal projection on H_0. We show that

$$(H_0, x, x, PA_1, \dots, PA_K)$$

is an HSR of $(X_N)_{N\in\mathbb{Z}}$. Let $v \in T$ with $\langle v; x\rangle = 0$. Because $A_i T \subset T$ we have $\langle A_i v; x\rangle \geq 0$ for all $i = 1, \dots, K$. Thus

$$0 \leq \sum_{i=1}^{K} \langle A_i v; x\rangle = \langle (A_1 + \cdots + A_K)v; x\rangle$$
$$= \langle \langle v; x\rangle x; x\rangle = \langle v; x\rangle = 0,$$

which implies that $\langle A_i v; x\rangle = 0$ for all $i = 1, \dots, K$, and all $v \in V$. Hence $A_i V \subset V$ for all $i = 1, \dots, K$. If $h \in H$, then $h - Ph \in V$, so

$$\langle A_{i_1} \cdots A_{i_m}(h - Ph); x\rangle = 0$$

for all $m \in \mathbb{N}$ and all $i_1, \dots, i_m \in \{1, \dots, K\}$, and hence

$$\langle A_{i_1} \cdots A_{i_m} Ph; x\rangle = \langle A_{i_1} \cdots A_{i_m} h; x\rangle.$$

Now we have ($h \in H_0$)

$$(PA_1 + \cdots + PA_K)h = P(A_1 + \cdots + A_K)h$$
$$= P\langle h; x\rangle x = \langle h; x\rangle Px = \langle h; x\rangle x,$$

and

$$\langle PA_{i_1}PA_{i_2} \cdots PA_{i_N}x; x\rangle = \langle A_{i_1}PA_{i_2} \cdots PA_{i_N}x; P^*x\rangle$$
$$= \langle A_{i_1}A_{i_2}PA_{i_3} \cdots PA_{i_N}x; x\rangle$$
$$= \cdots = \langle A_{i_1} \cdots A_{i_N}x; x\rangle,$$

which proves our lemma. □

Now we consider the case that the Hilbert space has dimension one or two.

THEOREM 4.3. *Let* $(H, x, y, A_1, \dots, A_K)$ *be an HSR of a one-dependent process* $(X_N)_{N\in\mathbb{Z}}$.

(a) *If* $\dim(H) = 1$, *then* $(X_N)_{N\in\mathbb{Z}}$ *is an i.i.d. sequence.*
(b) *If* $\dim(H) = 2$, *then* $(X_N)_{N\in\mathbb{Z}}$ *is a two-block-factor of an i.i.d. sequence.*

PROOF. (a) If $\dim H = 1$, then $A_i = (a_i)$, $i = 1, \dots, K$. We have

$$P[X_1 = i_1, \dots, X_N = i_N] = \langle A_{i_1} \cdots A_{i_N} y; x\rangle$$
$$= a_{i_1} \cdots a_{i_N}\langle y; x\rangle = a_{i_1} \cdots a_{i_N}$$
$$= P[X_1 = i_1] \cdots P[X_N = i_N].$$

(b) Theorem 4.1 implies that we may assume that $x = y$. If $\dim H = 2$, then we consider the closed convex cone spanned by the orbit of x under the operators $A_1, \ldots, A_K$:

$$T := \overline{co}\{\alpha A_{i_1} \cdots A_{i_N} x \colon \alpha \geq 0,\ N \in \mathbb{N},\ i_1, \ldots, i_N \in \{1, \ldots, K\}\}.$$

Note that $x \in T$, and that $A_i T \subset T\ \forall\ i = 1, \ldots, K$.

We choose an orthonormal basis of $\mathbb{R}^2$ such that $x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. The claim implies that there exist vectors $v = \begin{pmatrix} 1 \\ v_2 \end{pmatrix}$, $w = \begin{pmatrix} 1 \\ w_2 \end{pmatrix}$ such that $v_2 - w_2 > 0$ and

$$T = \overline{co}\{\alpha v, \alpha w \colon \alpha \geq 0\}.$$

Let $A_i = \begin{pmatrix} a_{11}^i & a_{12}^i \\ a_{21}^i & a_{22}^i \end{pmatrix}$ be the matrix of A_i $(i = 1, \ldots, K)$ with respect to the basis $\{v, w\}$. Because $A_i v, A_i w \in \overline{co}\{\alpha v, \alpha w \colon \alpha \geq 0\}$ it follows that $a_{j_1 j_2}^i \geq 0$ $\forall\ i = 1, \ldots, K$, $\forall\ j_1, j_2 \in \{1, 2\}$. With respect to the standard basis we have

$$(A_1 + \cdots + A_K)v = \left\langle \begin{pmatrix} 1 \\ v_2 \end{pmatrix}; \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = x.$$

On the other hand, we have

$$(A_1 + \cdots + A_K)v = \sum_{i=1}^{K} \left(a_{11}^i v + a_{21}^i w\right);$$

hence

$$\text{(3)} \qquad \begin{aligned} 1 = \langle x; x \rangle &= \left\langle \sum_{i=1}^{K} \left(a_{11}^i v + a_{21}^i w\right); x \right\rangle \\ &= 1 = \sum_{i=1}^{K} \left(a_{11}^i + a_{21}^i\right). \end{aligned}$$

Analogously (considering $(A_1 + \cdots + A_K)w$) we find that

$$\text{(4)} \qquad \sum_{i=1}^{K} \left(a_{12}^i + a_{22}^i\right) = 1.$$

Further, we have

$$\text{(5)} \qquad \begin{aligned} 0 = \left\langle x; \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle &= \left\langle \sum_{i=1}^{K} \left(a_{11}^i v + a_{21}^i w\right); \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle \\ &= 0 = \sum_{i=1}^{K} \left(a_{11}^i v_2 + a_{21}^i w_2\right). \end{aligned}$$

Analogously,

$$\text{(6)} \qquad \sum_{i=1}^{K} \left(a_{12}^i v_2 + a_{22}^i w_2\right) = 0.$$

Equations (1), (2), (3) and (4) imply that $\sum_{i=1}^{K} a_{11}^i = \sum_{i=1}^{K} a_{12}^i$ and $\sum_{i=1}^{K} a_{21}^i =$

$\sum_{i=1}^{K} a_{22}^{i}$. Let us define the matrix $S = \begin{pmatrix} 1 & 1 \\ v_2 & w_2 \end{pmatrix}$; then

$$S^{-1} = \begin{pmatrix} \sum_{i=1}^{K} a_{11}^{i} & \dfrac{1}{v_2 - w_2} \\ \sum_{i=1}^{K} a_{21}^{i} & \dfrac{-1}{v_2 - w_2} \end{pmatrix},$$

as is easily checked.

We note that A_i has matrix

$$S \begin{pmatrix} a_{11}^{i} & a_{12}^{i} \\ a_{21}^{i} & a_{22}^{i} \end{pmatrix} S^{-1}$$

with respect to the standard basis. So we have (with respect to the standard basis)

$$\begin{aligned} &\left\langle A_{i_1} \cdots A_{i_N} \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle \\ &= \left\langle S \begin{pmatrix} a_{11}^{i_1} & a_{12}^{i_1} \\ a_{21}^{i_1} & a_{22}^{i_1} \end{pmatrix} S^{-1} \cdots S \begin{pmatrix} a_{11}^{i_N} & a_{12}^{i_N} \\ a_{21}^{i_N} & a_{22}^{i_N} \end{pmatrix} S^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle \\ &= \left\langle \begin{pmatrix} a_{11}^{i_1} & a_{12}^{i_1} \\ a_{21}^{i_1} & a_{22}^{i_1} \end{pmatrix} \cdots \begin{pmatrix} a_{11}^{i_N} & a_{12}^{i_N} \\ a_{21}^{i_N} & a_{22}^{i_N} \end{pmatrix} S^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}; S^{*} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle \\ &= \left\langle \begin{pmatrix} a_{11}^{i_1} & a_{12}^{i_1} \\ a_{21}^{i_1} & a_{22}^{i_1} \end{pmatrix} \cdots \begin{pmatrix} a_{11}^{i_N} & a_{12}^{i_N} \\ a_{21}^{i_N} & a_{22}^{i_N} \end{pmatrix} \begin{pmatrix} \sum_{i=1}^{K} a_{11}^{i} \\ \sum_{i=1}^{K} a_{21}^{i} \end{pmatrix}; \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\rangle. \end{aligned}$$

By induction (on $N \in \mathbb{N}$) it now follows that $(X_N)_{N \in \mathbb{Z}}$ is a two-block-factor of an i.i.d. sequence $(Y_N)_{N \in \mathbb{Z}}$ of random variables that are uniformly distributed over the unit interval. We have $X_N = f(Y_N, Y_{N+1})$ with

$$f(t,s) = i \quad \Leftrightarrow$$

$$(t,s) \in \left[a_{11}^{1} + \cdots + a_{11}^{i-1}; a_{11}^{1} + \cdots + a_{11}^{i}\right) \times \left[0; a_{11}^{1} + \cdots + a_{11}^{K}\right)$$

or

$$\begin{aligned} (t,s) \in \big[&a_{11}^{1} + \cdots + a_{11}^{K} + a_{21}^{1} + \cdots + a_{21}^{i-1}; \\ &a_{11}^{1} + \cdots + a_{11}^{K} + a_{21}^{1} + \cdots + a_{21}^{i}\big) \times \left[0; a_{11}^{1} + \cdots + a_{11}^{K}\right) \end{aligned}$$

or

$$(t,s) \in \left[a_{12}^{1} + \cdots + a_{12}^{i-1}; a_{12}^{1} + \cdots + a_{12}^{i}\right) \times \left[a_{11}^{1} + \cdots + a_{11}^{K}; 1\right)$$

or

$$(t,s) \in \left[a_{12}^1 + \cdots + a_{12}^K + a_{22}^1 + \cdots + a_{22}^{i-1};\right.$$
$$\left. a_{12}^1 + \cdots + a_{12}^K + a_{22}^1 + \cdots + a_{22}^i\right) \times \left[a_{11}^1 + \cdots + a_{11}^K; 1\right). \quad \square$$

We generalize Theorem 4.3 to the case of more dimensions when there exists an invariant cone spanned by a finite number of linearly independent vectors.

THEOREM 4.4. *Let* $(\mathbb{R}^N, x_0, x_0, A_1, \dots, A_K)$ *be an HSR of a one-dependent process* $(X_N)_{N\in\mathbb{Z}}$. *Assume that there exist N linearly independent vectors* $v_1, \dots, v_N \in \mathbb{R}^N$ *with* $\langle v_i; x_0\rangle > 0$ *for all i and such that the cone*

$$T := \{\alpha_1 v_1 + \cdots + \alpha_N v_N : \alpha_1 \geq 0, \dots, \alpha_N \geq 0\}$$

is invariant; that is,

$$A_i T \subset T \quad \text{for all } i = 1, \dots, K.$$

Assume further that $x_0 \in T$. *Then* $(X_N)_{N\in\mathbb{Z}}$ *is a two-block-factor of an i.i.d. sequence.*

PROOF. Let $A_{i_0}^v = (a_{i,j}^{i_0})_{i,j=1}^N$ be the matrix of A_{i_0} with respect to $\{v_1, \dots, v_N\}$. Since $A_{i_0}T \subset T$, we have $A_{i_0}v_j = \sum_{i=1}^N a_{ij}^{i_0} v_i \in T$ (for all i_0, j). This implies that

$$a_{i,j}^{i_0} \geq 0 \quad \text{for all } i_0, i, j.$$

Let S be the matrix of $\{v_1, \dots, v_N\}$ with respect to the standard basis $\{x_0 = e_1, \dots, e_N\}$, so

$$S = (v_{ij})_{i,j=1}^N; \quad \text{that is,} \quad v_j = \sum_{i=1}^N v_{ij} e_i \qquad \forall j.$$

Let $R = S^{-1}$ be the matrix of coordinates of the standard basis with respect to $\{v_1, \dots, v_N\}$, so

$$R = (t_{ij})_{i,j=1}^N; \qquad e_j = \sum_{i=1}^N t_{ij} v_i \qquad \forall j.$$

Because $e_1 = x_0 = \sum_{i=1}^N t_{i1} v_i \in T$, we have

$$t_{i1} \geq 0 \quad \text{for all } i.$$

Because $\langle v_i; x_0\rangle > 0$, we can assume by multiplying the v_i that

$$v_{1i} = \langle v_i; x_0\rangle = 1 \quad \text{for all } i.$$

We have for all j,

$$(A_1 + \cdots + A_K)v_j = \sum_{i_0=1}^K \sum_{i=1}^N a_{ij}^{i_0} v_i$$

and $(A_1 + \cdots + A_K)v_j = \langle v_j; x_0\rangle x_0 = x_0$.

This implies that for all j,

$$1 = \langle x_0; x_0 \rangle = \sum_{i_0=1}^{K} \sum_{i=1}^{N} a_{ij}^{i_0} \langle v_i; x_0 \rangle = \sum_{i_0=1}^{K} \sum_{i=1}^{N} a_{ij}^{i_0}.$$

Because evidently,

$$x_0 = \sum_{i_0=1}^{K} \sum_{i=1}^{N} a_{ij}^{i_0} v_i = \sum_{i=1}^{N} t_{i1} v_i,$$

we have that $t_{i1} = \sum_{i_0=1}^{K} a_{ij}^{i_0}$ for all j and i (we make the crucial observation that this sum is independent of j).

Because A_{i_0} has matrix representation $SA_{i_0}^v R$ with respect to the standard basis, we have

$$\begin{aligned} P[X_1 = i_1, \dots, X_m = i_m] &= \langle SA_{i_1}^v R \cdots SA_{i_m}^v R e_1; e_1 \rangle \\ &= \langle SA_{i_1}^v \cdots A_{i_m}^v R e_1; e_1 \rangle \\ &= \langle A_{i_1}^v \cdots A_{i_m}^v R e_1; S^* e_1 \rangle \\ &= \left\langle A_{i_1}^v \cdots A_{i_m}^v \begin{pmatrix} a_{11}^1 + \cdots + a_{11}^K \\ \vdots \\ a_{N1}^1 + \cdots + a_{N1}^K \end{pmatrix}; \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \right\rangle. \end{aligned}$$

By induction (on m) it is easy to show (just as in the proof of Theorem 4.3) that this corresponds to a two-block-factor. □

REMARK. In Section 3 we described the HSR of a class of counter-examples of one-dependent processes that are not two-block-factors.Their Hilbert space is three-dimensional. Theorem 4.3 states that a two-dimensional HSR is always a two-block-factor. From Theorem 4.4 it follows that the crucial difference between two and three dimensions is apparently the geometry of cones. A closed convex cone in two dimensions is spanned by the convex hull of two linearly independent vectors. In three dimensions closed convex cones exist that are not spanned by the convex hull of three vectors, but of more than three vectors (a finite or even infinite number). Note that these vectors are the extreme points of a convex set. It seems that the difference between two-block-factors and non–two-block-factors is determined by the geometry of the invariant cone. We generalize Theorem 4.3(a) by showing that a one-dependent process is an i.i.d. sequence if the operators commute.

THEOREM 4.5. *Let $(H, x, y, A_1, \dots, A_K)$ be an HSR of a one-dependent process $(X_N)_{N \in \mathbb{Z}}$. If the operators $A_1, \dots, A_K$ commute (i.e., $A_i A_j = A_j A_i$ for all i, j), then $(X_N)_{N \in \mathbb{Z}}$ is an i.i.d. sequence.*

PROOF. We have

$$P[X_1 = i_1, \dots, X_N = i_N, X_{N+1} = j_1, \dots, X_{N+m} = j_m]$$

$$= \sum_{i=1}^{K} P[X_1 = i_1, \dots, X_N = i_N, X_{N+1} = j_1, \dots, X_{N+m} = j_m, X_{N+m+1} = i]$$

$$= \sum_{i=1}^{K} \langle A_{i_1} \cdots A_{i_N} A_{j_1} \cdots A_{j_m} A_i x; x \rangle$$

$$= \sum_{i=1}^{K} \langle A_{i_1} \cdots A_{i_N} A_i A_{j_1} \cdots A_{j_m} x; x \rangle$$

$$= \sum_{i=1}^{K} P[X_1 = i_1, \dots, X_N = i_N, X_{N+1} = i, X_{N+2} = j_1, \dots, X_{N+m+1} = j_m]$$

$$= P[X_1 = i_1, \dots, X_N = i_N] \cdot P[X_{N+2} = j_1, \dots, X_{N+m+1} = j_m],$$

and the theorem follows. □

5. Remarks.

Minimal zero-cylinders and minimal dimension. Let $(X_N)_{N \in \mathbb{Z}}$ be a one-dependent process over $\{1, \dots, K\}^{\mathbb{Z}}$. We call the cylinder $[i_1, \dots, i_N]$ a minimal zero-cylinder if $P[X_1 = i_1, \dots, X_N = i_N] = 0$ and if $P[X_1 = j_1, \dots, X_m = j_m] > 0$ for all $m < N$ and all $j_1, \dots, j_m \in \{1, \dots, K\}$. We call N the length of the minimal zero-cylinder.

Let $(H, x, y, A_1, \dots, A_K)$ be an HSR of a one-dependent process $(X_N)_{N \in \mathbb{Z}}$. We call $\dim(H)$ the minimal dimension of $(X_N)_{N \in \mathbb{Z}}$ if for all HSR $(H', x', y', A'_1, \dots, A'_K)$ of $(X_N)_{N \in \mathbb{Z}}$, we have $\dim(H') \geq \dim(H)$.

For two-valued one-dependent processes, if there is a zero-cylinder, then the length of the minimal zero-cylinder is greater than or equal to the minimal dimension (see [29]). Minimal zero-cylinders play an important role in the theory of two-valued one-dependent processes.

In [26] is proved that certain $0-1$-valued two-block-factors with minimal zero-cylinder [010] or [101] are extremal, that is, the probability of [13] is maximal over the class of all $0-1$-valued one-dependent processes with fixed probability of a one. In [8] it is shown that the minimal probability of [13] (over the class of all $0-1$-valued two-block-factors with fixed probability of one) is attained in processes with $[1^N]$ as minimal zero-cylinder (for some N). The counterexamples in [2] of one-dependent processes that are not two-block-factors have minimal zero-cylinder [111].

Minimal zero-cylinders $[i_1 \cdots i_N]$ of two-valued one-dependent processes with length $N \leq 7$ are symmetric, that is, $i_t = i_{N+1-t}$ for all t (see [29]).

Generalization of the HSR construction. When we replace condition (2) in the definition of HSR by the conditions

$$(A_1 + \cdots + A_K)^m h = \langle h; x \rangle y,$$

$$(A_1 + \cdots + A_K) y = y,$$

$$(A_1^* + \cdots + A_K^*) x = x,$$

we obtain the definition of HSR of an m-dependent process.

Analogously to Theorem 2.1, to every HSR corresponds an m-dependent process and analogously to Theorem 3.2, every m-dependent process with finite state space admits an HSR (see [29]). Theorems 3.4, 4.1, 4.3 and 4.5 can be generalized to m-dependent processes (see [29]).

There are more dependence structures (such as Markov, ergodicity, mixing and renewal) that can be translated to properties of operators in Hilbert space representations; see [30].

6. Conjectures and open problems.

1. The essential difference between two-block-factors and one-dependent processes that are not two-block-factors is determined by the geometry of the invariant cone. More research is necessary to investigate this.
2. A $0-1$ valued one-dependent process can have no other minimal zero-cylinders than [101], [010], $[1^N]$ and $[0^N]$, $N \in \mathbb{N}$. The minimal dimensions are 2, 2, N and N, respectively.
3. For any $N \in \mathbb{N}$, $N \geq 3$, there exists a one-dependent process that is not a two-block-factor, with minimal dimension equal to N, and without zero-cylinders.
4. For any $N \in \mathbb{N}$, $N \geq 3$, there exist a one-dependent process that is not a two-block-factor, with minimal dimension equal to N, and with a minimal zero-cylinder with length N.
5. For any $N \in \mathbb{N}$, $N \geq 1$, there exists a two-block-factor with minimal dimension equal to N, and without zero-cylinders.
6. For any $N \in \mathbb{N}$, $N \geq 1$, there exists a two-block-factor with minimal dimension equal to N, and with a minimal zero-cylinder with length N.
7. Under which conditions is a one-dependent Markov process necessarily a two-block factor?
8. Are one-dependent processes always functions of Markov processes, or even functions of one-dependent Markov processes?
9. Do there exist one-dependent m-block-factors ($m \geq 3$) that can not be written as a two-block-factor?
10. Is a one-dependent process with an m-dimensional HSR ($m \geq 3$) always an m-block-factor?
11. Do there exist two-dependent processes that are not two-block-factors of one-dependent processes?

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