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CONVOLUTION OF UNIMODAL DISTRIBUTIONS CAN PRODUCE ANY NUMBER OF MODES

By Ken-iti Sato

Nagoya University

For any positive integer n, there exists a unimodal distribution μ such that $\mu * \mu$ is n-modal. Furthermore, there is a unimodal distribution μ such that $\mu * \mu$ has infinitely many modes. Lattice analogues of the results are also given.

1. Introduction. K. L. Chung [1] points out that the convolution of two unimodal distributions is not necessarily unimodal (see also the Appendix in his translation of Gnedenko and Kolmogorov [4]). He gives an example of a unimodal distribution such that its convolution with itself is bimodal. Other examples are given by Ibragimov [5], Feller [3], Wolfe [9, 10], Dharmadhikari and Joag-dev [2], and others. But the resulting convolutions in these examples are either bimodal distributions or distributions such that one can only check that they are not unimodal. We will construct, in this note, a unimodal distribution μ such that $\mu * \mu$ is n-modal. Further we will give a unimodal distribution μ such that $\mu * \mu$ is ∞-modal.

After the discovery that unimodality is not preserved under convolution operation, probabilists looked for conditions under which the convolution of two unimodal distributions is again unimodal. Thus Ibragimov [5] introduces the notion of strong unimodality and establishes its equivalence with log-concavity of density functions, and Yamazato [11] finds a sufficient condition for unimodality of the convolution of two unimodal distributions which are one-sided in the opposite directions. Karlin [6] and several others study variation-diminishing properties in general. However, as far as the author knows, no attention has been paid to the problem of how complicated the convolution of unimodal distributions can be. The infinitely divisible case will be of future interest, because some Lévy processes get or lose unimodality as time passes, and properties of their distributions in the period of non-unimodality are not known (see Sato [8] for a survey).

2. Results. A distribution μ on the line is said to be *unimodal with mode* a if its distribution function $F(x)$ is convex on $(-\infty, a)$ and concave on (a, ∞). Let n be a positive integer. In this note we say that a distribution μ is *strictly* n*-modal* if μ is absolutely continuous with a continuous density function $f(x)$ and there are points

$$-\infty < b_1 < a_1 < b_2 < a_2 < \cdots < b_n < a_n < b_{n+1} < +\infty$$

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such that $f(x)$ vanishes on $(-\infty, b_1]$ and $[b_{n+1}, +\infty)$, strictly increases on $[b_i, a_i]$ for $i = 1, \dots, n$, and strictly decreases on $[a_i, b_{i+1}]$ for $i = 1, \dots, n$. We say that μ is strictly n-modal with *modes* $a_1, a_2, \dots, a_n$ and *bottoms* $b_1, b_2, \dots, b_{n+1}$. A distribution μ is said to be *strictly ∞-modal with modes* $a_1, a_2, \dots$ and *bottoms* $b_1, b_2, \dots$ if

$$-\infty < b_1 < a_1 < b_2 < a_2 < \cdots \to +\infty$$

and if μ is absolutely continuous with a continuous density $f(x)$ which vanishes on $(-\infty, b_1]$, strictly increases on $[b_i, a_i]$ for $i = 1, 2, \dots$, and strictly decreases on $[a_i, b_{i+1}]$ for $i = 1, 2, \dots$. For any $a > 0$ denote the uniform distribution on $[0, a]$ by μ_a. If $a_1, a_2, \dots$ are positive reals and $p_1, p_2, \dots$ are nonnegative reals satisfying $\sum_{i=1}^{\infty} p_i = 1$, then, obviously, the distribution $\mu = \sum_{i=1}^{\infty} p_i \mu_{a_i}$ is unimodal with mode 0.

THEOREM 1. *Let n be an integer ≥ 2 and let $a_1, \dots, a_n$ be positive reals satisfying $2a_i < a_{i+1}$ for $i = 1, \dots, n-1$. Let $p_1, \dots, p_n$ be positive reals such that $p_i \geq p_{i+1}$ for $i = 1, \dots, n-1$ and $\sum_{i=1}^{n} p_i = 1$. Let $\mu = \sum_{i=1}^{n} p_i \mu_{a_i}$. Then $\mu * \mu$ is strictly n-modal with modes $a_1, a_2, \dots, a_n$ and bottoms $0, 2a_1, 2a_2, \dots, 2a_n$.*

THEOREM 2. *Let $a_1, a_2, \dots$ be an infinite sequence of positive reals such that $2a_i < a_{i+1}$ for each i. Let $p_1, p_2, \dots$ be positive reals such that $p_i \geq p_{i+1}$ for each i and $\sum_{i=1}^{\infty} p_i = 1$. Let $\mu = \sum_{i=1}^{\infty} p_i \mu_{a_i}$. Then $\mu * \mu$ is strictly ∞-modal with modes $a_1, a_2, \dots$ and bottoms $0, 2a_1, 2a_2, \dots$.*

Proofs of these results are simple and elementary. They are given in the next section. In Section 4 we show that lattice analogues of these theorems are also true.

3. Proofs.

The basic fact that we use is the following.

LEMMA . *Let $a > 0$. The convolution $\mu_a * \mu_a$ is a triangular distribution with a continuous density $f(x)$ and*

$$f(x) = 0, a^{-2}x, 2a^{-1} - a^{-2}x, 0$$

on $(-\infty, 0], [0, a], [a, 2a], [2a, +\infty)$, respectively. If $0 < a_1 < a_2$, then $\mu_{a_1} * \mu_{a_2}$ is a trapezoidal distribution with a continuous density $f(x)$ such that

$$f(x) = 0, (a_1 a_2)^{-1} x, a_2^{-1}, (a_1 a_2)^{-1}(a_1 + a_2 - x), 0$$

on $(-\infty, 0], [0, a_1], [a_1, a_2], [a_2, a_1 + a_2], [a_1 + a_2, +\infty)$, respectively.

Proof of this lemma is straightforward.

PROOF OF THEOREM 1. Denote the density of $\mu_{a_i} * \mu_{a_j}$ by $f_{ij}(x)$. Since

$$(1) \qquad \mu * \mu = \sum_i p_i^2 \mu_{a_i} * \mu_{a_i} + 2 \sum_{i>j} p_i p_j \mu_{a_i} * \mu_{a_j},$$

the distribution $\mu * \mu$ has density

$$f(x) = \sum_i p_i^2 f_{ii}(x) + 2\sum_{i>j} p_i p_j f_{ij}(x). \tag{2}$$

The graph of f_{ii} is a polygonal line with vertices at $x = 0, a_i, 2a_i$. If $i > j$, then that of f_{ij} is a polygonal line with vertices at $x = 0, a_j, a_i, a_i + a_j$. Hence the graph of $f(x)$ is also a polygonal line and the x-coordinates of its vertices are, in increasing order,

$$0, a_1, 2a_1, a_2, a_2 + a_1, 2a_2, a_3, a_3 + a_1, a_3 + a_2, 2a_3, a_4, \dots,$$
$$a_n, a_n + a_1, \dots, a_n + a_{n-1}, 2a_n.$$

Note that we have used our assumption that $2a_i < a_{i+1}$ for each i. In the sequence there are i points between a_i and a_{i+1}. The derivative $f'(x)$ is a step function. Let us denote the value of f' on an interval I by $f'|I$. For convenience we denote $a_0 = 0$. We claim the following three facts:

(i) $f'|(2a_k, a_{k+1}) > 0$ for $0 \le k \le n-1$.

(ii) $f'|(a_k + a_l, a_k + a_{l+1}) < f'|(a_k + a_{l+1}, a_k + a_{l+2})$ for $n \ge k > l + 1 \ge 1$.

(iii) $f'|(a_k + a_{k-1}, 2a_k) < 0$ for $1 \le k \le n$.

These imply that, for each k, the function f is convex on (a_k, a_{k+1}), strictly decreasing on $(a_k, 2a_k)$, and strictly increasing on $(2a_k, a_{k+1})$. Hence, if (i), (ii) and (iii) are established, then $\mu * \mu$ is strictly n-modal with modes $a_1, \dots, a_n$ and bottoms $0, 2a_1, \dots, 2a_n$. Write $c_i = p_i/a_i$ for $i \ge 1$.

Proof of (i): We have

$$f'(x) = \sum_i p_i^2 f'_{ii}(x) + 2\sum_{i>j} p_i p_j f'_{ij}(x), \tag{3}$$

except at the points in the sequence. On the interval $(2a_k, a_{k+1})$, we see that f_{ii} vanishes for $i \le k$, f_{ij} vanishes for $k \ge i > j$ and f_{ij} is flat for $i \ge k + 1 > j$. Hence

$$f'|(2a_k, a_{k+1}) = \sum_{i>k} c_i^2 + 2\sum_{i>j\ge k+1} c_i c_j > 0.$$

Proof of (ii): Let $k > l \ge 0$ and denote $I_{kl} = (a_k + a_l, a_k + a_{l+1})$. On the interval I_{kl}, the functions f_{ii} and f_{ij} vanish for $k > i > j$, and f_{kj} vanishes for $j \le l$. Moreover, f_{ij} for $i \ge k + 1 > j$ are flat there. Thus we have, from (3),

$$\begin{aligned} f'|I_{kl} = &-c_k^2 + \left(c_{k+1}^2 + \cdots + c_n^2\right) \\ &+ 2\big[(-c_k c_{l+1} - \cdots - c_k c_{k-1}) + c_{k+2}c_{k+1} \\ &\quad + (c_{k+3}c_{k+1} + c_{k+3}c_{k+2}) \\ &\quad + (c_{k+4}c_{k+1} + c_{k+4}c_{k+2} + c_{k+4}c_{k+3}) \\ &\quad + \cdots + (c_n c_{k+1} + \cdots + c_n c_{n-1})\big]. \end{aligned} \tag{4}$$

If $k > l + 1$, then the expression of $f'|I_{k,l+1}$ is obtained from the right-hand side of (4) by deleting the term $-c_k c_{l+1}$. It follows that $f'|I_{kl} < f'|I_{k,l+1}$ for $n \geq k > l+1 \geq 1$.

Proof of (iii): Let $k \geq 1$. We get an expression for $f'|I_{k,k-1}$ by letting $l = k-1$ in (4). The term $(-c_k c_{l+1} - \cdots - c_k c_{k-1})$ does not exist in this case. Using the assumption $p_k \geq p_{k+1} \geq \cdots$, we have

$$\begin{aligned} f'|I_{k,k-1} \leq p_k^2\big\{-a_k^{-2} &+ \big(a_{k+1}^{-2} + a_{k+2}^{-2} + \cdots + a_n^{-2}\big) \\ &+ 2\big[a_{k+2}^{-1}a_{k+1}^{-1} + a_{k+3}^{-1}\big(a_{k+1}^{-1} + a_{k+2}^{-1}\big) \\ &+ a_{k+4}^{-1}\big(a_{k+1}^{-1} + a_{k+2}^{-1} + a_{k+3}^{-1}\big) \\ &+ \cdots + a_n^{-1}\big(a_{k+1}^{-1} + a_{k+2}^{-1} + \cdots + a_{n-1}^{-1}\big)\big]\big\}. \end{aligned}$$

Now recall the assumption that $2^i a_k < a_{k+i}$ for $i \geq 1$. Then

$$\begin{aligned} f'|I_{k,k-1} &\leq c_k^2\big\{-1 + (2^{-2} + 2^{-4} + \cdots + 2^{-2(n-k)}) \\ &\qquad + 2\big[2^{-2}2^{-1} + 2^{-3}(2^{-1} + 2^{-2}) + 2^{-4}(2^{-1} + 2^{-2} + 2^{-3}) \\ &\qquad + \cdots + 2^{-(n-k)}(2^{-1} + \cdots + 2^{-(n-k-1)})\big]\big\} \\ &< c_k^2\left(-1 + 2^{-2}\sum_{i=0}^{\infty} 2^{-2i} + 2^{-2}\sum_{i=0}^{\infty} 2^{-i}\sum_{j=0}^{i} 2^{-j}\right) \\ &= c_k^2(-1 + 3^{-1} + 3^{-1}2) \\ &= 0. \end{aligned}$$

The proof is complete. □

Proof of Theorem 2. We can proceed similarly to the proof of Theorem 1. The only difference is that now we deal with infinite sums instead of finite sums. The distribution μ is absolutely continuous with a density $f(x)$ and the expressions (1) and (2) still hold. Note that $|f_{ij}| \leq a_i^{-1} \leq a_1^{-1}$ and $|f'_{ij}| \leq (a_i a_j)^{-1} \leq a_1^{-2}$ for $i \geq j$. We see that the sums in the right-hand side of (2) are uniformly convergent and are, except at a sequence of points, termwise differentiable, yielding the expression (3). The exceptional points consist of $0, a_1, a_2, \ldots$ together with i points

$$a_i + a_1, a_i + a_2, \ldots, a_i + a_{i-1}, 2a_i$$

between a_i and a_{i+1} for each i. Thus the function $f(x)$ is continuous on the whole line and its graph is a polygonal line with vertices having x-coordinates at these points. The assertion of the theorem is obtained by checking the three facts (i), (ii) and (iii), where, in the statement, $0 \leq k \leq n-1$, $n \geq k > l + 1 \geq 1$ and $1 \leq k \leq n$ are replaced with $0 \leq k < \infty$, $\infty > k > l + 1 \geq 1$ and $1 \leq k < \infty$, respectively. The proof of these three facts is accomplished in the same way as before, except for an obvious change of some finite sums into infinite sums. This completes the proof. □

4. Lattice analogues. Let us consider lattice distributions, that is, distributions on the integers. In this section, for integers a, b with $a \le b$, the interval $[a, b]$ means the set of integers x satisfying $a \le x \le b$. The intervals $(-\infty, a]$ and $[b, +\infty)$ are understood similarly. Distributions in this section are exclusively lattice distributions and numbers a, b, a_i, b_i are integers. Unimodality and related properties are frequently investigated also for lattice distributions; see Keilson and Gerber [7] and Dharmadhikari and Joag-dev [2]. For any lattice distribution μ, denote $f(x) = \mu(\{x\})$ and $g(x) = f(x+1) - f(x)$ for integers x. A distribution μ is said to be unimodal with mode a if $g(x) \ge 0$ on $(-\infty, a-1]$ and $g(x) \le 0$ on $[a, +\infty)$. We say that a distribution μ is *strictly n-modal with modes* $a_1, \dots, a_n$ and *bottoms* $b_1, \dots, b_{n+1}$ if

$$b_1 < a_1 < b_2 < a_2 < \cdots < b_n < a_n < b_{n+1}$$

and if

$$g|(-\infty, b_1 - 1] = 0, \qquad g|[b_{n+1}, +\infty) = 0,$$

$$g|[b_i, a_i - 1] > 0 \quad \text{for } 1 \le i \le n,$$

$$g|[a_i, b_{i+1} - 1] < 0 \quad \text{for } 1 \le i \le n.$$

A distribution μ is *strictly ∞-modal with modes* $a_1, a_2, \dots$ and *bottoms* $b_1, b_2, \dots$ if

$$b_1 < a_1 < b_2 < a_2 < \cdots$$

and if

$$g|(-\infty, b_1 - 1] = 0,$$

$$g|[b_i, a_i - 1] > 0 \quad \text{for } i \ge 1,$$

$$g|[a_i, b_{i+1} - 1] < 0 \quad \text{for } i \ge 1.$$

For $a \ge 0$, denote by μ_a the lattice uniform distribution on $[0, a]$, that is, $f(x) = (a+1)^{-1}$ for $x \in [0, a]$.

The lattice analogues of Theorems 1 and 2 are as follows.

THEOREM 3. *Let $a_1, \dots, a_n$ be positive integers such that $2a_i + 1 < a_{i+1}$ for $1 \le i \le n-1$. Let $p_1, \dots, p_n$ be positive reals satisfying $p_1 \ge p_2 \ge \cdots \ge p_n$ and $\sum_{i=1}^{n} p_i = 1$, and let $\mu = \sum_{i=1}^{n} p_i \mu_{a_i}$. Then $\mu * \mu$ is strictly n-modal with modes $a_1, a_2, \dots, a_n$ and bottoms $-1, 2a_1 + 1, 2a_2 + 1, \dots, 2a_n + 1$.*

THEOREM 4. *Let $a_1, a_2, \dots$ be an infinite sequence of positive integers satisfying $2a_i + 1 < a_{i+1}$ for each i. Let $p_1, p_2, \dots$ be positive reals such that $p_i \ge p_{i+1}$ for each i and $\sum_{i=1}^{\infty} p_i = 1$. Let $\mu = \sum_{i=1}^{\infty} p_i \mu_{a_i}$. Then $\mu * \mu$ is strictly ∞-modal with modes $a_1, a_2, \dots$ and bottoms $-1, 2a_1 + 1, 2a_2 + 1, \dots$.*

The analogue of the lemma is as follows.

LEMMA. *Let* $0 < a < b$. *If* $\mu = \mu_a * \mu_a$, *then*

$$f(x) = 0, (a+1)^{-2}(x+1), (a+1)^{-2}(2a-x+1), 0,$$

$$g(x) = 0, (a+1)^{-2}, -(a+1)^{-2}, 0$$

on $(-\infty, -2], [-1, a-1], [a, 2a], [2a+1, +\infty)$, *respectively. If* $\mu = \mu_a * \mu_b$, *then*

$$f(x) = 0, (a+1)^{-1}(b+1)^{-1}(x+1), (b+1)^{-1}, (a+1)^{-1}(b+1)^{-1}(a+b-x+1), 0,$$

$$g(x) = 0, (a+1)^{-1}(b+1)^{-1}, 0, -(a+1)^{-1}(b+1)^{-1}, 0$$

on $(-\infty, -2], [-1, a-1], [a, b-1], [b, a+b], [a+b+1, +\infty)$, *respectively.*

Now proofs of Theorems 3 and 4 are done in the same way as in Section 3. Note that, by the lemma above, the function $g(x)$ for the distribution μ in Theorem 3 is constant on each of the intervals

$$(-\infty, -2], [-1, a_1 - 1], [a_1, 2a_1], [2a_1 + 1, a_2 - 1], [a_2, a_2 + a_1], [a_2 + a_1 + 1, 2a_2], [2a_2 + 1, a_3 - 1], \dots, [2a_n + 1, +\infty)$$

We can see that $g(x)$ vanishes on $(-\infty, -2]$ and $[2a_n + 1, +\infty)$, is positive on $[-1, a_1 - 1]$ and $[2a_i + 1, a_{i+1} - 1]$ for $1 \le i \le n-1$, and is negative on $[a_i, 2a_i]$ for $1 \le i \le n$. We have assumed that $2a_i + 1 < a_{i+1}$ in order to guarantee non-emptiness of the interval $[2a_i + 1, a_{i+1} - 1]$. If we assume only $2a_i < a_{i+1}$ instead of $2a_i + 1 < a_{i+1}$, the assertion becomes false.

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DEPARTMENT OF MATHEMATICS
COLLEGE OF GENERAL EDUCATION
NAGOYA UNIVERSITY
NAGOYA, 464-01
JAPAN