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A CONTINUOUS VERSION OF DE FINETTI'S THEOREM

BY L. ACCARDI AND Y. G. LU[1]

Università di Roma II

A continuous version of De Finetti's theorem is proved in which the role of the homogeneous product states is played by the independent increment stationary processes on the real line. The proof is based on a conditional, finite De Finetti's theorem (i.e., a result involving only a finite number of random variables and exchangeable conditional expectations rather than exchangeable probabilities).

Our technique of proof improves and simplifies a result of Freedman and includes a generalization of the quantum De Finetti's theorem as well as some more recent variants of it. The last section of the paper is an attempt to answer a question of Diaconis and Freedman.

1. Introduction. It is known that De Finetti's theorem characterizes the sequences of independent identically distributed (i.i.d) random variables as the extremal points of the sequences of exchangeable random variables, that is, invariant under permutations (we identify a process with its law).

Now it is well known that the independent increment stationary processes are the continuous analogue of the i.i.d sequences. Moreover, on the general class of *increment processes* (also called *additive processes*) on the real line, that is, stochastic processes $\{\xi_I\}_{I \in \mathcal{J}}$ indexed by the family $\mathcal{J}$ of bounded closed intervals of $\mathbf{R}$ and satisfying

$$\xi_{[a,b]} + \xi_{[b,c]} = \xi_{[a,c]}, \tag{1.1}$$

one can naturally define an action of the permutation group by permuting among themselves intervals which are the translates of one another (cf. Section 3 for a precise definition of this action). An increment process is called *exchangeable* if it is invariant under the above mentioned action of the permutation group.

It is natural to conjecture that, in analogy with the discrete index case, the extremal points of the exchangeable increment processes are the independent increment stationary processes. This conjecture (in fact a much more general one, which is restricted neither to real valued processes nor to one-dimensional index sets and includes a quantum probabilistic generalization) shall be proved in the present paper. The result follows as a corollary from a new proof of De Finetti's theorem (cf. Section 3) which unifies the classical and the quantum (Boson) case (cf. [21]). The idea of the proof puts together the approaches

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of [5] and [2] into a conditional finite De Finetti's theorem in the spirit of [6]. We also use a general result from operator algebras [cf. Theorem (3.2)] which shows that the simplex property (i.e., the uniqueness of the mixing measure) has to do with groups much more general than the symmetric group on $\mathbf{N}$, while the product structure of the extremals seems to be more strictly related to this group. Our approach also provides a unified setting for the various De Finetti type results discussed in [6] and for their quantum analogues (this point is discussed in Section 5).

On the other hand, our algebraic approach, when restricted to the classical probabilistic case, corresponds to De Finetti's theorem on the field (not σ-field) generated by the cylindrical sets, hence it does not include the subtle and deep results of Dubins and Freedman [7] which show that, when the Kolmogorov extension theorem does not hold, De Finetti's theorem, although universally valid on the cylindrical field as shown by Hewitt and Savage [11], might fail on the full σ-field.

After the completion of the first draft of this paper, we have discovered that a continuous version of De Finetti's theorem had already been proved in 1963 by Freedman [10] (who also refers to a previous work of Bühlman [4] which we were not able to find). Freedman's technique is completely different from ours and, in one respect, is more general since it yields analogous results for stationary Markov chains, but in another respect is less general, since it does not seem to allow a natural extension to processes with multidimensional index set or to quantum process. Moreover, even in the case of a one-dimensional index, we replace Freedman's condition of absence of fixed points of discontinuity with the (apparently) weaker condition (iii) of Theorem (4.1) which, in the classical case, amounts to the statement that, if (I_n) is a sequence of open subintervals of an interval $I \subset \mathbf{R}$, whose union is dense in I, then the σ-field $\mathscr{F}_I$ is generated by the σ-fields $\mathscr{F}_{I_n}$, $n \in \mathbf{N}$, (if J is an interval of $\mathbf{R}$, $\mathscr{F}_J$ denotes the σ-field generated by the increments of the process in J). An extended study of processes with exchangeable increments is due to Kallenberg [14–17], with emphasis on connections with stopping times and path properties. Also in this case, the techniques used seem to be different from ours.

Because of Hudson's result [13] according to which a locally normal exchangeable state is a mixture of normal states, our results also intersect (but do not include completely) the results of Dubins and Freedman [7] and of Diaconis and Freedman. Finally [6], the continuous De Finetti's theorem establishes in a rigorous way an interpretation, in terms of exchangeability of the Fock space (and more generally of continuous tensor products of Hilbert spaces), a fact which was shadowed in the deep analysis of several applications in physics of De Finetti's theorem due to Bach [3].

Our proof also applies to the recent extension of De Finetti's theorem due to Fannes, Lewis and Verbeure [8].

REMARK. Recently L. Pratelli informed us that he was able to adapt the technique of [13] to include the proof of the continuous De Finetti theorem for classical independent increment processes.

2. Basic notation and estimates. We shall use in the following a formalism which is familiar in quantum probability (cf. [1] for a survey) but, since we do not assume the reader to be familiar with the quantum probabilistic formalism, we briefly outline the interpretation, in terms of classical probability, of the notation we are going to use.

Let (X_n) be a sequence of random variables defined on a probability space $(\Omega, \mathscr{F}, P)$ and with values in a measurable space (S, θ). Denote $\mathscr{B} = L^\infty(S, \theta)$, $\tilde{\mathscr{A}} = L^\infty(\Omega, \mathscr{F}, P)$ (unless otherwise specified all the measurable functions we consider are complex valued). Endowed with the sup (ess sup) norm and with the involution given by complex conjugation ($f^*(x) = \overline{f(x)}$) both $\mathscr{B}$ and $\tilde{\mathscr{A}}$ are C^* algebras and the P-integral on $\tilde{\mathscr{A}}$ defines a normalized positive linear functional or, in the C^* terminology, a *state* φ on $\tilde{\mathscr{A}}$. Each random variable X_n defines a $*$-homomorphism

$$j_n\colon f \in \mathscr{B} \to j_n(f) := f \circ X_n \in \tilde{\mathscr{A}} \tag{2.0}$$

and the restriction of the state φ on the algebra $j_n(\mathscr{B})$ uniquely determines the distribution of X_n.

Denoting by $\mathscr{A}$ the norm closure in $\tilde{\mathscr{A}}$ of the algebra generated by the $j_n(\mathscr{B})$ for all natural integers n, $\mathscr{A}$ is a C^* subalgebra of $\tilde{\mathscr{A}}$ and the restriction of the state φ on $\mathscr{A}$ uniquely determines the restriction of the probability measure P on the subfield (not a σ-field) $\mathscr{F}_0$ of $\mathscr{F}$ generated by the cylindrical sets.

The abstract context in the following extends this situation to the case in which the algebras $\mathscr{B}$ and $\mathscr{A}$ are arbitrary C^* algebras (always with unit) and the embedding j_n need not be of the form (2.0). There are plenty of examples in physics (quantum spin systems, quantum electromagnetic field in a bounded region, ...) which motivate this extension.

For $N \in \mathbf{N}$, denote by $\mathscr{S}_N$ the permutation group on $\{1, \dots, N\}$. With the convention

$$\sigma(k) = k \quad \text{for } k > N,\ \sigma \in \mathscr{S}_N,$$

we consider $\mathscr{S}_N$ as acting on the natural integers $\mathbf{N}$. With this convention $\mathscr{S}_N \subseteq \mathscr{S}_{N+1}$ and

$$\mathscr{S}_\infty := \bigcup_{N \in \mathbf{N}} \mathscr{S}_N$$

is a group of transformations of $\mathbf{N}$, called the *symmetric group* on $\mathbf{N}$.

Let $\mathscr{A}, \mathscr{B}$ be two topological algebras and assume that their topologies are given by submultiplicative seminorms both denoted by $\|\cdot\|$. For each $n \in \mathbf{N}$ let there be given a continuous homomorphism $j_n\colon \mathscr{B} \to \mathscr{A}$ and an action of $\mathscr{S}_\infty$ by $*$-automorphisms of $\mathscr{A}$ such that for each $\pi \in \mathscr{S}_\infty$, for each $n_1, n_2, \dots, n_m \in \mathbf{N}$, $m \in \mathbf{N}$ and for each $b_1, b_2, \dots, b_m \in \mathscr{B}$ one has

$$\pi\big(j_{n_1}(b_1) \cdot \cdots \cdot j_{n_m}(b_m)\big) = j_{\pi(n_1)}(b_1) \cdot \cdots \cdot j_{\pi(n_m)}(b_m). \tag{2.1}$$

EXAMPLE 2.1. Let $\mathscr{B}$ be a C^* algebra, $\mathscr{A} := \otimes_{\mathbf{N}} \mathscr{B}$ the C^*-tensor product of countably many copies of $\mathscr{B}$ and $j_n\colon \mathscr{B} \to \mathscr{A}$ the canonical embedding on $\mathscr{B}$

into the nth factor of the product ($n \in \mathbf{N}$). There is a natural action of $\mathscr{S}_\infty$ on $\mathscr{A}$ by endomorphisms characterized by the property

$$\pi j_n = j_{\pi(n)}, \qquad \pi \in \mathscr{S}_\infty \tag{2.2}$$

and with this action $\mathscr{S}_\infty$ is identified with a group of automorphisms of $\mathscr{A}$.

In the case of $\mathscr{B} = L^\infty(S, \theta)$, where (S, θ) is a standard Borel space, the assignment of a state μ on $\mathscr{A}$ is equivalent to the assignment of a cylindrical measure on the sample space $\Omega := \prod_N S$, hence, by Kolmogorov's theorem, a probability measure on Ω. The state μ is exchangeable in the sense of Definition 2.2 if and only if the measure, associated to μ in the way previously described, is exchangeable in the usual sense of classical probability. Therefore the situation considered in the present Example 2.1 includes the formulation of the classical De Finetti's theorems.

DEFINITION 2.2. A state μ on $\mathscr{A}$ will be called *exchangeable* or symmetric or $\mathscr{S}_\infty$ *invariant* if it is a fixed point for the action of $\mathscr{S}_\infty$ on the states on $\mathscr{A}$, that is,

$$\mu \circ \pi = \mu, \qquad \forall\, \pi \in \mathscr{S}_\infty. \tag{2.3}$$

The set of all exchangeable states on $\mathscr{A}$ will be denoted $\mathscr{S}(\mathscr{A}; \mathscr{S}_\infty)$. It is a weak*-compact convex set. De Finetti's theorem states that, in the conditions of Example 2.1, $\mathscr{S}(\mathscr{A}; \mathscr{S}_\infty)$ is a Choquet simplex and its extremal points are exactly the homogeneous product states.

Define, for each natural integer N,

$$E_{S_N} = E_N = \frac{1}{N!} \sum_{\pi \in \mathscr{S}_N} \pi(\cdot). \tag{2.4}$$

PROPOSITION 2.3. *The maps E_N have the following properties*:

(i) *For each $N \in \mathbf{N}$, $\pi \in \mathscr{S}_N$,*

$$E_N = E_N \circ \pi = \pi \circ E_N. \tag{2.5}$$

(ii) *E_N is a norm one projection* [*i.e.*, $E_N^2 = E_N$; $E_N(1) = 1$].

(iii) *For each $N \in \mathbf{N}$ the fixed points of E_N coincide with the fixed points of $\mathscr{S}_N$.*

PROOF. (i) is a simple computation. (ii) and (iii) follow from (i) and (2.4). □

REMARK. The property (ii) implies that each map E_N enjoys the properties which, in the classical case, according to a result of Moy [19] characterize a conditional expectation as an operator on the bounded measurable functions. In the general case, Proposition 2.3 implies that E_N is an Umegaki conditional expectation on $\mathscr{A}$ (cf. [1] for the definition) and its range coincides with the algebra $\mathscr{A}(\mathscr{S}_N)$ of its fixed points which are also the fixed points of $\mathscr{S}_N$. Since

$\mathcal{S}_N \subseteq \mathcal{S}_{N+1}$ one has also

$$\mathcal{A}(\mathcal{S}_N) \supseteq \mathcal{A}(\mathcal{S}_{N+1}). \tag{2.6}$$

The exchangeable states can be characterized in terms of the E_N, in fact:

PROPOSITION 2.4. *A state μ on $\mathcal{A}$ is exchangeable if and only if*

$$\mu \circ E_N = \mu, \qquad N \in \mathbf{N}. \tag{2.7}$$

PROOF. If μ is exchangeable, then for each natural integer N,

$$\mu \circ E_N = \frac{1}{N!} \sum_{\pi \in \mathcal{S}_N} \mu \circ \pi = \mu. \tag{2.8}$$

Conversely if (2.7) holds then, using (2.5), one easily sees that for each N and $\pi \in \mathcal{S}_N$ one has

$$\mu \circ \pi = \mu \circ E_N \circ \pi = \mu \circ E_N = \mu. \tag{2.9}$$

Therefore, since each $\pi \in \mathcal{S}_\infty$ belongs to $\mathcal{S}_N$, for N large enough, μ is exchangeable. □

LEMMA 2.5. *The family (E_N) enjoys the following properties*:

$$E_N \circ E_M = E_N \quad \text{for } M \leq N, \tag{2.10a}$$

$$E_N \circ j_k = \frac{1}{N} \sum_{h=1}^{N} j_h \quad \text{if } k \leq N. \tag{2.10b}$$

PROOF. Equation (2.10a) follows from (2.5) because, for $M \leq N$, $\mathcal{S}_M \subseteq \mathcal{S}_N$. To prove (2.10b), notice that the number of permutations in $\mathcal{S}_N$ which map k to h $(h, k = 1, \ldots, N)$ is exactly $(N-1)!$. Then (2.4) implies that

$$E_N \circ j_k = \frac{1}{N} \sum_{h=1}^{N} \frac{1}{(N-1)!} \sum_{\substack{\pi \in \mathcal{S}_N \\ \pi_k = h}} \pi \circ j_k = \frac{1}{N} \sum_{h=1}^{N} j_h. \qquad \square$$

The following result is a finite form of the conditional De Finetti's theorem much in the spirit of [6] (which considers the nonconditional case). Notice that our conditional bound has the same order of magnitude, in m and in N, as the bound of [6].

LEMMA 2.6. *Let there be given $k < n_1 < n_2 < \cdots < n_m < N \in \mathbf{N}$, $m \in \mathbf{N}$. For each $a, b_1, \ldots, b_m \in \mathcal{B}$ one has*

$$\begin{aligned} &\| E_N\big(j_k(a) \cdot j_{n_1}(b_1) \cdot \cdots \cdot j_{n_m}(b_m)\big) \\ &\quad - E_N\big(j_k(a)\big) \cdot E_N\big(j_{n_1}(b_1)\big) \cdot \cdots \cdot E_N\big(j_{n_m}(b_m)\big) \| \\ &\quad \leq \frac{m^2}{N} \|a\| \cdot \|b_1\| \cdot \cdots \cdot \|b_m\|. \end{aligned} \tag{2.11}$$

In particular,

$$\lim_{N\to\infty} \| E_N\big(j_k(a)\cdot j_{n_1}(b_1)\cdot\cdots\cdot j_{n_m}(b_m)\big) - E_N\big(j_k(a)\big)\cdot E_N\big(j_{n_1}(b_1)\big)\cdot\cdots\cdot E_N\big(j_{n_m}(b_m)\big)\| = 0. \tag{2.12}$$

PROOF. The difference in the norm $\|\cdot\|$ of the left-hand side of (2.11) is equal to, by definition,

$$\begin{aligned} &\frac{1}{N(N-1)\cdots(N-m+1)} \\ &\times \sum_{1\le r_1\le N}\ \sum_{\substack{1\le r_2\le N\\ r_2\notin\{r_1\}}}\ \sum_{\substack{1\le r_3\le N\\ r_3\notin\{r_1,r_2\}}}\cdots\ \sum_{\substack{1\le r_m\le N\\ r_m\notin\{r_1,\dots,r_{m-1}\}}} j_{r_1}(b_1)\cdots j_{r_m}(b_m) \\ &-\frac{1}{N^m}\sum_{1\le r_1\le N}\ \sum_{1\le r_2\le N}\cdots\ \sum_{1\le r_m\le N} j_{r_1}(b_1)\cdots j_{r_m}(b_m). \end{aligned} \tag{2.13}$$

Using the identity $\{1,\dots,N\} = (\{1,\dots,N\}\setminus\{r\})\cup\{r\}$, one can rewrite the second term of (2.13) as

$$\begin{aligned} &\frac{1}{N^m}\sum_{1\le r_1\le N}\ \sum_{\substack{1\le r_2\le N\\ r_2\notin\{r_1\}}}\ \sum_{1\le r_3\le N}\cdots\ \sum_{1\le r_m\le N} j_{r_1}(b_1)\cdots j_{r_m}(b_m) \\ &+\frac{1}{N^m}\sum_{1\le r_1\le N}\ \sum_{1\le r_3\le N}\cdots\ \sum_{1\le r_m\le N} j_{r_1}(b_1)j_{r_1}(b_2)j_{r_3}(b_3)\cdots j_{r_m}(b_m). \end{aligned} \tag{2.14}$$

Notice the factor $j_{r_1}(b_2)$ [not $j_{r_2}(b_2)$] in the second term of (2.14) and also that the summation over the index r_2 is absent, so that in the second term there are only $m-1$ summations.

Similarly using the identity $\{1,\dots,N\} = (\{1,\dots,N\}\setminus\{r_1,r_2\})\cup\{r_1,r_2\}$ we see that the first term of (2.14) is equal to

$$\begin{aligned} &\frac{1}{N^m}\sum_{1\le r_1\le N}\ \sum_{\substack{1\le r_2\le N\\ r_2\notin\{r_1\}}}\ \sum_{\substack{1\le r_3\le N\\ r_3\notin\{r_1,r_2\}}}\ \sum_{1\le r_4\le N}\cdots\ \sum_{1\le r_m\le N} j_{r_1}(b_1)\cdots j_{r_m}(b_m) \\ &+\frac{1}{N^m}\sum_{1\le r_1\le N}\ \sum_{\substack{1\le r_2\le N\\ r_2\notin\{r_1\}}}\ \sum_{1\le r_4\le N}\cdots\ \sum_{1\le r_m\le N} j_{r_1}(b_1)j_{r_2}(b_2) \\ &\times\big[j_{r_1}(b_3)+j_{r_2}(b_3)\big]j_{r_4}(b_4)\cdots j_{r_m}(b_m). \end{aligned} \tag{2.15}$$

Notice that again the second term of (2.15) has only $m-1$ summations (r_3 is absent). Iterating this argument for the index r_4; then for the index $r_5,\dots$; and finally for the index r_m, the second term of (2.13) is found to be

equal to

$$\frac{1}{N^m}\sum_{1\le r_1\le N}\sum_{\substack{1\le r_2\le N\\ r_2\notin\{r_1\}}}\sum_{\substack{1\le r_3\le N\\ r_3\notin\{r_1,r_2\}}}\cdots\sum_{\substack{1\le r_m\le N\\ r_m\notin\{r_1,\dots,r_{m-1}\}}} j_{r_1}(b_1)\cdots j_{r_m}(b_m)$$

$$+\frac{1}{N^m}\sum_{1\le r_1\le N}\sum_{1\le r_3\le N}\cdots\sum_{1\le r_m\le N} j_{r_1}(b_1)j_{r_1}(b_2)j_{r_3}(b_3)\cdots j_{r_m}(b_m)$$

$$+\frac{1}{N^m}\sum_{1\le r_1\le N}\sum_{\substack{1\le r_2\le N\\ r_2\notin\{r_1\}}}\sum_{1\le r_4\le N}\cdots\sum_{1\le r_m\le N} j_{r_1}(b_1)j_{r_2}(b_2) \tag{2.16}$$

$$\times\big(j_{r_1}(b_3)+j_{r_2}(b_3)\big)j_{r_4}(b_4)\cdots j_{r_m}(b_m)$$

$$+\cdots+\frac{1}{N^m}\sum_{1\le r_1\le N}\sum_{\substack{1\le r_2\le N\\ r_2\notin\{r_1\}}}\cdots\sum_{\substack{1\le r_{m-1}\le N\\ r_{m-1}\notin\{r_1,\dots,r_{m-2}\}}} j_{r_1}(b_1)j_{r_2}(b_2)$$

$$\cdots\big(j_{r_1}(b_m)+j_{r_2}(b_m)+\cdots+j_{r_{m-1}}(b_m)\big).$$

Notice that for each $p = 2,\dots,m$, in the pth term of (2.16) one has only $m-1$ summations and therefore the norm of the pth term is majorized by

$$\frac{p-1}{N}\prod_{k=1}^{m}\|b_k\|. \tag{2.17}$$

This shows that the norm of the summation of the last $m-1$ terms is less than or equal to

$$\sum_{p=2}^{m}\frac{p-1}{N}\prod_{k=1}^{m}\|b_k\| = \frac{m(m-1)}{2N}\prod_{k=1}^{m}\|b_k\|. \tag{2.18}$$

The norm of the difference between the first term of (2.16) and the first term of (2.13) is majorized by

$$\left(\frac{1}{N(N-1)\cdots(N-m+1)}-\frac{1}{N^m}\right)N(N-1) \tag{2.19}$$

$$\cdots(N-m+1)\prod_{k=1}^{m}\|b_k\|,$$

where the factor $N(N-1)\cdots(N-m+1)$, in the numerator, arises because the r_1 summation has N terms, the r_2 summation $N-1,\dots,$ the r_m summation $N-m+1$.

Now (2.19) is equal to

$$\left(1-\left(1-\frac{1}{N}\right)\cdots\left(1-\frac{m-1}{N}\right)\right)\prod_{k=1}^{m}\|b_k\| \tag{2.19a}$$

and by induction one easily checks that

$$\left(1-\left(1-\frac{1}{N}\right)\cdots\left(1-\frac{m-1}{N}\right)\right)\leq\frac{m^2}{2N},\qquad 2\leq m\leq N. \tag{2.20}$$

Adding (2.18) to (2.19a) and taking into account (2.20), one ends the proof. □

3. De Finetti's theorem. Let φ be an exchangeable state on $\mathscr{A}$ and denote by $\{\mathscr{H},\pi,\Phi\}$ the cyclic representation associated to $\{\mathscr{A},\varphi\}$, that is, $\mathscr{H}$ is an Hilbert space and $\pi\colon \mathscr{A}\to\mathscr{B}(\mathscr{H})$ is the representation of $\mathscr{A}$ by bounded operators on $\mathscr{H}$, characterized by the properties:

(i) $\{\pi(a)\Phi\colon a\in\mathscr{A}\}$ is a dense subspace of $\mathscr{H}$ (cyclicity);
(ii) $\langle\pi(a)\Phi,\pi(b)\Phi\rangle=\varphi(a^*b)$ for all $a,b\in\mathscr{A}$.

Let U denote the unitary representation of $\mathscr{S}_\infty$ on $\mathscr{H}$ characterized by

$$U_\sigma\pi(a)\cdot\Phi=\pi(\sigma(a))\Phi,\qquad a\in\mathscr{A},\ \sigma\in\mathscr{S}_\infty. \tag{3.1}$$

Let $\mathscr{H}_{\mathscr{S}_\infty}$ denote the fixed space of U, that is,

$$\mathscr{H}_{\mathscr{S}_\infty}:=\{\xi\in\mathscr{H}\colon U_\sigma\xi=\xi;\ \forall\,\sigma\in\mathscr{S}_\infty\} \tag{3.2a}$$

and $P_\infty\colon\mathscr{H}\to\mathscr{H}_{\mathscr{S}_\infty}$ the orthogonal projection, and define

$$E_\infty(a):=E_\infty^\varphi(a)=P_\infty\pi(a)P_\infty,\qquad a\in\mathscr{A}. \tag{3.2b}$$

LEMMA 3.1. *Let φ be exchangeable. Then for any $k\in\mathbf{N}$, $n_1,\dots,n_k\in\mathbf{N}$ such that $n_1<\cdots<n_k$, and for any $b_1,\dots,b_k\in\mathscr{B}$ one has*

$$\varphi\big(j_{n_1}(b_1)\cdot\cdots\cdot j_{n_k}(b_k)\big)=\big\langle\Phi,E_\infty(b_1)\cdot\cdots\cdot E_\infty(b_k)\Phi\big\rangle. \tag{3.2c}$$

PROOF. For each $N\in\mathbf{N}$, let P_N denote the orthogonal projection onto the closure of $\pi(E_N(\mathscr{A}))\cdot\Phi$. One easily checks that P_N is characterized by the property

$$P_N\pi(a)\cdot\Phi=\pi(E_N(a))\cdot\Phi,\qquad\forall\,a\in\mathscr{A}.$$

Because of (2.5), $U_\sigma P_N=P_N$ for each $\sigma\in\mathscr{S}_N$, hence $P_N\geq P_\infty$. The sequence (P_N) is decreasing so its strong limit Q exists. We claim that $Q=P_\infty$. Assume that $Q\neq P_\infty$, then, since $Q\geq P_\infty$, there exists a nonzero vector ξ, orthogonal to $\mathscr{H}_{\mathscr{S}_\infty}$ and such that $Q\xi=\xi$. Therefore, for each $N\in\mathbf{N}$, $P_N\xi=P_NQ\xi=\xi$, hence $U_\sigma\xi=\xi$ for each $\sigma\in\mathscr{S}_N$. Since N is arbitrary, $\xi\in\mathscr{H}_{\mathscr{S}_\infty}$ so ξ must be zero, contrary to the assumption. □

Now let k, n_j, b_j be as in Lemma (3.1). Then the exchangeability of φ and (2.12) imply that

$$\begin{aligned}
&\varphi\big(j_{n_1}(b_1)\cdot\cdots\cdot j_{n_k}(b_k)\big)\\
&\quad=\lim_{N\to\infty}\varphi\Big(E_N\big(j_{n_1}(b_1)\cdot\cdots\cdot j_{n_k}(b_k)\big)\Big)\\
&\quad=\lim_{N\to\infty}\varphi\Big(E_N\big(j_{n_1}(b_1)\big)\cdot\cdots\cdot E_N\big(j_{n_k}(b_k)\big)\Big)\\
&\quad=\lim_{N\to\infty}\big\langle\Phi, P_N\pi\big(j_{n_1}(b_1)\big)P_N\cdot\cdots\cdot P_N\pi\big(j_{n_k}(b_k)\big)P_N\cdot\Phi\big\rangle\\
&\quad=\big\langle\Phi, E_\infty(b_1)\cdot\cdots\cdot E_\infty(b_k)\Phi\big\rangle.
\end{aligned}$$

Recall from [15], Definition (3.1.11), that the triple $\{\mathscr{A}, \mathscr{S}_\infty, \varphi\}$ is said to be $\mathscr{S}_\infty$-*abelian* if $P_\infty\pi(\mathscr{A})P_\infty$ is a *commuting* family of operators. Notice that in classical probability, $\mathscr{A}$ is a subalgebra of $L^\infty(\Omega, \mathscr{F}, P)$, which is an abelian algebra. Hence the condition of $\mathscr{S}_\infty$-abelianity is automatically satisfied.

THEOREM 3.2 (De Finetti). *Suppose that, for any exchangeable state φ on $\mathscr{A}$, the triple $\{\mathscr{A}, \mathscr{S}_\infty, \varphi\}$ is $\mathscr{S}_\infty$-abelian. Then the exchangeable states on $\mathscr{A}$ are a Choquet simplex whose extremal points φ have the following property: There exists a state φ_0 on $\mathscr{B}$ such that for any $k \in \mathbf{N}$, $n_1 < \cdots < n_k$, $n_j \in \mathbf{N}$ and $b_1, \ldots, b_k \in \mathscr{B}$,*

$$\varphi\big(j_{n_1}(b_1)\cdot\cdots\cdot j_{n_k}(b_k)\big) = \varphi_0(b_1)\cdot\varphi_0(b_2)\cdot\cdots\cdot\varphi_0(b_k). \tag{3.3}$$

PROOF. The $\mathscr{S}_\infty$-abelianity implies that the exchangeable states are a Choquet simplex (cf. Theorem (3.1.14) of [20]). The same condition implies that for an extremal state φ one has $\mathscr{H}_{\mathscr{S}_\infty} = \mathbf{C}\cdot\Phi$ ($\mathbf{C}$ is the complex field), and since for any $a \in \mathscr{A}$,

$$\varphi(a) = \lim_{N\to\infty}\varphi\big(E_N(a)\big) = \lim_{N\to\infty}\big\langle\Phi, P_N\pi(a)P_N\Phi\big\rangle = \big\langle\Phi, E_\infty(a)\Phi\big\rangle,$$

it follows that

$$E_\infty(a) = \varphi(a)P_\infty, \qquad a \in \mathscr{A}. \tag{3.3a}$$

If φ is exchangeable, then the state on $\mathscr{B}$ defined by $b \in \mathscr{B} \mapsto \varphi(j_n(b)) =: \varphi_0(b)$ does not depend on n; in fact if $n < k$ are fixed and $\pi \in \mathscr{S}_\infty$ is such that $\pi n = k$, then

$$\varphi(j_n(b)) = \varphi\big(\pi[j_n(b)j_k(1)]\big) = \varphi(j_k(b)).$$

From this, (3.2c) and (3.3a), (3.3) follows immediately. □

REMARK. Notice that, up to now, no commutativity condition has been introduced. In particular we do not require that $j_n(b)$ and $j_m(b')$, $b, b' \in \mathscr{B}$, commute or anticommute.

A stationary state φ on $\mathscr{A}$ is called *1-ergodic* if for each $b \in \mathscr{B}$ one has

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \pi(j_n(b)) = \varphi_1(b) := \varphi(j_1(b)), \tag{3.4}$$

the limit being meant strongly in the cyclic representation π of $\{\mathscr{A}, \varphi\}$. More generally, φ is called *k-ergodic* if for each $b_0, \dots, b_k \in \mathscr{B}$, $n_0, n_1, \dots, n_k \in \mathbf{N}$,

$$\begin{aligned} &\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N} \pi\big(j_{n+n_0}(b_0)\big) \cdot \pi\big(j_{n+n_1}(b_1)\big) \cdot \cdots \cdot \pi\big(j_{n+n_k}(b_k)\big) \\ &\quad = \varphi\big(j_{n_0}(b_0) \cdot j_{n_1}(b_1) \cdot \cdots \cdot j_{n_k}(b_k)\big). \end{aligned} \tag{3.5}$$

A state which is k-ergodic for each k is called *∞-ergodic*.

COROLLARY 3.3. *In the assumptions of De Finetti's theorem, let φ be an exchangeable state on $\mathscr{A}$. Then the following statements are equivalent*:

(i) *φ is 1-ergodic.*
(ii) *φ is extremal exchangeable.*
(iii) *φ is ∞-ergodic.*

PROOF. (i) $\Rightarrow$ (ii): Using (2.10b) and (2.11) we obtain, for every $k \in \mathbf{N}$, $n_1 < \cdots < n_k \in \mathbf{N}$, $b_1, \dots, b_k \in \mathscr{B}$ and any $N \in \mathbf{N}$,

$$\begin{aligned} &\varphi\big(j_{n_1}(b_1) \cdots j_{n_k}(b_{n_k})\big) \\ &\quad = \varphi\Big(E_N\big(j_{n_1}(b_1) \cdots j_{n_k}(b_{n_k})\big)\Big) \\ &\quad = \left\langle \Phi, \pi\left(\frac{1}{N} \sum_{h_1=1}^{N} j_{h_1}(b_1) \cdot \cdots \cdot \frac{1}{N} \sum_{h_k=1}^{N} j_{h_k}(b_k) \cdot \Phi \right\rangle + O\left(\frac{1}{N}\right). \end{aligned} \tag{3.6}$$

Since each sequence $(1/N)\sum_{h=1}^{N} j_h(b_m)$, $N \in \mathbf{N}$, is bounded, by 1-ergodicity the limit of (3.6) for $N \to \infty$ exists and is equal to $\varphi_0(b_1) \cdot \cdots \cdot \varphi_0(b_k)$. This implies (3.3), hence the extremality of φ.

(ii) $\Rightarrow$ (iii): By De Finetti's theorem, the extremal exchangeable states are homogeneous product states, hence a fortiori ergodic (cf. [1]).

(iii) $\Rightarrow$ (i): This is obvious.

REMARK. Let $\mathscr{A}$, $\mathscr{B}$ and j_n, $n \in \mathbf{N}$, be as in Section 2. Recall that any endomorphism u of $\mathscr{A}$ satisfying

$$u \circ j_n = j_{n+1}, \qquad \forall\, n \in \mathbf{N} \tag{3.7}$$

is called a *shift* on $\mathscr{A}$ and that a state φ on A is called u-ergodic if for every $x \in \mathscr{A}$,

$$s - \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \pi(u^n(x)) = \varphi(x), \tag{3.8}$$

the limit being meant in the strong topology on the cyclic space of $\{\mathscr{A}, \varphi\}$.

If $\mathscr{A}$ is such that the products $j_{n_1}(b_1)\cdots j_{n_k}(b_k)$; $k \in \mathbf{N}$, $n_1 < \cdots < n_k \in \mathbf{N}$; $b_1, \ldots, b_k \in \mathscr{B}$ are total, then we say that $\mathscr{A}$ *satisfies condition TOD* (the totally ordered products are dense). Clearly if condition TOD is satisfied, then at most one shift can exist on $\mathscr{A}$ and in this case the notions of u-ergodicity and ergodicity as defined above coincide.

4. A continuous version of De Finetti's theorem. Let $\mathscr{I}$ denote the family of bounded intervals of $\mathbf{R}$ (open, closed or half-open). Then $\mathbf{R}$ acts on $\mathscr{I}$ by translation and an interval J, which can be obtained by translation from an interval I, is called a *copy* of I.

Given a C^*-algebra $\mathscr{A}$ and a family of subalgebras $\mathscr{A}_I (I \in \mathscr{I})$ of $\mathscr{A}$ such that

$$I \subseteq J \Rightarrow \mathscr{A}_I \subseteq \mathscr{A}_J,$$

let $v\colon x \in \mathbf{R} \to v_x \in \mathrm{Aut}(\mathscr{A})$ be a representation of $\mathbf{R}$ by endomorphisms of $\mathscr{A}$ for which the family $(\mathscr{A}_I)$ is covariant, that is,

$$v_x(\mathscr{A}_I) = \mathscr{A}_{I+x}, \qquad \forall\, x \in \mathbf{R}. \tag{4.1}$$

We also assume that each v_x has a left inverse, that is, there exists an isomorphism $v_x^*\colon v_x(\mathscr{A}) \to \mathscr{A}$ such that

$$v_x^* v_x = \mathbf{1}, \qquad \forall\, x \in \mathbf{R}.$$

Denoting, for each $I \in \mathscr{I}$, by $\min(I)$ [resp., $\max(I)$] the left (resp., right) extremum of I, a partial order α is induced on $\mathscr{I}$ by the prescription

$$I \prec J \quad \Leftrightarrow \quad \min I \leq \min J.$$

A tessellation of $\mathbf{R}$ is a sequence (I_n) of intervals in $\mathbf{R}$ such that all the I_n are copies of a single interval and

$$\bigcup_n I_n = \mathbf{R}, \qquad I_n^0 \cap I_m^0 = \varnothing \text{ if } n \neq m,$$

where I^0 denotes the interior of I.

The symmetric group $\mathscr{S}_\infty$ acts in the obvious way on each tessellation $\mathscr{I} = (I_n)$ of $\mathbf{R}$. Let us denote by $\pi^{\mathscr{I}}$ this action, that is,

$$\pi^{\mathscr{I}}(I_n) = I_{\pi(n)}, \qquad \pi \in \mathscr{S}_\infty,\ n \in \mathbf{N}.$$

Suppose that for each tessellation $\mathscr{I}$ of $\mathbf{R}$ there exists an action, still denoted $\pi^{\mathscr{I}}$, of $\mathscr{S}_\infty$ by automorphisms of $\mathscr{A}$ with the property that

$$\pi^{\mathscr{I}} \mathscr{A}_{I_n} = \mathscr{A}_{I_{\pi(n)}}, \qquad \forall\, \pi \in \mathscr{S}_\infty.$$

A state φ on $\mathscr{A}$ is called *exchangeable* if it is invariant under the action of $\pi^{\mathscr{I}}$ of each tessellation $\mathscr{I}$. Clearly this property is preserved under convex combinations. The set of all $\pi^{\mathscr{I}}$ with $\pi \in \mathscr{S}_\infty$ will be denoted $\mathscr{S}_\infty^{\mathscr{I}}$.

EXAMPLE 4.0a. Let (X_t), $t \in \mathbf{R}$, be a real stationary, independent increment process on $\mathbf{R}$. If $I = [a, b]$ is an interval, define $X_I = X_b - X_a$. If

$\mathscr{I} = (I_n)$ is a tessellation of $\mathbf{R}$, for any $n \in \mathbf{N}$ and $F \in L^\infty(\mathbf{R}^n)$, define

$$\pi^{\mathscr{I}} F(X_{I_1}, \ldots, X_{I_n}) = F(X_{I_{\pi(1)}}, \ldots, X_{I_{\pi(n)}}).$$

If $(\Omega, \mathscr{F}, P)$ is the space of the X process and

$$\mathscr{A} := L^\infty(\Omega, \mathscr{F}, P) \qquad \mathscr{A}_I := L^\infty(\Omega, \mathscr{F}_I, P_I),$$

$\mathscr{F}_I :=$ the σ-algebra spanned by the $X_t - X_s$ with $(s,t) \subseteq I$, then all the above conditions are satisfied and the process (X_I), $I \in \mathscr{I}$, is exchangeable.

EXAMPLE 4.0b. A quantum example is as follows. Let $\mathscr{H}$ denote the Fock space over $L^2(\mathbf{R})$ and let $\mathscr{A} = \mathscr{B}(\mathscr{H})$. For $I \in \mathscr{I}$ denote by $\mathscr{H}_I$ the subspace of $\mathscr{H}$ spanned by the exponential vectors $\psi(f)$ with $f \in L^2(\mathbf{R})$ and supp $f \subseteq I$. Let finally $\mathscr{A}_I = \mathscr{B}(\mathscr{H}_I)$ and $v_x (x \in \mathbf{R})$ be the automorphism of $\mathscr{A}$ induced by the second quantization of the translation by x on $L^2(\mathbf{R})$. Then the covariance condition (4.1) is satisfied and $v_x^* = v_{-x}$. Let $\mathscr{I} = (I_n)$ be a tessellation. Then

$$I_n = I_0 + nl_0, \qquad n \in \mathbf{I},$$

where $l_0 := |I_0|$ is the length of I_0.

For each fixed finite interval $[m, n] \subseteq \mathbf{Z}$ we use the identifications

$$\begin{aligned}\mathscr{A} &\cong \mathscr{A}_{\min(I_m)]} \otimes \mathscr{A}_{I_m} \otimes \cdots \otimes \mathscr{A}_{I_n} \otimes \mathscr{A}_{[\max(I_n)} \\ &\cong \mathscr{A}_{\min(I_m)]} \otimes v_{ml_0}(\mathscr{A}_{I_0}) \otimes \cdots \otimes v_{nl_0}(\mathscr{A}_{I_0}) \otimes \mathscr{A}_{[\max(I_n)},\end{aligned}$$

where $\mathscr{A}_{t]} := \mathscr{A}_{(-\infty, t]}$; $\mathscr{A}_{[t} := \mathscr{A}_{[t, +\infty)}$. Define the action of $\mathscr{S}_{[m,n]}$ on $\mathscr{A}$ by requiring that each permutation in $\mathscr{S}_{[m,n]}$ acts trivially on the past of I_m and on the future of I_n and permutes the intervals (I_k), for $k = 1, \ldots, n$. Then

$$\begin{aligned}&\pi^{\mathscr{I}}\left(a_{\min(I_m)]} \cdot v_{ml_0}(b_m) \cdot \cdots \cdot v_{nl_0}(b_n) \cdot a_{[\max(I_n)}\right) \\ &\quad = a_{\min(I_m)]} \cdot v_{\pi(m)l_0}(b_m) \cdot \cdots \cdot v_{\pi(n)l_0}(b_n) \cdot a_{[\max(I_n)}\end{aligned}$$

for any $\pi \in \mathscr{S}_{[m,n]}$, $a_{\min(I_m)]} \in \mathscr{A}_{\min(I_m)]}$, $a_{[\max(I_n)} \in \mathscr{A}_{[\max(I_n)}$, $b_m, b_{m+1}, \ldots, b_n \in \mathscr{A}_{I_0}$. The state φ on $\mathscr{A}$ defined by

$$\varphi(a) = \langle \Phi, a\Phi \rangle,$$

where Φ is the Fock vacuum, is exchangeable.

THEOREM 4.1. *Let $\mathscr{A}$ and $(\mathscr{A}_I)$ be as in the beginning of this section, and suppose that for each exchangeable state φ on $\mathscr{A}$:*

(i) *For each tessellation $\mathscr{I} = (I_n)$, the triple $\{\mathscr{A}, \mathscr{S}_\infty^{\mathscr{I}}, \varphi\}$ is $\mathscr{S}_\infty^{\mathscr{I}}$-abelian.*

(ii) *For each interval I and subintervals $I_1, \ldots, I_n \subseteq I$ such that*

$$I_j \prec I_{j+1},\ j = 1, \ldots, n, \ \bigcup_{j=1}^{n} I_j = I,$$

the products

$$a_{I_1} \cdot \cdots \cdot a_{I_n} \quad \text{with } a_{I_j} \in \mathscr{A}_{I_j},\ j = 1, \ldots, n$$

are total in $\mathscr{A}_I$.

(iii) *If* (I_n) *is a sequence of subintervals of* I *such that* $I_n \prec I_{n+1}$ *and the closure of* $\cup_n I_n = I$, *then* $\cup_n \mathscr{A}_{I_n}$ *is dense in* $\mathscr{A}_I$.

Then the exchangeable states are a simplex whose extremal points φ *satisfy*

$$\varphi(a_{I_1} \cdot \cdots \cdot a_{I_n}) = \varphi(a_{I_1}) \cdot \cdots \cdot \varphi(a_{I_n}) \tag{4.2}$$

for each $n \in \mathbf{N}$, $a_{I_j} \in \mathscr{A}_{I_j}$ *and for any sequence of disjoint intervals* $I_1, \ldots, I_n$.

PROOF. We begin by showing that, for any $n \in \mathbf{N}$ and for any set of pairwise disjoint intervals $I_1, \ldots, I_n \in \mathscr{I}$, whose lengths are rational numbers, (4.2) holds.

Since the length of each interval I_k is rational, there exists a tessellation (J_n) of $\mathbf{R}$ such that the length l_0 of J_0 is a rational number and each interval I_k is the union of a family of (J_n).

From Theorem (3.2) and assumption (i) it follows that, if φ is extremal, then

$$\varphi(a_{J_0} a_{J_1} \cdot \cdots \cdot a_{J_M}) = \varphi(a_{J_0})\varphi(a_{J_1}) \cdot \cdots \cdot \varphi(a_{J_M})$$

for any $a_{J_k} \in \mathscr{A}_{J_k}$, $k = 1, \ldots, M$.

Because of the assumption (ii) it then follows that

$$\varphi(a_{I_1} \cdot a_{I_2} \cdot \cdots \cdot a_{I_n}) = \varphi(a_{I_1}) \cdot \cdots \cdot \varphi(a_{I_n})$$

for a set of a_{I_j} dense in $\mathscr{A}_{I_j}$, $j = 1, \ldots, n$. Thus (4.2) holds if the intervals I_j have rational lengths. By approximation and assumption (iii) it holds in all cases.

The simplex property remains to be proved.

By (iii), a state φ is exchangeable if and only if it is $\pi^{\mathscr{I}}$ invariant, for every tessellation $\mathscr{I} = (I_n)$ with rational intervals.

If $\mathscr{I}'$ and $\mathscr{I}''$ are two such tessellations, then there is a third one $\mathscr{I}$ which refines both of them. By condition (iii), if $\mathscr{I}$ refines $\mathscr{I}'$, then $\mathscr{A}_\infty^{\mathscr{I}'}$ can be identified with a subgroup of $\mathscr{A}_\infty^{\mathscr{I}}$. This implies that the union of the $\mathscr{A}_\infty^{\mathscr{I}}$ over all the tessellations with rational intervals is a group [the inductive limit of the family $(\mathscr{A}_\infty^{\mathscr{I}})$ for the given embeddings]. Therefore the simplex property follows from [20], Theorem 3.1.14. □

REMARK. Let T be a topological space, G a topological group acting on T, $(x, g) \in T \times G \to xg \in T$. The covariance condition (4.1), the notion of tessellation on $\mathbf{R}$ and the three conditions of Theorem (4.1) are easily rephrased in terms of G and T (the condition $I \prec J$ is replaced by $I^0 \cap J^0 = \varnothing$ and the family $\mathscr{I}$ of intervals by a family of open sets). Given these adjustments, the

proof of Theorem (4.1) applies with no changes to this more general situation. If μ is a G-invariant Borel measure on T, one can consider the Fock space (or equivalently the unit Gaussian process) over $L^2(T, \mu)$, and an easy adaptation of Example 4.0b provides an example of this more general situation.

5. Exchangeable conditional expectations. We conclude with a remark which might shed at least some partial light on an interesting question posed by Diaconis and Freedman. In [6] these authors show that, under various sets of conditions, a probability measure turns out to be a mixture of product measures and ask whether there might be a general principle underlying these sets of conditions. The core of paper [6] is the explicit determination of the factor and the mixing measures in a number of interesting situations, as well as the bounds on the finite dimensional approximations. These results by their very nature depend on the specific class of measures considered and the only common feature seems to be the order of magnitude of the approximation, which is m/N in the notation of the remark before Lemma 2.6 (k/n in the notation of [6]). As far as the qualitative picture is concerned, one may notice that in all the cases considered in [6], the various classes of measures are determined by:

(i) Assigning the conditional probabilities on a σ-field determined by a condition which is *symmetric* in the random variables (e.g., $\xi_1, \ldots, \xi_n \geq 0$; $\sum_{j=1}^n \xi_j = s$).

(ii) The conditional probabilities of point (i) are themselves invariant under finite permutations of the coordinates (the rotation invariance implies invariance under permutations of any family of orthogonal coordinates).

In algebraic language and in the notations of Section 2, we can rephrase (and generalize) conditions (i) and (ii) as follows: Let $\mathscr{A}_{[0,N]}$, $N \in \mathbf{N}$ denote the C^*algebra generated by $j_n(\mathscr{B})$ with $n = 1, \ldots, N$ and $\mathscr{A}_{[0,N]}(\mathscr{S}_N)$ denote the fixed point subalgebra of $\mathscr{A}_{[0,N]}$ under the action of $\mathscr{S}_N$. From the remark after Proposition 2.3 we know that $\mathscr{A}_{[0,N]}(\mathscr{S}_N)$ is the range of the restriction of the conditional expectation E_N, given by (2.4), on $\mathscr{A}_{[0,N]}$. A conditional expectation F_N: $\mathscr{A}_{[0,N]} \to \mathscr{A}_{[0,N]}(\mathscr{S}_N)$ is called *exchangeable* if

$$F_N \circ \pi = F_N, \qquad \forall\, \pi \in \mathscr{S}_N. \tag{5.1}$$

With this notation, the following problem is a generalization of the various problems considered in [6]: *Let there be given for each $N \in \mathbf{N}$ an algebra $\mathscr{C}_N \subset \mathscr{A}_{[0,N]}$ and an exchangeable conditional expectation F_N from $\mathscr{A}_{[0,N]}$ onto $\mathscr{C}_N$. Characterize all the states φ on $\mathscr{A}$ such that*

$$\varphi \circ F_N = \varphi, \qquad \forall\, N \in \mathbf{N}. \tag{5.2}$$

For example, $\mathscr{C}_N$ can be the algebra of all bounded measurable functions of the random variables $[\prod_{j=1}^N \chi_{[0,+\infty)}(\xi_j)] \cdot (\sum_{j=1}^N \xi_j) =: X$ (where $\xi_1, \ldots, \xi_n, \ldots$ are real valued random variables) and we can require (as in [6]) that, for any fixed

$s \in [0, +\infty)$, $F_N(\cdot)_X = s$ is the uniform distribution on the simplex $\{(t_1, \ldots, t_N) \in \mathbf{R}_+^N: \sum_{j=1}^N t_j = s\}$. It is obvious due to (5.1) and (5.2) that any φ satisfying (5.2) is an exchangeable state, hence, by De Finetti's theorem, a mixture of homogeneous products.

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CENTRO MATEMATICO V. VOLTERRA
DIPARTIMENTO DI MATEMATICA
UNIVERSITÀ DI ROMA II
VIA ORAZIO RAIMONDO
00173 ROMA
ITALY