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WEAK CONVERGENCE FOR REVERSIBLE RANDOM WALKS IN A RANDOM ENVIRONMENT

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Assign to each edge e of the square lattice $\mathbb{Z}^2$ a random bond conductivity $c(e)$. If $c(e)$ are stationary, ergodic and such that $0 < a < c(e) < b < \infty$ for all edges e, then there is a central limit theorem for the corresponding reversible random walk on the lattice which holds for almost all environments.

1. Introduction. Let $(\tau_x\colon x \in \mathbb{Z}^2)$ be a given group of ergodic measure preserving transformations of a probability space $(\Omega, \mathscr{F}, \mu)$, that is,

$$\tau_x\colon \Omega \to \Omega \text{ is measurable} \quad \text{for all } x \text{ in } \mathbb{Z}^2,$$

$$\mu(\tau_x A) = \mu(A) \quad \text{for all } A \text{ in } \mathscr{F} \text{ and all } x \text{ in } \mathbb{Z}^2,$$

$$\tau_x \circ \tau_y = \tau_{x+y} \quad \text{for all } x, y \text{ in } \mathbb{Z}^2,$$

$$\text{if } \tau_x A = A \quad \text{(up to null sets) for some } x \neq (0,0) \text{ then } \mu(A) = 0 \text{ or } 1.$$

Let $c_i(\omega)$ be two measurable functions such that

$$0 < a < c_i(\omega) < b < \infty \quad \text{for all } \omega \in \Omega,\ i = 1, 2. \tag{1.1}$$

For x in $\mathbb{Z}^2$, the conductivity of the edge between x and $x + e_i$ is $c_i(\tau_x\omega)$, where $e_1 = (1,0)$ and $e_2 = (0,1)$. We refer to ω as an environment since each ω in Ω determines a conductivity for all edges of $\mathbb{Z}^2$.

Electrical networks and reversible random walks have long been known to be related [see Doyle and Snell (1984)]. For $\omega \in \Omega$ fixed, we consider a Markov chain X_n on $\mathbb{Z}^2$, $X_0 = (0,0)$, whose transition probabilities $p(\omega; x, y)$, are given by

$$p(\omega; x, x + e_i) = \frac{c_i(\tau_x\omega)}{c(\tau_x\omega)} \quad \text{and} \quad p(\omega; x, x - e_i) = \frac{c_i(\tau_{x-e_i}\omega)}{c(\tau_x\omega)}, \tag{1.2}$$

where $c(\omega) = c_1(\omega) + c_2(\omega) + c_1(\tau_{-e_1}\omega) + c_2(\tau_{-e_2}\omega)$. These random walks are reversible since $c(\tau_x\omega)p(\omega; x, y) = c(\tau_y\omega)p(\omega; y, x)$ for all adjacent vertices x, y in $\mathbb{Z}^2$ (i.e., $|x_1 - y_1| + |x_2 - y_2| = 1$). And by Kolmogorov's existence theorem, these transition probabilities define a probability measure P_ω on $(\mathbb{Z}^2)^N$. The set of possible jumps will be denoted by $\Lambda = \{\pm e_1, \pm e_2\}$ and for z in Λ we abbreviate $p(\omega; 0, z)$ by $p(\omega; z)$.

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In Kozlov (1985), a general framework for obtaining central limit theorems for random walks in a random environment is described. In the case of reversible random walks on $\mathbb{Z}^d$, there is a $\delta > 0$ small enough so that if for all z in Λ there is a constant $\bar{p}(z)$ such that $|p(\omega; z) - \bar{p}(z)| < \delta$ μ-a.e. and if (1.1) holds, then $n^{-1/2}X_n$ converges weakly to a normal law for μ-almost all environments.

The purpose of this paper is to show that in the two-dimensional case, condition (1.1) is sufficient to obtain this CLT.

THEOREM 1. *If* (1.1) *holds for some real numbers a, b then for μ-a.a. ω, $n^{-1/2}X_n$ converges weakly to a centered normal law with covariance matrix σ_{ij} given in* (4.6).

The main step of the proof is to show that the cocycle which is the solution of Problem B has a moment of order strictly greater than 2.

The ellipticity condition (1.1) appears in Kozlov (1980), Papanicolaou and Varadhan (1982) and Golden and Papanicolaou (1983). It was used to obtain a CLT that holds in the mean for the continuous parameter case. The discrete case is studied in Künneman (1983). Also related is Astrauskas (1989), where different conditions are considered to obtain a CLT in the mean. In Lawler (1982), there is a CLT for almost all environments for balanced walks. In that case, there is no need for cocycles but the problem is to show the existence of an invariant measure. In both cases however, a CLT for martingales is used.

SOME NOTATION. $0 = (0, 0)$. For $x = (x_1, x_2)$, $|x| = (x_1^2 + x_2^2)^{1/2}$.

For f: $\mathbb{Z}^2 \to \mathbb{R}$, $\Delta_i f(x) = f(x + e_i) - f(x)$ and $\Delta_i^* f(x) = f(x - e_i) - f(x)$. Δ^2 is the discrete Laplacian; $\Delta^2 = 1/4(\Delta_1^*\Delta_1 + \Delta_2^*\Delta_2)$. For p, $1 \le p < \infty$, $L_p(\mathbb{Z}^2)$ is the space of functions f: $\mathbb{Z}^2 \to \mathbb{R}$ such that $\|f\|_p = (\Sigma_x |f(x)|^p)^{1/p}$ is finite. Σ_x will always be a summation over all x in $\mathbb{Z}^2$. The Fourier transform of f in $L_1(\mathbb{Z}^2)$ is $\hat{f}(t) = \Sigma_x e^{it\cdot x} f(x)$, $t \in T^2 = \{t = (t_1, t_2)$: $-\pi \le t_i < \pi\}$.

For $1 \le p < \infty$, $L_p(\Omega)$ is the space (of equivalence classes) of measurable functions f: $\Omega \to \mathbb{R}$ such that $\|f\|_p = (\int_\Omega |f(\omega)|^p \, d\mu)^{1/p}$ is finite. $L_\infty(\Omega)$ is the space of (essentially) bounded measurable functions f: $\Omega \to \mathbb{R}$. Often, we consider τ_x, $x \in \mathbb{Z}^2$, as a linear isometry τ_x: $L_p(\Omega) \to L_p(\Omega)$, $1 \le p < \infty$. $L_p^0(\Omega) = \{f$ in $L_p(\Omega)$: $\int_\Omega f \, d\mu = 0\}$.

For f in $L_1(\Omega)$, $\partial_i f(\omega) = f(\tau_{e_i}\omega) - f(\omega)$ and $\partial_i^* f(\omega) = f(\tau_{-e_i}\omega) - f(\omega)$. $\mathscr{E} = \{f$ in L_∞^0: $f = \partial_1 h$ for some h in $L_\infty(\Omega)\}$. Because of the ergodicity, $\mathscr{E}$ is dense in L_p^0, $1 \le p < \infty$. $\mathbf{L}_p = \{\mathbf{f} = (f_1, f_2)$: f_i are in $L_p(\Omega)\}$. The norm on $\mathbf{L}_p$ is $\|\mathbf{f}\|_p = (\int_\Omega (|f_1|^p + |f_2|^p) \, d\mu)^{1/p}$. $\mathbf{L}_p^0 = \{\mathbf{f}$ in $\mathbf{L}_p$: $\int f_i \, d\mu = 0\}$ and $\mathbf{E}_p$ is the closed subset of $\mathbf{L}_p^0$, $\mathbf{E}_p = \{\mathbf{f}$ in $\mathbf{L}_p^0$: $\partial_1 f_2 = \partial_2 f_1\}$. We note that

$$\{\mathbf{f}\colon f_i = \partial_i g \quad \text{for some } g \text{ in } L_\infty\} \text{ is a dense subset of } \mathbf{E}_p, \qquad 1 \le p < \infty.$$

There is a proof of this fact in Boivin and Derriennic (1991).

2. L_p theory of the discrete Laplacian.

PROBLEM A. Given $\mathbf{h} = (h_1, h_2)$ in $\mathbf{L}_p(\Omega)$, $p \geq 2$, find $\mathbf{g} = (g_1, g_2)$ in $\mathbf{E}_p$ such that $\Sigma_i \partial_i^* g_i = -\Sigma_i \partial_i^* h_i$.

For each $\mathbf{h} \in \mathbf{L}_2(\Omega)$, Problem A has a unique solution; it is the element $\mathbf{g}$ in $\mathbf{E}_2$ such that

$$\sum_i \int u_i g_i = -\sum_i \int u_i h_i \quad \text{for all } \mathbf{u} = (u_1, u_2) \text{ in } \mathbf{E}_2, \tag{2.1}$$

which is a Hilbert space. This is a special case of the Lax–Milgram lemma. [See for instance Gilbarg and Trudinger (1977), page 78.]

This defines a linear operator $\mathbf{K}$: $L_2 \to \mathbf{E}_2$. Then

$$\|\mathbf{Kh}\|_2 \leq \|\mathbf{h}\|_2 \quad \text{for all } \mathbf{h} \text{ in } L_2(\Omega). \tag{2.2}$$

To see why (2.2) holds, let $\mathbf{g} = \mathbf{Kh}$. Then by (1.5), there is a sequence u_n in L_∞ such that $\partial_i u_n \to g_i$ in L_2 norm as $n \to \infty$. Then,

$$\begin{aligned}\|\mathbf{g}\|_2\|\mathbf{h}\|_2 &\geq -\sum_i \int g_i h_i \, d\mu = \lim_n -\sum_i \int (\partial_i u_n) h_i \, d\mu \\ &= \lim_n -\sum_i \int u_n \partial_i^* h_i \, d\mu \\ &= \lim_n \int u_n \Big(\sum_i \partial_i^* g_i \Big) d\mu = \lim_n \sum_i \int (\partial_i u_n) g_i \, d\mu = \|\mathbf{g}\|_2^2.\end{aligned}$$

We can also write $\mathbf{K}$ as $\mathbf{Kh} = (K_1\mathbf{h}, K_2\mathbf{h})$, where K_i are linear operators, K_i: $\mathbf{L}_2 \to L_2^0$.

Spitzer (1976) showed how some concepts for random walks are related to classical potential theory. For example, the potential kernel $a(x)$, $x \in \mathbb{Z}^2$, of the simple symmetric random walk in the plane satisfies

$$\lim_{|x| \to \infty} (a(x) - c \ln |x|) = \text{constant}. \tag{2.3}$$

The purpose of this section is to add one more fact to this analogy.

PROPOSITION. *The operator* $\mathbf{K}$: $\mathbf{L}_p \to \mathbf{E}_p$ *is bounded for all* p, $2 \leq p < \infty$; *that is, there are finite constants* γ_p *that depend only on* p *such that*

$$\|\mathbf{Kh}\|_p \leq \gamma_p \|\mathbf{h}\|_p \quad \text{for all } \mathbf{h} \text{ in } \mathbf{L}_p \text{ and all } p,\ 2 \leq p < \infty. \tag{2.4}$$

Before giving the proof, we recall some useful facts, first from Spitzer (1976). The potential kernel $a(x)$ is given by

$$a(x) = \frac{1}{4\pi^2} \int_{T^2} \frac{1 - \cos r \cdot \theta}{1 - \varphi(\theta)} \, d\theta, \qquad x \in \mathbb{Z}^2, \tag{2.5}$$

where $\varphi(\theta) = (\cos\theta_1 + \cos\theta_2)/2$ and we have the estimate

$$\frac{|\theta|^2}{8} \le 1 - \varphi(\theta) \le \frac{|\theta|^2}{4}. \tag{2.6}$$

The important property of $a(x)$ is

$$-\Delta^2 a(x) = \begin{cases} 1, & \text{if } x = 0, \\ 0, & \text{if } x \ne 0. \end{cases} \tag{2.7}$$

The following discrete version of part of Theorem II.2.2 of Stein (1970) will also be useful. The proof is the same with obvious modifications and uses a discrete version of Theorem I.3.2, whose proof also holds with few changes.

LEMMA. *Let $s: \mathbb{Z}^2 \to \mathbb{R}$ be a function for which there is a constant B such that*

$$|s(x)| \le \frac{B}{|x|^2} \quad \text{and} \quad |\Delta_i s(x)| \le \frac{B}{|x|^3} \qquad \forall\, x \ne 0,\ i = 1, 2 \tag{2.8}$$

and

$$|\hat{s}(t)| \le B \quad a.e. \text{ on } T^2, \tag{2.9}$$

and for $f: \mathbb{Z}^2 \to \mathbb{R}$ with finite support, let

$$Sf(x) = \sum_{y \in \mathbb{Z}^2} s(x - y) f(y), \qquad x \in \mathbb{Z}^2. \tag{2.10}$$

(2.11) *Then $m\{x: Sf(x) \ge \lambda\} \le \dfrac{C}{\lambda} \sum |f(x)|$ for all f with finite support, where C is a constant that depends only on B.*

PROOF OF THE PROPOSITION. For h in $\mathscr{E}$, let us set

$$\tilde{K}_i h(\omega) = \sum_y k_i(y) h(\tau_y \omega), \qquad h \in \mathscr{E},\ i = 1, 2, \tag{2.12}$$

where $k_i(x) = (1/4)\Delta_i^* \Delta_1 a(x)$.

A short calculation from (2.5) gives

$$k_1(x) = -\frac{1}{4\pi^2} \int_{T^2} \frac{\cos x \cdot \theta \sin^2(\theta_1/2)}{1 - \varphi(\theta)}\, d\theta \tag{2.13}$$

For $k_2(x)$, we have

$$\begin{aligned} k_2(x) &= -\frac{1}{4\pi^2} \int_{T^2} \frac{\cos x \cdot \theta \cos((e_1 - e_2)/2 \cdot \theta) \sin(\theta_1/2) \sin(\theta_2/2)}{1 - \varphi(\theta)}\, d\theta \\ &\quad + \frac{1}{4\pi^2} \int_{T^2} \frac{\sin x \cdot \theta \sin((e_1 - e_2)/2 \cdot \theta) \sin(\theta_1/2) \sin(\theta_2/2)}{1 - \varphi(\theta)}\, d\theta \\ &= k_2'(x) + k_2''(x). \end{aligned} \tag{2.14}$$

STEP 1. The first step will be to show that there is a positive constant B such that for all $x \neq (0,0)$,

$$|k_1(x)| \le B|x|^{-2}, \qquad |\Delta_j k_1(x)| \le B|x|^{-3}, \qquad j = 1,2, \tag{2.15}$$

$$|k_1'(x)| \le B|x|^{-2}, \qquad |\Delta_j k_1'(x)| \le B|x|^{-3}, \qquad j = 1,2, \tag{2.16}$$

$$|k_2''(x)| \le B|x|^{-3}. \tag{2.17}$$

Both estimates in (2.15) follow from Theorem 1.6.5 of Lawler (1991). The first one also follows from the calculations of Stöhr (1949). Similarly, if (2.17) holds, (2.16) will also follow from Theorem 1.6.5 of Lawler. We will now show in some detail how to obtain (2.17).

Assume $x_1 \ge x_2 > 0$. Put

$$u(\theta) = \sin\left(\frac{\theta_1 - \theta_2}{2}\right)\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}, \qquad \psi(\theta) = (1-\varphi)^{-1}u$$

and

$$D = \{\theta \text{ in } \mathbf{T}^2 \colon |\theta_i| \le \pi x_1^{-1} \text{ for } i = 1,2\}.$$

Then

$$4\pi^2 k_2''(x) = \int_{\mathbf{T}^2\setminus D} \sin x\cdot\theta\psi(\theta)\,d\theta + \int_D \sin x\cdot\theta\psi(\theta)\,d\theta$$
$$= I_1 + I_2,$$

where

$$|I_2| < \int_D \tfrac{1}{8}|(\theta_1-\theta_2)\theta_1\theta_2|\cdot 8|\theta|^{-2}\,d\theta$$
$$< 2\int_D |\theta|\,d\theta < 2x_1^{-3},$$

and

$$I_1 = \left(\int_{\partial\mathbf{T}_l^2} - \int_{\partial\mathbf{T}_r^2} + \int_{\partial V_l} - \int_{\partial V_r}\right)\Bigg(-x_1^{-1}\cos x\theta\psi(\theta) + x_1^{-2}\sin x\theta\frac{\partial\psi}{\partial\theta_1} + x_1^{-3}\cos x\theta\frac{\partial^2\psi}{\partial\theta_1^2}\Bigg)\,d\theta_2$$
$$-\left(\int_{\mathbf{T}^2\setminus D} x_1^{-3}\cos x\theta\frac{\partial^3\psi}{\partial\theta_1^3}\,d\theta\right),$$

where $\partial\mathbf{T}_l^2 = \{\theta \text{ in } \mathbf{T}^2\colon \theta_1 = -\pi\}$, $\partial\mathbf{T}_r^2 = \{\theta \text{ in } \mathbf{T}^2\colon \theta_1 = \pi\}$,

$$\partial V_l = \{\theta \text{ in } D\colon \theta_1 = -\pi x_1^{-1}\} \quad\text{and}\quad \partial D_r = \{\theta \text{ in } D\colon \theta_1 = \pi x_1^{-1}\}.$$

Since $\psi(\pi,\theta_2) = \psi(-\pi,\theta_2)$, $\cos x(\pi,\theta_2) = \cos x(-\pi,\theta_2)$ and $\sin x(\pi,\theta_2) = \sin x(-\pi,\theta_2)$, the difference of the first two integrals is 0. From the expressions for the derivatives, we see that $|\partial^n\psi/\partial\theta_1^n| \le 8^n|\theta_1|^{n-1}$, $n = 0,1,2$. Hence the third and fourth integrals are bounded by $16x_1^{-3}$.

It remains to show that $\int_{\mathbf{T}^2\setminus D} \cos x\theta(\partial^3\psi/\partial\theta_1^3)\,d\theta$ is bounded in $x_1 \ge x_2 > 0$:

$$\frac{\partial^3\psi}{\partial\theta_1^3} = (1-\varphi)^{-1}\frac{\partial^3 u}{\partial\theta_1^3} + 3\frac{\partial}{\partial\theta_1}(1-\varphi)^{-1}\frac{\partial^2 u}{\partial\theta_1^2} + 3\frac{\partial^2}{\partial\theta_1^2}(1-\varphi)^{-1}\frac{\partial u}{\partial\theta_1} + u\frac{\partial^3}{\partial\theta_1^3}(1-\varphi)^{-1}.$$

A calculation of these derivatives shows that

$$(1-\varphi)^{-1}\frac{\partial^3 u}{\partial\theta_1^3}, \qquad \frac{\partial}{\partial\theta_1}(1-\varphi)^{-1}\frac{\partial^2 u}{\partial\theta_1^2} + 2\frac{\theta_1\theta_2}{|\theta|^4},$$

$$\frac{\partial^2}{\partial\theta_1^2}(1-\varphi)^{-1}\frac{\partial u}{\partial\theta_1} - \left(\frac{8\theta_1^3\theta_2}{|\theta|^6} + \frac{\theta_2^2}{|\theta|^4} - \frac{2\theta_1\theta_2}{|\theta|^4} - \frac{4\theta_1^2\theta_2^2}{|\theta|^6}\right)$$

and

$$u\frac{\partial^3}{\partial\theta_1^3}(1-\varphi)^{-1} - \left(\frac{12\theta_1^2\theta_2(\theta_1-\theta_2)}{|\theta|^6} - \frac{24(\theta_1-\theta_2)\theta_1^4\theta_2}{|\theta|^8}\right)$$

are integrable over $\mathbf{T}^2$.

Now if $f(\theta)$ is any of these rational functions, $\int_{\mathbf{T}^2\setminus D}\cos x\theta f(\theta)\,d\theta$ is bounded in $x_1 \ge x_2 > 0$. The calculations are similar in all cases. For example, for $f(\theta) = \theta_1\theta_2/|\theta|^4$,

$$\int_{-\pi}^{\pi}\int_J \cos x\theta f(\theta)\,d\theta = -4\int_0^{\pi}\int_{\pi x_1^{-1}}^{\pi} \sin x_1\theta_1 \sin x_2\theta_2 f(\theta)\,d\theta_1\,d\theta_2,$$

where

$$\begin{aligned}
J &= \left[-\pi, -\pi x_1^{-1}\right) \cup \left(\pi x_1^{-1}, \pi\right],\\
&= 2\int_0^{\pi}\int_0^{\pi x_1^{-1}} \sin x_1\theta_1 \sin x_2\theta_2 f\left(\theta_1 + \pi x_1^{-1}, \theta_2\right) d\theta_1\,d\theta_2\\
&\quad + 2\int_0^{\pi}\int_{\pi x_1^{-1}}^{\pi-\pi x_1^{-1}} \sin x_1\theta_1 \sin x_2\theta_2\left(f\left(\theta_1 + \pi x_1^{-1}, \theta_2\right) - f(\theta)\right) d\theta_1\,d\theta_2\\
&\quad - 2\int_0^{\pi}\int_{\pi-\pi x_1^{-1}}^{\pi} \sin x_1\theta_1 \sin x_2\theta_2 f(\theta)\,d\theta_1\,d\theta_2\\
&= I_1' + I_2' + I_3',
\end{aligned}$$

where

$$\begin{aligned}
|I_1'| &\le 2\int_0^{\pi}\int_0^{\pi x_1^{-1}} \left|\left(\theta_1 + \pi x_1^{-1}, \theta_2\right)\right|^{-2} d\theta_1\,d\theta_2\\
&\le \pi\int_0^{\pi x_1^{-1}} \left(\theta_1 + \pi x_1^{-1}\right)^{-1} d\theta_1 = \pi \ln 2,
\end{aligned}$$

and similarly for I_3':

$$
\begin{aligned}
|I_2'| &\le 2\int_0^{\pi}\int_{\pi x_1^{-1}}^{\pi-\pi x_1^{-1}} \left| f(\theta_1+\pi x_1^{-1},\theta_2) - f(\theta)\right| d\theta_1\, d\theta_2 \\
&\le 2\pi x_1^{-1}\int_0^{\pi}\int_{\pi x_1^{-1}}^{\pi-\pi x_1^{-1}} 5|\theta|^{-3}\, d\theta_1\, d\theta_2 \\
&\le 2\pi x_1^{-1}\int_0^{2\pi}\int_{\pi x_1^{-1}}^{\pi} 5\rho^{-2}\, d\rho\, d\alpha \\
&\le 20\pi x_1^{-1}(x_1-1) \le 20\pi.
\end{aligned}
$$

STEP 2. Using these estimates, we can see why the series (2.12) converges in norm for h in $\mathscr{E}$. Let $h(\omega) = h_0(\tau_{e_1}\omega) - h_0(\omega)$ for some $\tilde{h}$ in $L_\infty(\Omega)$. Then

$$
\tag{2.18}
\begin{aligned}
\tilde{K}_{i,n}h(\omega) &= \sum_{[-n,n]^2} k_i(y)h(\tau_y\omega) \\
&= \sum_{(n+1)\times[-n,n]} k_i(y-e_1)h_0(\tau_y\omega) \\
&\quad + \sum_{[-n+1,n]\times[-n,n]} \Delta_1^* k(y)h_0(\tau_y\omega) \\
&\quad - \sum_{(-n)\times[-n,n]} k_i(y)h_0(\tau_y\omega).
\end{aligned}
$$

And, $\|\tilde{K}_{i,n}h - \tilde{K}_{i,m}h\|_p \le C\|h_0\|_p(\int_{-n}^{n} ds/(n^2+s^2) + \int_{-m}^{m} ds/(m^2+s^2) + 2\pi\int_m^{\sqrt{2n}} d\rho/\rho^2)$ which converges to 0 as $n > m \to \infty$, $1 \le p \le \infty$.

For h in $\mathscr{E}$, we also have that $\partial_1\tilde{K}_2 h = \partial_2\tilde{K}_1 h$ and, by (2.7),

$$
\tag{2.19}
\partial_1^*\tilde{K}_1 h + \partial_2^*\tilde{K}_2 h = \sum_y \Delta^2\Delta_1 a(y)h(\tau_y\omega) = -\partial_1^* h,
$$

that is,

$$
\tag{2.20}
\left(\tilde{K}_1 h, \tilde{K}_2 h\right) = \mathbf{K}(h,0) \quad \text{for all } h \text{ in } \mathscr{E},\ i = 1,2.
$$

STEP 3. First we show that $\hat{k}(\theta)$ is essentially bounded in $\mathbf{T}^2 \setminus \{|\theta| > 1/10\}$, where $k(x)$ is $k_1(x)$ or $k_2(x)$. For all z in $\mathbb{Z}^2$, $(1 - e^{i\theta\cdot z})\hat{k}(\theta) = \sum_x e^{i\theta\cdot x}(k(x) - k(x-z))$. If $\theta_1 < 0$, take $z = e_1$, and if $\theta_1 > 0$, take $z = -e_1$; then $|\hat{k}(\theta)| < \sup_{1/10<|\theta_1|\le\pi}|1 - e^{i\theta_1}|^{-1}\sum_x B/|x|^3$ which is finite.

Now if $|\theta| \le 1/10$, write

$$(1 - e^{i\theta\cdot z})\hat{k}(\theta) = \lim_{R\to\infty}\Bigg[\sum_{|\theta|^{-1}<|x|<R} e^{i\theta\cdot x}(k(x) - k(x-z)) + \Bigg(\sum_{\substack{|x-z|\le|\theta|^{-1}\\ |x|\ge|\theta|^{-1}}} e^{i\theta\cdot x}k(x-z) - \sum_{\substack{|x-z|\ge|\theta|^{-1}\\ |x|\le|\theta|^{-1}}} e^{i\theta\cdot x}k(x-z)\Bigg) + \Bigg(\sum_{\substack{|x-z|\le R\\ |x|\ge R}} e^{i\theta\cdot x}k(x-z) - \sum_{\substack{|x-z|\ge R\\ |x|\le R}} e^{i\theta\cdot x}k(x-z)\Bigg)\Bigg]$$

$$= S_{1,R} + S_2 + S_{3,R}.$$

For $|\theta| \le 1/10$, choose a vertex $z = z(\theta)$ in $\mathbb{Z}^2$ such that $|z - \theta/3|\theta|^2| < 1$. Then

$$|S_{1,R}| < 9\int_{|\theta|^{-1}}^{\infty}(B|z|/\rho^3)\rho\,d\rho = 4B|z|\,|\theta|,$$

$$|S_2| < 9\int_{|\theta|^{-1}/2}^{|\theta|^{-1}}(B/\rho^2)\rho\,d\rho + 9\int_{|\theta|^{-1}}^{3|\theta|^{-1}/2}(B\rho/\rho^2)\,d\rho$$

and

$$|S_{3,R}| < 9\int_{R/2}^{2R}(B/\rho^2)\rho\,d\rho \text{ for } R > |z|.$$

Therefore $\operatorname{ess\,sup}_{|\theta|<1/10}|\hat{k}(\theta)| < C\sup_{1/3<t<1/5}|1 - e^{it}|^{-1}$, which is finite.

STEP 4. Define the operators S_i for $f\colon \mathbb{Z}^2 \to \mathbb{R}$ with finite support by $S_i f(x) = \sum k_i(x-y)f(y)$, $i = 1, 2$. Since $k_i(x)$ satisfy (2.9) and (2.10), by (2.12),

$$m\{x\colon |S_i f(x)| > \lambda\} \le \frac{C}{\lambda}\|f\|_1. \tag{2.21}$$

To transfer this inequality to the operators $\tilde{K}_i$, fix $\lambda > 0$ and for $f\colon \Omega \to \mathbb{R}$ and for any positive integer L, define

$$f_{\omega,L}(x) = \begin{cases} f(\tau_x\omega), & \text{if } x \in [-L, L]^2,\\ 0, & \text{otherwise.}\end{cases}$$

If f is in $\mathscr{E}$, by (2.18), there is an integer N such that $|S_i f_{\omega,L}(x) - \tilde{K}_i f(\tau_x\omega)| < \lambda/3$ whenever $x + [-N, N]^2 \subset [-L, L]^2$. For f in $\mathscr{E}$ and $\lambda > 0$ fixed, put

$$E = \{\omega\colon |\tilde{K}_i f(\omega)| > \lambda\},$$

$$A(\omega) = \{x\colon \tau_x\omega \in E,\ x \in [-L, L]^2\},$$

$$B(\omega) = \left\{x\colon |Sf_{\omega,L}(x)| > \frac{\lambda}{3},\ x \in [-L, L]^2\right\}.$$

Then

$$m(A(\omega)) \le m(B(\omega)) + 4NL \le \frac{C}{\lambda}\|f_{\omega,L}\|_1 + 4NL$$

and

$$\int m(A(\omega))\,d\mu = \int \sum_{[-L,L]^2} I_E(\tau_x\omega)\,d\mu = (2L+1)^2\mu(E).$$

Combine these two facts, divide by L^2 and let $L \to \infty$ to obtain

$$\mu\{\omega: |\tilde{K}_i f(\omega)| > \lambda\} \le \frac{C}{\lambda}\|f\|_1 \quad \text{for all } f \text{ in } \mathscr{E},\ \lambda > 0. \tag{2.22}$$

STEP 5. In Steps 3 and 4, we showed that $\tilde{K}_i$ are linear operators of weak-type (1, 1) and (2, 2). Therefore, by the interpolation theorem [Stein (1970)], there are finite constants γ_p such that $\|\tilde{K}_i h\|_p \le \gamma_p\|h\|_p$, $1 < p \le 2$, for all h in $\mathscr{E}$.

But τ_y are measure preserving, $k_1(y) = k_1(-y)$ and therefore, we see from (2.18) that $\int_\Omega(\tilde{K}_1 h)u\,d\mu = \int_\Omega h(\tilde{K}_1 u)\,d\mu$ for all h and u in $\mathscr{E}$. Now fix $p > 2$ and let $q = p/(p-1)$, $1 < q < 2$. Then, for any h in $\mathscr{E}$,

$$\|\tilde{K}_1 h\|_p = \sup_{\substack{\|u\|_q \le 1 \\ u \in \mathscr{E}}} \left|\int_\Omega (\tilde{K}_1 h)u\right| = \sup_{\substack{\|u\|_q \le 1 \\ u \in \mathscr{E}}} \left|\int_\Omega h(\tilde{K}_1 u)\right| \le \gamma_q\|h\|_p.$$

Equation (2.17) implies that $k_2''(x) \in L_1(\mathbb{Z}^2)$ and hence $\tilde{K}_2''$, the operator induced by $k_2''(x)$ on $\mathscr{E}$, is bounded for all p, $1 \le p < \infty$. But $\tilde{K}_2'$ is self-adjoint since $k_2'(y) = k_2'(-y)$ and since $\tilde{K}_2$ is bounded for p, $1 < p \le 2$, $\tilde{K}_2'$ is also bounded. Therefore the same reasoning applies to show that $\tilde{K}_2'$ is also bounded for p, $2 \le p < \infty$ and hence, so is $\tilde{K}_2$.

Because of (2.20), the density of $\mathscr{E}$ in L_p^0 and the fact that $\mathbf{K}(h_1, 0) = (0, 0)$ if h is constant, the proposition is proved for the case $h_2 = 0$. Similar calculations can be done if $\mathbf{h} = (0, h_2)$ and by linearity, the proof is complete. □

To obtain a useful corollary, fix a number $r > 2$ and combine the proposition with (2.2) by the interpolation theorem of Riesz [Stein and Weiss (1971), Chapter V, Theorem 1.3]

COROLLARY 1. *Let $r > p > 2$, then* $\|\mathbf{Kh}\|_p \le \gamma_r^{r/(r-2)((p-2)/2)}\|\mathbf{h}\|_p$.

3. The potential differences are in L_p for some $p > 2$. Although this section follows the proof of Meyers' theorem as it is found in Giaquinta (1983), we give details because it shows why the condition of the smallness of fluctuations of Kozlov (1985) is not necessary for the proof of the CLT for the two-dimensional random walks.

PROBLEM B. Find $\mathbf{f} = (f_1, f_2)$ in $\mathbf{E}_2$ such that

$$\partial_1^*(c_1(\omega)(1 + f_1(\omega))) + \partial_2^*(c_2(\omega) f_2(\omega)) = 0 \quad \text{a.e.} \tag{3.1}$$

Again, the existence and unicity of the solution follow from the Lax–Milgram lemma.

THEOREM 2. *There exists* $p_0 > 2$ *such that* $\mathbf{f} = (f_1, f_2)$, *the solution of Problem B, is in* $\mathbf{E}_{p_0}$.

PROBLEM C. Given $\mathbf{u}$ in $\mathbf{E}_2$, find $\mathbf{w}$ in $\mathbf{E}_2$ such that

$$\Sigma_i \partial_i^* w_i = \Sigma_i \partial_i^* \left(\left(1 - \frac{c_i}{b}\right) u_i \right) - \frac{\partial_1^* c_1}{b}. \tag{3.2}$$

By the Lax–Milgram lemma, this problem has a unique solution for every $\mathbf{u}$ in $\mathbf{E}_2$. This defines an operator $\mathbf{S}$: $\mathbf{E}_2 \to \mathbf{E}_2$, where $\mathbf{Su} = \mathbf{w}$.

PROOF OF THEOREM 2. First, we show that $\mathbf{S}$: $\mathbf{E}_p \to \mathbf{E}_p$ for any $p \geq 2$. Let $\mathbf{w} = \mathbf{Su}$, $\mathbf{g} = \mathbf{K}(c_1/b, 0)$ and hence $\mathbf{g} - \mathbf{w} = \mathbf{Kh}$, where $h_i = (1 - c_i/b)u_i$. Then, by the proposition, $\|\mathbf{w}\|_p \leq \|\mathbf{g}\|_p + \|\mathbf{g} - \mathbf{w}\|_p \leq \gamma_p/b\|c_1\|_p + (1 - a/b)\|\mathbf{u}\|_p$.

The second step is to show that for some $p_0 > 2$, there is a number β, $0 < \beta < 1$, such that $\|\mathbf{Su} - \mathbf{Su}'\|_{p_0} \leq \beta\|\mathbf{u} - \mathbf{u}'\|_{p_0}$ for all $\mathbf{u}, \mathbf{u}'$ in E_{p_0}. By Corollary 1,

$$\|\mathbf{Su} - \mathbf{Su}'\|_p \leq \gamma_r^{(r/(r-2))((p-2)/p)} \left(1 - \frac{a}{b}\right) \|\mathbf{u} - \mathbf{u}'\|_p$$

and therefore if p_0 is close to 2, we can find a constant $\beta < 1$.

Therefore the map $\mathbf{S}$ has a fixed point in $\mathbf{E}_{p_0}$ [see, for instance, Gilbarg and Trudinger (1977), page 69] and it is also the solution of Problem B. □

Let $\mathbf{f}$ be the solution of Problem B. Define a function F: $\Omega \times \mathbb{Z}^2 \to \mathbb{R}$ by

$$F(\omega, 0) = 0, \qquad F(\omega, e_i) = f_i(\omega), \tag{3.3}$$

$$F(\omega, x + y) = F(\omega, x) + F(\tau_x \omega, y), \qquad x, y \in \mathbb{Z}^2. \tag{3.4}$$

Because $\partial_1 f_2 = \partial_2 f_1$, the value of $F(\omega, x)$ obtained by (3.3) and (3.4) is independent of the path from 0 to x chosen.

In the terminology of ergodic theory, a function which satisfies (3.4) is called a cocycle. Therefore, if we combine Theorem 2 and the ergodic theorem for cocycles [Boivin and Derriennic (1991)], we obtain the following corollary.

COROLLARY 2. *Let* $F(\omega, x)$ *be the cocycle defined by* (3.3) *and* (3.4). *Then* $|x|^{-1}F(\omega, x) \to 0$ *a.e. as* $|x| \to \infty$.

Let $V(\omega, x) = x_1 + F(\omega, x)$. Then $V(\omega, x)$ is also a cocycle. It is a discrete version of the potential function introduced in Papanicolaou and Varadhan

(1982). In this formulation of the electrostatic problem, $v_1(\tau_x\omega) = 1 + f_1(\tau_x\omega)$ and $v_2(\tau_x\omega) = f_2(\omega)$ can be interpreted as the potential difference between x and $x + e_i$. Corollary 2 states that $|x|^{-1}(V(\omega, x) - x_1) \to 0$ a.e. as $|x| \to \infty$.

4. Proof of Theorem 1. We can now follow the arguments of Kozlov (1985) to complete the proof of this CLT. To keep the notation compatible with the original references, φ and ψ will now be used differently than they were in Section 2; they will be functions on $\Omega \times \Lambda$.

STEP 1 (Problem B′). Find $\mathbf{g} = (g_1, g_2)$ in $\mathbf{E}_2$ such that

$$\partial_1^*(c_1(\omega)g_1(\omega)) + \partial_2^*(c_2(\omega)(1 + g_2(\omega))) = 0 \quad \text{a.e.}$$

This problem also has a unique solution in $\mathbf{E}_{p_0}$ for some $p_0 > 2$. With $\mathbf{f}$ the solution of Problem B and $\mathbf{g}$ the solution of Problem B′, we construct the cocycle $\mathbf{F}\colon \Omega \times \mathbb{Z}^2 \to \mathbb{R}^2$ by

$$\mathbf{F}(\omega, \mathbf{0}) = \mathbf{0}, \qquad \mathbf{F}(\omega, e_i) = (f_i(\omega), g_i(\omega)), \tag{4.1}$$

$$\mathbf{F}(\omega, x + y) = \mathbf{F}(\omega, x) + \mathbf{F}(\tau_x\omega, y) \quad \text{for all } x, y \in \mathbb{Z}^2. \tag{4.2}$$

For a fixed environment $\omega \in \Omega$, let $\mathscr{G}_n$ be the σ-algebra generated by $X_1, \dots, X_n$. Then

$$\Phi_n = X_n + \mathbf{F}(X_n), \qquad n \geq 1, \Phi_0 = \mathbf{0} \tag{4.3}$$

is a martingale with respect to P_ω and $\mathscr{G}_n$. It can also be written as

$$z_j = X_{j+1} - X_j, \qquad \omega_j = \tau_{X_j}\omega,$$

$$\varphi(\omega, z) = z + \mathbf{h}_z(\omega),$$

where $\mathbf{h}_{e_i} = (f_i(\omega), g_i(\omega))$ and $\mathbf{h}_{-e_i}(\omega) = (-f_i(\tau_{-e_i}\omega), -g_i(\tau_{-e_i}\omega))$.

STEP 2. The Markov chain $\{(\omega_j, z_j)\}$ with state space $\Omega \times \Lambda$ and initial distribution $p(\omega; z)c(\omega)$ is stationary and ergodic. The transition operator of the chain is $Qf(\omega, z) = \Sigma_{y \in \Lambda} p(\tau_z\omega; y) f(\tau_z\omega, y)$ for $f \in L_\infty(\Omega \times \Lambda)$, $Q^*f(\omega, z) = p(\omega; z)\Sigma_{y \in \Lambda} f(\tau_{-y}\omega, y)$ and hence $\int_\Omega \Sigma_{z \in \Lambda} Qf(\omega, z)g(\omega, z)\, d\mu = \int_\Omega \Sigma_{z \in \Lambda} f(\omega, z)Q^*g(\omega, z)\, d\mu$ for all $f, g \in L_\infty(\Omega \times \Lambda)$.

Let $g(\omega, z) = p(\omega; z)c(\omega)$. Then $Q^*g = g$ by (1.2). This implies that the chain is stationary.

To show the ergodicity, suppose $Qf = f$ for some $f \in L_\infty(\Omega \times \Lambda)$. Put $g(\omega) = f(\tau_{-z}\omega, z)$, which is independent of $z \in \Lambda$ since $f(\tau_{-z}\omega, z) = \Sigma_{y \in \Lambda} f(\omega, y)p(\omega; y)$. Then $f(\omega, z) = g(\tau_z\omega)$, $z \in \Lambda$, and $g(\omega) = \Sigma_{y \in \Lambda} f(\omega, y)p(\omega; y) = Q_1 g(\omega)$, where Q_1 is the transition operator of the chain $\{\omega_j\}$. That is, for $h \in L_\infty(\Omega)$, $Q_1 h(\omega) = \Sigma_{y \in \Lambda} p(\omega; y)h(\tau_y\omega)$ and $Q_1^* h(\omega) = \Sigma_{y \in \Lambda} p(\tau_{-y}\omega; y)h(\tau_{-y}\omega)$.

But

$$\int_{\Omega}\sum_{y\in\Lambda} c(\omega)p(\omega;y)\big(g(\tau_y\omega)-g(\omega)\big)^2\,d\mu$$

$$= \int_{\Omega}\sum_{\Lambda} c(\tau_{-y}\omega)p(\tau_{-y}\omega;y)g^2(\omega)\mu(d\omega)$$

$$-2\int_{\Omega}\sum_{\Lambda} c(\omega)p(\omega;y)g^2(\omega)\mu(d\omega)$$

$$+\int_{\Omega}\sum_{\Lambda} c(\omega)p(\omega;y)g^2(\omega)\mu(d\omega) \quad \text{since } Q_1 g = g$$

$$= \int_{\Omega} Q_1^* c(\omega)g^2(\omega)\mu(d\omega) - \int_{\Omega} c(\omega)g^2(\omega)\mu(d\omega) = 0.$$

And by (1.1) and the ergodicity of τ_y, $y \neq 0$, g, and hence f, are constant.

STEP 3. It is now possible to show that for each t in $\mathbb{R}^2$, the martingale $t\Phi_n$ satisfies the conditions of the CLT of Brown (1971), that is, for μ-a.a. environments,

$$s_n^{-2}V_n^2 \to 1 \quad \text{in probability as } n \to \infty, \tag{4.4}$$

where $V_n^2 = \sum_{j=1}^n E(\varphi_t^2(\omega_{j-1}, z_{j-1})|g_{j-1})$, $s_n^2 = E_\omega V_n^2$, E_ω is the integration with respect to P_ω and $\varphi_t = t \cdot \varphi$, and

$$s_n^{-2}\sum_{j=1}^n E_\omega\big[\varphi_t^2(\omega_{j-1}, z_{j-1})I\big(|\varphi_t(\omega_{j-1}, z_{j-1})| \geq \varepsilon|s_n|\big)\big] \to 0 \tag{4.5}$$

in probability as $n \to \infty$ for all $\varepsilon > 0$. [$I(A)$ is the indicator function of the set A.]

By the ergodic theorem and Step 2,

$$n^{-1}V_n^2 = n^{-1}\sum_{j=1}^n \psi(\omega_{j-1}) \to \int_{\Omega}\psi(\omega)c(\omega)\mu(d\omega) \qquad c(\omega)\mu P_\omega\text{-a.e.}$$

as $n \to \infty$. (where $\psi(\omega) = \sum_{y\in\Lambda}\varphi_t^2(\omega, y)p(\omega; y)$).
Equation (4.4) follows after applying Fubini's theorem and (1.1).

Similarly, for every real number l,

$$n^{-1}\sum_{j=1}^n \varphi_t^2(\omega_{j-1}, z_{j-1})I\big(|\varphi_t(\omega_{j-1}, z_{j-1})| \geq \varepsilon l\big)$$

$$\to \int_{\Omega}\sum_{\Lambda}\varphi_t^2(\omega, z)I(|\varphi_t(\omega, z)| \geq \varepsilon l)p(\omega; z)c(\omega)\mu(d\omega)$$

$c\mu P_\omega$-a.e. as $n \to \infty$. And by Fubini's theorem, (1.1) and because $|s_n| \to \infty$ as $n \to \infty$ and $\varphi_t(\cdot, z) \in L_2$ for all z in Λ, (4.5) follows.

Therefore for μ-a.a. ω and for all t with t_1 and t_2 rational, $n^{-1/2}t \cdot \Phi_n$ converges weakly to a normal law with mean 0 and hence $n^{-1/2}\Phi_n$ converges weakly to a normal law with mean **0** and covariance matrix

$$(4.6)\qquad \sigma_{ij} = \int_\Omega \sum_\Lambda \varphi_i(\omega, z)\varphi_j(\omega, z)p(\omega; z)c(\omega)\mu(d\omega) \qquad i = 1, 2.$$

STEP 4. Fix ω (in the set of measure one for which (4.6) and Corollary 2 are true). Then for $\delta > 0$ fixed, there is a number $R(\delta)$ such that $|F(x)| < \delta|x|$ for all x, $|x| > R(\delta)$.

Put $M(\delta) = \sup_{|x| < R(\delta)}|F(x)|$:

$$\begin{aligned}|F(X_n)| &= |F(\Phi_n - F(X_n))| \\ &\le \max(\delta|\Phi_n| + \delta|F(X_n)|, M(\delta)) \\ &\le \max\left(\frac{\delta}{1-\delta}|\Phi_n|, M(\delta)\right).\end{aligned}$$

But by (4.6), $P_\omega(\delta(1-\delta)^{-1}n^{-1/2}|\Phi_n|) > \varepsilon \to P_\omega(N(0, \sigma^2) > (1-\delta)\delta^{-1}\varepsilon)$ as $n \to \infty$ for all $\delta > 0$. Therefore $P_\omega(n{-1/2}|\Phi_n - X_n| > \varepsilon) = P_\omega(n^{-1/2}|F(X_n)| > \varepsilon) \to 0$ as $n \to \infty$ for all $\varepsilon > 0$. □

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