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AN ORDERED PHASE WITH SLOW DECAY OF CORRELATIONS IN CONTINUUM $1/r^2$ ISING MODELS

BY LUIZ RENATO G. FONTES[1]

Instituto de Matemática e Estatística

For continuum $1/r^2$ Ising models, we prove that the critical value of the long range coupling constant (inverse temperature), above which an ordered phase occurs (for strong short range cutoff), is exactly 1. This leads to a proof of the existence of an ordered phase with slow decay of correlations. Our arguments involve comparisons between continuum and discrete Ising models, including (quenched and annealed) site diluted models, which may be of independent interest.

1. Introduction. The model to be discussed below is an Ising model in one dimension with long range, translation invariant, ferromagnetic pair interaction. However, unlike the usual case, its configurations are ± 1 valued functions *on the real line* $\mathbf{R}$ rather than on the discrete one dimensional lattice $\mathbf{Z}$. Thus, it is a continuous time stochastic process, to be more precisely defined in the next section.

Such a process, which is called the *continuum Ising model*, arises in the study of a quantum mechanical model of the motion of a particle subjected to a field. The quantum mechanical energy operator, known as the spin-boson Hamiltonian for this model, can be analysed by Feynman–Kac techniques, leading to the reexpression of quantities related to the quantum model in terms of continuum Ising quantities (see [11], [8] and [10]).

Two parameters, α and ε, and a function W enter the model. The parameter α is the long range coupling constant which can also be interpreted as the inverse temperature; ε can be related to the inverse of the short range coupling strength. $W = W(r)$ is defined on $[0, \infty)$, nonnegative and decays at infinity as $1/r^2$.

In previous work ([11] and [10]), the case of $1/r^2$ long range interactions (corresponding to the *ohmic* case of the quantum model) has been studied with, among other results, the following rigorous description of the phase diagram (Theorem 2 in [10]). For $\alpha \leq 1$ and any $\varepsilon > 0$, the model shows no spontaneous magnetization, whereas if $\alpha > 2$, then for small ε (large short range coupling force), there is spontaneous magnetization.

The strategy applied to get these results is to use the FK representation of the Ising model, which in the continuum case leads to a continuum bond percolation model, and then adapt the results existing for the discrete FK model, obtained in [2], [9] and [1].

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Here, we establish the existence of long range order (in the strong form known as *long long range order*, which implies spontaneous magnetization) for $\alpha > 1$, thus closing a gap in the phase picture.

Indeed, what we do is establish a comparison between the continuum model and the discrete $1/r^2$ Ising model at inverse temperature α_ε and nearest neighbor coupling J_ε, with α_ε close to α and J_ε large when ε is small. We then quote the results for the discrete model obtained in [6].

As in the discrete case, long long range order for $\alpha > 1$ leads to the existence of an intermediate phase (at least for $1 < \alpha < 2$) with slow decay of correlations. Here, we prove lower bounds for the decay of the truncated two-point function in the ordered phase. In the disordered phase, lower and upper bounds for the two-point function were obtained in [10].

Upper bounds in the ordered phase remain to be obtained for the continuum model, unlike for the discrete case, for which they were derived in [6].

Our results are stated and proved in the next 5 sections, one for the description of the model and statement of results, one for each of three steps of the comparison with the discrete model and the last one for the lower bounds on the truncated two-point function.

2. The model. For T positive, let Ω_T be the space of functions σ_t defined in the interval $[-T, T]$ and taking values in $\{-1, +1\}$, which have only a finite number of flips (and are right continuous, say). Ω_T is the set of configurations of the continuum system.

Let $P^f_{\varepsilon,T}$ be the measure on Ω_T such that the flip points form a Poisson process with rate $\varepsilon > 0$ and σ_{-T} equals $+1$ or -1 with equal probabilities (free boundary conditions). $P^+_{\varepsilon,T}$ will denote the measure on Ω_T such that the flips form a Poisson process in $[-T, T]$ with rate ε, *conditioned on having only an even number of flips in* $[-T, T]$, and starting at $+1$ (i.e., $\sigma_{-T} = +1$), which corresponds to plus boundary conditions.

Now, let $W(t)$ be a nonnegative bounded (piecewise) continuous function decaying like $1/t^2$ at infinity, that is, $t^2W(t) \to 1$, as $t \to \infty$. It defines the continuum ferromagnetic couplings and will be kept fixed throughout.

The finite volume continuum Ising measures with free and plus boundary conditions are defined as follows:

$$d\rho^*_T(\sigma) = \frac{1}{Z^*_T}\, dP^*_{\varepsilon,T}(\sigma) \exp^{-\alpha H^*(\sigma)}, \tag{2.1}$$

where $* = f$ or $+$, and

$$H^f(\sigma) = -\tfrac{1}{4}\int_{-T}^{T}\int_{-T}^{T} W(|t-s|)\sigma_t\sigma_s\, dt\, ds, \tag{2.2}$$

$$H^+(\sigma) = H^f(\sigma) - \tfrac{1}{2}\int_{\mathbf{R}\setminus[-T,T]}\int_{-T}^{T} W(|t-s|)\sigma_t\, dt\, ds \tag{2.3}$$

are the Hamiltonians and Z^*_T is a normalizing constant.

We denote by ρ^* the infinite volume limit ($T \to \infty$) of ρ_T^* (which exists by standard arguments) and by $\langle \cdot \rangle^*$ and $\langle \cdot \rangle_T^*$ the expectations w.r.t. ρ^* and ρ_T^*, respectively.

DEFINITION 2.1. *For given* α *and* ε long long range order *is said to occur if there are positive constants* ν *and* μ *such that for all* $T > 0$,

$$\langle \sigma_0 \sigma_t \rangle_T^f \geq \nu^2 \quad \textit{for all } |t| \leq \mu T.$$

Let M denote the spontaneous magnetization of $\langle \cdot \rangle^+$, that is,

$$M = \langle \sigma_0 \rangle^+,$$

and $G^T(t)$ its truncated two-point function, that is,

$$G^T(t) = \langle \sigma_0 \sigma_t \rangle^+ - M^2.$$

We now state the main results.

THEOREM 1. *If* $\alpha > 1$, *then long long range order occurs for* ε *small enough*.

THEOREM 2. *If for given* α *and* ε *long long range order occurs, then, for any* $\delta > 0$, *there exists some* $C > 0$ *so that*

$$G^T(t) \geq C/|t|^{2\gamma} \quad \textit{for all } t \geq 1, \tag{2.4}$$

where $\gamma = \min(1, \alpha - 1 + \delta)$.

Our proof of Theorem 1 will rely on a comparison between the continuum model and an ordinary $1/r^2$ discrete Ising model at a slightly bigger temperature (for ε small), in such a way that the correlations of the former are bigger than those of the latter. We then quote the corresponding discrete result to get the continuum one. The comparison to the discrete model will be carried out in three steps, one in each of the next three sections. We describe them briefly now.

Previous attempts at proving the occurrence of spontaneous magnetization in the continuum model for $1 < \alpha \leq 2$ were based on either using the sequence of discrete Ising models approximating the continuum one (see the beginning of next section), which met the difficulty of conflicting limits of the couplings (of the discrete models) in short and medium distances (the nearest neighbor ones going to infinity while the medium distance ones go to 0), or by discretizing the a priori measure $P_{\varepsilon,T}^f$ together with the interaction function W. In the latter process too much coupling at short distances is lost, so that a comparison with a discrete $1/r^2$ Ising model is obtained, but not with J (the discrete model nearest neighbor interaction) as large as needed.

Our approach is the following. Consider the free b.c. Ising measure. In Section 3, we modify the model by partitioning $[-T, T]$ into intervals I_k, $k = -K, -K + 1, \dots, K - 1$, of length $L = T/K$. We then consider the model obtained similarly as the continuum Ising model, but with $P^f_{\varepsilon,T}(\sigma)$ replaced by $\prod_k P^{(k)}(\sigma_k)$, where σ_k is a continuum configuration in I_k and $P^{(k)}$ is a measure similar to $P^f_{\varepsilon,T}$, but on configurations restricted to I_k. Since the (positive) couplings among intervals existing (implicitly) in the original a priori measure are not present in the product (independent) one, the correlations of the original model are bigger.

We then identify the modified model as a sort of annealed site diluted Ising model. The dilution variables are indicators of nonoccurrence of flips of σ_k in the intervals I_k (subjected to an independent coin tossing; see the precise meaning below). We establish the identification by conditioning on these variables.

We derive in Section 4 a correlation inequality between this model and an ordinary discrete quenched site diluted Ising model, using the GKS and Harris–FKG inequalities, showing that the correlations of the latter are smaller. The interactions for the quenched model will be given by $J_{ij} = \int_{I_i}\int_{I_j} W(|t-s|)\,dt\,ds$ and the dilution by a family of Bernoulli random variables (λ_k) with mean $1 - O(\varepsilon)$ ($\lambda_k = 0$ means the site i is diluted).

Now, the correlation of the quenched model is the expected value of the (random) correlation obtained from the (random) configuration of the λ_k's, so we would like to establish some sort of convexity of this random correlation (as a function of the configuration in $[0,1]^{2K}$ of the λ_k's) in order to apply Jensen's inequality. In Section 5 this is done by first lowering the λ_k's somewhat (by replacing them by $\eta(\lambda_k)$, with $0 \le \eta(x) \le x$, $\eta(x) \to 1$ as $x \to 1$, which lowers the correlations). Applying Jensen's and the previous inequalities, we then have that the continuum Ising correlations are bigger than those of a (nonrandom) discrete Ising model with inverse temperature $\alpha_\varepsilon = [\eta(1 - O(\varepsilon))]^2\alpha$ and interactions given by J_{ij} above. Notice that this is a $1/r^2$ model whose nearest neighbor coupling tends to infinity as L gets large. So, by starting with $\alpha > 1$, L large and ε small enough, the conditions for applying the discrete result will be satisfied (see below) and we get the continuum model by the comparison inequalities.

The arguments for proving Theorem 2, to be presented in the Section 6, are essentially those for the discrete case with a few modifications to account for the extra randomness of the continuum system.

3. Comparison to an annealed site-diluted model. We begin this section by representing the continuum Ising measure (2.1) introduced in the last section as a weak limit of discrete Ising measures. (This is a well known result, the arguments for which we sketch here for completeness.) This representation is then used to derive a comparison between the continuum model and a sort of annealed site diluted Ising model.

Consider a discrete Ising model on the lattice $\Lambda \equiv \delta Z$, where $\mathbf{Z}$ is the set of integers and δ is a positive number, with interactions J_{ij} given by

$$J_{i,i+\delta} = \tfrac{1}{2}|\log \varepsilon\delta|, \qquad i \in \Lambda,$$
$$J_{ij} = \alpha\delta^2 W(i-j), \qquad i,j \in \Lambda,\ |i-j| > \delta$$

and Hamiltonian

$$H_\delta(\sigma) = -\tfrac{1}{4}\sum_{i,j} J_{ij}\sigma_i\sigma_j. \tag{3.1}$$

The finite volume (in $[-T,T]$) Ising measure so defined is denoted by $\nu^*_{T,\delta}$ and its expectation by $\langle\,\cdot\,\rangle^*_{T,\delta}$, $* = f, +$, where the appropriate boundary conditions are used.

We make the discrete configurations into continuum ones by setting $\sigma_t = \sigma_i$ for $t \in [i, i+\delta)$, $i \in \Lambda$. If A is a set of (distinct) points $\{t_1,\dots,t_n\}$ in $[-T,T]$, let σ_A denote the product $\prod_{i=1}^n \sigma_{t_i}$.

We can write H_δ as the sum $H_\delta^{(1)} + H_\delta^{(2)}$, where

$$H_\delta^{(1)}(\sigma) = -\tfrac{1}{4}\sum_{|i-j|=\delta} J_{ij}\sigma_i\sigma_j,$$

$$H_\delta^{(2)}(\sigma) = -\tfrac{1}{4}\sum_{|i-j|>\delta} J_{ij}\sigma_i\sigma_j.$$

Notice that $H_\delta^{(2)}(\sigma) \to H^*(\sigma)$ as $\delta \to 0$, with H^* the Hamiltonian of the continuum system given by (2.2) and (2.3). Also, the measure

$$\frac{e^{-H_\delta^{(1)}(\sigma)} \times \text{counting measure}}{\text{normalization}}$$

is that of a Markov chain which, as $\delta \to 0$, converges weakly to the Poisson measure $P^*_{\varepsilon,T}$ entering into the continuum Ising measure (2.1).

It follows that $\nu^*_{T,\delta}$ converges weakly to ρ^*_T as $\delta \to 0$. In particular, $\langle\sigma_A\rangle^*_{T,\delta} \to \langle\sigma_A\rangle^*_T$ as $\delta \to 0$.

From now on, we write $H^f(\sigma)$ as $H(\sigma)$.

For K, N positive integers, let $L = T/K$ and $\delta = L/N$. Consider the discrete Ising model in Λ with interactions given by

$$J^\circ_{ij} = J_{ij},$$

for $|i-j| > \delta$ and for $|i-j| = \delta$, but $i \neq kL$, and given by

$$J^\circ_{kL,\,kL+\delta} = 0,$$

for $k \in \{-K,\dots,K\}$. Denote it by $\nu^{\circ,f}_{T,\delta}$ and its expectations by $\langle\,\cdot\,\rangle^{\circ,f}_{T,\delta}$. Since $J_{ij} \geq J^*_{ij}$, $\forall\, i,j$, we have by the GKS inequalities (see [4] and [7]) that

$$\langle\sigma_A\rangle^f_{T,\delta} \geq \langle\sigma_A\rangle^{\circ,f}_{T,\delta}. \tag{3.2}$$

Now, as $N \to \infty$, the measures $\nu^{\circ,f}_{T,\delta}$ converge weakly to the measure

$$d\rho^{\circ,f}_T(\sigma) = \frac{1}{Z'}e^{-\alpha H(\sigma)}\prod_{k=-K}^{K-1} dP^{(k)}(\sigma_k), \tag{3.3}$$

where H is given by (2.2), σ_k is a continuum configuration in the interval $I_k \equiv [kL,(k+1)L)$ and $P^{(k)}$ is the measure in the set of those configurations such that the flips form a Poisson process of rate ε and the initial distribution assigns equal probabilities to ± 1. (By an abuse of notation, we will omit the superscript from $P^{(k)}$ from now on.)

Denote expectations w.r.t. (3.3) by $\langle \cdot \rangle'$ (the dependence on T is omitted). From (3.2) we conclude that

$$\langle \sigma_A \rangle_T^f \geq \langle \sigma_A \rangle'. \tag{3.4}$$

(Concerning the preceding discussion, see also [11].)

Now, let N_i denote the number of jumps of σ_i in the interval I_i. We can write the measure P as

$$\begin{aligned} dP(\sigma_i) &= dP(\sigma_i | N_i = 0)P(N_i = 0) + dP(\sigma_i | N_i > 0)P(N_i > 0) \\ &= (1 - \varepsilon')\, dP_1(\sigma_i) + \varepsilon'\, d\overline{P}(\sigma_i), \end{aligned}$$

where

$$\begin{aligned} dP_1(\sigma_i) &= dP(\sigma_i | N_i = 0) = \tfrac{1}{2}(\delta_{-1}(\sigma_i) + \delta_1(\sigma_i)), \\ d\overline{P}(\sigma_i) &= dP(\sigma_i | N_i > 0), \end{aligned}$$

$\varepsilon' = 1 - e^{-\varepsilon L}$ is approximately εL as $\varepsilon \downarrow 0$ and $\delta_u(\cdot)$ is the Dirac delta measure at the constant function u.

We rewrite P further as

$$dP(\sigma_i) = (1 - 2\varepsilon')\, dP_1(\sigma_i) + 2\varepsilon'\, dP_0(\sigma_i),$$

where $P_0 = (1/2)(P_1 + \overline{P})$, and ε is so small that $2\varepsilon' \leq 1$.

Now, we can write the correlations

$$\begin{aligned} \langle \sigma_A \rangle' &= \frac{1}{Z'} \int \sigma_A e^{-\alpha H(\sigma)} \prod_i dP(\sigma_i) \\ &= \frac{1}{Z'} \int \sigma_A e^{-\alpha H(\sigma)} \prod_i ((1 - \rho)\, dP_1(\sigma_i) + \rho\, dP_0(\sigma_i)), \end{aligned}$$

where A is a set of (distinct) points $\{t_1, \dots, t_n\}$ in $[-T, T]$. We have rewritten $2\varepsilon'$ as ρ.

Expanding the product, we see that the integral can be viewed as an expectation with respect to a family of i.i.d. random variables $\lambda = (\lambda_i)$, where λ_i has a Bernoulli distribution with parameter $1 - \rho$, as follows:

$$\langle \sigma_A \rangle' = \frac{1}{Z'} E_\lambda \left(\int \sigma_A e^{-\alpha H(\sigma)} \prod_i dP_{\lambda_i}(\sigma_i) \right). \tag{3.5}$$

This model is a (sort of) annealed site diluted Ising model. In this case, dilution applies to configurations in an interval and means that there can be flips in the configuration in that interval.

We have thus shown that the correlations for $\langle \cdot \rangle^f$ are bigger than the corresponding ones of an annealed model.

4. Comparison to a quenched site-diluted model. In this section we derive a comparison between the annealed site diluted model of the last section and a regular quenched one. From now on, we restrict attention to sets $A = \{t_1, \dots, t_n\}$ such that there are no two t_i's in the same I_j.

We write the expectation in (3.5) as

$$E_\lambda\left(\int \sigma_A e^{-\alpha H(\sigma)} \prod_i dP_{\lambda_i}(\sigma_i)\right) = E_\lambda\left(\frac{\int \sigma_A e^{-\alpha H(\sigma)} \prod_i dP_{\lambda_i}(\sigma_i)}{Z_\lambda(H)} Z_\lambda(H)\right),$$

where $Z_\lambda(H) = \int e^{-\alpha H(\sigma)} \prod_i dP_{\lambda_i}(\sigma_i)$. We denote the quotient inside the expectation sign above by $\langle \sigma_A \rangle_{\lambda, H}$.

LEMMA 4.1.

$$\int \sigma_A e^{-\alpha H(\sigma)} \prod_i dP_{\lambda_i}(\sigma_i) \geq 0.$$

PROPOSITION 4.1. *$Z_\lambda(H)$ is increasing in λ (w.r.t. the usual partial ordering).*

PROPOSITION 4.2.

$$\langle \sigma_A \rangle_{\lambda, H} \geq \langle \sigma_A \rangle_{\lambda, \overline{H}},$$

where

$$\overline{H}(\sigma) = \sum_{i,j} \lambda_i \lambda_j \int_{I_i} \int_{I_j} W(|t-s|) \sigma_t \sigma_s \, dt \, ds. \tag{4.1}$$

PROOF OF LEMMA 4.1. Expanding the exponential, we obtain

$$\int \sigma_A e^{-\alpha H(\sigma)} \prod_i dP_{\lambda_i}(\sigma_i) = \sum_n C_n \int \sigma_A H^n(\sigma) \prod_i dP_{\lambda_i}(\sigma_i),$$

where C_n are positive numbers.

Expanding the nth power, the r.h.s. can be expressed as

$$\sum_n C_n \int \sigma_A \int_{-T}^{T} \int_{-T}^{T} \cdots \int_{-T}^{T} \int_{-T}^{T} \prod_{j=1}^{n} \left(W(|t_j - s_j|) \sigma_{t_j} \sigma_{s_j} \, dt_j \, ds_j\right) \prod_i dP_{\lambda_i}(\sigma_i).$$

Moving the integral w.r.t. σ inside, we get

$$\sum_n C_n \int\int \cdots \int\int \prod_j \left(W(|t_j - s_j|) \, dt_j \, ds_j\right) \left(\int \sigma_A \prod_j \sigma_{t_j} \sigma_{s_j} \prod_i dP_{\lambda_i}(\sigma_i)\right). \tag{4.2}$$

The expectation $\int \sigma_A \prod_j \sigma_{t_j} \sigma_{s_j} \prod_i dP_{\lambda_i}(\sigma_i)$ factors into $\prod_i \int \sigma_{A_i} \, dP_{\lambda_i}(\sigma_i)$, where A_i is a set of points in the interval I_i, for all i.

Now, if $|A_i|$ is odd ($|\cdot|$ denotes the cardinality), then $\int \sigma_{A_i}\, dP_{\lambda_i}(\sigma_i) = 0$, by the symmetry $\sigma_i \to -\sigma_i$ of P_0 and P_1. If $|A_i|$ is even and $\lambda_i = 1$, then $\int \sigma_{A_i}\, dP_{\lambda_i}(\sigma_i) = 1$. If $\lambda_i = 0$, we have

$$\int \sigma_{A_i}\, dP_0(\sigma) = \tfrac{1}{2}\int \sigma_{A_i}\, dP_1(\sigma) + \tfrac{1}{2}\int \sigma_{A_i}\, d\bar{P}$$

$$= \tfrac{1}{2}\left(1 + \int \sigma_{A_i}\, d\bar{P}\right).$$

Now, $\int \sigma_{A_i}\, d\bar{P} \geq -1$. Therefore, $\int \sigma_{A_i}\, dP_0(\sigma) \geq 0$.

We conclude that $\int \sigma_{A_i}\, dP_{\lambda_i}(\sigma_i) \geq 0$, for all i. Thus, we see that (4.2) is nonnegative and the lemma is proved. □

PROOF OF PROPOSITION 4.1. Do the same steps as in the last proof and notice that

$$\int \sigma_{A_i}\, dP_0(\sigma) \leq \int \sigma_{A_i}\, dP_1(\sigma).$$

Indeed, both integrals are 0 if $|A_i|$ is odd and, if $|A_i|$ is even, the r.h.s. is 1, which is the most the l.h.s. can be. This and positivity prove the proposition. □

PROOF OF PROPOSITION 4.2. We will use the following terminology. Let $\Lambda = \{i\colon \lambda_i = 1\}$ and $\Lambda^c = \{i\colon \lambda_i = 0\}$. We call an interval I_i either a 1-interval or a 0-interval depending on whether $i \in \Lambda$ or $i \in \Lambda^c$. Also, let J_{ij} denote the integral $\int_{I_i}\int_{I_j} W(|t-s|)\, dt\, ds$.

First, notice that if any of the elements of A, say t_{j_0}, belongs to a 0-interval, say I_{i_0}, then $\langle \sigma_A \rangle_{\lambda, \bar{H}} = 0$. This is because under $\bar{H}$, all the 0-intervals get disconnected from the rest of the system, so that $\langle \sigma_A \rangle_{\lambda, \bar{H}}$ factors into terms, one of which is $\int \sigma_{t_{j_0}}\, dP_0(\sigma_{i_0})$ (it is here that the restriction on A made at the beginning of the section enters). This integral vanishes (by symmetry), making the product vanish. Thus, we need only consider A's all of whose elements belong to 1-intervals.

We change notation here and write S_t instead of σ_t for t's belonging to 1-intervals. Further, since S_t is constant in each 1-interval I_i, we write S_i instead. Notice that $\langle S_A \rangle_{\lambda, \bar{H}}$ is the correlation of a discrete Ising model (in $\mathbf{Z} \cap [-K, K]$) with interactions $\bar{J}_{ij} \equiv \lambda_i \lambda_j J_{ij}$. Now, the proof:

$$Z_\lambda(H)\langle S_A \rangle_{\lambda, H} = \int S_A e^{-\alpha(H_1(S) + H_2(\sigma) + H_{12}(S,\sigma))} \prod_{i \in \Lambda} dP_1(S_i) \prod_{i \in \Lambda^c} dP_0(\sigma_i)$$

$$= \int S_A e^{-\alpha H_{12}(S,\sigma)}\, dQ(S)\, d\tilde{Q}(\sigma),$$

where

$$H_1(S) = -\tfrac{1}{4}\sum_{i,j\in\Lambda} J_{ij}S_iS_j\left(= -\tfrac{1}{4}\sum_{i,j}\lambda_i\lambda_jJ_{ij}S_iS_j\right),$$

$$H_2(\sigma) = -\tfrac{1}{4}\sum_{i,j\in\Lambda^c}\int_{I_i}\int_{I_j} W(|t-s|)\sigma_t\sigma_s\,dt\,ds,$$

$$H_{12}(S,\sigma) = -\tfrac{1}{4}\sum_{i\in\Lambda} S_i\psi_i(\sigma),$$

with

$$\psi_i(\sigma) = \sum_{j\in\Lambda^c}\int_{I_i}\int_{I_j} W(|t-s|)\sigma_t\,dt\,ds,$$

$$dQ(S) = e^{-\alpha H_1(S)}\prod_{i\in\Delta} dP_1(S_i),$$

$$d\tilde{Q}(\sigma) = e^{-\alpha H_2(\sigma)}\prod_{i\in\Lambda^c} dP_0(\sigma_i).$$

Notice that Q is the unnormalized Ising measure with interactions $\bar{J}_{ij}$.

We expand the exponential:

$$\begin{aligned} Z_\lambda(H)\langle S_A\rangle_{\lambda,H} &= \int S_A \exp\left(\frac{\alpha}{4}\sum_{i\in\Lambda} S_i\psi_i(\sigma)\right) dQ(S)\,d\tilde{Q}(\sigma) \\ &= \sum_n C_n\int S_A\left(\sum_{i\in\Lambda} S_i\psi_i(\sigma)\right)^n dQ(S)\,d\tilde{Q}(\sigma) \\ &= \sum_n C_n\sum_{(m)\in\Gamma_n} C_{(m)}\int S_A\prod_i S_i^{m_i}\,dQ(S)\int\prod_i \psi_i^{m_i}(\sigma)\,d\tilde{Q}(\sigma), \end{aligned}$$

where $\Gamma_n = \{(m)\equiv(m_1,\dots,m_n)|m_i\ge 0,\ \Sigma_i m_i = n\}$ and $C_{(m)}$ are positive numbers.

Now, by the GKS inequality, the first integral is bigger than

$$\frac{1}{Z''}\int S_A\,dQ(S)\int\prod_i S_i^{m_i}\,dQ(S),$$

where $Z'' = \int dQ(S)$ is a normalizing factor.

Also, the second integral is positive (this is proved like Lemma 4.1 by expanding $\Pi_i\psi_i^{m_i}(\sigma)$ as well as the exponential in $\tilde{Q}$ and checking that everything is positive, which is done as before).

Notice now that $(1/Z'')\int S_A\,dQ(S) = \langle S_A\rangle_{\lambda,\bar{H}}$ and that this quantity does not depend on (m) or n. We thus have

$$\begin{aligned} Z_\lambda(H)\langle S_A\rangle_{\lambda,H} &\ge \langle S_A\rangle_{\lambda,\bar{H}}\sum_n C_n\sum_{\Gamma_n} C_{(m)}\int\prod_i\left(S_i\psi_i(\sigma)\right)^{m_i} dQ(S)\,d\tilde{Q}(\sigma) \\ &= \langle S_A\rangle_{\lambda,\bar{H}}\int e^{-\alpha H(\sigma)}\prod_i dP_{\lambda_i}(\sigma_i). \end{aligned}$$

Now Proposition 4.2 follows by the fact that the last integral is $Z_\lambda(H)$. □

Proposition 4.2 and Lemma 4.1 imply the inequality

$$E_\lambda(\langle\sigma_A\rangle_{\lambda,H}Z_\lambda(H)) \geq E_\lambda(\langle\sigma_A\rangle_{\lambda,\bar{H}}Z_\lambda(H)). \tag{4.3}$$

We need one last result in this section.

PROPOSITION 4.3. *$\langle\sigma_A\rangle_{\lambda,\bar{H}}$ is increasing in λ.*

PROOF. Since $\langle\sigma_A\rangle_{\lambda,\bar{H}}$ is always nonnegative and vanishes whenever there is an element of A in a 0-interval, we need only check the proposition for A's without elements in 0-intervals. But this case follows by monotonicity in the couplings for correlations of the ordinary discrete Ising ferromagnet. □

Now, Propositions 4.1 and 4.3 imply, via the Harris–FKG inequality (see [5]), the following inequality for (3.5):

$$\begin{aligned}\frac{1}{Z'}E_\lambda(\langle\sigma_A\rangle_{\lambda,H}Z_\lambda(H)) &\geq \frac{1}{Z'}E_\lambda(\langle\sigma_A\rangle_{\lambda,\bar{H}}Z_\lambda(H))\\ &\geq \frac{1}{Z'}E_\lambda(\langle\sigma_A\rangle_{\lambda,\bar{H}})E_\lambda(Z_\lambda(H))\\ &= E_\lambda(\langle\sigma_A\rangle_{\lambda,\bar{H}}),\end{aligned}$$

since $E_\lambda(Z_\lambda(H))$ is by definition Z'.

We have thus obtained the following comparison:

$$\langle\sigma_A\rangle' \geq E_\lambda(\langle\sigma_A\rangle_{\lambda,\bar{H}}). \tag{4.4}$$

The latter expression is the correlation of a quenched site diluted Ising model.

REMARK. For the case of regular (discrete) annealed site diluted models (i.e., those where the spins at the sites are independently randomly diluted), a similar inequality follows by the same arguments.

5. Comparison to a standard discrete Ising model. In this section, a general inequality relating a quenched (site) diluted Ising model and an ordinary undiluted Ising model is derived. We then apply it to the quenched model at the end of the last section to complete the comparison of the continuum to the discrete model and prove Theorem 1.

For an ordinary (discrete) Ising model in a finite volume Λ with ferromagnetic pair interactions J_{ij} and Hamiltonian

$$H(\sigma) = -\tfrac{1}{4}\sum_{i,j} J_{ij}\sigma_i\sigma_j,$$

let $\langle\cdot\rangle_{\alpha,J}$ denote the expectation w.r.t. the Ising measure (with free or + b.c.) at inverse temperature α.

Let $(\lambda_i)_{i\in\Lambda}$ be a sequence of independent nonnegative random variables which are less than one and have means (ξ_i). Let $\langle\cdot\rangle_{\alpha,\tilde{J}}$ denote the (random) expectation w.r.t. the Ising measure with inverse temperature α and interactions $\bar{J}_{i,j} = \lambda_i \lambda_j J_{ij}$. We define the quenched site diluted Ising measure with expectation $\langle \cdot \rangle^q_{\alpha,J}$ to be the mean of the random measure w.r.t. the distribution of the λ's, that is,

$$\langle \cdot \rangle^q_{\alpha,J} \equiv E_\lambda(\langle \cdot \rangle_{\alpha,\bar{J}}).$$

We prove the following result.

PROPOSITION 5.1. *Let A be a set of points in Λ. Then*

$$\langle \sigma_A \rangle^q_{\alpha,J} \geq \langle \sigma_A \rangle_{\alpha,\tilde{J}}, \tag{5.1}$$

where $\tilde{J}_{ij} = \eta_i(\xi_i)\eta_j(\xi_j)J_{ij}$, and η_i is an increasing continuous function in $[0,1]$ which is 0 at 0 and 1 at 1, for each $i \in \Lambda$. [η_j does not depend on the distribution of the λ_i's; it does depend on α and the J_{ij}'s, specifically on $(\alpha/2)\Sigma_j J_{ij}$—see (5.5).]

PROOF. For $i \in \Lambda$, let $\eta_i(x)$ be a nonnegative function in $[0,1]$ such that $\eta_i(x) \leq x$. (The specific form of η_i will be chosen below.) By monotonicity of Ising correlations in the couplings, we have

$$\langle \sigma_A \rangle_{\alpha,\bar{J}} \geq \langle \sigma_A \rangle_{\alpha,J^*}, \tag{5.2}$$

where $J^*_{ij} = \eta_i(\lambda_i)\eta_j(\lambda_j)J_{ij}$.

Let $N = |\Lambda|$ and write $\langle \sigma_A \rangle_{\alpha,J^*}$ as

$$\langle \sigma_A \rangle_{\alpha,J}(\eta_1(\lambda_1), \dots, \eta_N(\lambda_N)).$$

Now suppose that

$$\frac{\partial^2}{\partial x^2}\langle \sigma_A \rangle_{\alpha,J}(\gamma_1, \dots, \gamma_{i-1}, \eta_i(x), \gamma_{i+1}, \dots, \gamma_N) \geq 0 \tag{5.3}$$

for all $0 \leq \gamma_i \leq 1$, $i \in \Lambda$ and all x in $(0,1)$. Then we get (5.1) by successively applying Jensen's inequality to $\langle \sigma_A \rangle^q_{\alpha,J}$.

By differentiating $\langle \sigma_A \rangle_{\alpha,J}$ as above, we obtain the following expression:

$$\begin{aligned}
&\left\{\eta_i'' - 2(\eta_i')^2\left\langle \frac{\alpha}{4}\sum_{j\in\Lambda}\gamma_j J_{ij}\sigma_i\sigma_j\right\rangle\right\}\\
&\quad\times\left[\left\langle \sigma_A\left(\frac{\alpha}{4}\sum_{j\in\Lambda}J_{ij}\gamma_j\sigma_i\sigma_j\right)\right\rangle - \langle\sigma_A\rangle\left\langle\frac{\alpha}{4}\sum_{j\in\Lambda}J_{ij}\gamma_j\sigma_i\sigma_j\right\rangle\right]\\
&\quad+(\eta_i')^2\left[\left\langle \sigma_A\left(\frac{\alpha}{4}\sum_{j\in\Lambda}J_{ij}\gamma_j\sigma_i\sigma_j\right)^2\right\rangle - \langle\sigma_A\rangle\left\langle\left(\frac{\alpha}{4}\sum_{j\in\Lambda}J_{ij}\gamma_j\sigma_i\sigma_j\right)^2\right\rangle\right],
\end{aligned}$$

where primes mean differentiation w.r.t. x. We have omitted the argument of η_i and the subscripts of the Ising expectation signs.

The expressions in square brackets are nonnegative, by the GKS inequality, so that we only need the expression in braces to be positive. We use the boundedness of the σ's and γ's to bound it below by

$$\eta_i'' - \Gamma_i(\eta_i')^2, \tag{5.4}$$

where $\Gamma_i = (\alpha/2)\Sigma_{j\in\Lambda}J_{ij}$.

Setting (5.4) to 0 and solving the differential equation with boundary conditions 0 at 0 and 1 at 1, we obtain

$$\eta_i = \zeta(\Gamma_i, x) \equiv \frac{1}{\Gamma_i}\log(1-(1-e^{-\Gamma_i})x)^{-1}, \tag{5.5}$$

which satisfies all the conditions above. □

REMARK 1. The above proposition holds for both free and + b.c.

REMARK 2. A similar result is valid for quenched *bond* models with a similar proof.

REMARK 3. These results can be used to derive lower bounds for the critical temperature of diluted models in cases more general than, for example, those studied in [8]. For the cases studied in this reference, our bounds are weaker.

We are ready now to prove Theorem 1.

PROOF OF THEOREM 1. The results of this and previous sections give us the following comparison between the correlations of the continuum model (in $[-KL, KL]$, configurations denoted by the letter σ) and those of the discrete one (in $\{-K,\dots,K\}$, configurations denoted by S):

$$\langle\sigma_A\rangle^f \geq \langle S_{A^*}\rangle_{\tilde{J}}^f, \tag{5.6}$$

where A is a set of points in $[-KL, KL]$ such that no two are in the same interval I_i $(= iL, (i+1)L])$, $i \in \{-K,\dots,K\}$, and A^* is the set of integers $i \in \{-K,\dots,K\}$ such that there is a point of A in I_i. Here $\tilde{J}_{ij} = (\zeta(\Gamma, 1-\rho))^2 J_{ij}$, with $J_{ij} = \int_{I_i}\int_{I_j} W(|t-s|)\,dt\,ds$ and $\Gamma = (\alpha/2)\Sigma_j J_{ij}$ (which does not depend on i, by translation invariance, and is finite, due to the decay of W —notice that by applying Proposition 5.1 directly, we obtain (5.6) but with the finite sum for Γ; we can then replace it by the infinite sum due to the monotonicity of both ζ and the Ising correlations).

Notice that the model in the r.h.s. of (5.6) is a (one dimensional) $1/r^2$ Ising model at inverse temperature $\alpha\zeta^2(\Gamma, 1-\rho)$. Notice also that $J_{ij} = J_{ij}(L)$ is such that, denoting

$$\int_i^{i+1}\int_j^{j+1}\frac{1}{|t-s|^2}\,dt\,ds$$

by J'_{ij} for $|i-j| > 1$, we have that, as $L \to \infty$, J_{ij}/J'_{ij} converges to 1 uniformly in i, j such that $|i-j| \geq 2$, and $J_{i,i+1}$ (which does not depend on i) goes to ∞.

We want to use the result of [6] (Theorem 3.4) stating that, as $J'_{i,i+1} \to \infty$,

$$\langle S_0 S_x\rangle_{J'}^f \to 1$$

uniformly in the volume and in x inside the volume, provided $\alpha > 1$, to prove the following corresponding continuum result: As $\varepsilon \to 0$,

$$\langle \sigma_0 \sigma_t \rangle_T^f \to 1$$

uniformly in the volume and in t, provided $\alpha > 1$.

We proceed as follows. Given $\alpha > 1$ and $\delta > 0$, let $\bar{\alpha}$ be such that $1 < \bar{\alpha} < \alpha$. By the discrete result just quoted, there exists J such that for the model with nearest neighbor interactions bigger than J, long range interactions given by J'_{ij} and inverse temperature $\bar{\alpha}$, we have

$$\langle S_0 S_x \rangle_{J'}^f \geq 1 - \delta$$

uniformly in the volume and in x. Now, let L be so big that $J_1 \equiv J_{i,i+1} > J$ and also $\alpha J_{ij} > \bar{\alpha} J'_{ij}$ for i, j with $|i - j| \geq 2$. Next, make ε so small that ρ is so small that ζ is so close to 1 that $\zeta^2 \alpha J_1 > \bar{\alpha} J$ and $\zeta^2 \alpha J_{ij} > \bar{\alpha} J'_{ij}$. By applying the comparison (5.6), we get

$$\langle \sigma_0 \sigma_t \rangle_T^f \geq 1 - \delta \tag{5.7}$$

uniformly in the volume and for $|t| > L$. If necessary we can take ε smaller so that (5.7) holds uniformly in t. The theorem is now proven. □

6. Slow decay of correlations. In this last section, we use the FK representation of the continuum Ising model to derive lower bounds for the truncated two point function, proving Theorem 2. It is done almost exactly in the same way as has been done in [6] for discrete FK models, with a few modifications (to account for the extra randomness of the continuum case). For this reason we will be a bit sketchy, referring the reader to the discrete results for missing details (also to [10] for definitions and properties of continuum FK measures).

We start by defining the FK measures. Let θ_i, $i = 1, 2, \ldots$ be the points of a Poisson process of rate ε on the real line and $\omega_j = (s_j, t_j)$, $j = 1, 2, \ldots$ the points of a Poisson process in $R_0^2 = \{(s, t): s \leq t\}$ with density $\Delta = \alpha W(t - s)$. Denote these (random) sets of points by θ and ω, respectively, and call them configurations. (We will alternatively use the terminology θ-points for θ.) The ω_j's will be given the meaning of occupied bonds linking s_j and t_j.

Consider the partition of R (resp., of the interval $[-T, T]$, for $T > 0$) into intervals, produced by the points θ_i in $[-T, T]$. Call those θ-intervals. Say that two disjoint intervals I and J are linked (denoted $I \frown J$) if there is an occupied bond linking two points, one in each interval. (If there are none, we denote this by $I \not\frown J$. Notice that the two infinite intervals of the partition of R are linked with probability 1.) Two θ-intervals I and J are *connected* if there is a sequence of θ-intervals $I_0, \ldots, I_n$ with $I_0 = I$ and $I_n = J$, so that $I_i \frown I_{i-1}$, $i = 1, \ldots, n$. Two points s and t are connected (denoted $s \leftrightarrow t$) if either they belong to the same θ-interval or belong to distinct connected θ-intervals. A *cluster* is a maximal union of connected θ-intervals.

Let $C_T^w(\theta, \omega)$ [resp., $C_T^f(\theta, \omega)$] be the number of distinct connected clusters obtained with the θ-intervals of the partition of R (resp., of $[-T, T]$). We define the finite volume, continuum FK measures with parameter q as follows:

$$(6.1)\qquad dP^*_{q,\Delta,T}(\theta,\omega) = \frac{1}{N}\, dP(\theta)\, dP'(\omega) q^{C^*_T(\theta,\omega)},$$

where $* = w$ or f for the *wired* and *free* cases (see [3]), P and P' are the Poisson processes mentioned and N is the normalizing factor. (We will drop some subscripts sometimes.)

The infinite volume measure exists (by standard arguments) and is denoted $P^*_{q,\Delta}$. Notice that for $q = 1$, $P^w_{1,\Delta} = P^f_{1,\Delta}$ is an independent (continuum) percolation model. We list the properties of the FK measures we will need.

1. $P^*_{q,\Delta}$ is a strong FKG measure, that is, for any region A in $R \times R^2_0$ and f, g increasing functions in the configurations [w.r.t. the partial order $(\theta,\omega) \prec (\theta',\omega')$ whenever $\theta \supset \theta'$ and $\omega \subset \omega'$], we have

$$P^*_{q,\Delta}(fg|\mathscr{A}) \geq P^*_{q,\Delta}(f|\mathscr{A}) P^*_{q,\Delta}(g|\mathscr{A}),$$

where $\mathscr{A}$ is the σ-algebra generated by the configurations in A.

For the properties below, we use the notation $P(\cdot)$ for the expectation w.r.t. the measure P.

2. $P^*_{q,\Delta}(f) \geq P^*_{q',\Delta'}(f)$, for $q' \geq q$, $\Delta \geq \Delta'$ and f increasing.
3. $P^w_{q,\Delta}(f) \geq P^f_{q,\Delta}(f)$, for f increasing.
4. We have the following representation of continuum Ising correlations (where the notation $0 \leftrightarrow \infty$ means that the cluster of the origin is infinite):

$$M = \langle\sigma_0\rangle^+ = P^w_2(0 \leftrightarrow \infty),$$
$$\langle\sigma_s\sigma_t\rangle^* = P^*_2(s \leftrightarrow t).$$

We conclude that

$$\tau(t) = \langle\sigma_0\sigma_t\rangle^+ - M^2 = P^f_2(0 \leftrightarrow t) - \big(P^w_2(0 \leftrightarrow \infty)\big)^2$$
$$\geq P^w_2(0 \leftrightarrow t, 0 \not\leftrightarrow \infty, t \not\leftrightarrow \infty) \equiv \tau'(t).$$

So, all we need to prove Theorem 2 is to derive the same bounds for τ'. We do that in the following propositions.

As in [6], we begin with an estimate for the *self similar* percolation case, that is, the $q = 1$ case with

$$(6.2)\qquad W(t) = \tilde{W}(t) \equiv \frac{1}{t^2} 1_{\{t>1\}}.$$

For ξ a real number, let

$$T_\xi = \inf\{\theta_i : \theta_i \geq \xi\}, \qquad S_\xi = \sup\{\theta_i : \theta_i \leq \xi\}.$$

Define

$$\mu^\xi_1 = T_\xi,$$
$$\mu^\xi_{n+1} = T_{\mu^\xi_n}, \qquad n \geq 1,$$
$$\nu^\xi_1 = S_\xi,$$
$$\nu^\xi_{n+1} = S_{\nu^\xi_n}, \qquad n \geq 1,$$

that is, μ^ε_n is the nth θ-point after ξ and ν^ξ_n is the nth θ-point before ξ.

Notice that $\mu_n^\xi = \xi + Y_n$ and $\nu_n^\xi = \xi - W_n$, where Y_n and W_n are random variables each having a gamma distribution with parameters n and ε. In particular, $E\mu_n^\xi = \xi + n/\varepsilon$ and $E\nu_n^\xi = \xi - n/\varepsilon$.

We say that an interval $[\xi', \xi]$ is *dissociated* if there is no occupied bond from $[S_{\xi'}, T_\xi]$ to its complement. Below, we use the notation $\{I \not\leftrightarrow \infty\}$ for an interval I none of whose points are in an infinite cluster.

PROPOSITION 6.1. *For L a positive integer, let $F_L = \{\exists$ an integer $k \in [1, L) | [0, \mu_k^L) \not\leftarrow (\mu_k^L, \infty)\}$.*

If $\alpha > 1$, then there exist constants C and C' so that in the self similar case (6.2),

$$P_1(F_L) \geq C/L^{\alpha-1} \quad \textit{for all } L, \tag{6.3}$$

$$P_1([0, L] \not\leftrightarrow \infty) \geq C'/L^{2(\alpha-1)} \quad \textit{for all } L. \tag{6.4}$$

PROOF. Define

$$F_L^* = \left\{\exists \text{ an integer } k' \in [1, L) \middle| [\nu_k^0, L] \not\leftarrow (-\infty, \nu_k^0)\right\},$$

$$H_L = \left\{(\nu_L^0, 0) \not\leftarrow (L, \infty)\right\},$$

$$H_L^* = \left\{(L, \mu_L^L) \not\leftarrow (-\infty, 0)\right\}.$$

Then

$$P_1([0, L] \not\leftrightarrow \infty) \geq P_1(F_L \cap F_L^* \cap H_L \cap H_L^*)$$
$$\geq P_1^2(F_L) P_1^2(H_L),$$

with the second inequality due to the FKG property (all of the events are decreasing).

Now,

$$P_1(H_L) = E\left\{\exp\left(-\alpha \int_{\nu_L^0}^0 \int_L^\infty (t-s)^{-2}\, dt\, ds\right)\right\}$$
$$= E\left\{\exp\left(-\alpha \int_{-Y}^0 \int_1^\infty (t-s)^{-2}\, dt\, ds\right)\right\}$$
$$\geq \exp\left(-\alpha \int_{-EY}^0 \int_1^\infty (t-s)^{-2}\, dt\, ds\right),$$

where the last inequality is Jensen's inequality and the expectation E is w.r.t. a gamma random variable Y.

The last expression is positive and does not depend on L. It follows that (6.3) implies (6.4).

To derive (6.3), let

$$\mathcal{N} = \#\left\{k \in [1, L) \cap Z \colon [0, \mu_k^L] \not\leftarrow (\mu_k^L, \infty)\right\}.$$

F_L is the event that $\mathcal{N} > 0$. We compute the expected value of $\mathcal{N}$:

$$\begin{aligned}
E_1(\mathcal{N}) &= \sum_{k=1}^{L-1} P_1\big([0,\mu_k^L] \not\leftrightarrow (\mu_k^L,\infty)\big)\\
&= \sum_1^{L-1} E\left\{\exp\left[-\alpha\left(\int_0^{\mu_k^L}\int_{\mu_k^L+1}^{\infty}(t-s)^{-2}\,dt\,ds\right.\right.\right.\\
&\qquad\left.\left.\left.+\int_{\mu_k^L}^{\mu_k^L+1}\int_0^{t-1}(t-s)^{-2}\,ds\,dt\right)\right]\right\}\\
&\geq \text{const.}\sum_1^{L-1} E\left(\frac{1}{(\mu_k^L)^{\alpha}}\right) \geq \text{const.}\sum_1^{L-1}\frac{1}{(E\mu_k^L)^{\alpha}}\\
&= \text{const.}\sum_1^{L-1}\frac{1}{(L+k/\varepsilon)^{\alpha}} \geq \text{const.}\,L^{1-\alpha},
\end{aligned}$$

where the second inequality follows by Jensen's inequality and E is expectation w.r.t. μ_k^L.

Now,

$$P_1(\mathcal{N}>0) = \frac{E_1(\mathcal{N})}{E_1(\mathcal{N}\,|\,\mathcal{N}>0)}.$$

Let $X = \inf\{k' \in [1,L)\cap Z\colon [0,\mu_{k'}^L] \not\leftrightarrow (\mu_{k'}^L,\infty)\}$. Then,

$$\begin{aligned}
E_1(\mathcal{N}\,|\,\mathcal{N}>0) &= \sum_{k'=1}^{L-1}\int_L^{\infty} P_1\big(X=k',\mu_{k'}^L\in dt\,|\,\mathcal{N}>0\big)E_1\big(\mathcal{N}\,|\,X=k',\mu_{k'}^L=t\big)\\
&= \sum_{k'=1}^{L-1}\int_L^{\infty} P_1\big(X=k',\mu_{k'}^L\in dt\,|\,\mathcal{N}>0\big)\\
&\qquad\times\left(1+\sum_{k=1}^{L-k'} P_1\big((t,\mu_k^t] \not\leftrightarrow (\mu_k^t,\infty)\big)\right)\\
&\leq \sum_{k'=1}^{L-1}\int_L^{\infty} P_1\big(X=k',\mu_{k'}^L\in dt\,|\,\mathcal{N}>0\big)\\
&\qquad\times\left(1+\sum_{k=1}^{\infty} E\left\{\exp\left(-\alpha\int_t^{\mu_k^t}\int_{\mu_k^t+1}^{\infty}(r-s)^{-2}\,dr\,ds\right)\right\}\right)\\
&\leq \sum_{k'=1}^{L-1}\int_L^{\infty} P_1\big(X=k',\mu_{k'}^L\in dt\,|\,\mathcal{N}>0\big)\\
&\qquad\times\left(1+\sum_{k=1}^{\infty} E\left(\frac{1}{(\mu_k^t-t+1)^{\alpha}}\right)\right)\\
&= \sum_{k'=1}^{L-1}\int_L^{\infty} P_1\big(X=k',\mu_{k'}^L\in dt\,|\,\mathcal{N}>0\big)\left(1+\sum_{k=1}^{\infty} E\left(\frac{1}{(Y_k+1)^{\alpha}}\right)\right)\\
&\leq 1+\text{const.}\sum_1^{\infty}\frac{1}{k^{\alpha}} = \text{const.}
\end{aligned}$$

Combining all these inequalities, we get (6.3). □

In the next proposition, we omit the subscript Δ in P^*.

PROPOSITION 6.2. *For* $|t| \le L$,

$$\tau'(t) \ge P^f_{2,L}(0 \leftrightarrow t) P^w_2([-L, L] \nleftrightarrow \infty).$$

PROOF. Let $\Lambda_L = \{\text{bonds } s, t \text{ with } |s|, |t| \le L\}$. Then,

$$\tau'(t) \ge P^w_2(0 \leftrightarrow t \text{ by bonds in } \Lambda_L | [-L, L] \nleftrightarrow \infty) P^w_2([-L, L] \nleftrightarrow \infty).$$

The first probability on the r.h.s. can be estimated by first noticing that the conditioning event only depend on bonds in Λ^c_L and points of θ outside $[-L, L]$. Proceed now exactly as in the proof of Proposition 2.1 in [6], by conditioning further on such configurations, expressing the infinite volume measure as the proper limit of the finite volume ones, and then using the strong FKG property to conclude that

$$\begin{aligned} &P^w_2(0 \leftrightarrow t \text{ by bonds in } \Lambda_L | [-L, L] \nleftrightarrow \infty) \\ &\quad \ge \lim_{L' \to \infty} P^w_{2,L'}(0 \leftrightarrow t \text{ by bonds in } \Lambda_L | \nexists \text{ occupied bonds in } \Lambda^c_L) \\ &\quad = P^f_2(0 \leftrightarrow t). \end{aligned}$$

□

PROPOSITION 6.3. *If* $\alpha > 1$, *then for any* $\delta > 0$, *there exists some* $C' > 0$ *so that*

$$P^w_{2,\Delta}([0, L] \nleftrightarrow \infty) \ge C'/L^{2(\alpha-1)+\delta} \quad \textit{for all } L \ge 1.$$

PROOF. Let $\hat{W} = W \cdot \chi(|t| < R) + \tilde{W} \cdot \chi(|t| > R)$, for $\tilde{W}$ given by (6.2), where χ is the indicator function of a set. Given δ, choose $\hat{\alpha} > \alpha$ so that $2(\hat{\alpha} - 1) = 2(\alpha - 1) + \delta$, and R so that $\hat{\alpha} W > \alpha \tilde{W}$, for $t > R$. Let $\hat{\alpha}\hat{W} = H_\delta$. We then have

$$P^w_{2,\Delta}([0, L] \nleftrightarrow \infty) \ge P_{1,\hat{\Delta}}([0, L] \nleftrightarrow \infty),$$

by the monotonicity properties of the FK measures.

Exactly as in Proposition 6.1,

$$P_{1,H_\delta}([0, L] \nleftrightarrow \infty) \ge \text{const.}\, P^2_{1,H_\delta}(F_L).$$

Define $\hat{F}_L = \{\exists\ \xi \in [L, 2L] \cap Z |$ there is no occupied bond longer than R linking $[0, T_\xi]$ to $(T_\xi, \infty)\}$.

Since $\hat{F}_L$ does not involve the short bonds distinguishing between P_{1,H_δ} and $P_{1,\hat{\alpha}\tilde{W}}$, we have

$$P_{1,H_\delta}(\hat{F}_L) = P_{1,\hat{\alpha}\tilde{W}}(\hat{F}_L) \ge P_{1,\hat{\alpha}\tilde{W}}(F_L) \ge C/L^{\hat{\alpha}-1},$$

by Proposition 6.1.

Now, conditioning,

$$\begin{aligned} P_{1,H_\delta}(F_L) &= P_{1,H_\delta}(\hat{F}_L)P_{1,H_\delta}(F_L|\hat{F}_L) \\ &\geq P_{1,H_\delta}(\hat{F}_L)E\{e^{-\hat{\alpha}\int_{T_\xi-R}^{T_\xi}\int_{T_\xi}^{T_\xi+R}W(t-s)\,dt\,ds}\}, \\ &= \text{const.}\ P_{1,H_\delta}(\hat{F}_L). \end{aligned}$$

To complete the proof of Theorem 2 we need the following result.

PROPOSITION 6.4. *There is a constant $C > 0$ such that*

$$\tau'(t) \geq C/t^2 \quad \text{for } t \text{ large}.$$

PROOF. We consider the event Λ_t that the θ-intervals I_0, I_t containing the origin and the point t, respectively, are connected to each other but to no other θ-interval. Condition on θ and observe that the resulting measure is a discrete FK measure. Follow the steps of [6] to find

$$\begin{aligned} P_2^w(\Lambda_t|\theta) &\geq (1 - e^{-\alpha\int_{I_0}\int_{I_t}W(|t-s|)\,dt\,ds}) \\ &\quad \times e^{-\alpha\int_{I_0}\int_{I_0^c}W(|t-s|)\,dt\,ds}e^{-\alpha\int_{I_t}\int_{I_t^c}W(|t-s|)\,dt\,ds} \\ &\equiv f(I_0, I_t). \end{aligned}$$

Now, there exist constants $0 < a < b < \infty$, $0 < c < 1$, such that

$$P_2^w(a < |I_0| < b, a < |I_t| < b) \geq c,$$

where a, b, c do not depend on t.

We conclude that for t large enough,

$$\begin{aligned} P_2^w(\Lambda_t) &\geq P_2^w(f(I_0, I_t)1(a < |I_0| < b, a < |I_t| < b)) \\ &\geq \frac{\text{const}}{t^2}P_2^w(a < |I_0| < b, a < |I_t| < b) \\ &= \text{const}/t^2. \end{aligned}$$

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INSTITUTO DE MATEMÁTICA E ESTATÍSTICA, USP
CAIXA POSTAL 20570
SÃO PAULO, SP 01498
BRAZIL