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THE SPECTRAL MEASURE OF A REGULAR STATIONARY RANDOM FIELD WITH THE WEAK OR STRONG COMMUTATION PROPERTY

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It is shown that a regular stationary random field on $\mathbf{Z}^2$ exhibits the weak (strong) commutation property if and only if its spectral density is the squared modulus of a weakly (strongly) outer function in the Hardy space $H^2(\mathbf{T}^2)$ of the torus. Applications to prediction are discussed.

0. Introduction. Two important new tools in the prediction theory for stationary random fields are the notions of "weak commutation" and "strong commutation." Kallianpur and Mandrekar [9] introduced the latter, and showed that a field possessing this property admits a fourfold Wold-type decomposition. Later, Kallianpur, Miamee and Niemi [10] demonstrated that the decomposition occurs under the less restrictive weak commutation property. They developed the associated spectral theory as well. In other work, Chiang [4], Korezlioglu and Loubaton [11, 13] and Soltani [17] further explored the connections between the commutation properties, Wold-type decompositions, regularity conditions, questions of multiplicity and moving-average representations; Miamee and Niemi [15] relate a commutation condition to the angle between half-planes of a random field.

From [11], Propositions III.6 and V.5, we find that the issue of whether a stationary random field has the weak commutation property rests entirely with its "regular" component, in precisely the sense described later. A complete treatment of the regular component is given in this article; taken with known results, this provides exact spectral criteria for an arbitrary wide sense stationary random field on the integer lattice to exhibit weak commutation. The same approach is used to derive an analogous description of regular strongly commutative fields. This investigation proceeds entirely within the spectral domain, in contrast with the extant literature. The main result is that the commutation properties are linked to factorizations of the spectral density, by what shall be called weakly and strongly outer functions in the Hardy class $H^2(\mathbf{T}^2)$.

1. Preliminaries. Let $\mathbf{D}$ be the unit disc in the complex plane $\mathbf{C}$, and $\mathbf{T}$ the unit circle. Let $d\sigma$ be normalized Lebesgue measure on the torus $\mathbf{T}^2$. The spectrum of a stationary random field on the integer lattice $\mathbf{Z}^2$ is a finite,

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nonnegative Borel measure μ on $\mathbf{T}^2$. Consider the Hilbert space $L^2(\mu)$. It is spanned by the collection of functions $\{e^{ims+int}: (m,n) \in \mathbf{Z}^2\}$. As such, every subset S of $\mathbf{Z}^2$ generates a natural subspace of $L^2(\mu)$: Define $\mathscr{M}(S)$ to be the closed linear span in $L^2(\mu)$ of $\{e^{ims+int}: (m,n) \in S\}$. Let P_S be the orthoprojection operator of $L^2(\mu)$ onto $\mathscr{M}(S)$.

These spaces $\mathscr{M}(S)$ are spectral isomorphs of spaces associated with the random field. Because of their relevance to prediction theory for fields, certain generating sets S are of particular interest. Among them are the "right half-plane" $R = \{(m,n) \in \mathbf{Z}^2: m \geq 0\}$, and the "top half-plane" $T = \{(m,n) \in \mathbf{Z}^2: n \geq 0\}$.

The next definition is motivated by the notion of "pure nondeterminism" for random processes.

DEFINITION. The space $L^2(\mu)$ is *regular* if

$$\bigcap_{m=0}^{\infty} e^{ims}\mathscr{M}(R) = \bigcap_{n=0}^{\infty} e^{int}\mathscr{M}(T) = (0).$$

Other types of regularity are of course possible, and indeed have been studied (see [4, 8–13, 17]). The one given here has a straightforward characterization, and seems to be a natural hypothesis in the analysis that follows.

Now we present the objects of interest.

DEFINITION. The space $L^2(\mu)$ has the *weak commutation property* (WCP) if $P_R P_T = P_T P_R$. If, in addition

$$\mathscr{M}(R) \cap \mathscr{M}(T) = \mathscr{M}(R \cap T), \tag{1.1}$$

then $L^2(\mu)$ has the *strong commutation property* (SCP).

These are spectral versions of the original definitions given in [15] and [10]. An immediate consequence of (WCP) is that the product $P_R P_T$ is the orthoprojection operator of $L^2(\mu)$ onto $\mathscr{M}(R) \cap \mathscr{M}(T)$. For deeper results, and applications to prediction theory, see [1, 4, 10, 12, 13]. Sufficient conditions for (1.1) are provided in [9] and [11]. A commutation condition properly intermediate to (WCP) and (SCP) is investigated in [11, 13]; it does not seem to yield to the present techniques.

Our analysis of (WCP) and (SCP) will bring in the theory of functions in $\mathbf{D}^2$ and $\mathbf{T}^2$, in particular the Hardy class $H^2(\mathbf{T}^2)$. The relevant background can be found in [5] and [16]. In addition, we shall be concerned with the following properties.

DEFINITION. A function f in $H^2(\mathbf{T}^2)$ is *weakly outer* if $f(\cdot, e^{it})$ is outer in $H^2(\mathbf{T})$ for $[dt]$-almost every fixed e^{it}, and $f(e^{is}, \cdot)$ is outer in $H^2(\mathbf{T})$ for $[ds]$-almost every fixed e^{is}.

A function f in $H^2(\mathbf{T}^2)$ is *strongly outer* if the family of functions $\{e^{ims+int}f(e^{is}, e^{it}): (m, n) \in R \cap T\}$ spans $H^2(\mathbf{T}^2)$.

Beurling's theorem for Hardy classes on $\mathbf{T}$ provides some motivation for considering strong outerness; the defining condition for weak outerness appears in [11] in connection with canonical representations for stationary fields. It is known that strongly outer functions are outer, while outer functions are weakly outer [10, 11]. Examples of these functions are given in Section 3. Further development of the theory of weakly and strongly outer functions appears in [2]. In any case, this establishes the right environment for investigating a variety of prediction problems. Our principal result, for instance, is the following theorem.

1.1 THEOREM. *The space $L^2(\mu)$ is regular and weakly (strongly) commutative if and only if $d\mu = |f|^2\, d\sigma$, where f is weakly (strongly) outer in $H^2(\mathbf{T}^2)$.*

The "weak" version is a slight improvement over [13], Theorem 2.2.1, which characterizes weakly commutative fields under the additional assumptions of "strong nondeterminism" (which is stronger than regularity) and "joint innovation spaces of dimension 1." The latter hypothesis was shown to be redundant in [4], Theorem 3.3. Both works combine spectral- and time-domain analysis of the associated random field, particularly in the use of moving-average representations. For closely related results on both the "strong" and "weak" versions of Theorem 1.1 see [14], which uses the theory of doubly commuting isometries. On the other hand, the proof given in the next section relies only on complex analysis in two variables.

Theorem 1.1 gives a complete characterization of the *regular* spaces $L^2(\mu)$ which enjoy the weak and strong commutation properties. An exact description of all weakly commutative fields is now possible. Let $d\mu = w\, d\sigma + d\eta$, and let μ_1 and μ_2 be the first and second marginals of μ.

1.2 THEOREM. *The space $L^2(\mu)$ is weakly commutative if and only if one of the following conditions holds*:

(i) $w = |h|^2$, *where h is weakly outer in $H^2(\mathbf{T}^2)$.*
(ii) $\int \log w(e^{is}, e^{it})\, ds = -\infty$, *a.e.* $[d\mu_2]$.
(iii) $\int \log w(e^{is}, e^{it})\, dt = -\infty$, *a.e.* $[d\mu_1]$.

The proof follows from the assertions of [11], Proposition III.6, Theorem III.12, Proposition V.5, and Theorem 1.1.

2. Proof of the principal result. The strategy for proving Theorem 1.1 is to transform the conditions (WCP) and (SCP) on $L^2(\mu)$ into conditions involving operators on $L^2(\sigma)$. In the latter space, the orthonormality of the functions $\{e^{int+int}: (m,n) \in \mathbf{Z}^2\}$ facilitates manipulation of these operators, which leads to representing them in concrete form.

We begin with a spectral description of regularity.

2.1 LEMMA. *The space $L^2(\mu)$ is regular if and only if $d\mu = w\, d\sigma$, where the weight function w satisfies*

$$\begin{aligned} &\int_{\mathbf{T}} \log w(e^{is}, e^{it})\, ds > -\infty, \quad a.e.\ [dt], \\ &\int_{\mathbf{T}} \log w(e^{is}, e^{it})\, dt > -\infty, \quad a.e.\ [ds]. \end{aligned} \tag{2.1}$$

PROOF. This follows from two applications of [12] Theorem 3. □

Lemma 2.1, or rather the work of which it is an immediate consequence (see also [3, 10]), is evidently a generalization of the Szegö–Kolmogorov–Kreĭn alternative from the one-variable theory. Here, too, a spectral factorization results: If (2.1) holds, define

$$\begin{aligned} h(z_1, e^{it}) &= \exp \int_{\mathbf{T}} \frac{e^{i\theta} + z_1}{e^{i\theta} - z_1} \log w(e^{i\theta}, e^{it}) \frac{d\theta}{4\pi}, \qquad z_1 \in \mathbf{D}, \\ k(e^{is}, z_2) &= \exp \int_{\mathbf{T}} \frac{e^{i\theta} + z_2}{e^{i\theta} - z_2} \log w(e^{is}, e^{i\theta}) \frac{d\theta}{4\pi}, \qquad z_2 \in \mathbf{D}. \end{aligned} \tag{2.2}$$

Extracting radial limits, we find that

$$\begin{aligned} &h(\cdot, e^{it}) \text{ is outer in } H^2(\mathbf{T}) \quad \text{for } [dt]\text{-almost every } e^{it}; \\ &k(e^{is}, \cdot) \text{ is outer in } H^2(\mathbf{T}) \quad \text{for } [ds]\text{-almost every } e^{is}; \\ &w = |h|^2 = |k|^2, \quad \text{a.e. } [\sigma]. \end{aligned} \tag{2.3}$$

The functions h and k will be the vehicle for bringing in the space $L^2(\sigma)$. In preparation for this step, define $\mathscr{H}(S)$ to be the subspace of $L^2(\sigma)$ spanned by $\{e^{ims+int}: (m,n) \in S\}$, and let Q_S be the orthoprojection operator of $L^2(\sigma)$ onto $\mathscr{H}(S)$. With that, the half-plane spaces are related through the following lemma.

2.2 LEMMA. *The transformation $g \to hg$ is a unitary operator from $L^2(\mu)$ onto $L^2(\sigma)$ which maps $\mathscr{M}(R)$ onto $\mathscr{H}(R)$; the transformation $g \to kg$ is a unitary operator from $L^2(\mu)$ onto $L^2(\sigma)$ which maps $\mathscr{M}(T)$ onto $\mathscr{H}(T)$.*

PROOF. If $f \in \mathscr{M}(R)$, then $\int |hf|^2\, d\sigma = \int |f|^2 w\, d\sigma < \infty$, so that $hf \in L^2(\sigma)$. Choose trigonometric polynomials f_j which tend to f in $\mathscr{M}(R)$. For $m < 0$, we have

$$(2.4)\qquad \begin{aligned}\int_{\mathbf{T}_2} hf_j e^{-ims-int}\,d\sigma &= \int_{\mathbf{T}}\left(\int_{\mathbf{T}} hf_j e^{-ims}\,\frac{ds}{2\pi}\right)e^{-int}\,\frac{dt}{2\pi}\\ &= \int_{\mathbf{T}} 0\cdot e^{-int}\,\frac{dt}{2\pi}\\ &= 0.\end{aligned}$$

This shows that $hf \in \mathscr{H}(R)$, and hence that $h\mathscr{M}(R) \subset \mathscr{H}(R)$.

Conversely, suppose that the polynomials F_j converge to F in $\mathscr{H}(R)$. From the sublemma below, we see that $h^{-1}F_j \in \mathscr{M}(R)$ for each j. Moreover,

$$\|h^{-1}F_j - h^{-1}F\|_{L^2(w)} = \|F_j - F\|_{L^2(\sigma)} \to 0.$$

This yields the fact $h^{-1}F \in \mathscr{M}(R)$, and so $\mathscr{H}(R) \subset h\mathscr{M}(R)$.

Now observe that for $f \in L^2(w)$ and $G \in L^2(\sigma)$,

$$\begin{aligned}\langle hf, G\rangle_{L^2(\sigma)} &= \int (hf)(\overline{G})\,d\sigma\\ &= \int (f)\left(\overline{h^{-1}G}\right)h\bar{h}\,d\sigma\\ &= \int (f)\left(\overline{h^{-1}G}\right)w\,d\sigma\\ &= \langle f, h^{-1}G\rangle_{L^2(w)}.\end{aligned}$$

This verifies the first set of assertions; the rest is proved in a similar way. □

The preceding argument makes use of the fact below, a version of Beurling's theorem with $\mathbf{D}$ replaced by $\mathbf{D}^2$.

2.3 Sublemma. $h^{-1} \in \mathscr{M}(R)$.

Proof. Let $\mathscr{H}_0$ be the span in $\mathscr{H}(R)$ of $\{he^{ims+int}: (m,n) \in R\}$. Evidently $\mathscr{H}_0$ is a subspace of $\mathscr{H}(R)$. Suppose that l_0 is a bounded linear functional on $\mathscr{H}(R)$ which annihilates $\mathscr{H}_0$. The Hahn–Banach theorem extends l_0 to some functional l on $L^2(\sigma)$. This l in turn has the representation

$$l(F) = \int_{\mathbf{T}^2} F\Phi\,d\sigma$$

for some Φ in $L^2(\sigma)$.

In particular, for $(m,n) \in R$,

$$\begin{aligned}0 &= l(e^{ims+int}h)\\ &= \int_{\mathbf{T}^2} e^{ims+int}h\Phi\,d\sigma\\ &= \int_{\mathbf{T}} e^{int}\left(\int_{\mathbf{T}} e^{ims}h\Phi\frac{ds}{2\pi}\right)\frac{dt}{2\pi}.\end{aligned}$$

Hence for almost every fixed t, and for $m = 0, 1, 2, \ldots,$ the integral $\int_{\mathbf{T}} e^{ims} h \Phi \, ds/2\pi$ vanishes. The F. and M. Riesz theorem now asserts that for almost every fixed t, the function $h(\cdot, e^{it})\Phi(\cdot, e^{it})$ is in the Hardy class $H^1(\mathbf{T})$, with mean 0. But $h(\cdot, e^{it})$ is outer, which forces $e^{-is}\Phi(e^{is}, e^{it})$ to be of Nevanlinna class in e^{is}. Consequently, for all $(m, n) \in R$,

$$\begin{aligned} \int_{\mathbf{T}^2} e^{ims+int}\Phi \, d\sigma &= \int_{\mathbf{T}} \left(\int_{\mathbf{T}} e^{ims}\Phi \frac{ds}{2\pi} \right) e^{int} \frac{dt}{2\pi} \\ &= \int_{\mathbf{T}} 0 \cdot e^{int} \frac{dt}{2\pi} \\ &= 0. \end{aligned} \tag{2.5}$$

Thus l annihilates all of $\mathscr{H}(R)$. We conclude that $\mathscr{H}_0 = \mathscr{H}(R)$.

Now since $1 \in \mathscr{H}(R)$, there are polynomials p_j in $\mathscr{H}(R)$ for which $hp_j \to 1$ in $\mathscr{H}(R)$. Then $\|h^{-1} - p_j\|_{L^2(w)} = \|1 - h_j p\|_{L^2(\sigma)} \to 0$. Hence $h^{-1} \in \mathscr{M}(R)$. □

Armed with the above correspondences between the half-plane spaces, we can represent the projections P_R and P_T in concrete form.

2.4 Lemma. *For all F in $L^2(d\sigma)$,*

$$\begin{aligned} P_R(h^{-1}F) &= h^{-1}Q_R F \\ P_T(k^{-1}F) &= k^{-1}Q_T F. \end{aligned} \tag{2.6}$$

Proof. First, note that $h^{-1}Q_R F \in \mathscr{M}(R)$, so that $P_R(h^{-1}Q_R F) = h^{-1}Q_R F$. Next, for any $G \in \mathscr{H}(R)$,

$$\begin{aligned} \langle h^{-1}Q_{R^c}F, h^{-1}G \rangle_{L^2(w)} &= \int_{\mathbf{T}^2} (h^{-1}Q_{R^c}F)\left(\overline{h^{-1}G}\right) w \, d\sigma \\ &= \int_{\mathbf{T}^2} (Q_{R^c}F)\overline{G} \, d\sigma \\ &= 0. \end{aligned}$$

That is, $h^{-1}Q_{R^c}F \in \mathscr{M}(R)^{\perp_w}$. With that,

$$\begin{aligned} P_R(h^{-1}F) &= P_R\left(h^{-1}[Q_R F + Q_{R^c}F]\right) \\ &= P_R\left(h^{-1}Q_R F\right) + P_R\left(h^{-1}Q_{R^c}F\right) \\ &= h^{-1}Q_R F + 0. \end{aligned}$$

The other claim is analogously verified. □

Thus, (WCP) can be expressed in terms of objects associated with $L^2(\sigma)$. To do this, let ϕ be the (unimodular) function h/k on $\mathbf{T}^2$; and for bounded u, let M_u be the operator $M_u F = uF$ on $L^2(\sigma)$.

2.5 LEMMA. *$L^2(w)$ has the weak commutation property if and only if*

$$Q_R M_\phi Q_T M_{\bar{\phi}} = M_\phi Q_T M_{\bar{\phi}} Q_R. \tag{2.7}$$

PROOF. By Lemma 2.4, $P_R P_T = P_T P_R$ if and only if

$$(h^{-1}Q_R h)(k^{-1}Q_T k) = (k^{-1}Q_T k)(h^{-1}Q_R h).$$

This, in turn, is equivalent to

$$Q_R(h/k)Q_T(k/h) = (h/k)Q_T(k/h)Q_R.$$

But $h/k = \phi$, and $k/h = \bar{h}/\bar{k} = \bar{\phi}$. The result follows. □

Condition (2.7) imposes a severe restriction on w via h and k. Suppose that it holds, and apply the operator $M_\phi Q_T M_{\bar{\phi}} Q_R$ to the functions e^{int} and $e^{i(n-1)t}$ in $\mathscr{H}(R)$. By (2.7), this yields functions in $\mathscr{H}(R)$. Hence $\mathscr{H}(R)$ also contains

$$\begin{aligned} &e^{-int}M_\phi Q_T M_{\bar{\phi}} e^{int} - e^{-i(n-1)t}M_\phi Q_T M_{\bar{\phi}} e^{i(n-1)t} \\ &= \phi\left(e^{-int}Q_T e^{int} - e^{-i(n-1)t}Q_T e^{i(n-1)t}\right)\bar{\phi} \\ &= \phi(e^{is}, e^{it})e^{-int}\int_{\mathbf{T}} \bar{\phi}(e^{is}, e^{i\theta})e^{in\theta}\frac{d\theta}{2\pi}. \end{aligned}$$

Next, apply $M_\phi Q_T M_{\bar{\phi}}$ to $e^{-is}e^{int}$ and $e^{-is}e^{i(n-1)t}$, which belong to $\mathscr{H}(R^c)$. By (2.7), this yields functions in $\mathscr{H}(R^c)$. Hence $\mathscr{H}(R^c)$ contains

$$\begin{aligned} &e^{-int}M_\phi Q_T M_{\bar{\phi}} e^{-is+int} - e^{-i(n-1)t}M_\phi Q_T M_{\bar{\phi}} e^{-is+i(n-1)t} \\ &= \phi(e^{is}, e^{it})e^{-is-int}\int_{\mathbf{T}} \bar{\phi}(e^{is}, e^{i\theta})e^{in\theta}\frac{d\theta}{2\pi}. \end{aligned}$$

This shows that $\phi(e^{is}, e^{it})e^{-int}\int_{\mathbf{T}}\bar{\phi}(e^{is}, e^{i\theta})e^{in\theta}\, d\theta/2\pi$ lies in $\mathscr{H}(R) \cap e^{is}\mathscr{H}(R^c)$, that is, it is a function of e^{it} only. Since $|\phi| = 1$ a.e. $[\sigma]$, there is an n for which $A(e^{is}) = \int_{\mathbf{T}}\bar{\phi}(e^{is}, e^{i\theta})e^{in\theta}\, d\theta/2\pi$ does not vanish a.e. $[ds]$. We can now write

$$A(e^{is})\phi(e^{is}, e^{it}) = B(e^{it})$$

or

$$B(e^{it})^{-1}h(e^{is}, e^{it}) = A(e^{is})^{-1}k(e^{is}, e^{it}). \tag{2.8}$$

Note that since $|h| = |k|$ a.e. $[\sigma]$, we can choose A and B to be unimodular functions.

Let $f(e^{is}, e^{it}) = B(e^{it})^{-1}h(e^{is}, e^{it})$. Then $f \in \mathscr{H}(R)$, and $f(\cdot, e^{it})$ is outer in $H^2(\mathbf{T})$ for $[dt]$-almost every fixed e^{it}. From (2.8), we see that $f \in \mathscr{H}(T)$ and $f(e^{is}, \cdot)$ is outer in $H^2(\mathbf{T})$ for $[ds]$-almost every fixed e^{is}. This shows that f is weakly outer in $H^2(\mathbf{T}^2)$. Moreover, $|f|^2 = |h|^2 = |k|^2 = w$.

Equation (2.8) is essentially the same as [13], equation (2.2.1), which concerns the issue of commutation, but under hypotheses not needed here.

2.6 PROOF OF THEOREM 1.1. If $L^2(\mu)$ is regular, then by Lemma 2.1, $d\mu = w\,d\sigma$, where w satisfies (2.1). If, in addition, (WCP) holds, then $w = |f|^2$, where f is weakly outer in $H^2(\mathbf{T}^2)$.

Conversely, if f is weakly outer in $H^2(\mathbf{T}^2)$, then $\log|f| \in L^1(T^2)$ ([17], Theorem 1.1a). In particular, the conditions (2.1) hold, so that $L^2(|f|^2)$ is regular. With $w = |f|^2$; and h and k defined by (2.2), we find that

$$f(e^{is}, e^{it}) = \alpha(e^{it})h(e^{is}, e^{it}),$$
$$f(e^{is}, e^{it}) = \beta(e^{is})k(e^{is}, e^{it}),$$

for some unimodular, univariate functions α and β. In this case, $\phi = h/k = \beta/\alpha$ so that

$$\begin{aligned} Q_R M_\phi Q_T M_{\bar\phi} &= Q_R(\beta/\alpha)Q_T(\alpha/\beta) \\ &= \alpha^{-1}Q_R\beta\beta^{-1}Q_T\alpha \\ &= \alpha^{-1}Q_RQ_T\alpha \\ &= \alpha^{-1}Q_TQ_R\alpha \\ &= \alpha^{-1}Q_T\beta\beta^{-1}Q_R\alpha \\ &= (\beta/\alpha)Q_T(\alpha/\beta)Q_R \\ &= M_\phi Q_T M_{\bar\phi} Q_R. \end{aligned}$$

That is, (2.7) holds, and $L^2(|f|^2)$ has (WCP).

Now assume that (SCP) holds in the regular space $L^2(\mu)$. In particular, (WCP) holds, yielding a weakly outer f in $H^2(\mathbf{T}^2)$ for which $d\mu = |f|^2\,d\sigma$. But from (1.1) we deduce that

$$\begin{aligned} f\mathscr{M}(R \cap T) &= f(\mathscr{M}(R) \cap \mathscr{M}(T)) \\ &= (f\mathscr{M}(R)) \cap (f\mathscr{M}(T)) \\ &= \mathscr{H}(R) \cap \mathscr{H}(T) \\ &= H^2(\mathbf{T}^2). \end{aligned} \tag{2.9}$$

That is, f is strongly outer.

Finally, observe that if f is strongly outer in $H^2(\mathbf{T}^2)$, then f is weakly outer, so that $L^2(|f|^2)$ is regular and (WCP) holds. Moreover,

$$\begin{aligned} \mathscr{M}(R \cap T) &= f^{-1}H^2(\mathbf{T}^2) \\ &= f^{-1}(\mathscr{H}(R) \cap \mathscr{H}(T)) \\ &= \mathscr{M}(R) \cap \mathscr{M}(T), \end{aligned}$$

giving (1.1).

This completes the proof. □

3. Examples and applications. Theorem 1.1 provides a vehicle for applying function theory to prediction problems, and a natural medium in which to extend results concerning processes on **Z**. Consider, for instance, these versions of Szegö's infimum:

3.1 PROPOSITION. (i) *If $L^2(w)$ is regular and strongly commutative, then*

$$\inf \int_{\mathbf{T}^2} |1 + p|^2 w \, d\sigma = \exp \int_{\mathbf{T}^2} \log w \, d\sigma,$$

where the infimum is taken over analytic polynomials p with zero constant term.

(ii) *If $L^2(w)$ is regular and weakly commutative, then*

$$\inf \int_{\mathbf{T}^2} |1 + e^{is} q_1 + e^{it} q_2|^2 w \, d\sigma = \exp \int_{\mathbf{T}^2} \log w \, d\sigma,$$

where the infimum is taken over q_1 and q_2 in the space $\mathscr{M}(R) \cap \mathscr{M}(T)$.

PROOF. In the case (i), $w = |f|^2$ for some strongly outer f in $H^2(\mathbf{T}^2)$. Now

$$\begin{aligned} \inf \int_{\mathbf{T}^2} |1 + p|^2 w \, d\sigma &= \inf \int |1 + p|^2 |f|^2 \, ds \\ &= \inf \int |f + fp|^2 \, ds \\ &= |\hat{f}(0,0)|^2 \\ &= \exp \int \log w \, d\sigma. \end{aligned}$$

As for case (ii), we have $w = |f|^2$ for some weakly outer f in $H^2(T^2)$. From the representations in Lemma 2.2, it follows that $f(\mathscr{M}(R) \cap \mathscr{M}(T)) = \mathscr{H}^2(\mathbf{T}^2)$. Consequently,

$$\begin{aligned} &\inf \int |1 + e^{is} q_1 + e^{it} q_2|^2 w \, d\sigma \\ &= \inf \int |f + e^{is} q_1 f + e^{it} q_2 f|^2 \, d\sigma \\ &= |\hat{f}(0,0)|^2 \\ &= \exp \int \log w \, d\sigma. \end{aligned}$$

□

Another application is concerned with the dependence between two subspaces, rather than a subspace and a fixed vector. More specifically, if $\mathscr{M}$ and $\mathscr{N}$ are subspaces of a Hilbert space, we define the cosine of the angle between $\mathscr{M}$ and $\mathscr{N}$ to be the quantity

$$c(\mathscr{M}, \mathscr{N}) = \sup |\langle x, y \rangle|,$$

where the supremum is taken over x in the unit ball of $\mathscr{M}$, and y in the unit ball of $\mathscr{N}$. Helson and Szegö [8] studied the behavior of $c(\mathscr{P}, \mathscr{F})$, where $\mathscr{P}$ and $\mathscr{F}$ are the past and (one-step) future of a stationary process on $\mathbf{Z}$. The following is an analogous treatment of a random field, in which the past and future are replaced by the spaces $\mathscr{M}(R^c) \cap \mathscr{M}(T^c)$ and $\mathscr{M}(R) \cap \mathscr{M}(T)$. The cosine of the angle between them is found to be the norm of a Hänkel-type operator.

3.2 PROPOSITION. *Suppose that $L^2(w)$ is regular and weakly commutative, so that $w = |h|^2$ for some weakly outer h in $H^2(\mathbf{T}^2)$. Then*

$$c(\mathscr{M}(R) \cap \mathscr{M}(T), \mathscr{M}(R^c) \cap \mathscr{M}(T^c)) = \|Q_{R^c \cap T^c} M_{\bar{h}/h} Q_{R \cap T}\|.$$

PROOF.

$$c(\mathscr{M}(R) \cap \mathscr{M}(T), \mathscr{M}(R^r) \cap \mathscr{M}(T^r)) = \sup\left|\int G_1 \bar{G}_2 w \, d\sigma\right|,$$

where G_1 and G_2 vary over the unit balls of $\mathscr{M}(R) \cap \mathscr{M}(T)$ and $\mathscr{M}(R^c) \cap \mathscr{M}(T^c)$, respectively,

$$= \sup\left|\int (hG_1)(h\bar{G}_2)(\bar{h}/h) \, d\sigma\right|$$

$$= \sup\left|\int g_1 \bar{g}_2 (\bar{h}/h) f \, d\sigma\right|,$$

where g_1 and g_2 vary over the unit balls of $\mathscr{H}(R \cap T)$ and $\mathscr{H}(R^c \cap T^c)$, respectively. But then the last expression is $\|Q_{R^c \cap T^c} M_{\bar{h}/h} Q_{R \cap T}\|$. □

Here, if strong commutation holds as well, then the conclusion of Proposition 3.2 would be $c(\mathscr{M}(R \cap T), \mathscr{M}(R^c \cap T^c)) = \|Q_{R^c \cap T^c} M_{\bar{h}/h} Q_{R \cap T}\|$. In any case, note that $Q_{R \cap T}$ and $Q_{R^c \cap T^c}$ are the orthoprojection operators of $L^2(\sigma)$ onto $H^2(\mathbf{T}^2)$ and $\overline{(e^{is} + e^{it}) H^2(\mathbf{T}^2)}$, respectively; hence the operator $Q_{R \cap T} M_{\bar{h}/h} Q_{R^c \cap T^c}$ is indeed of Hänkel type.

Soltani ([17], Theorem 4.3) showed that the condition $\mathscr{M}(R) \cap \mathscr{M}(T) = \mathscr{M}(R \cap T)$ in $L^2(\mu)$ is one of the key criteria for determining whether a stationary field with spectral measure μ has a quarter-plane moving-average representation. The full statement is paraphrased below so as to reveal the role of (SCP). A weak version is stated as well. Both of these were established previously ([4], Theorems 5.1 and 5.2), using techniques differing from those of the present (see also [11], Proposition V.12).

3.3 THEOREM. *Let $\{X_{mn}\}$ be a stationary random field with spectral measure μ. The following are equivalent*:

(a) *$L^2(\mu)$ is regular and strongly (weakly) commutative*;

(b) *there exist a white noise $\{Y_{mn}\}$ and coefficients $\{a_{mn}\}$ such that*:

(i) $\Sigma |a_{mn}|^2 < \infty$;

(ii) $X_{mn} = \Sigma_{j=0}^{\infty} \Sigma_{k=0}^{\infty} a_{jk} Y_{m+j, n+k}$ *for all m and n*;

(iii) $\text{span}\{X_{jk}: j \geq m, k \geq n\} = \text{span}\{Y_{jk}: j \geq m, k \geq n\}$ *for all* m *and* n:

> ($\text{span}\{X_{jk}: j \geq m, k \in \mathbf{Z}\} = \text{span}\{Y_{jk}: j \geq m, k \in \mathbf{Z}\}$ *and* $\text{span}\{X_{jk}: j \in \mathbf{Z}, k \geq n\} = \text{span}\{Y_{jk}: j \in \mathbf{Z}, k \geq n\}$ *for all* m *and* n.)

In either case, the result follows simply from identifying X_{mn} with $h(e^{is}, e^{it})e^{ims+int}$, and Y_{mn} with $e^{ims+int}$, in the space $L^2(\mathbf{T}^2)$, where $h(e^{is}, e^{it}) = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} a_{jk} e^{ijs+ikt}$ is the strongly (weakly) outer factor for w.

Finally, we furnish some examples of $H^2(\mathbf{T}^2)$ functions with the outerness properties.

3.4 PROPOSITION. (i) *The function* $f(e^{is}, e^{it}) = e^{is} + e^{it} + 3$ *is strongly outer in* $H^2(\mathbf{T}^2)$.

(ii) *The function* $g(e^{is}, e^{it}) = \exp((e^{is} + e^{it} + 2)/(e^{is} + e^{it} - 2))$ *is outer, but not strongly outer in* $H^2(\mathbf{T}^2)$.

(iii) *The function* $h(e^{is}, e^{it}) = e^{is} + e^{it}$ *is weakly outer, but not outer.*

PROOF. Since $1/f$ can be estimated uniformly by analytic polynomials, the function $1 = f(1/f)$ lies in the span of $\{fe^{ims+int}: (m, n) \in R \cap T\}$ in $L^2(\sigma)$. Hence that span is $H^2(\mathbf{T}^2)$, and f is strongly outer. (See [16], Theorem 4.4.9, for further examples.)

For verification of claim (ii), see [16], Theorem 4.4.8.

The function h is obviously weakly outer; it fails to be outer since $z_1 + z_2$, its harmonic extension into $\mathbf{D}^2$, vanishes at some point of $\mathbf{D}^2$. □

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