

THE MOMENT PROBLEM FOR POLYNOMIAL FORMS IN NORMAL RANDOM VARIABLES¹

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Let Y be a random variable defined by a polynomial $p(W)$ of degree n in finitely many normally distributed variables. This paper studies which such variables Y are “determinate,” i.e., have probability laws uniquely determined by their moments. Extending results of Berg, which applied to powers of a single normal variable, we prove that (a) Y is determinate if $n = 1, 2$ or if $n = 4$, with the essential support of the law of Y strictly smaller than the real line, and (b) Y is not determinate either if n is odd ≥ 3 or if n is even ≥ 6 such that $p(\mathbf{w})$ attains a finite minimum value. Some other polynomials $Y = p(\mathbf{W})$ with even degree $n \geq 4$ are proved not to be determinate.

1. Introduction. A great deal is known about the classical problem of criteria under which a probability distribution μ on the real line is uniquely determined by its moments, together with the related problem of when the polynomials form a dense set of elements of $L^2(\mathbb{R}, \mathcal{B}, \mu)$. This paper provides a nearly complete classification of mean-square convergent polynomial forms of finite degree in (a stationary and ergodic random sequence of) normal variables, as to whether their probability laws are uniquely determined by their moments.

2. Preliminary survey.

2.1. *Moment problems.* It is well known that every finite positive Borel measure μ with a finite moment generating function

$$m(t) = \int e^{tx} \mu(dx) < \infty, \quad 0 \leq |t| \leq t_0, t_0 > 0$$

is uniquely determined by its moments

$$m_k \equiv \left. \frac{d^k}{dt^k} m(t) \right|_{t=0}, \quad k = 0, 1, 2, \dots$$

More generally, if μ satisfies Carleman’s condition

$$(1) \quad \sum_{k=0}^{\infty} (m_{2k})^{-1/2k} = \infty,$$

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then [Shohat and Tamarkin (1943), pages 19–20] μ is determinate (uniquely determined by its moments) in the “Hamburger sense,” that is, among all finite and positive Borel measures on \mathbb{R} . Slightly less well known is the following.

LEMMA 1 (Carleman, Chihara). *Suppose that the finite positive nonatomic measure μ is supported on $[0, \infty)$ and satisfies the Carleman condition*

$$(2) \quad \sum_{k=0}^{\infty} (m_k)^{-1/2k} = \infty.$$

Then μ is determinate in the Hamburger sense.

PROOF. Shohat and Tamarkin [(1943), pages 19–20] prove from (2) that μ is uniquely determined by its moments in the “Stieltjes sense,” that is, among positive measures supported on the half-line $[0, \infty)$. By a theorem of Chihara (1968), all measures μ which are determinate in the Stieltjes but not the Hamburger sense have atoms. The lemma follows. \square

Whenever a measure μ is determinate, it is easy to prove [see Corollary 2.3.3., page 45 of Akhiezer (1965)] that

$$(3) \quad L^2(\mathbb{R}, \mathcal{B}, \mu) = \text{closed linear span } \{1, x, x^2, \dots\}.$$

A simple necessary condition [Akhiezer (1965), pages 87–88] for a finite positive Borel measure μ either to be determinate or alternatively to satisfy (3) is

$$(4) \quad \int_{-\infty}^{\infty} \frac{1}{1+t^2} \ln \left(\frac{d\mu}{d\lambda}(t) \right) dt = -\infty,$$

where λ denotes the Lebesgue measure on \mathbb{R} , and where $d\mu/d\lambda$ is defined as the Radon–Nikodym density of the absolutely continuous component of μ with respect to Lebesgue measure. Condition (4) and the following corollary are the tools we use to establish moment-indeterminacy for measures μ .

COROLLARY 1. *Suppose X is a random variable which has a density $f(t)$ with respect to Lebesgue measure satisfying*

$$\int_0^{\infty} \frac{-\ln(f(t^2))}{1+t^2} dt < \infty \quad \text{and} \quad f(t) = 0 \quad \text{for } t < 0.$$

Then the probability law of X is indeterminate.

PROOF. Let W be a random variable on a possibly larger probability space, with law supported on all of \mathbb{R} , defined as $\varepsilon\sqrt{|X|}$ ($= \varepsilon\sqrt{X}$ a.s.) where ε is a random variable independent of X such that $P\{\varepsilon = 1\} = P\{\varepsilon = -1\} = \frac{1}{2}$. Then the density of W with respect to Lebesgue measure is a.e. $f_W(s) = |s|f(s^2)$. By hypothesis, this density violates (4). Therefore the probability measure with density f_W is indeterminate, and there exists [Akhiezer (1965), page 87] a

function $g \in L^2(\mathbb{R}, f_W(s) ds)$ for which

$$\int g(t)t^m f_W(t) dt = 0, \text{ for } m = 0, 1, 2, \dots$$

By the symmetry of $f_W(t)$, the same equalities hold for all m if $g(t)$ is replaced by $g(-t)$, and for $k = 0, 1, 2, \dots$,

$$\int_{-\infty}^{\infty} \frac{g(t) + g(-t)}{2} t^{2k} f_W(t) dt = 0 = \int_{-\infty}^{\infty} \frac{g(t) - g(-t)}{2} t^{2k+1} f_W(t) dt.$$

Thus at least one of the even functions $(g(t) + g(-t))/2$ and $t(g(t) - g(-t))/2$ on $[0, \infty)$, which we denote by $h(\cdot)$, is nontrivial and has the property

$$E(h(W)W^{2k}) = E(h(\sqrt{|X|})X^k) = 0, \text{ for } k = 0, 1, 2, \dots$$

In the last step we have also used the a.s. nonnegativity of X . It now follows that the nonzero function $h(\sqrt{|s|})$ is orthogonal to all polynomials in $L^2(\mathbb{R}, f(s) ds)$, and the probability law with density f is indeterminate. \square

2.2. *Polynomial forms in normal variables.* In this paper, we are interested in measures μ which are probability laws for (mean-square convergent) polynomial forms $p(Z_1, Z_2, \dots)$ of a stationary and ergodic Gaussian sequence $\mathbf{Z} = \{Z_j: j = 0, \pm 1, \pm 2, \dots\}$ (of variables with mean 0 and variance 1), defined on a probability space (Ω, \mathcal{F}, P) . Such polynomial forms, say of degree $d < \infty$, can be defined as the mean-square limits of sequences of polynomials of degree d in finitely many of the variables Z_j . Another way in which polynomial forms arise is as (constants plus finite sums of) finite-order multiple Wiener–Itô integrals $I_k(f_k)$, $k \geq 1$. Here $I_k(\cdot)$ denotes the k th order homogeneous multiple Wiener–Itô integration operator [see Major (1981) or Kallianpur (1980), Chapter 6]; f_k is any integrand from the space $L^2_{k, \text{sym}}$ of Hermitian-symmetric and permutation-symmetric elements of the complex space $L^2([-\pi, \pi]^k, \sigma^{\otimes k})$; and $\sigma^{\otimes k}$ is the k -fold Cartesian product of the spectral measure σ of the sequence \mathbf{Z} . (Note that σ is nonatomic because of the assumed ergodicity of \mathbf{Z} .) It is known [Major (1981), Theorem 4.1] that every square-integrable measurable function Y of the variables Z_j has a unique mean-square-convergent representation of the form $E(Y) + \sum_{k \geq 1} I_k(f_k)$. The finite-order multiple Wiener–Itô integrals arise naturally in the theory of nonlinear prediction for nonlinear functionals of stationary Gaussian processes [Kallianpur (1980); cf. Slud (1991)], and in the ergodic theory of such processes [Kornfeld, Fomin and Sinai (1982)]. For an application of the multiple Wiener–Itô integrals to central limit theory, see Chambers and Slud (1989).

Suppose that the square-integrable mean-0 random variable Y on the probability space (Ω, \mathcal{F}, P) has the form

$$(5) \quad Y = \sum_{k=1}^n I_k(f_k), \quad n < \infty.$$

A central idea of the present paper is to represent Y as a polynomial of degree

n in some normal variable U with random coefficients which are independent of U .

LEMMA 2. Each mean-0 polynomial form $p(Z_1, Z_2, \dots)$ of degree n , that is, each square-integrable random variable Y on (Ω, \mathcal{F}, P) which can be represented in the form (5) with $I_n(f_n) \neq 0$, can be written in the form

$$(6) \quad Y = q_n U^n + \sum_{k=1}^{n-1} q_k(\mathbf{Z}) U^k + q_0(\mathbf{Z}),$$

where q_n is a nonzero constant, and U is a standard normal random variable obtained as a nontrivial linear form in the variables Z_j , with U independent of the square-integrable random variables $\{q_k(\mathbf{Z}): k = 0, \dots, n - 1\}$.

PROOF. The general finite-order Wiener-Itô integral expansion $Y = \sum_{k=1}^n I_k(f_k)$ is known [Major (1981), page 12] to have the form

$$(7) \quad \sum_{\alpha: \alpha_1 + \alpha_2 + \dots \leq n} c_\alpha \prod_{i \geq 1} H_{\alpha_i}(U_i),$$

where $\sum_\alpha c_\alpha^2 (\alpha_1! \cdot \alpha_2! \cdot \dots) = E(Y^2) < \infty$, and $H_k(\cdot)$ denotes the Hermite polynomial of degree k normalized to have leading coefficient 1, and where the independent standard-normal random variables $\{U_i\}$ form a complete orthonormal system in the range-space of $I_1(\cdot)$. Assume that $H_{\alpha_1}(U_1) \cdots H_{\alpha_m}(U_m)$ with $\alpha_1 + \dots + \alpha_m = n$ appears in the expansion (7) with a nonzero coefficient. Possibly after an orthogonal linear transformation from (U_1, \dots, U_m) to (U'_1, \dots, U'_m) , and a reordering of indices if necessary, there is no loss of generality in assuming that $\alpha_1 = n$. Then Y can be regarded conditionally given (U_2, U_3, \dots) as a polynomial of degree precisely n in U_1 . Since U_1 is independent of (U_2, U_3, \dots) and the coefficients of U_1^k depend on \mathbf{Z} only through (U_2, U_3, \dots) , we have proved the representation (6) with coefficients independent of $U \equiv U_1$. Finally, observe by the Fubini theorem and square-integrability of Y that for almost every value of U , the right-hand side of (6) is a square-integrable function of (U_2, U_3, \dots) . It follows easily that each of the random coefficients $q_k(\mathbf{Z})$ for $k = 0, \dots, n - 1$ is square-integrable. \square

2.3. *Some known results.* The previously known results on moment-determinacy of variables Y of the form (5) are as follows. Berg (1988) proves that the law of a monomial Z^d in a single normal variable Z is determinate if $d = 1, 2$, or 4 , and is indeterminate otherwise. Using a result of McKean (1973) ("the Eidlin-Linnik-type" tail-probability bounds), Nualart, Üstünel and Zakai (1988) prove that for all homogeneous finite-order multiple Wiener-Itô integrals $I_k(f_k)$, the absolute moments $m_r = E|I_k(f_k)|^r$ satisfy the Carleman condition (1) (indeed, have finite moment generating function) for $k \leq 2$ and violate the condition for $k > 2$. The same methods show that in the case $k = 4$, the law μ of $Y = I_4(f_4)$ satisfies the Carleman condition (2). Moreover, $I_4(1) = H_4(Z)/4!$ [Major (1981), Theorem 4.5], which is evidently a continuously distributed random variable since there are only three values of

Z for which $H_4(Z)$ is 0. Since the polynomial $H_4(z) + 6 = (z^2 - 3)^2$ has minimum value 0, it follows by Lemma 1 that the law of $H_4(Z) + 6$, and therefore of $H_4(Z)$, is determinate in the Hamburger sense.

3. Classification of moment-determinacy. The main results of this paper classify the moment-determinacy of laws of polynomial forms $Y = p(Z_1, Z_2, \dots)$ of finite degree n in Gaussian variables Z_i from an ergodic and stationary sequence.

THEOREM. *Suppose that the random variable $Y = p(Z_1, Z_2, \dots)$ on the probability space (Ω, \mathcal{F}, P) is a polynomial form of degree $n \geq 1$, that is, has the form (5) with $I_n(f_n) \neq 0$. Let $\text{supp}(Y)$ denote the support of the law μ of Y , and suppose $q_n > 0$ in (6). Then we have the following:*

- (i) *is determinate if $n = 1$ or 2 , or if $n = 4$ and $\text{supp}(Y) \neq \mathbb{R}$.*
- (ii) *μ is indeterminate if n is odd and greater than or equal to 3.*
- (iii) *If $n \geq 6$ is even and Y can be expressed as a polynomial $R(\mathbf{W})$ in finitely many normal variables W_1, \dots, W_m with $R(\mathbf{w}_0) = \inf(\text{supp}(Y)) > -\infty$ for some $\mathbf{w}_0 \in \mathbb{R}^m$, then μ is indeterminate.*
- (iv) *If $n \geq 4$ is even, if $Y = R(\mathbf{W})$ can be expressed as a polynomial in finitely many normal variables, and if $R(\mathbf{W})$ has leading term $W_j^d g(\mathbf{W})$ with respect to some variable W_j , where $d \geq 3$ and $g(\mathbf{W})$ does not involve W_j and either d is odd or $g(\mathbf{W})$ can take on negative values, then μ is indeterminate.*

REMARKS. (a) If the degree n of $p(\cdot)$ is 1 or 2, then Y has a finite moment generating function [McKean (1973) and Nualart, Üstünel and Zakai (1988)], so that the law of Y is determinate by the Carleman criterion (1). More generally, for polynomial forms Y of arbitrary degree n , McKean's (1973) extension of the "Eidlin-Linnik" bounds immediately implies that there exists a constant C such that for all positive integers k ,

$$[E|Y|^k]^{1/2k} \leq Ck^{n/4}.$$

(b) If $p(\cdot)$ depends on only a single variable Z_1 , then cases (i)–(iii) are exhaustive. The classification of moment-determinacy is then the same as that given by Berg (1988) for powers Z^n .

(c) For polynomials $p(\cdot)$ in several (but finitely many) variables, the classification of moment-determinacy is more complicated in depending not only on degree but on support (and perhaps other properties as well: cf. Proposition 1 below). Note that cases like $Y = Z_1^4 + Z_2$ with even degree and $\text{supp}(Y) = \mathbb{R}$ do occur. Moreover, we give examples in Section 4 of two-variable polynomials $Y = p(Z_1, Z_2)$ not covered by (i)–(iv) which do not satisfy (1) or (2) but do satisfy (4) and violate the hypothesis of Corollary 1.

(d) The assumption that the variables Z_j appearing in $p(\cdot)$ are taken from an ergodic and stationary sequence is nonrestrictive if there are only finitely many of them, since an orthogonal linear change of variables could then be chosen to make them independent and standard normal (or degenerate at 0).

The proof of the theorem will follow from four lemmas exploiting the special properties of polynomial forms. We begin by giving an explicit representation (and a simple proof of existence) for the density of Y with respect to Lebesgue measure, a density which was first shown to exist by Shigekawa [(1980), page 286] using the machinery of the Malliavin calculus.

LEMMA 3. *Let $Y = p(Z_1, Z_2, \dots)$ be a polynomial form of degree n , as in the theorem, and let its representation (6) with properties given in Lemma 2 be*

$$(8) \quad Y \equiv Q(U, \mathbf{V}) = q_n U^n + \sum_{k=1}^{n-1} q_k(\mathbf{Z}) U^k + q_0(\mathbf{Z}),$$

where $\mathbf{V} \equiv (q_k(\mathbf{Z}), k = 0, \dots, n - 1)$ is independent of U . The density of Y with respect to Lebesgue measure exists and is equal to

$$(9) \quad f_Y(t) = E \left((2\pi)^{-1/2} \sum_{u: Q(u, \mathbf{V})=t} e^{-(1/2)u^2} \left| \frac{\partial Q}{\partial u}(u, \mathbf{V}) \right|^{-1} \right).$$

PROOF. Observe first that $Q(u, \mathbf{V})$ given by (8), considered as a polynomial in u for fixed \mathbf{V} , can for some value of t and $u = u_0$ (possibly depending on \mathbf{V} and t) satisfy the equalities

$$Q(u_0, \mathbf{V}) = t \quad \text{and} \quad \frac{\partial Q}{\partial u}(u_0, \mathbf{V}) = 0$$

for at most n values of t , since $Q(\cdot, \mathbf{V})$ has an inflection point or extremum at each such u_0 . For fixed \mathbf{V} and all other values of t , the variable

$$(10) \quad (2\pi)^{-1/2} \sum_{u: Q(u, \mathbf{V})=t} e^{-(1/2)u^2} \left| \frac{\partial Q}{\partial u}(u, \mathbf{V}) \right|^{-1}$$

is well defined and finite, since $Q(\cdot, \mathbf{V})$ is smooth and at most n -to-1 as a mapping on \mathbb{R} . Indeed, according to (a slight extension of) the standard change-of-variable formula for smooth monotonic functions of real random variables with densities, expression (10) is the conditional density of Y given \mathbf{V} , and so for each value \mathbf{V} must integrate to 1 in the variable t with respect to Lebesgue measure. It follows by the Fubini–Tonelli theorem that $f_Y(t)$ given by (9) is a density with respect to Lebesgue measure. Since expression (9) is the expectation over \mathbf{V} of the conditional density of Y given \mathbf{V} , it is the unconditional density of Y . \square

The polynomial property of $Q(\cdot, \mathbf{V})$ is used further through the following bounds, and through an estimation of volumes of inverse images contained in Lemma 6.

LEMMA 4. Let $Q(u) = q_0 + q_1u + \dots + q_nu^n$ be a polynomial, and let $t \in \mathbb{R}$. Then for a universal constant C_n which does not depend on Q ,

$$\begin{aligned} \sup\{|u|: Q(u) = t\} &\leq \left(\sum_{j=0}^n |q_j|/|q_n|\right) (1 + |t/q_n|^{1/n}), \\ (11) \quad \sup\left\{\left|\frac{dQ}{du}(u)\right|: Q(u) = t\right\} &\leq C_n \left(\sum_{j=0}^n \frac{|q_j|}{|q_n|}\right)^n (|q_n| + |q_n|^{1/n}|t|^{(n-1)/n}). \end{aligned}$$

PROOF. If $Q(u) = t$, then for some $\theta \in [-1, 1]$,

$$|t| = q_n|u|^n + \theta \sum_{j=0}^{n-1} |q_j| \max(1, |u|^{n-1}),$$

so that, in the case $|u| \geq 1 + \sum_{j=0}^{n-1} |q_j|/|q_n|$,

$$\begin{aligned} |u| &\leq |t|^{1/n} |q_n|^{-1/n} \left(1 - \sum_{j=0}^{n-1} |q_j| |q_n|^{-1} |u|^{-1}\right)^{-1/n} \\ &\leq |t/q_n|^{1/n} \left(\sum_{j=0}^n |q_j|/|q_n|\right)^{1/n}, \end{aligned}$$

which immediately implies the first bound in (11). To obtain the second bound, use the first to say that for any u for which $Q(u) = t$,

$$\begin{aligned} \left|\frac{dQ}{du}(u)\right| &\leq \sum_{j=0}^n j|q_j| \left[\left(\sum_{k=0}^n \frac{|q_k|}{|q_n|}\right) \left(1 + \left|\frac{t}{q_n}\right|^{1/2}\right)\right]^{j-1} \\ &< n \left(\sum_{j=0}^n \frac{|q_j|}{|q_n|}\right)^n \left(1 + \left|\frac{t}{q_n}\right|^{1/n}\right)^{n-1}, \end{aligned}$$

proving the second part of (11) with $C_n = n2^{n-1}$. \square

A combination of ideas from Lemmas 3 and 4 yields a useful lower bound on the density found in Lemma 3, a bound which finds application in proofs of indeterminacy via criterion (4).

LEMMA 5. In the notation of (8), let $A_t(\mathbf{V})$ denote $\{u: Q(u, \mathbf{V}) = t\}$ for fixed \mathbf{V} and t , where the degree n of $p(\mathbf{Z}) = Y = Q(U, \mathbf{V})$ is at least 1. Then there exist constants K_1 and K_2 which may depend on n and Q but not on t , such that

$$\begin{aligned} (12) \quad -\ln f_Y(t) &\leq -\ln P\{A_t(\mathbf{V}) \neq \emptyset\} + K_1 \\ &\quad + \log(1 + |t|^{(n-1)/n}) + K_2(1 + |t|^{1/n})^2. \end{aligned}$$

PROOF. According to (9),

$$-\ln f_Y(t) \leq -\ln E \left(\frac{1}{\sqrt{2\pi}} I_{[A_t(\mathbf{V}) \neq \phi]} e^{-(1/2)\sup\{u^2: u \in A_t(\mathbf{V})\}} \inf_{u \in A_t(\mathbf{V})} \left| \frac{\partial Q}{\partial u}(u, \mathbf{V}) \right|^{-1} \right).$$

Next, by the conditional Jensen's inequality,

$$\begin{aligned} -\ln f_Y(t) &\leq -\ln P\{A_t(\mathbf{V}) \neq \phi\} \\ &\quad - E \left(\ln \left[\frac{1}{\sqrt{2\pi}} e^{-(1/2)\sup\{u^2: u \in A_t(\mathbf{V})\}} \right. \right. \\ &\quad \left. \left. \times \inf_{u \in A_t(\mathbf{V})} \left| \frac{\partial Q}{\partial u}(u, \mathbf{V}) \right|^{-1} \right] \middle| A_t(\mathbf{V}) \neq \phi \right). \end{aligned}$$

Then by (11), with the notation $M(\mathbf{V}) = \sum_{j=0}^n |q_j(\mathbf{Z})|/|q_n|$ for fixed \mathbf{V} ,

$$\begin{aligned} -\ln f_Y(t) &\leq -\ln P\{A_t(\mathbf{V}) \neq \phi\} + E \left[\frac{1}{2} [M(\mathbf{V})(1 + |t/q_n|^{1/n})]^2 \right. \\ &\quad \left. + \ln(C_n M^n(\mathbf{V})\sqrt{2\pi} (|q_n| + |q_n|^{1/n}|t|^{(n-1)/n})) \middle| A_t(\mathbf{V}) \neq \phi \right]. \end{aligned}$$

Since $Y = Q(U, \mathbf{V})$ does possess finite moments of all orders [McKean (1973)], it follows from the Fubini-Tonelli theorem that

$$\text{for every } m \geq 1, \quad E\{[Q(U, \mathbf{V})]^{2m} | U\} < \infty \text{ almost surely.}$$

By (8) and Fubini-Tonelli, the moments of each coefficient $q_k(\mathbf{Z})$ are finite, for $k = 0, \dots, n - 1$. Therefore all moments of $M(\mathbf{V})$ are finite, and (12) follows immediately from the last bound on $-\ln f_Y(t)$. \square

LEMMA 6. Assume that $n \geq 2$ is even and that $p(\mathbf{Z}) = Q(U, \mathbf{V})$ in Lemma 3 is a polynomial form $R(\mathbf{W})$ in a finite number m of independent $\mathcal{N}(0, 1)$ random variables W_1, W_2, \dots, W_m . Let ν_m denote the multivariate-normal law $\mathcal{N}(\mathbf{0}, I_m)$ on \mathbb{R}^m , with I_m the identity matrix. Assume also that the (constant) leading coefficient q_n of $Q(\cdot, \mathbf{V})$ is positive and that for a fixed real t_0 , there exists \mathbf{z}_0 for which $p(\mathbf{z}_0) = t_0$. Express the vector $\mathbf{V} = (q_0(\mathbf{Z}), q_1(\mathbf{Z}), \dots, q_{n-1}(\mathbf{Z}))$ as a polynomial vector function $\mathbf{V} = \psi(\mathbf{W})$ of the m -vector (W_1, W_2, \dots, W_m) , with degree at most n . Let $\mathbf{w}_0 \in \mathbb{R}^m$ be such that $\psi(\mathbf{w}_0) = \mathbf{v}_0 = (q_0(\mathbf{z}_0), q_1(\mathbf{z}_0), \dots, q_{n-1}(\mathbf{z}_0))$. Then constants c_1 and c_2 , not depending on t_0 , exist such that for all $s > 0$,

$$\begin{aligned} \ln P\{A_{t_0+s}(\mathbf{V}) \neq \phi\} &= \ln \left[\nu_m \{ \mathbf{w} \in \mathbb{R}^m : A_{t_0+s}(\psi(\mathbf{w})) \neq \phi \} \right] \\ (13) \quad &\geq c_1 + m \ln^- \left(\frac{s}{1 + |t_0|} \right) - c_2 \ln(1 + \|\mathbf{w}_0\|) - \|\mathbf{w}_0\|^2, \end{aligned}$$

where $\ln^-(x)$ denotes $\min\{\ln(x), 0\}$.

PROOF. All notation not defined in the statement of the lemma are as in Lemmas 3 and 5. Observe first that the family of measurable sets $\{\mathbf{w}: A_t(\psi(\mathbf{w})) \neq \phi\}$ is increasing in t by the intermediate value theorem. Moreover, if $A_t(\psi(\mathbf{w}))$ is nonempty then so is $A_{t+\delta}(\psi(\mathbf{w}))$ whenever

$$\sum_{j=0}^{n-1} |\psi(\mathbf{w})_j - \psi(\mathbf{w}')_j| \cdot |u|^j \leq \delta \quad \text{for every } u \text{ in } A_t(\psi(\mathbf{w})).$$

By the first part of (11), all u in $A_t(\psi(\mathbf{w}))$ must satisfy

$$|u| \leq M(\psi(\mathbf{w}))(1 + |t/q_n|^{1/n}),$$

where we recall that $M(\mathbf{v}) = 1 + \sum_{j=1}^n |v_j|/q_n$, and we conclude that if $A_t(\psi(\mathbf{w})) \neq \phi$ and $\|\psi(\mathbf{w}) - \psi(\mathbf{w}')\|_1 \leq \delta[M(\psi(\mathbf{w}))(1 + |t/q_n|^{1/n})]^{-n}$, then

$$(14) \quad A_{t+\delta}(\psi(\mathbf{w}')) \neq \phi,$$

where $\|u\|_1$ for $u = (u_0, u_1, \dots, u_{n-1}) \in \mathbb{R}^n$ denotes $\sum_j |u_j|$. Now fix $t = t_0$ to be the hypothesized value such that $p(\mathbf{z}_0) = t_0$, take \mathbf{w}_0 an element of \mathbb{R}^m such that $\psi(\mathbf{w}_0) = \mathbf{v}_0 = (q_0(\mathbf{z}_0), q_1(\mathbf{z}_0), \dots, q_{n-1}(\mathbf{z}_0))$, and fix $u_0 \in A_{t_0}(\psi(\mathbf{w}_0))$. We learn from (14) with $s = \delta > 0$ that

$$(15) \quad \left\{ \mathbf{w}: \|\psi(\mathbf{w}) - \mathbf{v}_0\|_1 \leq s \left[M(\mathbf{v}_0)(1 + |t_0/q_n|^{1/n}) \right]^{-n} \right\} \\ \subset \left\{ \mathbf{w}: A_{t_0+s}(\psi(\mathbf{w})) \neq \phi \right\}.$$

Each of the n polynomials $\psi(\mathbf{w})_j$ in \mathbf{w} has a representation

$$(16) \quad \psi(\mathbf{w})_j = \sum_{\alpha} C^{(j)}(\alpha_1, \alpha_2, \dots, \alpha_m) \prod_{i=1}^m H_{\alpha_i}(w_i),$$

where the summation is over m -tuples α with $\sum_{i=1}^m \alpha_i \leq n$, and there are $\binom{m+n}{n}$ such m -tuples. Since the W_i are independent $\mathcal{N}(0, 1)$, the variables $H_k(W_i)/\sqrt{k!}$ for distinct (i, k) are orthonormal, and

$$(17) \quad \sum_{\alpha} [C^{(j)}(\alpha_1, \alpha_2, \dots, \alpha_m)]^2 \alpha_1! \cdots \alpha_m! = E\{q_{j-1}(\mathbf{Z})\}^2 < \infty.$$

Recall that $\psi(\mathbf{w}_0) = \mathbf{v}_0$; next apply the representation (16) to $\psi(\mathbf{w})_j$ and to the difference $\psi(\mathbf{w})_j - \psi(\mathbf{w}_0)_j$, and (17) and the Cauchy-Schwarz inequality to the resulting sums, to obtain for all \mathbf{w} and j ,

$$(18) \quad |\psi(\mathbf{w})_j|^2 \leq \left[E\{q_{j-1}(\mathbf{Z})\}^2 \right] \cdot \sum_{\alpha} \left[\prod_{i=1}^m H_{\alpha_i}(w_i) \right]^2 \frac{1}{\alpha_1! \cdots \alpha_m!}, \\ |\psi(\mathbf{w})_j - (\mathbf{v}_0)_j|^2 \leq \left[E\{q_{j-1}(\mathbf{Z})\}^2 \right] \\ \times \sum_{\alpha} \left[\prod_{i=1}^m H_{\alpha_i}(w_i) - \prod_{i=1}^m H_{\alpha_i}((\mathbf{w}_0)_i) \right]^2 \frac{1}{\alpha_1! \cdots \alpha_m!}.$$

We must next estimate the relative sizes of terms $H_k(z)$ and z . There exists a constant K_0 such that for all $r > 0$ and $k = 0, 1, \dots, n$,

$$(19) \quad \max\{|H_k(z)|, |H'_{k+1}(z)|: |z| \leq r, k = 0, \dots, n\} \leq K_0(1+r)^k.$$

Now by (18) and (19), for constants C and C' which can depend on $m, p(\cdot)$, and $R(\cdot)$, for all w and w_0 ,

$$M(\mathbf{v}_0) \leq C(1 + \|\mathbf{w}_0\|^n)$$

and

$$\|\psi(\mathbf{w}) - \mathbf{v}_0\|_1 \leq C'\|\mathbf{w} - \mathbf{w}_0\|(1 + (\|\mathbf{w} - \mathbf{w}_0\| + \|\mathbf{w}_0\|)^{n-1}).$$

Now choose

$$\varepsilon \equiv \min \left\{ 1, \frac{s}{C^n C' (1 + (1 + \|\mathbf{w}_0\|)^{n-1}) (1 + \|\mathbf{w}_0\|^n)^n (1 + |t_0/q_n|^{1/n})^n} \right\}.$$

Then according to (15) and our bounds on $M(\mathbf{v}_0)$ and $\|\psi(\mathbf{w}) - \mathbf{v}_0\|_1$, $\|\mathbf{w} - \mathbf{w}_0\| \leq \varepsilon$ implies $A_{t_0+s}(\psi(\mathbf{w})) \neq \phi$. Note that for some constant $c^* < 1$,

$$\varepsilon \geq \min \left\{ 1, c^* \left(\frac{s}{1 + |t_0|} \right) (1 + \|\mathbf{w}_0\|)^{-n(n+1)} \right\},$$

so that

$$\ln(\varepsilon) \geq \ln(c^*) + \ln \left(\frac{s}{1 + |t_0|} \right) - n(n+1)\ln(1 + \|\mathbf{w}_0\|).$$

Finally, fix \mathbf{w}_0 and estimate the ν_m probability of the ball $\{\mathbf{w}: \|\mathbf{w} - \mathbf{w}_0\| \leq \varepsilon\}$ from below as the product of its volume (= constant times ε^m) by the lower bound $(2\pi)^{-m/2} \exp(-(\|\mathbf{w}_0\| + \varepsilon)^2/2)$ of the ν_m density on the ball. Then (13) follows by taking logarithms and collecting terms, using $(\|\mathbf{w}_0\| + \varepsilon)^2/2 \leq \|\mathbf{w}_0\|^2 + 1$. \square

PROOF OF THEOREM. As mentioned in remark (a) following the statement of the theorem, the cases with $n = 1$ or 2 have been treated by Nualart, Üstünel and Zakai (1988). In the case where $n = 4$ and $\text{supp}(Y) \neq \mathbb{R}$, with $q_4 > 0$ in (8) for some choice of U and \mathbf{V} , the intermediate value theorem applied to $Q(\cdot, \mathbf{V})$ implies that the support of the law of Y is a half-line $[c, \infty)$. The Eidlin-Linnik-type upper bound of McKean (1973) cited in remark (a) immediately implies that the Carleman condition (2) holds in the form

$$\sum_{k=0}^{\infty} \{E(Y - c)^k\}^{-1/2k} < \infty.$$

By Lemma 1, the law of $Y - c$, or equivalently of Y itself, is uniquely determined by its moments as a measure on \mathbb{R} . This proves case (i).

Suppose next that $n > 2$ is odd [case (ii)]. Then $A_t(\mathbf{V}) \neq \phi$ for every fixed t and \mathbf{V} , and the random variable $Y = Q(U, \mathbf{V})$ obviously has the entire real line for its support. Integrating (12) against $(1 + t^2)^{-1} dt$ on \mathbb{R} shows that the law

of Y does not satisfy the necessary condition (4) for determinacy. The same idea works in case (iv), but the proof is a little more difficult: This case of the theorem is an immediate corollary of Proposition 1(β) stated and proved below.

Finally, suppose that $n \geq 6$ is even and that $Y = Q(U, \mathbf{V}) = R(\mathbf{W})$ is expressible as a polynomial in an m -vector of normal variables \mathbf{W} , which may without loss of generality be assumed to have independent components with mean 0 and variance 1. In Lemma 3, there is no loss of generality in taking the leading coefficient q_n to be positive (otherwise replace Y by $-Y$). Now in case (iii), assume also that $\text{supp}(Y) = [t_0, \infty)$, $t_0 > -\infty$, and that there exists $\mathbf{w}_0 \in \mathbb{R}^m$ with $R(\mathbf{w}_0) = t_0$. If $A_{t_0}(\cdot)$ is as defined in Lemma 5, then $A_{t_0}(\psi(\mathbf{w}_0)) \neq \phi$. Put $s = t - t_0$ in Lemma 6, and substitute (13) into (12), to find that the density $f_Y(t)$ [which is 0 on $(-\infty, t_0)$] satisfies, for all $t > t_0$,

$$(20) \quad -\ln f_Y(t) \leq K_1 + \ln(1 + |t|^{(n-1)/n}) + K_2(1 + |t|^{1/n})^2 - c_1 - m \ln \left(\frac{s}{1 + |t_0|} \right) + c_2 \ln(1 + \|\mathbf{w}_0\|) + \|\mathbf{w}_0\|^2.$$

Corollary 1 applied to the random variable $X = Y - t_0$ shows that X and therefore Y has indeterminate law. Case (iii) is proved. \square

The technique of proof via Lemma 6 which has been used in part (iii) of the theorem can yield a more refined result in some cases not covered by the theorem. However, the examples of Section 4 show that the same technique cannot be used in all such cases.

PROPOSITION 1. *Assume that $Y = R(\mathbf{W})$ is a polynomial of even degree $n \geq 4$ in finitely many (m) independent standard-normal random variables W_i , and (without loss of generality) assume that the leading coefficient q_n in its representation (6) is positive. Suppose that $C > 0$ and $0 < \gamma < \frac{1}{2}$ are constants and that one of the following conditions holds:*

(α) $n \geq 6$, $\text{supp}(Y) = [t_1, \infty)$, $t_1 > -\infty$, and for all $N > 0$ there exists $\mathbf{w}_N \in \mathbb{R}^m$ for which $\|\mathbf{w}_N\| \leq CN^\gamma$ and $R(\mathbf{w}_N) \leq t_1 + 1/N$;

(β) $\text{supp}(Y) = \mathbb{R}$, and for all $N \geq 0$ there exists $\mathbf{w}_N \in \mathbb{R}^m$ for which $\|\mathbf{w}_N\| \leq CN^\gamma$ and $R(\mathbf{w}_N) \leq -N$.

Then the law of Y is indeterminate.

PROOF. First, in case (α), for all $N > 0$ and $t > t_1 + 1/N$, put $t_0 = t_1 + 1/N$, $s = t - t_0$ and $\mathbf{w}_0 = \mathbf{w}_N$ in Lemma 6, and combine Lemmas 5 and 6 as in deriving (20) to obtain for some constants K_i ,

$$-\ln f_Y(t) \leq K_1 + K_2(1 + |t|^{1/n})^2 + K_3 \ln \left(t - t_1 - \frac{1}{N} \right) + \|\mathbf{w}_N\|^2.$$

Therefore, with possibly different constants K'_i , for $s > 1/N$,

$$-\ln f_{Y-t_1}(t) \leq K'_1 + K'_2(1 + |s|^{1/n})^2 + K'_3 \ln\left(s - \frac{1}{N}\right) + \|\mathbf{w}_N\|^2.$$

Apply this bound first for $N = 1$ and $s > 1$, and then successively for all integers $N > 1$ and $s \in (1/N, 1/(N - 1)]$ using the bound $\|\mathbf{w}_N\|^2 \leq C^2 N^{2\gamma}$. Recall that $2\gamma < 1$ and $n \geq 6$ to check that $-(1 + s^2)^{-1} \ln f_{Y-t_1}(s^2)$ is Lebesgue integrable on (t_1, ∞) , and Corollary 1 implies that the law of Y is indeterminate.

Similarly in case (β) , we put $t_0 = -N$, $s = t - t_0$ for $t > -N$, and $\mathbf{w}_0 = \mathbf{w}_N$ in Lemma 6 to obtain as in (20) for some constants K_i , for $t > -N$,

$$-\ln f_Y(t) \leq K_1 + K_2(1 + |t|^{1/n})^2 + K_3 \ln(t + N) + \|\mathbf{w}_N\|^2.$$

Applying this bound first for $N = 0$ and $t > 0$, and then successively for all positive integers N and $t \in (-N, -N + 1]$, using $\|\mathbf{w}_N\|^2 \leq C^2 N^{2\gamma}$, we conclude that $-(1 + t^2)^{-1} \ln f_Y(t^2)$ is Lebesgue integrable on \mathbb{R} . Since (4) is violated, the law of Y is indeterminate. \square

As mentioned in the proof of the theorem, case (β) of the proposition holds in case (iv) of the theorem. The requirement that Y contain both a term $q_n U^n$ and a leading term $W_j^d g(\mathbf{W})$ in W_j , where U and W_j are jointly normal and not perfectly correlated, implies condition (β) with $\gamma = 1/d$. This explains why polynomials like $Y = Z_1^4 + Z_2$ or $Z_1^4 - Z_2^2$ are not covered by the theorem, and a calculation along the lines of Example 2 shows that these polynomials cannot be proved either determinate or indeterminate by the methods of this paper.

Throughout this section, we have focussed on polynomial forms in normally distributed random variables. It is clear that some of the same techniques apply with nonnormal distributions such as gamma. For example, the theorem for the case of a polynomial $Y = p(Z)$ in a single normal variable follows immediately from Corollary 1 and Lemma 5 (since A_t is then a nonrandom set), and the steps of Lemmas 3 and 5 apply with straightforward modifications if Z is distributed with the $\Gamma(\alpha, \lambda)$ density

$$\lambda^\alpha t^{\alpha-1} e^{-\lambda t} / \Gamma(\alpha) \quad \text{for } t > 0.$$

The result of Targhetta (1990), that Z^n for $Z \sim \Gamma(\alpha, \lambda)$ has indeterminate law for $n > \max(2, 2\alpha)$, can thus be improved slightly and generalized to arbitrary polynomials in a single gamma variable. The result is:

PROPOSITION 2. *Suppose that $Y = p(Z)$ is either a monomial Z^β or a polynomial of degree n in the $\Gamma(\alpha, \lambda)$ distributed random variable Z . Then the law of Y is determinate if β or n is less than or equal to 2, and is indeterminate otherwise.*

PROOF. The determinate cases follow immediately from the fact that Z^β has a finite moment generating function on an interval around 0 if $\beta \leq 1$,

together with the Carleman condition (2) and the fact that the law of either Z^β with $\beta \leq 2$ or $p(Z)$ with degree 2 has support equal to a half-line $[c, \infty)$. The indeterminate cases follow by condition (4) and Corollary 1 together with the inequality

$$-\ln f_Y(t) \leq c_1 + c_2 \ln(1 + |t|^{(n-1)/n}) + c_3 |t|^{1/n}, \quad t > \inf(\text{supp}(Y))$$

(which is valid also for monomials if $\beta > 2$ replaces n), proved exactly as in Lemma 5 by the method of Lemmas 3 and 4. \square

4. Examples not covered by the theorem. If $Y = p(Z)$ is a polynomial of even degree 4 or larger in finitely many independent $\mathcal{N}(0, 1)$ random variables, then it may happen that neither the theorem nor Proposition 1 applies to establish the determinacy or indeterminacy of its law. We give two examples, with $\text{supp}(Y) = [0, \infty)$ and \mathbb{R} , respectively.

EXAMPLE 1. Let $Y = p(Z_1, Z_2) \equiv [(Z_2^4 + 1)Z_1^2 - 2Z_1Z_2^2 + 1](Z_2^2 + 1)$, where Z_1 and Z_2 are independent standard-normal. For every fixed value of z_2 , the minimum of $p(z_1, z_2)$ occurs at $z_1 = z_2^2/(1 + z_2^4)$ and is equal to $-(1 + z_2^2)/(1 + z_2^4)$. Thus $\text{supp}(Y) = [0, \infty)$. The asymptotic behavior of the moments EY^k is the same as that of the moments of $S = (Z_2^4 + 1)Z_1^2(Z_2^2 + 1)$. Since the law of S is indeterminate according to part (iii) of the theorem, it follows that the moments of S and hence those of Y do not satisfy (2). We calculate in the following lemma that $-\ln f_Y(t)$ is asymptotically equal to a constant multiplied by $1/|t|$ as t converges to 0, so that (4) holds and the hypothesis of Corollary 1 is violated. Therefore it is not clear whether Y has a determinate law in this example. Note that this example corresponds to the case $\gamma = \frac{1}{2}$ in the setting of Proposition 1(α). Examples of the same type can be constructed with higher even degrees by replacing Z_2^4 with Z_2^{2k} , $k > 2$.

LEMMA 7. For $Y = [(Z_2^4 + 1)Z_1^2 - 2Z_1Z_2^2 + 1](Z_2^2 + 1)$, as t decreases to 0,

$$\ln f_Y(t) = -1/(2t) + o(1/t).$$

PROOF. For $p(z_1, z_2) \equiv [(z_2^4 + 1)z_1^2 - 2z_1z_2^2 + 1](z_2^2 + 1)$, $t > 0$, and z_2 fixed,

$$\begin{aligned} & \{z_1: p(z_1, z_2) \leq t\} \\ &= \left\{ z_1: \left| z_1 - \frac{z_2^2}{(1 + z_2^4)} \right|^2 \leq \frac{1}{(z_2^2 + 1)(z_2^4 + 1)} \left(t - \frac{z_2^2 + 1}{z_2^4 + 1} \right) \right\}. \end{aligned}$$

Since $Z_1 = Z_2^2/(Z_2^4 + 1) \pm h(Y, Z_2)$, where the function h is defined by

$$h(t, z) \equiv \sqrt{(t(z^4 + 1) - z^2 - 1)} / \{(z^4 + 1)\sqrt{z^2 + 1}\},$$

the density of Y at $t > 0$ is given in terms of the standard normal density $\varphi(\cdot)$

by

$$\int I_{[(z_2^4+1)t \geq (z_2^2+1)]} \frac{\varphi(z_2)}{2(z_2^2+1)(z_2^4+1)h(t, z_2)} \times \left\{ \varphi\left(h(t, z_2) + \frac{z_2^2}{(z_2^4+1)}\right) + \varphi\left(-h(t, z_2) + \frac{z_2^2}{(z_2^4+1)}\right) \right\} dz_2,$$

which for small positive t is asymptotically equal to

$$\int I_{[(z^4+1)t \geq (z^2+1)]} \frac{1}{2\pi} \frac{z^2}{1+z^4} \frac{\exp\left\{-\frac{1}{2}\left[z^2 + z^4/(1+z^4)^2\right]\right\}}{(1+z^2)(1+z^4)} dz.$$

Since the inequality $(z^4 + 1)t \geq (z^2 + 1)$ for small t requires that z exceed a quantity asymptotically equal to $1/\sqrt{t}$, the logarithm of the last expression is easily shown to be asymptotically equal to $-1/(2t)$. \square

EXAMPLE 2. Let $Y = p(Z_1, Z_2) \equiv (Z_1^2 - 1)(Z_2^2 + 1)$, where Z_1 and Z_2 are independent standard-normal. For every fixed value of z_2 , the minimum of $p(z_1, z_2)$ occurs at $z_1 = 0$ and is equal to $-(1 + z_2^2)$. Thus $\text{supp}(Y) = \mathbb{R}$. It is easy to check [and follows from the lower bounds of Nualart, Üstünel and Zakai (1988)] that the sequence of moments $m_k = E(Y^k)$ does not satisfy (1). Moreover, we calculate in the following lemma that $-\ln f_Y(t)$ is asymptotically equal to a constant multiplied by $|t|$ as t converges to $-\infty$, so that (4) holds. Therefore it is not clear whether Y has determinate law in this example. Similar examples can be produced for arbitrary even degree $n \geq 4$ simply by replacing $(z_1^2 - 1)$ by polynomials $(H_{2m}(z_1) - c)$ for $m \geq 2$, where $c > \min\{H_{2m}(z) : z \in \mathbb{R}\}$, in which case the inequalities of Nualart, Üstünel and Zakai (1988) again show that $Y = (H_{2m}(Z_1) - c)(1 + Z_2^2)$ does not satisfy (1).

LEMMA 8. For $Y = (Z_1^2 - 1)(Z_2^2 + 1)$, as t approaches $-\infty$,

$$\ln f_Y(t) = t/2 + o(t).$$

PROOF. For fixed t and z_2 ,

$$\{z_1 : (z_1^2 - 1)(z_2^2 + 1) \leq t\} = \left\{z_1 : |z_1| \leq \sqrt{1 + t/(z_2^2 + 1)}\right\}$$

so that

$$f_Y(t) = \int_{\sqrt{|t|-1}}^{\infty} \frac{\varphi(z)}{(z^2+1)\sqrt{1+t/(z^2+1)}} \times \left\{ \varphi\left(\sqrt{1 + \frac{t}{z^2+1}}\right) + \varphi\left(-\sqrt{1 + \frac{t}{z^2+1}}\right) \right\} dz,$$

from which it follows easily that $\ln f_Y(t)$ is asymptotic to $-t/2$. \square

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