The Annals of Probability
1993, Vol. 21, No. 4, 2189–2199

COUPLING AND INVARIANT MEASURES FOR THE HEAT EQUATION WITH NOISE[1]

BY CARL MUELLER

University of Rochester

We consider the heat equation with a noise term, on a finite interval with periodic boundary conditions. We show how to construct coupled solutions to the equation. As applications, we prove the uniqueness of invariant measures and the triviality of bounded harmonic functions.

1. Introduction. Our goal is to develop a coupling technique for solutions $u(t, x)$, $t \geq 0$, $x \in S^1 \equiv \mathbb{R}(\text{mod}\, 2\pi)$ to the equation

$$u_t = Du_{xx} - \alpha u + a(u) + b(u)\dot{W}, \tag{1.1}$$

where $\dot{W} = \dot{W}(t, x)$ is a two-parameter white noise, and $D > 0$ and $\alpha \geq 0$ are constants. As an application of coupling, we prove the existence of a unique invariant measure.

Strictly speaking, a coupling of the Markov processes $X(t), Y(t)$ is a realization $[X(t), Y(t)]$ on a common probability space such that for some stopping time $\tau(w)$, we have $X(t) = Y(t)$ for $t > \tau(w)$. Sometimes coupling means that $X(t)$ and $Y(t)$ approach each other asymptotically. We aim to show coupling in the strict sense for pairs of solutions to (1.1) starting from different initial conditions. Coupling dates back to Doeblin's (1938) work on Markov chains and it is one of the main tools in particle systems [see Liggett (1985)]. In particular, it is used to study invariant measures of particle systems.

Recently, Sowers (1992b) has shown, for $b(u)$ small enough, that there is a unique invariant measure for (1.1). His proof is short and uses semigroup theory. Coupling allows us to prove that there is a unique invariant measure even for large $b(u)$. This parallels the situation in particle systems as described in Liggett (1985) where for small parameter values, general semigroup arguments give a unique invariant measure. For larger parameter values, other techniques such as coupling are used. One might hope that these techniques will be useful for other problems in stochastic partial differential equations.

As far as we know, this is the first use of coupling for stochastic partial differential equations. Generally speaking, qualitative properties of stochastic partial differential equations are not well understood. Some work on parabolic equations has been done by Krylov and Rozovskii (1982), Walsh (1986), Shiga (1992), Sowers (1992a, b) and the author (1991a, b, c). Konno and Shiga (1988)

Received August 1991; revised October 1992.
[1]Supported by NSF and NSA grants.
AMS 1991 *subject classifications*. 60H15, 35R60, 35K05.
Key words and phrases. Coupling, invariant measures, heat equation, white noise.

have shown that $u_t = u_{xx} + u^{1/2}\dot{W}$ gives the density of super-Brownian motion, an object of much current attention. To give rigorous meaning to (1.1), we interpret it in the weak sense, as in Walsh (1986). Walsh proves existence, uniqueness, and the strong Markov property for solutions. We interpret (1.1) as an integral equation:

$$
\begin{aligned}
u(t,x) = &\int_{S^1} G(t,x,y)u(0,y)\,dy \\
&+ \int_0^t \int_{S^1} G(t-s,x,y)[-\alpha + a(u(s,y))]\,dy\,ds \\
&+ \int_0^t \int_{S^1} G(t-s,x,y)b(u(s,y))\dot{W}(s,y)\,dy\,ds,
\end{aligned} \tag{1.2}
$$

where the integral against $\dot{W}$ is given meaning in terms of Walsh's theory of martingale measures. Here, $G(t,x,y)$ is the fundamental solution of the heat equation on S^1,

$$
\frac{\partial}{\partial t}G(t,x,y) = D\frac{\partial^2}{\partial y^2}G(t,x,y),
$$
$$
G(0,x,y) = \delta(x-y).
$$

We make several assumptions about the coefficients $a(u), b(u)$ and about $u(0,y)$:

$$
\begin{aligned}
&\text{(i)} && u(0,x) \text{ is continuous.} \\
&\text{(ii)} && \text{For some constants } L_0, L > 0 \text{ we have} \\
& && L_0 \le b(u) \le L. \\
& && |b(u) - b(v)| \le L|u-v|, \\
&\text{(iii)} && |a(u) - a(v)| \le L|u-v|, \\
& && a(u) \text{ is nonincreasing.}
\end{aligned} \tag{1.3}
$$

The assumptions on $a(u)$ are needed to show that an invariant measure exists. Under these conditions, we have the following theorems.

THEOREM 1.1. *Let $u^1(0,x)$ and $u^2(0,x)$ satisfy* (1.3)(i), *and assume* (1.3) *holds. We can construct $u^1(t,x), u^2(t,x)$ on a common probability space, both satisfying* (1.2), *such that for some stopping time τ, we have $u^1(t,x) = u^2(t,x)$ for $t \ge \tau$ and $x \in S^1$.*

COROLLARY 1.1A. *For the process defined by* (1.2) *and* (1.3), *the invariant σ-field is trivial, and all bounded harmonic functions are constant.*

THEOREM 1.2. *Assume that* (1.3) *holds, and that $\alpha > 0$. There is a unique invariant measure μ for the process defined by* (1.2). *Furthermore, let μ_t denote the measure induced by $u(t,\cdot)$. Then μ_t converges to μ as $t \to \infty$, in the total variation norm.*

Sowers [(1992b), Section 3], proves that at least one stationary distribution exists for the process defined by (1.2), provided $\alpha > 0$. Observe that Theorem 1.1 immediately gives uniqueness. Indeed, it is easily seen that even if the initial conditions $u^1(0,x), u^2(0,x)$ are random, there is still a stopping time τ giving coupling. Now let $u^1(0,x), u^2(0,x)$ have distributions given by two invariant measures, Q_1 and Q_2, respectively. Of course, $u^1(t,x)$ and $u^2(t,x)$ also have these distributions if $t > 0$. Thus, the total variation distance between Q_1 and Q_2 is bounded by

$$P\left\{\sup_{x\in S^1} |u^1(t,x) - u^2(t,x)| \neq 0\right\} \le P(t < \tau) \to 0 \quad \text{as } t \to \infty.$$

A similar argument proves Corollary 1.1A.

2. Preliminary results. In this section we give some lemmas which will prove useful later. The first is due to Sowers [(1992b), Section 3].

LEMMA 2.1. *Assume that $\alpha > 0$. There is at least one stationary distribution for the process u defined by* (1.2).

The next result, also from Sowers (1992a), is an estimate on the maximum of $|u(t,x)|$. Fix $T > 0$, and for $\varphi(t,x)$ a function on $[0,T] \times S^1$, define

$$\|\varphi\|_\infty = \sup_{[0,T]\times S^1} |\varphi(t,x)|.$$

LEMMA 2.2. *Suppose that $u(t,x)$ satisfies* (1.2) *and* (1.3). *There are constants K_0, K_1 depending only on $\sup_x |u(0,x)|, T$, such that*

$$P\{\|u\|_\infty \ge z\} \le \exp\left[\frac{-K_1 z^2}{\sup_{x\in\mathbb{R}} b(x)^2}\right],$$

provided $z > K_0$.

PROOF. Referring to equation (1.2), we denote the terms on the right-hand side as (I), (II) and (III). (I) is bounded by the maximum principle, and since $|a(u)|$ is bounded, (II) is also bounded. Proposition A.2 of Sowers (1992a) implies that for some constants K_0, K_1, we have

$$P\{\|(\mathrm{III})\|_\infty > z\} \le \exp\left[\frac{-K_1 z^2}{\sup_{x\in\mathbb{R}} b(x)^2}\right]$$

for all $z > 0$ such that $z^2/\sup_{x\in\mathbb{R}} b(x)^2 \ge K_0$. This proves Lemma 2.2, possibly for different K_0 and K_1. □

To show that $u(t,x)$ does not wander off to ∞, we need the following.

Let $u(t,x), v(t,x)$ both satisfy (1.2), perhaps with different initial conditions and different noises $\dot{W}$.

LEMMA 2.3. *Assume* $\alpha > 0$. *With probability* 1,

$$\liminf_{t\to\infty} \sup_{x\in S^1} (|u(t,x)| \vee |v(t,x)|) < \infty,$$

where $\vee$ *means* max.

PROOF. By (2.24) of Sowers (1992b) and by Sowers' comment between (2.23) and (2.24), we have

$$\sup_{t>0} E \sup_{x\in S^1} |u(t,x)| < \infty.$$

Then by Fatou's lemma, we have

$$\begin{aligned} E\Big[\liminf_{t\to\infty} \sup_{x\in S^1} (|u(t,x)| \vee |v(t,x)|)\Big] &\le \sup_{t>0} E\Big[\sup_{x\in S^1} (|u(t,x)| \vee |v(t,x)|)\Big] \\ &\le \sup_{t>0}\Big[E \sup_{x\in S^1} |u(t,x)| + E \sup_{x\in S^1} |v(t,x)|\Big] \\ &< \infty, \end{aligned}$$

since u and v both satisfy (1.2). This proves Lemma 2.3. □

Our next result is an easy modification of Theorem 3.2 of Walsh (1986). We omit the proof.

LEMMA 2.4. *Equation* (1.2) *is equivalent to the following. For all* $\varphi \in C^2([0,\infty) \times S^1)$, *and for all* $t > 0$, *we have*

$$\begin{aligned} &\int_{S^1} [u(t,x)\varphi(t,x) - u(0,x)\varphi(0,x)]\, dx \\ &\quad = \int_0^t \int_{S_1} \bigg\{ u(s,x)\bigg[\frac{\partial\varphi(s,x)}{\partial t} + \frac{\partial^2\varphi(s,x)}{\partial x^2} - \alpha\varphi(s,x)\bigg] \\ &\qquad\qquad + a(u(s,x))\varphi(s,x)\bigg\}\, dx\, ds \\ &\quad + \int_0^t \int_{S^1} \varphi(t,x) b(u(t,x)) W(dx\, ds). \end{aligned}$$

3. Proof of Theorem 1.1. In this section we construct a pair of coupled processes $u^1(t,x)$, $u^2(t,x)$, each of which satisfies (1.1). Of course, the white noise appearing in (1.1) will be different for u^1 and u^2.

First we assume that

$$u^1(0,x) \ge u^2(0,x). \tag{3.1}$$

Let $\dot{W}_1(t,x)$ and $\dot{W}_2(t,x)$ be independent space-time white noises on $(t,x) \in [0,\infty) \times S^1$. Consider the following equations:

$$
\begin{aligned}
&u_t = u_{xx} - \alpha u + a(u) + b(u)\dot{W}_1,\\
&v_t = v_{xx} - \alpha v + a(v) + b(v)\Big[(1 - |u - v| \wedge 1)^{1/2}\dot{W}_1\\
&\qquad\qquad + (|u - v| \wedge 1)^{1/2}\dot{W}_2\Big],\\
&u(0,x) = u^1(0,x), \qquad v(0,x) = u^2(0,x).
\end{aligned} \tag{3.2}
$$

The rigorous meaning of these equations is given via integral equations similar to (1.2).

Consider the difference

$$\Delta(t,x) = u(t,x) - v(t,x). \tag{3.3}$$

Roughly speaking, $\Delta(t,x)$ "more or less" satisfies

$$\Delta_t = \Delta_{xx} + \Delta^{1/2}\dot{W},$$

where $\dot{W}$ is yet another white noise. This equation is related to the super-Brownian motion, as explained by Konno and Shiga (1988). It is known that solutions become identically zero after some finite time. But if $\Delta(t,x) = 0$ for $x \in S^1$, then u and v are coupled. More precisely, $\Delta(t,x)$ satisfies

$$
\begin{aligned}
\Delta_t &= \Delta_{xx} + a(u) - a(v)\\
&\quad + \left[(b(u) - b(v))^2 + 2b(u)b(v)\frac{|\Delta| \wedge 1}{1 + (1 - |\Delta| \wedge 1)^{1/2}}\right]^{1/2}\dot{W}(t,x).
\end{aligned} \tag{3.4}
$$

LEMMA 3.1. *If* (3.1) *holds, then equations* (3.2) *have solutions* $u(t,x)$ *and* $v(t,x)$ *with* $u(t,x) \geq v(t,x)$ *for all* $t \geq 0$, $x \in S^1$, *almost surely.*

REMARK. We make no claim of uniqueness for $v(t,x)$. Uniqueness would hold if the coefficients in the second equation of (3.2) were Lipschitz functions of v, but they are not. We merely construct a pair u, v which satisfy (3.2).

PROOF OF LEMMA 3.1. Similar comparison theorems were proved by Kotelenez (1992), Mueller [(1991b), Theorem 3.1], Shiga (1992) and Donati-Martin and Pardoux (1992), among others. Since the argument is standard, we omit some details.

We construct approximations to (3.2), and then show tightness. For $0 \leq x \leq 1$, let

$$
\begin{aligned}
f_n(x) &= \left(x + \frac{1}{n}\right)^{1/2} - \left(\frac{1}{n}\right)^{1/2},\\
g_n(x) &= \left[1 - f_n(x)^2\right]^{1/2}.
\end{aligned}
$$

Note that $f_n(x), g_n(x)$ are Lipschitz functions on $[0,1]$ for $n \geq 1$, that

$f_n(x)^2 + g_n(x)^2 = 1$, that $f_n(0) = 0$, and that $f_n(x) \to x^{1/2}$, $g_n(x) \to (1-x)^{1/2}$ uniformly as $n \to \infty$, for $x \in [0,1]$.

Let $u(t,x), v^n(t,x)$ satisfy

$$\begin{aligned} u(0,x) &= u^1(0,x), \\ v^n(0,x) &= u^2(0,x), \\ u_t &= u_{xx} + a(u) + b(u)\dot{W}_1, \\ v_t^n &= v_{xx}^n + a(v^n) \\ &\quad + b(v^n)\left[g_n(|u - v^n| \wedge 1)\dot{W}_1 + f_n(|u - v^n| \wedge 1)\dot{W}_2\right]. \end{aligned} \tag{3.5}$$

Since f_n and g_n are Lipschitz, (3.5) has a unique solution (u, v^n). Also, note that both u and v^n satisfy (1.2), with different white noises, of course. By discretizing space, as in Mueller (1991b) or Shiga (1992), for example, one can show that with probability 1, $u(t,x) \ge v^n(t,x)$ for all $t \ge 0$, $x \in S^1$. The argument is so similar to the proofs of the comparison results mentioned earlier that we omit the details. Briefly, the discrete approximation consists of a finite system of stochastic differential equations. Standard comparison theorems show that $u(t,x) \ge v^n(t,x)$ for the approximation, and this carries over to the limit.

Tightness is also easy, given results of Sowers (1992a). We combine his Propositions 1 and A.2. First, here is some notation. As in Section 2, fix $T > 0$ and let $\varphi(t,x)$ be a function on $[0,T] \times S^1$. For $x, y \in S^1$, let $d(x,y)$ be the length of the shortest arc from x to y. Let

$$r[(t,x),(s,y)] = \left[(t-s)^2 + d(x,y)^2\right]^{1/2}.$$

Define

$$[\varphi]_\kappa = \sup\left\{ \frac{|\varphi(t,x) - \varphi(s,y)|}{r[(t,x),(s,y)]^{\kappa/2}} : \right.$$
$$\left. (t,x),(s,y) \in [0,T] \times S^1, (t,x) \ne (s,y) \right\}.$$

Recalling that $\|\varphi\|_\infty$ was defined in Section 2, let

$$\|\varphi\|_\kappa = \|\varphi\|_\infty + [\varphi]_\kappa.$$

Sowers shows that, if $u(t,x)$ satisfies (1.2) and (1.3), then there are constants K_0, K_1 depending only on $u(0,x), T$ such that

$$P\{\|u\|_\kappa \ge z\} \le \exp\left[\frac{-K_1 z^2}{\sup_{x \in \mathbb{R}} b(x)^2}\right], \tag{3.6}$$

provided $z > K_0$.

To apply this result, note that by the Arzela–Ascoli theorem, for all $z > 0$, $\{\varphi\colon \|\varphi\|_\kappa \le z\}$ is a compact set in the uniform topology on $[0,T] \times S^1$. Thus, by (3.6), the sequence $v^n(t,x)(t \in [0,T])$ is tight for all $T > 0$. Hence the vector

(u, v^n) is tight. Let (u, v) be a limit point. A standard argument, which we will again omit, shows that (u, v) satisfies (3.2), and that $u(t, x) \geq v(t, x)$ a.s. This proves Lemma 3.1. □

Recall that $\Delta(t, x) = u(t, x) - v(t, x)$. Let

$$U(t) = \int_{S^1} \Delta(t, x)\, dx. \tag{3.7}$$

We define the following filtration:

$$\mathscr{F}_t = \sigma\left\{ \int_0^\infty \int_{S^1} \varphi(s, x) W_1(ds\, dx) + \int_0^\infty \int_{S^1} \psi(s, x) W_2(ds\, dx) : \right.$$
$$\left. \varphi, \psi \in L^2([0, \infty) \times S^1),\ \varphi, \psi \text{ are supported on } [0, t] \times S^1 \right\}.$$

So $\mathscr{F}_t$ is the σ-field generated by $\dot{W}_1, \dot{W}_2$ up to time t.

Lemma 3.2.

$$U(t) = U(0) + \int_0^t C(s)\, ds + M(t), \tag{3.8}$$

where $(M(t), \mathscr{F}_t)$ *is a continuous martingale. Furthermore,* $C(s) \leq 0$ *and*

$$\langle M \rangle(t) = \int_0^t \int_{S^1} \Bigg[\big(b(u(s, x)) - b(v(s, x))\big)^2 + 2b(u(s, x)) b(v(s, x)) \times \frac{|\Delta(s, x)| \wedge 1}{1 + (1 - |\Delta(s, x)| \wedge 1)^{1/2}} \Bigg]\, dx\, ds.$$

Proof. We apply Lemma 2.4 to both equations in (3.2), with $\varphi(t, x) = 1$. Then we subtract. We gather the terms involving stochastic integrals against $\dot{W}_1$ or $\dot{W}_2$ into $M(t)$, and gather the remaining terms [except $U(t)$ and $U(0)$] into $\int_0^t C(s)\, ds$. We find

$$C(s) = \int_{S^1} [\alpha(v(s, x) - u(s, x)) + a(u(s, x)) - a(v(s, x))]\, dx.$$

Then (1.3) implies $C(s) \leq 0$. A short calculation gives the formula for $\langle M \rangle$. This proves the lemma. □

Next, Lemma 2.4 and (1.3) imply that

$$\begin{aligned} \frac{d\langle M \rangle(t)}{dt} &\geq L_0^2 \int_{S^1} [\Delta(t, x) \wedge 1]\, dx \\ &= L_0^2 \int_{S^1} \frac{\Delta(t, x)}{\Delta(t, x) \vee 1}\, dx \\ &\geq L_0^2 \frac{U(t)}{\sup_{x \in S^1} \Delta(t, x) \vee 1}. \end{aligned}$$

Thus, there exists an adapted process $V(t)$ such that

$$\begin{aligned} \frac{d\langle M\rangle(t)}{dt} &= U(t)V(t), \\ V(t) &\geq \frac{L_0^2}{\sup_{x \in S^1} \Delta(t,x) \vee 1}. \end{aligned} \tag{3.9}$$

Next, let

$$\begin{aligned} \varphi(t) &= \int_0^t V(s)\,ds, \\ X(t) &= U(\varphi^{-1}(t)). \end{aligned} \tag{3.10}$$

We wish to show that $X(t)$ is defined for all $t \geq 0$; in other words, that $\lim_{t\to\infty} \varphi(t) = \infty$.

LEMMA 3.3. *With probability* 1,

$$\lim_{t\to\infty} \varphi(t) = \int_0^\infty \frac{1}{\sup_{x \in S^1} \Delta(t,x) \vee 1}\,dt = \infty.$$

PROOF. Trouble could come if $\sup_{x \in S^1} \Delta(t,x)$ wandered off to ∞ as $t \to \infty$. But Lemma 2.3 implies that

$$\begin{aligned} \liminf_{t\to\infty} \sup_{x\in S^1} \Delta(t,x) &\leq 2 \liminf_{t\to\infty} \sup_{x\in S^1} |u(t,x)| \vee |v(t,x)| \\ &< \infty \quad \text{a.s.} \end{aligned}$$

Furthermore, Lemma 2.2 states that when $|u(t,x)|$ and $|v(t,x)|$ become small, they have a positive chance of remaining small for some time.

Now we give the details. Assume that Lemma 3.3 is false, so that

$$P\left\{ \int_0^\infty \frac{1}{\sup_{x \in S^1} \Delta(t,x) \vee 1}\,dx < \infty \right\} > \varepsilon > 0$$

for some $\varepsilon > 0$. Then, for each $\delta > 1$, we can choose a deterministic time $R(\delta)$ such that

$$P\left\{ \int_{R(\delta)}^\infty \frac{1}{\sup_{x \in S^1} \Delta(t,x) \vee 1}\,dx < \delta \right\} > \frac{\varepsilon}{2}.$$

Therefore, for each stopping time $\tau > R(\delta)$ we have

$$P\left\{ \int_\tau^{\tau+1} \frac{1}{\sup_{x \in S^1} \Delta(t,x) \vee 1}\,dx < \delta \right\} > \frac{\varepsilon}{2}$$

and hence

$$P\left\{ \sup_{\substack{\tau < t < \tau+1 \\ x \in S^1}} \Delta(t,x) > \frac{1}{\delta} \right\} > \frac{\varepsilon}{2}. \tag{3.11}$$

Let $\tau = \tau(\delta, M)$ be the first time $t \geq R(\delta)$ such that $\sup_{x \in S^1}|u(t,x)| \vee |v(t,x)| < M$. If there is no such time, let $\tau = R(\delta)$.

Using Lemma 2.3, choose M such that

$$P\Big\{\liminf_{t\to\infty} \sup_{x\in S^1} (|u(t,x)| \vee |v(t,x)|) < M\Big\} > 1 - \frac{\varepsilon}{8}.$$

In other words, for all $\delta > 0$,

$$P\Big\{\sup_{x\in S^1} (|u(\tau,x)| \vee |v(\tau,x)|) < M\Big\} > 1 - \frac{\varepsilon}{8}. \tag{3.12}$$

Using Lemma 2.2, choose $\delta > 0$ such that if $\sup_{x \in S^1}(|u(0,x)| \vee |v(0,x)|) < M$, then

$$\begin{aligned} P\left\{\sup_{\substack{0\leq t\leq 1\\ x\in S^1}} \Delta(t,x) > \frac{1}{\delta}\right\} &\leq P\left\{\sup_{\substack{0\leq t\leq 1\\ x\in S^1}} |u(t,x)| > \frac{1}{2\delta}\right\} \\ &\quad + P\left\{\sup_{\substack{0\leq t\leq 1\\ x\in S^1}} |v(t,x)| > \frac{1}{2\delta}\right\} \\ &< \frac{\varepsilon}{8}. \end{aligned} \tag{3.13}$$

Finally, using the strong Markov property at τ, (3.12) and (3.13), we find

$$\begin{aligned} &P\left\{\sup_{\substack{\tau\leq t\leq \tau+1\\ x\in S^1}} \Delta(t,x) > \frac{1}{\delta}\right\} \\ &\quad\leq P\left\{\sup_{\substack{\tau\leq t\leq \tau+1\\ x\in S^1}} \Delta(t,x) > \frac{1}{\delta} \,\middle|\, \sup_{x\in S^1} |u(\tau,x)| \vee |v(\tau,x)| < M\right\} \\ &\qquad + P\Big\{\sup_{x\in S^1} (|u(\tau,x)| \vee |v(\tau,x)|) \geq M\Big\} \\ &\quad\leq \frac{\varepsilon}{8} + \frac{\varepsilon}{8} = \frac{\varepsilon}{4}. \end{aligned}$$

But this contradicts (3.11), and so Lemma 3.3 is proved. □

Now we return to the time-changed process $X(t)$ defined in (3.10). By Lemma 3.2 and the definition of the time change, we have

$$X(t) = U(0) + \int_0^t \tilde{C}(s)\,ds + \int_0^t X^{1/2}(s)\,dB(s), \qquad \tilde{C}(s) \leq 0$$

for some Brownian motion $B(s)$. Let

$$Y(t) = 2X^{1/2}(t).$$

We wish to apply Itô's lemma to $Y(t)$, but the function $f(x) = 2x^{1/2}$ is not differentiable at $x = 0$. If applicable, Itô's lemma would give

$$dY = dB + \left(\frac{\tilde{C}}{Y} - \frac{1}{Y}\right) dt \leq dB, \tag{3.14}$$
$$Y(0) = 2U(0)^{1/2}.$$

Let $\tau(n)$ be the first time $t \geq 0$ that $Y(t) = 1/n$. If there is no such time, let $\tau(n) = \infty$. Then (3.14) is valid for $t \leq \tau(n)$. Letting $n \to \infty$, we see that (3.14) is valid as long as $Y(t) \geq 0$. Therefore,

$$Y(t) \leq 2U(0)^{1/2} + B(t),$$

and so $Y(t)$ hits 0 with probability 1. But when $Y(t) = 0$, we have $U(\varphi^{-1}(t)) = 0$ and hence $u(\varphi^{-1}(t), x) - v(\varphi^{-1}(t), x) = \Delta(\varphi^{-1}(t), x) = 0$ for all $x \in S^1$. Thus, coupling has been achieved.

Now we prove the general case of Theorem 1, so we no longer assume that $u(0, x) \geq v(0, x)$. For this part, we go back to the notation $u^1(t, x)$, $u^2(t, x)$ for the processes to be coupled. Consider the following set of equations:

$$v_t = v_{xx} + a(v) + b(v)\dot{W}_1, \qquad v(0, x) = \max_{i=1,2} (u^i(0, x)),$$
$$u^i_t = u^i_{xx} + a(u^i) + b(u^i)\left[(1 - |v - u^i| \wedge 1)^{1/2}\right]\dot{W}_1 \tag{3.15}$$
$$+ (|v - u^i|)^{1/2}\dot{W}_2, \qquad i = 1, 2.$$

Arguing as in Lemma 3.1, we may find a set of solutions to (3.15) which satisfy $v(t, x) \geq u^i(t, x)$ for $i = 1, 2$ and for all $(t, x) \in [0, \infty) \times S^1$. By the previous case, there are times σ_i, $i = 1, 2$ such that $v(t, x)$ and $u^i(t, x)$ couple at time σ_i. Then, $u^1(t, x)$ and $u^2(t, x)$ couple at $\sigma = \max(\sigma_1, \sigma_2)$. This proves Theorem 1.1. □

Acknowledgment. I wish to thank the referee for suggesting improvements.

REFERENCES

DOEBLIN, W. (1938). Exposé de la théorie des chaines simples constantes de Markov á un nombre fini d'etats. *Rev. Math. Union Interbalkanique* **2** 77–105.

DONATI-MARTIN, C. and PARDOUX, E. (1992). White noise driven SPDEs with reflection. Preprint.

KONNO, N. and SHIGA, T. (1988). Stochastic partial differential equations for some measure-valued diffusions. *Probab. Theory Related Fields* **79** 201–225.

KOTELENEZ, P. (1992). Comparison methods for a class of function valued stochastic partial differential equations. *Probab. Theory Related Fields.* **93** 1–19.

KRYLOV, N. V. and ROZOVSKII, B. L. (1982). Stochastic partial differential equations and diffusion processes. *Russian Math. Surveys* **37** 81–105. [Originally in *Uspekhi Mat. Nauk* **37** 75–95].

LIGGETT, T. M. (1985). *Interacting Particle Systems*. Springer, New York.

MUELLER, C. (1991a). Limit results for two stochastic partial differential equations. *Stochastics* **37** 175–199.

Mueller, C. (1991b). On the support of solutions to the heat equation with noise. *Stochastics* **37** 225–246.

Mueller, C. (1991c). Long time existence for the heat equation with a noise term. *Probab. Theory Related Fields* **90** 505–518.

Shiga, T. (1992). Two contrastive properties of solutions for one-dimensional stochastic partial differential equations. Preprint.

Sowers, R. (1992a). Large deviations for a reaction-diffusion equation with non-Gaussian perturbations. *Ann. Probab.* **20** 504–537.

Sowers, R. (1992b). Large deviations for the invariant measure of a reaction-diffusion equation with non-Gaussian perturbations. *Probab. Theory Related Fields*. **92** 393–421.

Walsh, J. B. (1986). An introduction to stochastic partial differential equations. *École d'Éte de Probabilites de Saint-Flour XIV*. *Lecture Notes in Math*. **1180**. Springer, New York.

Department of Mathematics
University of Rochester
Rochester, New York 14627