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# TOWARD A GENERAL LAW OF THE ITERATED LOGARITHM IN BANACH SPACE[1]

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A general bounded law of the iterated logarithm for Banach space valued random variables is established. Our result implies: (a) the bounded LIL of Ledoux and Talagrand, (b) a bounded LIL for random variables in the domain of attraction of a Gaussian law and (c) new LIL results for random variables outside the domain of attraction of a Gaussian law in cases where the classical norming sequence $\{\sqrt{nLLn}\}$ does not work. Basic ingredients of our proof are an infinite-dimensional Fuk–Nagaev type inequality and an infinite-dimensional version of Klass's $K$-function.

**1. Introduction.** Let $B$ denote a real separable Banach space with norm $\|\cdot\|$ and assume that $X, X_1, X_2, \dots$ are iid $B$-valued random variables with $0 < E\|X\| < \infty$ and $EX = 0$. As usual, let $S_n := \sum_{j=1}^n X_j$, $n \ge 1$ and write $Lt$ to denote $\log(t \vee e)$, $t \ge 0$. The function $L(Lt)$ will be written as $LLt$, and $B^*$ stands for the topological dual of $B$.

Ledoux and Talagrand (1988) obtained the following characterization for the *bounded* law of the iterated logarithm (LIL) in Banach space.

THEOREM A. *A random variable* $X\colon \Omega \to B$ *satisfies the bounded LIL, that is,*

$$\limsup_{n\to\infty} \|S_n\|/\sqrt{nLLn} < \infty \quad a.s., \tag{1.1}$$

*if and only if the following three conditions are fulfilled*:

$$E\|X\|^2/LL\|X\| < \infty, \tag{1.2}$$

$$Ef(X)^2 < \infty, \qquad f \in B^*, \tag{1.3}$$

$$\left\{S_n/\sqrt{nLLn}\right\} \text{ is bounded in probability}. \tag{1.4}$$

Moreover, they showed that in type 2 spaces condition (1.4) follows from (1.2). This means that in such Banach spaces the bounded LIL holds if and only if conditions (1.2) and (1.3) are satisfied. Recall that a Banach space is called a type 2 space if there exists a constant $C$ such that for *any* sequence

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$\{Y_n\}$ of independent mean zero random variables,

$$(1.5) \qquad E\|Y_1 + \cdots + Y_n\|^2 \le C\left(E\|Y_1\|^2 + \cdots + E\|Y_n\|^2\right), \qquad n \ge 2.$$

It is well known that finite-dimensional spaces and Hilbert spaces are type 2 spaces.

Starting with the work of Feller (1968), a number of authors have shown that one can prove more general LIL type results for real-valued random variables when using different norming sequences. [See, e.g., Kesten (1972), Klass (1976, 1977) and Pruitt (1981).] In particular it is possible to obtain LIL type results for random variables which are only in the domain of attraction of the normal distribution. An infinite-dimensional version of the latter result is due to Kuelbs and Zinn (1983).

THEOREM B. *Let $X$ be a random variable in the domain of attraction of a Gaussian random variable $Z$, that is, there exists a sequence $a_n \nearrow \infty$ such that*

$$(1.6) \qquad \mathscr{L}(S_n/a_n) \text{ converges weakly to } \mathscr{L}(Z).$$

*Then we have*

$$(1.7) \qquad 0 < \limsup_{n\to\infty} \|S_n\|/a_{[n/LLn]}LLn < \infty \quad a.s.$$

*if and only if*

$$(1.8) \qquad \sum_{n=1}^{\infty} P\left\{\|X\| > a_{[n/LLn]}LLn\right\} < \infty.$$

This result was later improved by Kuelbs (1985) to a functional LIL with specified cluster set. Einmahl (1989) has finally established a strong invariance principle under condition (1.8) which implies the functional LIL of Kuelbs (1985).

Theorem B is more general than Theorem A in the sense that it admits more general norming sequences. On the other hand, if one specializes Theorem B to the sequence $a_n = \sqrt{n}$, one only obtains the LIL for random variables satisfying the central limit theorem (CLT) and condition (1.2). This is of course much more restrictive than the conditions of Theorem A. It is natural now to ask whether it is possible to establish a *general* LIL, which implies both Theorem A and Theorem B, and can also be applied to random variables outside the domain of attraction of a Gaussian law when the norming sequence $\{\sqrt{nLLn}\}$ does not work. It is the main purpose of the present paper to show that this is indeed possible. It will also turn out that in some situations, which are not covered by the previous theory, we have LIL type results of an entirely different character. We shall show that for any $\alpha > 1/2$ there exist random variables $X: \Omega \to B$ such that

$$(1.9) \qquad 0 < \limsup_{n\to\infty} \|S_n\|/\sqrt{n}\,(LLn)^{\alpha} < \infty \quad \text{a.s.}$$

and, at the same time, for any $f \in B^*$,

$$f(S_n)/\sqrt{n}(LLn)^{\alpha} \to 0 \quad \text{a.s.} \tag{1.10}$$

This means that we have a truly infinite-dimensional LIL behavior in this situation. If we project the properly normalized partial sum sequence onto a *fixed* finite-dimensional space, we obtain almost sure convergence to zero. Nevertheless, the original sequence does not converge to zero. It is interesting that a behavior of this type cannot occur in any of the cases covered by Theorems A and B.

We shall show that it is possible to obtain such a general LIL by an appropriate extension of Klass's universal LIL to Banach space. The $K$-function introduced by Klass (1976) will be of fundamental importance for our work. It is defined as follows. Let $\xi$ be a real-valued random variable with $0 < E|\xi| < \infty$. Define the strictly increasing absolutely continuous function $G(y)$, $y > 0$ by

$$G(y) = y^2 / \int_0^y E|\xi| 1\{|\xi| > u\}\, du \tag{1.11}$$

and let $K(\cdot)$ be the inverse function of $G$.

Then it follows from Theorem (1.1) of Klass (1976) and Theorem 7 of Klass (1980) that if $E\xi = 0$ and $\{\xi_n\}$ is a sequence of independent copies of $\xi$,

$$\tfrac{1}{2} E\left|\sum_1^n \xi_j\right| \le K(n) \le 2E\left|\sum_1^n \xi_j\right|, \qquad n \ge 1. \tag{1.12}$$

Moreover, we have in this case the following:

THEOREM C [Klass (1976, 1977)].

$$1 \le \limsup_{n\to\infty} \sum_1^n \xi_j / [K(n/LLn) LLn] \le 1.5 \quad a.s.$$

*if and only if*

$$\sum_{n=1}^{\infty} P\{\xi > K(n/LLn) LLn\} < \infty.$$

Using (1.12), one can infer both (1.1) for real-valued random variables with finite variances and Theorem B for real-valued random variables in the domain of attraction of the standard normal distribution from Theorem C. Thus, in order to prove the desired general LIL, we have to find an analogue of the $K$-function in Banach space.

**2. Statement of the main results.** Noting that we have $E|f(X)| < \infty$, $f \in B^*$, since $E\|X\| < \infty$, for any functional $f \in B^*$ with $E|f(X)| > 0$, let $K_f(\cdot)$ be the $K$-function corresponding to the real-valued random variable

$f(X)$. Set

$$\tilde{K}(y) := \sup\{K_f(y): \|f\| \le 1,\ E|f(X)| > 0\}, \qquad y > 0. \tag{2.1}$$

Denoting the $K$-function corresponding to $\|X\|$ by $\overline{K}(\cdot)$, we readily obtain

$$\tilde{K}(y) \le \overline{K}(y) < \infty, \qquad y > 0. \tag{2.2}$$

Using (2.1) and (2.2) in combination with relations (2.3) and (2.4) of Klass (1976), we find that

$$\tilde{K}(y)/y \searrow 0 \tag{2.3}$$

and

$$\tilde{K}(y)/\sqrt{y} \nearrow \sup\{Ef(X)^2: \|f\| \le 1\}^{1/2} \quad \text{(possibly infinite)}. \tag{2.4}$$

Finally, note that as a consequence of (1.12) we have

$$\tilde{K}(n) \le 2\sup\{E|f(S_n)|: \|f\| \le 1\} \le 2E\|S_n\|, \qquad n \ge 1. \tag{2.5}$$

THEOREM 1. *Let $X$ be a $B$-valued random variable with mean zero and $0 < E\|X\| < \infty$. Let $\{c_n\}$ be a sequence of positive real numbers satisfying*

$$c_n/n^\alpha \text{ is nondecreasing for some } \alpha > 0 \tag{2.6}$$

*and*

$$c_n \ge \tilde{K}(n/LLn)LLn. \tag{2.7}$$

*Then we have*

$$\limsup_{n\to\infty} \|S_n\|/c_n < \infty \quad a.s.,$$

*if and only if*

$$\{S_n/c_n\} \text{ is bounded in probability} \tag{2.8}$$

*and*

$$\sum_{n=1}^{\infty} P\{\|X\| > c_n\} < \infty. \tag{2.9}$$

COROLLARY 1. *Let $B$ be a type 2 space and assume that $X$ is as in Theorem 1. Let $\{c_n\}$ be a sequence of positive real numbers satisfying* (2.7),

$$c_n/\sqrt{n} \text{ is nondecreasing} \tag{2.6$'$}$$

*and*

$$c_n/n \text{ is nonincreasing}. \tag{2.6$''$}$$

*Then we have*

$$\limsup_{n\to\infty} \|S_n\|/c_n < \infty \quad a.s.,$$

*if and only if*

$$\sum_{n=1}^{\infty} P\{\|X\| > c_n\} < \infty.$$

REMARK 1. A straightforward application of the closed graph theorem shows that if condition (1.3) is satisfied we have $\sigma^2 := \sup\{Ef(X)^2: \|f\| \le 1\} < \infty$. It thus follows from (2.4) that $\tilde{K}(n) \le \sigma\sqrt{n}$. Applying Theorem 1 with $c_n = \sigma\sqrt{nLLn}$, we obtain (1.1) for all random variables satisfying the (optimal) conditions (1.2)–(1.4).

REMARK 2. If $X$ satisfies assumption (1.6), it follows that

$$E\|S_n\| \sim a_n E\|Z\| \quad \text{as } n \to \infty. \tag{2.11}$$

Recalling (2.5), we readily obtain Theorem B from Theorem 1.

REMARK 3. A closely related result is Theorem 3 of Kuelbs and Zinn (1983). This result, however, is only applicable to sequences satisfying

$$c_n \ge \overline{K}(n/LLn)LLn.$$

In view of (2.2) this is a more restrictive assumption than (2.7). As a matter of fact, in many cases of interest one has $\overline{K}(y)/\tilde{K}(y) \to \infty$ as $y \to \infty$. An example of such a situation is given by random variables satisfying (1.3) and $E\|X\|^2 = \infty$.

Our next result is to demonstrate that we have LIL behavior with respect to the sequence $\{\tilde{K}(n/LLn)LLn\}$, thereby showing that condition (2.7) in Theorem 1 is sharp.

THEOREM 2. *Let X be a B-valued random variable with* $0 < E\|X\| < \infty$ *and* $EX = 0$. *Then we have*

$$\limsup_{n\to\infty} \|S_n\|/\tilde{K}(n/LLn)LLn \ge \sqrt{2} \quad a.s. \tag{2.12}$$

Combining Theorem 1 and Theorem 2, we find that if $X$ is a $B$-valued r.v. satisfying

$$\sum_{n=1}^{\infty} P\{\|X\| > \tilde{K}(n/LLn)LLn\} < \infty \tag{2.13}$$

and

$$\{S_n/\tilde{K}(n/LLn)LLn\} \text{ is bounded in probability,} \tag{2.14}$$

then we have LIL behavior.

To be more specific, we have in this case

$$\sqrt{2} \leq \limsup_{n\to\infty} \|S_n\|/\tilde{K}(n/LLn)LLn < \infty \quad \text{a.s.} \tag{2.15}$$

REMARK 4. Applying Theorem 2 to iid real-valued mean zero random variables, we see that we always have

$$\limsup_{n\to\infty} \left| \sum_{j=1}^{n} \xi_j \right| / K(n/LLn)LLn \geq \sqrt{2} \quad \text{a.s.}$$

Thus we can get a better lower bound in the above two-sided situation than in the one-sided LIL result of Klass (1976), where the constant 1 is optimal. From the subsequent Theorem 3 it will also follow that the upper bound 1.5 in Theorem C, which is optimal in the one-sided case [see Klass (1984)], can be replaced by $\sqrt{2}$ in the two-sided case. This shows that we always have in this situation the same constant as in the Hartman–Wintner LIL.

Given Theorem 2, it is natural now to ask whether one can say more about the value of the lim sup in (2.15). In general, this is a very difficult problem which is still unsolved in the classical situation (dealing with the norming sequence $\sqrt{nLLn}$). However, as in Theorem 5.1 of Ledoux and Talagrand (1990), one can determine the exact value of the lim sup for random variables satisfying a somewhat more restrictive condition than (2.14).

THEOREM 3. *Let $X$ be a mean zero random variable with $0 < E\|X\| < \infty$ satisfying* (2.13) *and*

$$S_n/\tilde{K}(n/LLn)LLn \to_P 0. \tag{2.16}$$

*Then we have*

$$\limsup_{n\to\infty} \|S_n\|/\tilde{K}(n/LLn)LLn = \sqrt{2} \quad a.s.$$

COROLLARY 2. *Let $B$ be a type 2 Banach space, and let $X$ be a $B$-valued mean zero r.v. with $0 < E\|X\| < \infty$. Then we have*

$$\limsup_{n\to\infty} \|S_n\|/\tilde{K}(n/LLn)LLn = \sqrt{2} \quad a.s.$$

*if and only if*

$$\sum_{n=1}^{\infty} P\{\|X\| > \tilde{K}(n/LLn)LLn\} < \infty.$$

Our last result in this section is to demonstrate that in any infinite-dimensional Hilbert space there exist random variables with the properties (1.9) and (1.10).

THEOREM 4. *Let $H$ be an infinite-dimensional Hilbert space with scalar product $(\cdot,\cdot)$, and let $\alpha > 1/2$ be fixed. One can find mean zero random* *variables* $X\colon \Omega \to H$ *such that*

$$0 < \limsup_{n\to\infty} \|S_n\|/\sqrt{n}\,(LLn)^{\alpha} < \infty \quad a.s. \tag{2.17}$$

*and, at the same time, for any* $y \in H$,

$$(S_n, y)/\sqrt{n}\,(LLn)^{\alpha} \to 0 \quad a.s. \tag{2.18}$$

The remaining part of the paper is organized as follows. Theorem 1 will be proved in Section 3. We shall first establish an infinite-dimensional Fuk–Nagaev type inequality (see Theorem 5), which might be of independent interest. Our proof makes use of some of the ideas developed by Ledoux and Talagrand (1988). It is based on a randomization argument in combination with a refinement of a classical martingale argument due to Yurinskii (1976). But given the recent work of Talagrand (1989) and Ledoux and Talagrand (1989), we can now use a randomization by Rademacher random variables rather than a Gaussian randomization. Making appropriate use of some of the basic properties of the $\tilde{K}$-function, we can then infer Theorem 1 from our Fuk–Nagaev type inequality via Borel–Cantelli. The proof of Theorem 2 will be carried out in Section 4. As we are now dealing with a two-sided situation, we can give a much easier proof than that of Klass (1977) for the corresponding lower bound in Theorem C. In order to prove Theorem 3 in Section 5, we use a similar argument as in Theorem 5.1 of Ledoux and Talagrand (1990). But employing a double truncation argument will enable us to utilize an entropy argument based on a randomization by Rademacher random variables, thus avoiding the somewhat less natural Gaussian randomization. Section 6 is finally devoted to the proof of Theorem 4.

## 3. Proofs of Theorem 1 and Corollary 2.

3.1. *A Fuk–Nagaev type inequality in Banach space.* The purpose of this part of the paper is to prove the following result.

THEOREM 5. *Let* $Y_1, \dots, Y_n$ *be independent B-valued random variables such that for some* $p > 2$, $E\|Y_j\|^p < \infty$, $1 \le j \le n$. *Then for* $t > 0$,

$$P\left\{\left\|\sum_{j=1}^{n} Y_j\right\| \ge t + 37p^2 E\left\|\sum_{j=1}^{n} Y_j\right\|\right\} \le 16\exp\left(-t^2/144\Lambda_n\right) + C_1 \sum_{j=1}^{n} E\|Y_j\|^p/t^p,$$

*where* $\Lambda_n := \sup\{\sum_{j=1}^{n} Ef^2(Y_j)\colon \|f\| \le 1\}$ *and* $C_1$ *is a constant depending on p only.*

A related inequality if $p = 3$ is due to Yurinskii [(1976), Theorem 5.1]. The main advantage of the above inequality is that the exponential term is defined in terms of the weak second moments rather than the strong second moments. This improvement will be crucial for the applications we have in mind. Our proof will decisively use the Ledoux and Talagrand (1988) refinement of Yurinskii's martingale argument [cf. Proof of (3.7)]. Let us also mention that there is an entirely different (and somewhat easier) method available for proving analogues of Theorem 5 in Hilbert space (see, e.g., Lemma 6 of Einmahl (1991)].

In order to prove Theorem 5, it is enough to show that if $Z_1, \dots, Z_n$ are independent *mean zero* random variables, then

$$P\left\{\left\|\sum_{j=1}^n Z_j\right\| \ge t + 18p^2 E\left\|\sum_{j=1}^n Z_j\right\|\right\} \le 16\exp\left(-t^2/144\tilde{\Lambda}_n\right) + C_2 \sum_{j=1}^n E\|Z_j\|^p/t^p, \tag{3.1}$$

where $\tilde{\Lambda}_n := \sup\{\Sigma_{j=1}^n Ef^2(Z_j): \|f\| \le 1\}$ and $C_2$ is a constant depending on $p$ only.

To see this, note that if we set $Z_j = Y_j - EY_j$, $1 \le j \le n$, we have

$$\begin{aligned} &P\left\{\left\|\sum_{j=1}^n Y_j\right\| \ge t + 37p^2 E\left\|\sum_{j=1}^n Y_j\right\|\right\} \\ &\quad \le P\left\{\left\|\sum_{j=1}^n Z_j\right\| + E\left\|\sum_{j=1}^n Y_j\right\| \ge t + 37p^2 E\left\|\sum_{j=1}^n Y_j\right\|\right\} \\ &\quad \le P\left\{\left\|\sum_{j=1}^n Z_j\right\| \ge t + 18p^2 E\left\|\sum_{j=1}^n Z_j\right\|\right\}, \end{aligned}$$

where we use the trivial inequality $E\|\Sigma_{j=1}^n Z_j\| \le 2E\|\Sigma_{j=1}^n Y_j\|$. Taking into account that $\tilde{\Lambda}_n \le \Lambda_n$ and $E\|Z_j\|^p \le 2^p E\|Y_j\|^p$, we readily obtain Theorem 1 from (3.1). Thus, it remains to show (3.1).

For convenient reference later on we now state a number of results which we need for the proof of (3.1). We shall assume from now on that $\varepsilon_j$, $1 \le j \le n$ are independent Rademacher random variables which are also independent of the $Z_j$'s.

FACT 1 [Talagrand (1989)]. Let $z_1, \dots, z_n$ be points in $B$. Then we have

$$P\left\{\left\|\sum_{j=1}^n \varepsilon_j z_j\right\| \ge t + 2E\left\|\sum_{j=1}^n \varepsilon_j z_j\right\|\right\} \le 4\exp\left(-t^2/8\sigma_n^2(z_1, \dots, z_n)\right), \tag{3.2}$$

where $\sigma_n^2(z_1, \dots, z_n) = \sup\{\Sigma_{j=1}^n f^2(z_j): \|f\| \le 1\}$.

FACT 2 [Ledoux and Talagrand (1990)]. Let $z_1, \dots, z_n$ be points in B. Then for $p \ge 1$,

$$E\left(\sup_{\|f\|\le 1}\left\|\sum_{j=1}^n \varepsilon_j f^2(z_j)\right\|\right)^p \le 6^p E\left\|\sum_{j=1}^n \varepsilon_j \|z_j\| z_j\right\|^p.$$

FACT 3. For $p \ge 1$,

$$2^{-p}E\left\|\sum_{j=1}^n \varepsilon_j Z_j\right\|^p \le E\left\|\sum_{j=1}^n Z_j\right\|^p \le 2^p E\left\|\sum_{j=1}^n \varepsilon_j Z_j\right\|^p. \tag{3.3}$$

Fact 3 can be easily inferred from Corollary 4.2 of Hoffmann-Jørgensen (1974).

FACT 4 [Hoffmann-Jørgensen (1974)]. Let $X_1, \dots, X_n$ be independent symmetric $B$-valued random variables. Then for $p \ge 1$,

$$E\left\|\sum_{j=1}^n X_j\right\|^p \le 2\cdot 3^p E\left(\max_{1\le j\le n}\|X_j\|^p\right) + 2(3t_0)^p, \tag{3.4}$$

where $t_0 = \inf\{t\colon P\{\|\sum_{j=1}^n X_j\| \ge t\} \le 1/(8\cdot 3^p)\}$.

In particular for $p > 1$,

$$E\left\|\sum_{j=1}^n X_j\right\|^p \le 2\cdot 3^p E\max_{1\le j\le n}\|X_j\|^p + 2(24\cdot 3^p)^p\left(E\left\|\sum_{j=1}^n X_j\right\|\right)^p. \tag{3.5}$$

Next we need the following simple symmetrization inequality, the straightforward proof of which is omitted.

LEMMA 1. *For $s > 0$,*

$$P\left\{\left\|\sum_{j=1}^n Z_j\right\| \ge s + 2E\left\|\sum_{j=1}^n Z_j\right\|\right\} \le 4P\left\{\left\|\sum_{j=1}^n \varepsilon_j Z_j\right\| \ge \frac{s}{2}\right\}.$$

In the sequel the distribution of $(Z_1, \dots, Z_n)$ will be denoted by $Q_n$, and we write $\mathbf{z}$ for $n$-tuples in $B$ with $z_j$, $1 \le j \le n$ denoting the components of $\mathbf{z}$. Constants which depends on $p$ only will be denoted by $C_i$, $i \ge 1$. Finally, set $T_n := \sum_{j=1}^n Z_j$.

Using Lemma 1 in conjunction with Fubini's theorem, we readily obtain that

$$\begin{aligned} &P\{\|T_n\| \ge t + 18p^2E\|T_n\|\} \\ &\quad\le 4P\left\{\left\|\sum_{j=1}^n \varepsilon_j Z_j\right\| \ge \frac{t}{2} + (9p^2-1)E\|T_n\|\right\} \\ &\quad= 4\int P\left\{\left\|\sum_{j=1}^n \varepsilon_j z_j\right\| \ge \frac{t}{2} + (9p^2-1)E\|T_n\|\right\} Q_n(d\mathbf{z}). \end{aligned}$$

Consider the events

$$F_n := \left\{ \mathbf{z} \in B^n \colon E\left\| \sum_1^n \varepsilon_j z_j \right\| \le 4p^2 E\|T_n\| + \frac{t}{12} \right\},$$

$$G_n := \left\{ \mathbf{z} \in B^n \colon \max_{1 \le j \le n} \|z_j\| \le t/16p^2 \right\},$$

and set $H_n := F_n \cap G_n$.

Then it is obvious that the foregoing probability is bounded above by

$$4\int_{H_n} P\left\{ \left\| \sum_{j=1}^n \varepsilon_j z_j \right\| \ge \frac{t}{3} + 2E\left\| \sum_1^n \varepsilon_j z_j \right\| + (p^2 - 1)E\|T_n\| \right\} Q_n(d\mathbf{z}) + 4Q_n(H_n^c).$$

In view of Fact 1, it is clear that this term is

$$\le 16\int_{H_n} \exp\left( -\frac{(t + E\|T_n\|)^2}{72\sigma_n^2(z_1, \dots, z_n)} \right) Q_n(d\mathbf{z}) + 4Q_n(H_n^c).$$

We set $Z_j' := Z_j 1\{\|Z_j\| \le t/16p^2\}$, $1 \le j \le n$, and we denote the distribution of $(Z_1', \dots, Z_n')$ by $Q_n'$.

It is plain that

$$\sigma_n^2(z_1, \dots, z_n) \le \Lambda_n' + \sup_{\|f\| \le 1} \left| \sum_1^n \left\{ f^2(z_j) - Ef^2(Z_j') \right\} \right|,$$

where $\Lambda_n' := \sup\{\sum_1^n Ef^2(Z_j') \colon \|f\| \le 1\}$.

Thus, using the trivial inequalities

$$\exp\left( -\frac{s}{x + y} \right) \le \exp\left( -\frac{s}{2x} \right) + \exp\left( -\frac{s}{2y} \right) \le \exp\left( -\frac{s}{2x} \right) + C_3\left( \frac{y}{s} \right)^p,$$

$s, x, y > 0$, we finally obtain

$$\begin{aligned} &P\{\|T_n\| \ge t + 18p^2 E\|T_n\|\} \\ &\quad \le 16\exp(-t^2/144\Lambda_n') \\ &\qquad + C_4 E \sup_{\|f\| \le 1} \left| \sum_1^n \left\{ f^2(Z_j') - Ef^2(Z_j') \right\} \right|^p / (t + E\|T_n\|)^{2p} + 4Q_n(H_n^c). \end{aligned}$$

We now claim that

$$E \sup_{\|f\| \le 1} \left| \sum_1^n \left\{ f^2(Z_j') - Ef^2(Z_j') \right\} \right|^p \le C_5 (t + E\|T_n\|)^p \sum_1^n E\|Z_j\|^p \tag{3.6}$$

and

$$Q_n(H_n^c) \le C_6 \sum_{j=1}^n E\|Z_j\|^p / t^p. \tag{3.7}$$

Combining the last two relations with the above inequality, we obtain (3.1) and consequently Theorem 5. It thus remains to prove (3.6) and (3.7).

PROOF OF (3.6). We first note that one can show by using the same argument as in the proof of Fact 3 that

$$E \sup_{\|f\|\le 1} \left| \sum_{j=1}^{n} \left\{ f^2(Z_j') - Ef^2(Z_j') \right\} \right|^p \le 2^p E \sup_{\|f\|\le 1} \left| \sum_{j=1}^{n} \varepsilon_j f^2(Z_j') \right|^p .$$

Using Fact 2 in conjunction with Fubini's theorem, we see that the last term equals

$$\begin{aligned} 2^p \int E \sup_{\|f\|\le 1} \left| \sum_{j=1}^{n} \varepsilon_j f^2(z_j) \right|^p Q_n'(d\mathbf{z}) &\le 12^p \int E \left\| \sum_{j=1}^{n} \varepsilon_j \|z_j\| z_j \right\|^p Q_n'(d\mathbf{z}) \\ &= 12^p E \left\| \sum_{j=1}^{n} \varepsilon_j \|Z_j'\| Z_j' \right\|^p . \end{aligned}$$

We now apply inequality (3.5) with $X_j = \varepsilon_j \|Z_j'\| Z_j'$, $1 \le j \le n$. It follows that

$$E \left\| \sum_{j=1}^{n} \varepsilon_j \|Z_j'\| Z_j' \right\|^p \le C_7 E\left( \max_{1\le j\le n} \|Z_j'\|^{2p} \right) + C_8 \left( E \left\| \sum_{j=1}^{n} \varepsilon_j \|Z_j'\| Z_j' \right\| \right)^p .$$

Since we trivially have

$$E\left( \max_{1\le j\le n} \|Z_j'\| \right)^{2p} \le t^p \sum_{j=1}^{n} E\|Z_j\|^p,$$

it remains to show that

$$E \left\| \sum_{j=1}^{n} \varepsilon_j \|Z_j'\| Z_j' \right\| \le C_9 \left( t + E\|T_n\| \right) \left( \sum_{j=1}^{n} E\|Z_j\|^p \right)^{1/p} . \tag{3.8}$$

Using Fubini's theorem in combination with Lemma 4.1 of Hoffmann-Jørgensen (1974) (where we set $a_j = \|z_j\|$, $X_j = \varepsilon_j z_j$), we find that

$$\begin{aligned} E \left\| \sum_{j=1}^{n} \varepsilon_j \|Z_j'\| Z_j' \right\| &= \int E \left\| \sum_{j=1}^{n} \|z_j\| \varepsilon_j z_j \right\| Q_n'(d\mathbf{z}) \\ &\le \int \left( \max_{1\le j\le n} \|z_j\| \right) E \left\| \sum_{j=1}^{n} \varepsilon_j z_j \right\| Q_n'(d\mathbf{z}), \end{aligned}$$

which is, on account of the Hölder inequality,

$$\begin{aligned} &\le \left( E \max_{1\le j\le n} \|Z_j'\|^p \right)^{1/p} \left( E \left\| \sum_1^n \varepsilon_j Z_j' \right\|^{p/(p-1)} \right)^{(p-1)/p} \\ &\le \left( \sum_{j=1}^{n} E\|Z_j\|^p \right)^{1/p} \left( E \left\| \sum_1^n \varepsilon_j Z_j' \right\|^{p/(p-1)} \right)^{(p-1)/p} . \end{aligned}$$

Using inequality (3.5), we get

$$\left(E\left\|\sum_{j=1}^{n}\varepsilon_j Z_j'\right\|^{p/(p-1)}\right)^{(p-1)/p}$$
$$\le C_{10}\left\{\left(E\max_{1\le j\le n}\|Z_j'\|^{p/(p-1)}\right)^{(p-1)/p}+E\left\|\sum_{j=1}^{n}\varepsilon_j Z_j'\right\|\right\}$$
$$\le C_{10}\left\{t+E\left\|\sum_{j=1}^{n}\varepsilon_j Z_j\right\|\right\}\le 2C_{10}\{t+E\|T_n\|\}.$$

We see that (3.8) holds true, and the proof of (3.6) is complete. □

PROOF OF (3.7). By Markov's inequality, we have

$$Q_n(G_n^c)\le C_{11}\sum_{j=1}^{n}E\|Z_j\|^p/t^p.$$

Noting that $Q_n(F_n^c\cap G_n)\le Q_n'(F_n^c)$, it is clear that it suffices to show

$$Q_n'(F_n^c)\le C_{12}\sum_{j=1}^{n}E\|Z_j\|^p/t^p. \tag{3.9}$$

In order to prove (3.9), we need the following lemma.

LEMMA 2. *If $s\ge t/16p^2$, we have*

$$Q_n'\left\{\mathbf{z}\in B^n: E\left\|\sum_{j=1}^{n}\varepsilon_j z_j\right\|\ge 3s\right\}\le Q_n'\left\{\mathbf{z}\in B^n: E\left\|\sum_{j=1}^{n}\varepsilon_j z_j\right\|\ge s\right\}^2.$$

The proof of Lemma 2 is very similar to that of the Hoffmann-Jørgensen inequality and it is therefore omitted.

Let $m$ be the unique integer satisfying

$$2^{m-1}<p-1\le 2^m. \tag{3.10}$$

Note that (3.10) implies that $3^m\le(4/3)p^2$. We thus can apply Lemma 2 $m$ times and we obtain

$$\begin{aligned}Q_n'(F_n^c)&\le Q_n'\left\{\mathbf{z}\in B^n: E\left\|\sum_{j=1}^{n}\varepsilon_j z_j\right\|>3^{-m}\left(4p^2E\|T_n\|+t/12\right)\right\}^{2^m}\\&\le Q_n'\left\{\mathbf{z}\in B^n: E\left\|\sum_{j=1}^{n}\varepsilon_j z_j\right\|>3E\|T_n\|+t/16p^2\right\}^{p-1}.\end{aligned}\tag{3.11}$$

Recalling Fact 3, it is easy to see that

$$E\left\|\sum_{j=1}^{n}\varepsilon_j Z_j'\right\| \le E\left\|\sum_{j=1}^{n}\varepsilon_j Z_j\right\| \le 2E\|T_n\|$$

and we can infer from (3.11)

$$\begin{aligned}Q_n'(F_n^c) &\le Q_n'\Bigg\{\mathbf{z}\in B^n\colon E\left\|\sum_{j=1}^{n}\varepsilon_j z_j\right\| - E\left\|\sum_{j=1}^{n}\varepsilon_j Z_j'\right\| \\ &\qquad \ge \tfrac{1}{2}E\left\|\sum_{j=1}^{n}\varepsilon_j Z_j'\right\| + t/16p^2\Bigg\}^{p-1}.\end{aligned} \tag{3.12}$$

Put $h_n(z_1,\dots,z_n) = E\|\sum_{j=1}^{n}\varepsilon_j z_j\|$, $\mathbf{z}\in B^n$, and set

$$d_j := E\big[h_n(Z_1',\dots,Z_n')\|\mathscr{F}_j\big] - E\big[h_n(Z_1',\dots,Z_n')\|\mathscr{F}_{j-1}\big],\qquad 1\le j\le n,$$

where $\mathscr{F}_0 = \{\phi,\Omega\}$ and $\mathscr{F}_j = \sigma\{Z_1',\dots,Z_j'\}$, $1\le j\le n$. Then $d_j$, $1\le j\le n$, is a martingale difference sequence, and we obtain from (3.12)

$$\begin{aligned}Q_n'(F_n^c) &\le P\left\{\sum_{j=1}^{n} d_j \ge \tfrac{1}{2}E\left\|\sum_{j=1}^{n}\varepsilon_j Z_j'\right\| + t/16p^2\right\}^{p-1} \\ &\le C_{13}\left(\sum_{j=1}^{n} Ed_j^2\right)^{p-1}\Bigg/\left(E\left\|\sum_{j=1}^{n}\varepsilon_j Z_j'\right\| + t\right)^{2p-2}.\end{aligned} \tag{3.13}$$

Further, set $\bar{h}_j(z_1,\dots,z_n) := E\|\sum_{k=1}^{n}\varepsilon_k z_k\| - E\|\sum_{k\ne j}\varepsilon_k z_k\|$, $f_j := \bar{h}_j(Z_1',\dots,Z_n')$, $1\le j\le n$.

Observing that

$$0\le f_j\le \|Z_j'\| \tag{3.14}$$

and

$$d_j = E\big[f_j\|\mathscr{F}_j\big] - E\big[f_j\|\mathscr{F}_{j-1}\big],\qquad 1\le j\le n, \tag{3.15}$$

we easily get from the Hölder inequality

$$\begin{aligned}\sum_{j=1}^{n} Ed_j^2 &\le \sum_{j=1}^{n} Ef_j^2 \\ &\le \left(\sum_{j=1}^{n} Ef_j\right)^{(p-2)/(p-1)}\left(\sum_{1}^{n} E\|Z_j'\|^p\right)^{1/(p-1)}.\end{aligned}$$

Moreover, it is obvious that

$$\sum_{j=1}^{n} Ef_j = \sum_{j=1}^{n}\left\{E\left\|\sum_{k=1}^{n}\varepsilon_k Z_k'\right\| - E\left\|\sum_{k\neq j}\varepsilon_k Z_k'\right\|\right\}$$

$$\leq nE\left\|\sum_{k=1}^{n}\varepsilon_k Z_k'\right\| - E\left\|\sum_{j=1}^{n}\sum_{k\neq j}\varepsilon_k Z_k'\right\|$$

$$= E\left\|\sum_{k=1}^{n}\varepsilon_k Z_k'\right\|.$$

Recalling (3.13), we readily obtain (3.9), thereby establishing (3.7). Theorem 5 has been proved. □

3.2. *Conclusion of the proof of Theorem* 1. Using Ottaviani's inequality [see, e.g., Lemma 6.2 of Ledoux and Talagrand (1991)], we immediately obtain the following maximal inequality from Theorem 5.

LEMMA 3. *Let $Y_1, \dots, Y_n$ be independent B-valued random variables such that for some $p > 2$, $E\|Y_j\|^p < \infty$, $1 \leq j \leq n$. Suppose that*

$$E\left\|\sum_{j=k}^{n} Y_j\right\| \leq b_n, \qquad 1 \leq k \leq n. \tag{3.16}$$

*Then we have, for $t \geq 0$,*

$$P\left\{\max_{1\leq k\leq n}\left\|\sum_{1}^{k} Y_j\right\| \geq t + 38p^2 b_n\right\}$$

$$\leq 22\exp\left(-t^2/144\Lambda_n\right) + \tfrac{4}{3}C_1\sum_{j=1}^{n} E\|Y_j\|^p/t^p.$$

The next lemma easily follows from Lemma 4.4 of Berger (1991). It will come in handy when checking the above condition (3.16).

LEMMA 4. *Let $Y_1, \dots, Y_n$ be independent B-valued random variables satisfying*

$$\|Y_j\| \leq \beta_n, \qquad 1 \leq j \leq n \tag{3.17}$$

*and*

$$P\left\{\left\|\sum_{j=k}^{m} Y_j\right\| \geq \gamma_n\right\} \leq \frac{1}{10}, \qquad 1 \leq k \leq m \leq n. \tag{3.18}$$

*Then we have*

$$E\left\|\sum_{j=k}^{m} Y_j\right\| \leq 8\beta_n + 7\gamma_n, \qquad 1 \leq k \leq m \leq n. \tag{3.19}$$

Notice that in the two above lemmas we do *not* assume that $EY_j = 0$, $1 \le j \le n$.

We further need the following simple, but nevertheless very useful lemma.

LEMMA 5. *Let $\xi: \Omega \to [0, \infty)$ be a random variable such that*

$$\sum_{n=1}^{\infty} P\{\xi > c_n\} < \infty, \tag{3.20}$$

*where $\{c_n\}$ is a sequence satisfying* (2.6). *Then we have*:

(a) $\sum_{n=1}^{\infty} E\xi^p 1\{\xi \le c_n\}/c_n^p < \infty$

*provided that $p > \alpha^{-1}$.*

(b) $\sum_{n=1}^{\infty} P\{\xi > \varepsilon c_n\} < \infty$, $\varepsilon > 0$.

PROOF. (a) can be shown by a standard argument as used, for instance, in the proof of Lemma 1 of Einmahl (1988). Part (b) follows from (a) by the same argument as in Lemma 1 of Einmahl (1989). □

We finally record an additional property of the $K$-function, which one can obtain from its definition via integration by parts.

For any functional $f \in B^*$ with $E|f(X)| > 0$,

$$\begin{aligned} K_f^2(y) &= yE\Big(f(X)^2 1\big\{|f(X)| \le K_f(y)\big\}\Big) \\ &\quad + yK_f(y)E\Big(|f(X)| 1\big\{|f(X)| > K_f(y)\big\}\Big). \end{aligned} \tag{3.21}$$

Relation (3.21) trivially implies the following two inequalities:

$$Ef(X)^2 1\big\{|f(X)| \le K_f(y)\big\} \le K_f^2(y)/y \tag{3.22}$$

and

$$E|f(X)| 1\big\{|f(X)| > K_f(y)\big\} \le K_f(y)/y. \tag{3.23}$$

We are now in a position to prove Theorem 2. It is trivial that conditions (2.8) and (2.9) are necessary. Thus, we need only to show that these conditions are also sufficient.

Set $X_j' := X_j 1\{\|X_j\| \le c_j\}$, $j \ge 1$, and denote the corresponding partial sums by $S_n'$, $n \ge 1$.

On account of (2.9) we have

$$\sum_{n=1}^{\infty} P\{X_n \ne X_n'\} < \infty, \tag{3.24}$$

which trivially implies

$$\sum_{j=1}^{n} (X_j - X_j') = O(1) \quad \text{a.s.} \tag{3.25}$$

Thus, it is enough to show

$$\limsup_{n\to\infty} \|S'_n\|/c_n < \infty \quad \text{a.s.} \tag{3.26}$$

Let $t_0 > 0$ be chosen in a way such that

$$P\left\{\left\|\sum_{j=1}^{n} X_j\right\| \geq t_0 c_n\right\} \leq 1/20, \qquad n \geq 1. \tag{3.27}$$

Define the subsequences $\{m_k\}$, $\{n_k\}$ as follows: $m_1 := 1$, $m_k := \min\{m \geq m_{k-1}: c_m \geq 2c_{m_{k-1}}\}$, $k \geq 2$, $n_k := m_{k+1} - 1$, $k \geq 0$.

We now claim that

$$\sum_{k=1}^{\infty} P\left\{\max_{m_k \leq n \leq n_k} \|S'_n - S'_{n_{k-1}}\| \geq \overline{C} c_{n_k}\right\} < \infty, \tag{3.28}$$

where $\overline{C} = (266t_0 + 308)(2 + \alpha^{-1})^2$.

Using the Borel–Cantelli lemma and the definition of the sequence $\{m_k\}$, we readily obtain from (3.28),

$$\limsup_{n\to\infty} \|S'_n\|/c_n \leq 4\overline{C} \quad \text{a.s.},$$

which of course implies (3.26) and consequently Theorem 2.

In order to prove (3.28), we first note that for $m_k \leq m < n \leq n_k$, if $k$ is large enough,

$$\begin{aligned} &P\{\|S'_n - S'_m\| \geq t_0 c_{n_k}\} \\ &\quad \leq P\left\{\left\|\sum_{m+1}^{n} X_j\right\| \geq t_0 c_{n_k}\right\} + \sum_{j=m_k}^{\infty} P\{X_j \neq X'_j\} \\ &\quad \leq P\left\{\left\|\sum_{j=1}^{n-m} X_j\right\| \geq t_0 c_{n-m}\right\} + 1/20 \leq 1/10, \end{aligned}$$

where we use (3.24) and the fact that $\{c_n\}$ is increasing.

We now can infer from Lemma 4 that

$$E\|S'_{n_k} - S'_m\| \leq 8c_{n_k} + 7t_0 c_{n_k}, \qquad m_k \leq m \leq n_k, \tag{3.29}$$

provided $k$ is large enough.

Applying Lemma 3 with $p = 2 + \alpha^{-1}$, we obtain for large enough $k$,

$$\begin{aligned} &P\left\{\max_{m_k \leq n \leq n_k} \|S'_n - S'_{n_{k-1}}\| \geq \overline{C} c_{n_k}\right\} \\ &\quad \leq 22 \exp\left(-\frac{16}{9} c_{n_k}^2 \Big/ \sum_{m_k}^{n_k} \sigma_j^2\right) + \overline{C}_1 c_{n_k}^{-p} \sum_{j=m_k}^{n_k} E\|X'_j\|^p, \end{aligned} \tag{3.30}$$

where $\sigma_j^2 := \sup\{\sigma_j^2(f): \|f\| \leq 1\}$, $\sigma_j^2(f) := Ef^2(X)1\{\|X\| \leq c_n\}$, $f \in B^*$ and

$\overline{C}_1$ is a positive constant. Thus, in order to prove (3.28) it is enough to show

$$\sum_{k=1}^{\infty}\left(\sum_{j=m_k}^{n_k} E\|X_j'\|^p\right)/c_{n_k}^p < \infty, \tag{3.31}$$

$$\sum_{k=1}^{\infty} \exp\left(-16c_{n_k}^2/9\sum_{m_k}^{n_k}\sigma_j^2\right) < \infty. \tag{3.32}$$

Relation (3.31) follows from Lemma 5(a). As to (3.32), note first that for functionals $f \in B^*$ with $\|f\| \le 1$ and $E|f(X)| > 0$,

$$\begin{aligned}\sigma_j^2(f) &\le Ef^2(X)1\{|f(X)| \le K_f(j/LLj)\}\\ &\quad + Ef^2(X)1\{|f(X)| > K_f(j/LLj), \|X\| \le c_j\}\\ &=: \nu_{j,1}(f) + \nu_{j,2}(f).\end{aligned}$$

Inequality (3.22) now implies

$$\nu_{j,1}(f) \le K_f^2(n_k/LLn_k)LLn_k/n_k \le c_{n_k}^2/(n_k LLn_k), \qquad j \le n_k. \tag{3.33}$$

On the other hand, using inequality (3.23) in combination with the Hölder inequality, we get

$$\begin{aligned}\nu_{j,2}(f) &\le \left(E|f(X)|1\{|f(X)| > K_f(j/LLj)\}\right)^{(p-2)/(p-1)}\left(E\|X_j'\|^p\right)^{1/(p-1)}\\ &\le \left(K_f(j/LLj)LLj/j\right)^{(p-2)/(p-1)}\left(E\|X_j'\|^p\right)^{1/(p-1)}\\ &\le c_j^2 j^{(2-p)/(p-1)}\left(E\|X_j'\|^p/c_j^p\right)^{1/(p-1)}.\end{aligned}$$

Combining this last bound for $\nu_{j,2}(f)$ with (3.33), and again using the Hölder inequality, we find that

$$\begin{aligned}\sum_{m_k}^{n_k}\sigma_j^2 &\le c_{n_k}^2\left((LLn_k)^{-1} + \left(\sum_{m_k}^{n_k} j^{-1}\right)^{(p-2)/(p-1)}\left(\sum_{m_k}^{n_k} E\|X_j'\|^p/c_j^p\right)^{1/(p-1)}\right)\\ &\le c_{n_k}^2\left((LLn_k)^{-1} + \frac{n_k}{m_k}\left(\sum_{m_k}^{n_k} c_j^{-p}E\|X_j'\|^p\right)^{1/(p-1)}\right).\end{aligned}$$

Noting that on account of (2.6)

$$\left(\frac{n_k}{m_k}\right)^{\alpha} \le \frac{c_{n_k}}{c_{m_k}} \le 2,$$

we finally obtain

$$\sum_{m_k}^{n_k}\sigma_j^2 \le c_{n_k}^2\left((LLn_k)^{-1} + 2^{1/\alpha}\left(\sum_{m_k}^{n_k} c_j^{-p}E\|X_j'\|^p\right)^{1/(p-1)}\right).$$

Using the trivial inequality

$$\exp(-s/(x+y)) \le \exp(-3s/4x) + \exp(-s/4y), \qquad s, x, y > 0,$$

we can conclude

$$\begin{aligned}\exp&\left(-16c_{n_k}^2/9\sum_{m_k}^{n_k}\sigma_j^2\right)\\ &\le \exp(-4LLn_k/3) + \exp\left(-\tfrac{4}{9}2^{1/\alpha}\left(\sum_{m_k}^{n_k}c_j^{-p}E\|X_j'\|^p\right)^{1/(p-1)}\right)\\ &\le (Ln_k)^{-4/3} + \overline{C}_2\sum_{m_k}^{n_k}c_j^{-p}E\|X_j'\|^p,\end{aligned}$$

where $\overline{C}_2$ is a positive constant. If $\sum_{k=1}^{\infty}(Ln_k)^{-4/3} < \infty$, relation (3.32) immediately follows from (3.31). But since it can happen that the foregoing series is divergent, we have to use a slightly different argument.

To that end, set $\mathbb{N}_1 := [k \ge 1\colon n_k \le 2^{k/2}\}$, $\mathbb{N}_2 := \mathbb{N} - \mathbb{N}_1$. It is easy to see that if $\|f\| \le 1$, then

$$\begin{aligned}\sum_{m_k}^{n_k}\sigma_j^2(f)/c_{n_k}^2 &\le \sum_{m_k}^{n_k}E\|X_j'\|^2/c_{n_k}^2\\ &\le n_kE\|X\|/c_{n_k}.\end{aligned}$$

Noting that $c_{n_k} \ge 2^{k-1}c_1$, it is plain that the last term is

$$\le \overline{C}_3 2^{-k/2} \quad \text{if } k \in \mathbb{N}_1,$$

where $\overline{C}_3$ is a positive constant.

We now obtain, for an appropriate choice of $k_0$,

$$\begin{aligned}\sum_{k=k_0}^{\infty}&\exp\left(-16c_{n_k}^2/9\sum_{m_k}^{n_k}\sigma_j^2\right)\\ &\le \sum_{k\in\mathbb{N}}\exp\left(-16\cdot 2^{k/2}/9\overline{C}_3\right) + \sum_{k\in\mathbb{N}_2}(Ln_k)^{-4/3} + \overline{C}_2\sum_{j=1}^{\infty}c_j^{-p}E\|X_j'\|^p,\end{aligned}$$

where all series are convergent. Theorem 1 has been proved. □

3.3. *Proof of Corollary* 1. It is enough to prove the following lemma.

LEMMA 6. *Let X be a mean zero random variable taking values in a type* 2 *Banach space. Let* $\{c_n\}$ *be a sequence of positive real numbers satisfying*

(3.34) $$c_n/\sqrt{n} \nearrow \infty$$

*and*

(3.35) $c_n/n$ *is nonincreasing.*

*Assume that*

$$\sum_{n=1}^{\infty} P\{\|X\| > c_n\} < \infty. \tag{3.36}$$

*Then*

$$E\|S_n\|/c_n \to 0 \quad \textit{as } n \to \infty.$$

PROOF. Let $\tilde{X}$ be the symmetrization of $X$, and let $\tilde{X}_n$, $n \geq 1$ be independent copies of $\tilde{X}$. From (3.36) and Lemma 5(b) it is immediate that we also have

$$\sum_{n=1}^{\infty} P\{\|\tilde{X}\| \geq c_n\} < \infty. \tag{3.37}$$

Setting $Y_j^n := \tilde{X}_j 1\{\|\tilde{X}_j\| \leq c_n\}$, $1 \leq j \leq n$, and $T_n^n := \sum_{j=1}^n Y_j^n$, we have by (1.5),

$$E\|T_n^n\| \leq \left(E\|T_n^n\|^2\right)^{1/2} \leq C^{1/2} n^{1/2} \left(E\|Y_n^n\|^2\right)^{1/2}.$$

If we set $p_j := P\{c_{j-1} < \|\tilde{X}\| \leq c_j\}$, $j \geq 1$, where $c_0 := 0$, then (3.37) clearly implies

$$\sum_{j=1}^{\infty} j p_j < \infty. \tag{3.38}$$

Using (3.34), we then can infer

$$nE\|Y_1^n\|^2/c_n^2 \leq n \sum_{j=1}^{n} p_j c_j^2/c_n^2 \leq n c_{j_0}^2/c_n^2 + \sum_{j=j_0}^{\infty} j p_j.$$

Choosing $j_0$ in a way such that $\sum_{j=j_0}^{\infty} j p_j \leq \varepsilon^2/C$, it is plain that for large enough $n$,

$$E\|T_n^n\| \leq 2\varepsilon c_n. \tag{3.39}$$

Setting $\tilde{S}_n := \sum_{j=1}^n \tilde{X}_j$, we trivially have for large enough $n$,

$$E\|\tilde{S}_n\| \leq 2\varepsilon c_n + E\|\tilde{S}_n - T_n^n\| \leq 2\varepsilon c_n + nE\|\tilde{X}\| 1\left\{\|\tilde{X}\| \geq c_n\right\}.$$

Recalling (3.35) and using the subsequent Lemma 7, we finally obtain, for large enough $n$,

$$E\|\tilde{S}_n\| \leq 3\varepsilon c_n.$$

Noting that $E\|S_n\| \leq E\|\tilde{S}_n\|$, we see that the assertion of Lemma 6 is true. □

LEMMA 7. *Let $\xi$: $\Omega \to [0, \infty)$ be a random variable satisfying*

$$\sum_{n=1}^{\infty} P\{\xi > d_n\} < \infty,$$

*where $\{d_n\}$ is a sequence of positive real numbers such that $d_n/n$ is nonincreasing.*

*Then we have*

$$nE\xi 1\{\xi > d_n\} = o(d_n) \quad \text{as } n \to \infty.$$

PROOF. Set $q_j := P\{d_{j-1} < \xi \le d_j\}$, $j \ge 1$, where $d_0 = 0$. Then it is obvious that

$$\sum_{j=1}^{\infty} jq_j < \infty.$$

Furthermore, it is easy to see that

$$\begin{aligned} nE\xi 1\{\xi > d_n\}/d_n &\le n \sum_{n+1}^{\infty} d_j q_j / d_n \\ &\le \sum_{n+1}^{\infty} jq_j \to 0 \quad \text{as } n \to \infty, \end{aligned}$$

where we use the fact that

$$d_j \le jd_n/n, \qquad j \ge n. \qquad \square$$

**4. Proof of Theorem 2.** In order to prove Theorem 2, it is enough to consider random variables satisfying the two additional assumptions

$$\sum_{n=1}^{\infty} P\{\|X\| > \tilde{K}(n/LLn)LLn\} < \infty \tag{4.1}$$

and

$$\limsup_{n \to \infty} P\{\|S_n\| \ge \sqrt{2}\,\tilde{K}(n/LLn)LLn\} \le 1/2. \tag{4.2}$$

For if (4.1) were not true, it would follow that

$$\limsup_{n \to \infty} \|S_n\|/\tilde{K}(n/LLn)LLn = \infty \quad \text{a.s.},$$

which trivially implies the assertion of Theorem 2. To justify the additional assumption (4.2), note that if this were not the case, we would have

$$\begin{aligned} &P\left\{\limsup_{n \to \infty} \|S_n\|/\tilde{K}(n/LLn)LLn \ge \sqrt{2}\right\} \\ &\qquad \ge \limsup_{n \to \infty} P\{\|S_n\| \ge \sqrt{2}\,\tilde{K}(n/LLn)LLn\} \ge 1/2. \end{aligned}$$

Then by the 0-1 law of Hewitt–Savage, it would follow that

$$\limsup_{n \to \infty} \|S_n\|/\tilde{K}(n/LLn)LLn \ge \sqrt{2} \quad \text{a.s.}$$

This shows that Theorem 2 is trivially true if either of the conditions (4.1) or (4.2) fails. So from now on we may and actually do assume that these two conditions hold. Our proof is divided into several steps.

STEP 1. Let $m_j \nearrow \infty$ be a subsequence of $\mathbb{N}$ such that for some constants $a_1, a_2 > 1$ and all $j \geq 1$,

$$a_1 < m_{j+1}/m_j \leq a_2. \tag{4.3}$$

We shall show, for any $0 < \varepsilon < 1/4$,

$$\sum_{j=1}^{\infty} P\left\{\|S_{m_j}\| \geq (\sqrt{2} - \varepsilon)\tilde{K}(m_j/LLm_j)LLm_j\right\} = \infty. \tag{4.4}$$

To that end, pick a functional $f_j$ with $\|f_j\| = 1$ such that

$$K_{f_j}(m_j/LLm_j) \geq \left((\sqrt{2} - \varepsilon)/(\sqrt{2} - \varepsilon/2)\right)\tilde{K}(m_j/LLm_j)$$

and set $\beta_j := K_{f_j}(m_j/LLm_j)LLm_j$, $j \geq 1$.

It is obvious that

$$\begin{aligned} &P\left\{\|S_{m_j}\| \geq (\sqrt{2} - \varepsilon)\tilde{K}\left(\frac{m_j}{LLm_j}\right)LLm_j\right\} \\ &\quad \geq P\left\{|f_j(S_{m_j})| \geq \left(\sqrt{2} - \frac{\varepsilon}{2}\right)\beta_j\right\}, \qquad j \geq 1. \end{aligned} \tag{4.5}$$

To further simplify the notation, set

$$Y_{n,j} := f_j(X_n), \qquad Z_{n,j} := Y_{n,j}1\{|Y_{n,j}| \leq 2\beta_j\}, \qquad n \geq 1.$$

Then it is clear that

$$\begin{aligned} &P\left\{|f_j(S_{m_j})| \geq (\sqrt{2} - \varepsilon/2)\beta_j\right\} \\ &\quad \geq P\left\{\sum_{n=1}^{m_j} Z_{n,j} \geq (\sqrt{2} - \varepsilon/2)\beta_j\right\} - m_j P\left\{|Y_{1,j}| > 2\beta_j\right\} \\ &\quad \geq P\left\{\sum_{n=1}^{m_j} Z_{n,j} \geq (\sqrt{2} - \varepsilon/2)\beta_j\right\} \\ &\qquad - m_j P\left\{\|X\| > \tilde{K}(m_j/LLm_j)LLm_j\right\}. \end{aligned} \tag{4.6}$$

Recalling (4.1) and (4.3), it is easy to see that

$$\sum_{j=1}^{\infty} m_j P\left\{\|X\| \geq \tilde{K}(m_j/LLm_j)LLm_j\right\} < \infty. \tag{4.7}$$

Thus, in order to verify (4.4), it is enough to show

$$\sum_{j=1}^{\infty} P\left\{\sum_{n=1}^{m_j} Z_{n,j} \geq (\sqrt{2} - \varepsilon/2)\beta_j\right\} = \infty. \tag{4.8}$$

STEP 2. Applying Lemma 7 [with $\xi = \|X\|$, $d_j = \tilde{K}(j/LLj)LLj$], we can find a $j_0 = j_0(\varepsilon)$ such that for $j \geq j_0$,

$$(4.9)\qquad m_j E\|X\|1\left\{\|X\| > \tilde{K}\left(\frac{m_j}{LLm_j}\right)LLm_j\right\} \leq \frac{\varepsilon}{8}\tilde{K}\left(\frac{m_j}{LLm_j}\right)LLm_j,$$

and we can infer that

$$(4.10)\qquad m_j EZ_{1,j} \leq m_j E\|X\|1\{\|X\| \geq 2\beta_j\} \leq \frac{\varepsilon}{4}\beta_j, \qquad j \geq j_0.$$

Setting $\bar{Z}_{n,j} := Z_{n,j} - EZ_{n,j}$, $1 \leq n \leq m_j$, we get for $j \geq j_0$,

$$(4.11)\quad P\left\{\sum_{n=1}^{m_j} Z_{n,j} \geq (\sqrt{2} - \varepsilon/2)\beta_j\right\} \geq P\left\{\sum_{n=1}^{m_j} \bar{Z}_{n,j} \geq (\sqrt{2} - \varepsilon/4)\beta_j\right\}.$$

Using a well-known nonuniform bound on the rate of convergence in the central limit theorem [see, e.g., Theorem 13 on page 125 of Petrov (1975)], we obtain

$$(4.12)\qquad \begin{aligned} &P\left\{\sum_{n=1}^{m_j} \bar{Z}_{n,j} \geq (\sqrt{2} - \varepsilon/4)\beta_j\right\} \\ &\quad \geq 1 - \Phi\left((\sqrt{2} - \varepsilon/4)\beta_j/\sqrt{m_j}\sigma_j\right) - Am_j E|\bar{Z}_{1,j}|^3/\beta_j^3, \end{aligned}$$

where $\Phi(t)$, $-\infty < t < \infty$, is the distribution function of a standard normal r.v., $\sigma_j^2 = \mathrm{Var}(Z_{1,j})$ and $A$ is an absolute constant.

We now claim

$$(4.13)\qquad \sum_{j=1}^{\infty} m_j E|\bar{Z}_{1,j}|^3/\beta_j^3 < \infty$$

and

$$(4.14)\qquad \sum_{j=1}^{\infty}\left(1 - \Phi\left((\sqrt{2} - \varepsilon/4)\beta_j/\sqrt{m_j}\sigma_j\right)\right) = \infty.$$

In view of (4.8), (4.11) and (4.12), it is clear that (4.4) follows from (4.13) and (4.14).

STEP 3 [Proof of (4.13)]. First note that by the $c_r$-inequality,

$$E|\bar{Z}_{1,j}|^3 \leq 8E|Z_{1,j}|^3.$$

Recalling the definition of $Z_{1,j}$, we can conclude

$$(4.15)\qquad \begin{aligned} E|Z_{1,j}|^3 &\leq E|f_j(X)|^3 1\left\{\|X\| \leq \tilde{K}(m_j/LLm_j)LLm_j\right\} \\ &\quad + E|f_j(X)|^3 1\left\{|f_j(X)| \leq 2\beta_j,\ \|X\| > \tilde{K}(m_j/LLm_j)LLm_j\right\} \\ &\leq E\|X\|^3 1\left\{\|X\| \leq \tilde{K}(m_j/LLm_j)LLm_j\right\} \\ &\quad + 8\beta_j^3 P\left\{\|X\| > \tilde{K}(m_j/LLm_j)LLm_j\right\}. \end{aligned}$$

Employing a similar argument to that of the proof of Lemma 7.1 of Pruitt (1981), one can infer from (4.1) that

$$\sum_{j=1}^{\infty} m_j E\|X\|^3 1\left\{\|X\| \le \tilde{K}(m_j/LLm_j)LLm_j\right\}/\beta_j^3 < \infty. \tag{4.16}$$

Using the above bounds for $E|\bar{Z}_{1,j}|^3$ in conjunction with (4.7) and (4.16), we readily obtain (4.13).

STEP 4 [Proof of (4.14)]. We first derive a lower bound for $\sigma_j^2$. Set $\alpha_j := K_{f_j}(m_j/LLm_j)$. Then, using (4.10) and the property (3.21) of the $K$-function, we get for $j \ge j_0$,

$$\begin{aligned}
\sigma_j^2 &\ge Ef_j^2(X)1\{|f_j(X)| \le \alpha_j\} + Ef_j^2(X)1\{\alpha_j < |f_j(X)| \le 2\beta_j\} - \varepsilon^2\beta_j^2/16m_j^2 \\
&\ge \alpha_j^2 LLm_j/m_j - \alpha_j E|f_j(X)|1\{|f_j(X)| > \alpha_j\} \\
&\qquad + \alpha_j E|f_j(X)|1\{\alpha_j < |f_j(X)| \le 2\beta_j\} - \varepsilon^2\beta_j^2/16m_j^2 \\
&= \beta_j^2/m_j LLm_j - \alpha_j E|f_j(X)|1\{|f_j(X)| > 2\beta_j\} - \varepsilon^2\beta_j^2/16m_j^2 \\
&\ge \beta_j^2/m_j LLm_j - \varepsilon\beta_j^2/4m_j LLm_j - \varepsilon^2\beta_j^2/16m_j^2 \\
&\ge (1 - \varepsilon/3)\beta_j^2/m_j LLm_j.
\end{aligned}$$

Using the fact that

$$1 - \Phi(x) \sim \exp(-x^2/2)/x\sqrt{2\pi} \quad \text{as } x \to \infty,$$

we obtain for large enough $j$,

$$1 - \Phi\left((\sqrt{2} - \varepsilon/4)\beta_j/\sqrt{m_j}\sigma_j\right) \ge \tfrac{1}{4}(LLm_j)^{-1/2}(Lm_j)^{-1}.$$

In view of (4.3) it is now clear that the series in (4.14) is divergent.

STEP 5. Let $0 < \varepsilon < 1$ be fixed, and set $n_k := \sum_{j=1}^k m_j$, where $m_j := [(1 + 3\varepsilon^{-2})^j]$, $j \ge 1$.

Consider the events

$$F_k := \left\{\|S_{n_k} - S_{n_{k-1}}\| \ge (\sqrt{2} - \varepsilon)\tilde{K}(m_k/LLm_k)LLm_k\right\}$$

and

$$G_k := \left\{\|S_{n_{k-1}}\| \le \varepsilon\tilde{K}(m_k/LLm_k)LLm_k\right\}, \qquad k \ge 1.$$

Then we have for large enough $k$,

$$P(G_k) \ge \tfrac{1}{3}. \tag{4.17}$$

To prove (4.17), note that by the definition of $n_k$,

$$m_k \ge 3m_{k-1}/\varepsilon^2 \ge 3(1 - \varepsilon^2/3)n_{k-1}/\varepsilon^2 \ge 2n_{k-1}/\varepsilon^2.$$

Recalling (2.4), we obtain

$$P(G_k^c) \le P\Big\{\|S_{n_{k-1}}\| \ge \sqrt{2}\,\tilde{K}(n_{k-1}/LLm_k)LLm_k\Big\}$$
$$\le P\Big\{\|S_{n_{k-1}}\| \ge \sqrt{2}\,\tilde{K}(n_{k-1}/LLn_{k-1})LLn_{k-1}\Big\},$$

where the latter inequality is true on account of $n_{k-1} \le m_k$ and (2.3). Recalling assumption (4.2), we get (4.17).

Next note that on account of (4.4),

$$\sum_{k=1}^{\infty} P(F_k) = \infty.$$

We are now in a position to apply Lemma 3.4 of Pruitt (1981), and we find that

$$P(F_k \cap G_k \text{ infinitely often}) = 1. \tag{4.18}$$

Since, by the definition of $n_k$,

$$n_k \le m_k(1-\varepsilon^2/3)^{-1},$$

we obtain from (2.3)

$$F_k \cap G_k \subset \Big\{\|S_{n_k}\| \ge (\sqrt{2}-2\varepsilon)\tilde{K}(m_k/LLm_k)LLm_k\Big\}$$
$$\subset \Big\{\|S_{n_k}\| \ge (\sqrt{2}-3\varepsilon)\tilde{K}(n_k/LLn_k)LLn_k\Big\}.$$

This means, in view of (4.18),

$$\limsup_{n\to\infty} \|S_n\|/\tilde{K}(n/LLn)LLn \ge \sqrt{2}-3\varepsilon \quad \text{a.s.} \tag{4.19}$$

Since $\varepsilon$ can be made arbitrarily small, the last relation clearly implies Theorem 2. □

## 5. Proofs of Theorem 3 and Corollary 2.

5.1. *A double truncation argument.* We set

$$Z_j' := X_j 1\Big\{\|X_j\| \le \gamma_j/(LLj)^2\Big\}, \qquad \bar{Z}_j' := Z_j' - EZ_j',$$
$$Z_j'' := X_j 1\Big\{\gamma_j/(LLj)^2 < \|X_j\| \le \gamma_j\Big\}, \qquad \bar{Z}_j'' := Z_j'' - EZ_j'',$$
$$Z_j''' := X_j 1\Big\{\|X_j\| > \gamma_j\Big\}, \qquad \bar{Z}_j''' := Z_j''' - EZ_j''',$$

where $\gamma_j := \tilde{K}(j/LLj)LLj$, $j \ge 1$.

The purpose of this subsection is to prove the following:

LEMMA 8. *Under the assumptions of Theorem 3, we have*

$$\sum_{j=1}^{n} \big(X_j - \bar{Z}_j'\big)/\gamma_n \to 0, \quad a.s.$$

Lemma 8 then enables us to reduce the proof of Theorem 3 to studying the (bounded) sums $\sum_{j=1}^n \bar{Z}_j'$, $n \ge 1$ (see 5.2).

In order to prove Lemma 8, we need some further lemmas.

LEMMA 9. *Under the above assumptions, we have the following*:

(a) $E\|S_n\|/\gamma_n \to$ *as* $n \to \infty$.
(b) $E\|\sum_{j=1}^n \bar{Z}_j'\|/\gamma_n \to 0$ *as* $n \to \infty$.
(c) $E\|\sum_{j=1}^n \bar{Z}_j''\|/\gamma_n \to 0$ *as* $n \to \infty$.

PROOF. Recall that assumption (2.15) implies by Lemma 5(b) and Lemma 7, for any $\varepsilon > 0$,

$$nP\{\|X\| > \varepsilon\gamma_n\} \to 0 \quad \text{as } n \to \infty \tag{5.1}$$

and

$$nE\|X\|1\{\|X\| > \varepsilon\gamma_n\}/\gamma_n \to 0 \quad \text{as } n \to \infty. \tag{5.2}$$

Thus, setting $S_{k,n} := \sum_{j=1}^k X_j 1\{\|X_j\| \le \varepsilon\gamma_n\}$, $1 \le k \le n$, $S_{0,n} := 0$, we have for $0 \le k < m \le n$ and large enough $n$,

$$\begin{aligned} P\{\|S_{m,n} - S_{k,n}\| \ge \varepsilon\gamma_n\} &\\ \le P\{\|S_m - S_k\| \ge \varepsilon\gamma_n\} + nP\{\|X\| \ge \varepsilon\gamma_n\} &\\ = P\{\|S_{m-k}\| \ge \varepsilon\gamma_n\} + nP\{\|X\| \ge \varepsilon\gamma_n\} \le \tfrac{1}{10}. & \end{aligned}$$

This implies via Lemma 4

$$E\|S_{n,n}\| \le 15\varepsilon\gamma_n. \tag{5.3}$$

Combining (5.2) and (5.3), we obtain for large enough $n$,

$$\begin{aligned} E\|S_n\| &\le E\|S_{n,n}\| + nE\|X\|1\{\|X\| > \varepsilon\gamma_n\} \\ &\le 16\varepsilon\gamma_n. \end{aligned}$$

This shows that (a) is true. In order to verify (b), note that by symmetrization and (3.3),

$$\begin{aligned} E\left\|\sum_{j=1}^n \bar{Z}_j'\right\| &\le 2E\left\|\sum_{j=1}^n \varepsilon_j Z_j'\right\| \\ &\le 2E\left\|\sum_{j=1}^n \varepsilon_j Z_j\right\| \\ &\le 4E\|S_n\| \end{aligned}$$

so that (b) follows from (a). The proof of (c) is similar. □

LEMMA 10. *Under the above assumptions we have*

$$\sum_{j=1}^n \bar{Z}_j'''/\gamma_n \to 0 \quad a.s. \tag{5.4}$$

PROOF. Note that by assumption (2.15) and Borel–Cantelli,

$$\sum_{j=1}^{n} Z_j'''' = O(1) \quad \text{a.s.} \tag{5.5}$$

Furthermore, it is easy to see that for $1 \le m \le n$,

$$\left\| \sum_{j=1}^{n} EZ_j'''' \right\| \le mE\|X\| + \sum_{m+1}^{n} E\|X\|1\{\|X\| \ge \gamma_j\}.$$

Recalling (5.2), we can find for any $\delta > 0$ an $m = m(\delta)$ such that the last term is bounded above by

$$mE\|X\| + \delta \sum_{m+1}^{n} \gamma_j/j,$$

which in turn is

$$\begin{aligned} &\le mE\|X\| + \delta\gamma_n\left(\sum_{j=1}^{n} j^{-1/2}\right)n^{-1/2} \\ &\le mE\|X\| + 2\delta\gamma_n, \end{aligned}$$

where we have used the fact that $\gamma_j/\sqrt{j}$ is nondecreasing.

We can infer that

$$\limsup_{n\to\infty} \left\| \sum_{j=1}^{n} EZ_j'''' \right\| / \gamma_n \le 2\delta. \tag{5.6}$$

Since $\delta$ can be made arbitrarily small, we obtain (5.4) from (5.5) and (5.6). □

LEMMA 11. *Under the above assumptions we have*

$$\sum_{j=1}^{n} \bar{Z}_j''/\gamma_n \to 0 \quad a.s. \tag{5.7}$$

PROOF. Set $m_k := 2^{k-1}$, $n_k := 2^k - 1$, $k \ge 1$.

Using relations (2.3) and (2.4), it is then easy to see that it is enough to prove

$$\sum_{k=1}^{\infty} P\left\{ \max_{m_k \le m \le n_k} \left\| \sum_{m_k}^{m} \bar{Z}_j'' \right\| \ge \varepsilon\gamma_{n_k} \right\} < \infty. \tag{5.8}$$

Noting Lemma 9(c), a straightforward application of Lemma 3 yields, for large enough $k$,

$$\begin{aligned} &P\left\{ \max_{m_k \le m \le n_k} \left\| \sum_{m_k}^{m} \bar{Z}_j'' \right\| \ge \varepsilon\gamma_{n_k} \right\} \\ &\qquad \le 22 \exp\left( \frac{-\varepsilon^2\gamma_{n_k}^2}{576\sum_{m_k}^{n_k}\sigma_j^2} \right) + \frac{256}{3}C_1\varepsilon^{-3}\Delta_k, \end{aligned}$$

where $\sigma_j^2 := \sup\{Ef^2(Z_j''): \|f\| \le 1\}$, and $\Delta_k := \sum_{m_k}^{n_k} E\|Z_j''\|^3/\gamma_{n_k}^3$, $k \ge 1$. We have by Lemma 5(a)

$$\sum_{k=1}^{\infty} \Delta_k < \infty. \tag{5.9}$$

Thus, it remains to be shown that

$$\sum_{k=1}^{\infty} \exp\left(-\varepsilon^2\gamma_{n_k}^2 \Big/ \left(576 \sum_{m_k}^{n_k} \sigma_j^2\right)\right) < \infty. \tag{5.10}$$

Note that we trivially have

$$\sum_{m_k}^{n_k} \sigma_j^2/\gamma_{n_k}^2 \le \Delta_k (LLn_k)^2 \tag{5.11}$$

and also by the proof of Theorem 1 (where we set $c_n = \gamma_n$, $p = 3$),

$$\sum_{m_k}^{n_k} \sigma_j^2/\gamma_{n_k}^2 \le (LLn_k)^{-1} + A \cdot \Delta_k^{1/2}, \tag{5.12}$$

where $A > 0$ is an absolute constant.

Relation (5.12) in particular implies that for large enough $k$,

$$\sum_{m_k}^{n_k} \sigma_j^2/\gamma_{n_k}^2 \le 1. \tag{5.13}$$

Using the same argument as in the proof of (3.9) of Einmahl (1989), we get from (5.11)–(5.13) for large enough $k$,

$$\begin{aligned}
&\exp\left(-\varepsilon^2\gamma_{n_k}^2/576 \sum_{m_k}^{n_k} \sigma_j^2\right) \\
&\le 1152\varepsilon^{-2}\left(\sum_{m_k}^{n_k} \sigma_j^2\right)\gamma_{n_k}^{-2} \exp\left(-\varepsilon^2\gamma_{n_k}^2/1152 \sum_{m_k}^{n_k} \sigma_j^2\right) \\
&\le 1152\varepsilon^{-2}\left(\sum_{m_k}^{n_k} \sigma_j^2\right)\gamma_{n_k}^{-2} \exp\left(-\varepsilon^2/\left(1152(LLn_k)^{-1} + A'\Delta_k^{1/2}\right)\right) \\
&\le 1152\varepsilon^{-2}\Delta_k(LLn_k)^2(Ln_k)^{-\varepsilon^2/2304} + 1152\varepsilon^{-2}\exp\left(-\varepsilon^2/2A'\Delta_k^{1/2}\right) \\
&\le C_\varepsilon \cdot \Delta_k.
\end{aligned}$$

In view of (5.9), it is now clear that (5.10) and consequently Lemma 11 hold true. □

Combining Lemma 10 and Lemma 11, we obtain Lemma 8.

5.2. *Conclusion of the proof of Theorem* 3 *and Corollary* 2. As in Ledoux and Talagrand [(1991), page 199], let $D$ be a countable subset of the unit ball of $B^*$ such that

$$\|x\| = \sup\{|f(x)|: f \in D\}.$$

Following their proof of Theorem 8.2, we shall use a finite-dimensional approximation argument which in combination with an appropriate entropy argument will enable us to prove Theorem 3.

To that end we set for any $f, g \in D$,

$$d_2^n(f, g) = (LLm_n)^{1/2}\left\{\sum_{i=1}^{m_n} E(f-g)^2(\bar{Z}_i')\right\}^{1/2} / \gamma_{m_n},$$

where $m_n \sim \rho^n$, and $\rho > 1$ is fixed.

Finally, put $\alpha_n := E\|\sum_{i=1}^{m_n} \varepsilon_i \bar{Z}_i'\|/\gamma_{m_n}$, and recall that, on account of Lemma 9(b) and (3.3),

$$\alpha_n \to 0 \quad \text{as } n \to \infty. \tag{5.14}$$

From property (3.21) of the $K$-function, we can infer that for large $n$,

$$E\big(f(\bar{Z}_i')\big)^2 \le \gamma_{m_n}^2 / m_n LLm_n, \ \|f\| \le 1, \qquad 1 \le i \le m_n. \tag{5.15}$$

Using (5.14) and (5.15), it is fairly easy to modify the proof of Lemma 8.3 of Ledoux and Talagrand (1991), so as to obtain, for $\varepsilon > 0$ and large enough $n$,

$$N(D, d_2^n; \varepsilon) \le \exp(\alpha_n LLm_n), \tag{5.16}$$

where $N(D, d_2^n; \varepsilon)$ is the minimal number of functionals $g$ in $D$ such that for any $f$ in $D$ there exists a $g$ with $d_2^n(f, g) < \varepsilon$.

Let now $D_n$ be a subset of $D$ with cardinality $N(D, d_2^n; \varepsilon)$, which has the above property. We then can find for any functional $f \in D$ a functional $g_n(f) \in D_n$ such that

$$d_2^n(f, g_n(f)) < \varepsilon. \tag{5.17}$$

Setting $D_n' := \{f - g_n(f): f \in D\}$, it is plain that

$$\left\|\sum_{j=1}^{m_n} \bar{Z}_j'\right\| \le \sup_{g \in D_n}\left|\sum_{j=1}^{m_n} g(\bar{Z}_j')\right| + \sup_{h \in D_n'}\left|\sum_{j=1}^{m_n} h(\bar{Z}_j')\right|. \tag{5.18}$$

Because of (5.17) we have

$$\sup_{h \in D_n'} \sum_{j=1}^{m_n} Eh^2(\bar{Z}_j') \le \varepsilon^2 \gamma_{m_n}^2 / (LLm_n).$$

Noting that the proof of Lemma 3 also applies to the seminorm $\sup_{h \in D_n'}|h(\cdot)|$ and recalling Lemma 9(b), we readily obtain

$$\sum_{n=1}^{\infty} P\left\{\max_{1 \le k \le m_n} \sup_{h \in D_n'}\left|\sum_{j=1}^{k} h(\bar{Z}_j')\right| \ge 13\varepsilon\gamma_{m_n}\right\} < \infty, \tag{5.19}$$

where we use the fact that

$$(5.20)\qquad \sum_{n=1}^{\infty} \gamma_{m_n}^{-3} \sum_{j=1}^{m_n} E\|\bar{Z}'_j\|^3 < \infty.$$

Relation (5.20) easily follows from Lemma 5.

Using, for example, Lemma 1.6 of Ledoux and Talagrand (1991) (with $a = 2\gamma_{m_n}/(LLm_n)^2$ and $b^2 = \gamma_{m_n}^2/LLm_n$), we obtain for any $g$ in $D_n$,

$$P\left\{\left|\sum_{j=1}^{m_n} g(\bar{Z}'_j)\right| \ge (\sqrt{2} + \varepsilon)\gamma_{m_n}\right\} \le 2\exp(-(1+\varepsilon)LLm_n),$$

provided $n$ is large enough.

This last inequality in conjunction with (5.16) and Ottaviani's inequality clearly implies

$$(5.21)\qquad \sum_{n=1}^{\infty} P\left\{\max_{1\le k\le m_n} \sup_{g\in D_n} \left|\sum_{j=1}^{k} g(\bar{Z}'_j)\right| \ge (\sqrt{2} + 2\varepsilon)\gamma_{m_n}\right\} < \infty.$$

Combining (5.18), (5.19) and (5.21), after an appropriate choice of $\rho > 1$, we get

$$\limsup_{n\to\infty} \left\|\sum_{j=1}^{n} \bar{Z}'_j\right\| / \gamma_n \le (\sqrt{2} + 16\varepsilon) \quad \text{a.s.}$$

Recalling Lemma 8, we readily obtain the assertion of Theorem 3. Corollary 2 immediately follows from Theorem 3 via Lemma 6. □

**6. Proof of Theorem 4.** Let $\{\xi_k\}$ be a sequence of *independent* random variables satisfying for $k \ge 1$,

$$P\{\xi_k = \pm\exp(\exp(k^2))\} = c_\beta k^{2\beta}\exp(-2\exp(k^2))/2,$$

$$P\{\xi_k = \pm\exp(\exp(m^2))\} = c_\beta m^{-2}\exp(-2\exp(m^2))/2,$$

$$m = k+1, k+2, \dots,$$

$$P\{\xi_k = 0\} = 1 - c_\beta k^{2\beta}\exp(-2\exp(k^2)) - c_\beta \sum_{m=k+1}^{\infty} m^{-2}\exp(-2\exp(m^2)),$$

where $\beta := 2\alpha - 1$ and $c_\beta > 0$ has to be chosen in such a way that

$$\left(\max_k k^{2\beta}\exp(-2\exp(k^2))\right) + \sum_{m=2}^{\infty} m^{-2}\exp(-2\exp(m^2)) \le 1/c_\beta.$$

Let $\{e_k\}$ be a complete orthonormal system in $H$, and consider the random variable $X = \sum_{k=1}^{\infty}\xi_k e_k$. Let $\{X_n\}$ be independent copies of $X$.

We first note that

$$(6.1)\qquad E\|X\|^2/(LL\|X\|)^{\beta+1} < \infty.$$

To see (6.1), observe that

$$E\|X\|^2/(LL\|X\|)^{\beta+1} \le \sum_{k=1}^{\infty} E\xi_k^2/(LL|\xi_k|)^{\beta+1}$$
$$= c_\beta \sum_{k=1}^{\infty} \left\{ k^{-2} + \sum_{m=k+1}^{\infty} m^{-(4+2\beta)} \right\}$$
$$\le c_\beta \sum_{k=1}^{\infty} k^{-2} + c_\beta \sum_{m=2}^{\infty} m^{-(3+2\beta)} < \infty.$$

Next we claim that

$$\tilde{K}(n) \approx \sqrt{n}\,(LLn)^{\beta/2} \quad \text{as } n \to \infty. \tag{6.2}$$

That is, for some positive constants $\rho_1, \rho_2$,

$$\liminf_{n\to\infty} \tilde{K}(n)/\sqrt{n}\,(LLn)^{\beta/2} \ge \rho_1 \tag{6.3}$$

and

$$\limsup_{n\to\infty} \tilde{K}(n)/\sqrt{n}\,(LLn)^{\beta/2} \le \rho_2. \tag{6.4}$$

From (6.1) and (6.2) it is easy to see that

$$\sum_{n=1}^{\infty} P\{\|X\| \ge \tilde{K}(n/LLn)\,LLn\} < \infty. \tag{6.5}$$

Using Corollary 2 in conjunction with (6.2), we get (2.17). From (2.17) and Kolmogorov's 0-1 law we can infer that there exists a positive constant $K$ such that for the sums $S_n := \sum_{j=1}^{n} X_j$, $n \ge 1$,

$$\limsup_{n\to\infty} \|S_n\|/\sqrt{n}\,(LLn)^{\alpha} \le K \quad \text{a.s.} \tag{6.6}$$

This in turn implies, for any $z \in H$,

$$\limsup_{n\to\infty} |(S_n, z)|/\sqrt{n}\,(LLn)^{\alpha} \le K\|z\| \quad \text{a.s.} \tag{6.7}$$

If now $y = \sum_{j=1}^{\infty} y_j e_j$ is a fixed vector, we can find for $\varepsilon > 0$ an $N$ such that

$$y = \sum_{j=1}^{N} y_j e_j + \sum_{j=N+1}^{\infty} y_j e_j := y_N + z_N$$

and

$$\|z_N\| \le \varepsilon/K.$$

In view of (6.7) we have

$$\limsup_{n\to\infty} |(S_n, z_N)|/\sqrt{n}\,(LLn)^{\alpha} \le \varepsilon \quad \text{a.s.} \tag{6.8}$$

On the other hand, noticing that $E\xi_k^2 < \infty$, $1 \le k \le N$, we can infer from the

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$$\limsup_{n\to\infty} |(S_n, y_N)|/\sqrt{nLLn} < \infty \quad \text{a.s.} \tag{6.9}$$

Combining (6.8) and (6.9), we obtain

$$\limsup_{n\to\infty} |(S_n, y)|/\sqrt{n}(LLn)^{\alpha} \le \varepsilon \quad \text{a.s.,} \tag{6.10}$$

thereby establishing (2.18).

Thus, it remains to prove (6.3) and (6.4).

PROOF OF (6.3). Let $G_f(\cdot)$ be the $G$-function [see (1.11)] corresponding to the random variable $\xi = f(X)$, $f \in H^*$. Then using integration by parts, it is easy to see that

$$G_f(t) = t^2/\big(H_f(t) + M_f(t)\big), \qquad t > 0,$$

where

$$H_f(t) := Ef(X)^2 1\{|f(X)| \le t\},$$
$$M_f(t) := tE|f(X)|1\{|f(X)| > t\}, \qquad t > 0.$$

Further set for $k \ge 1$,

$$H_k(t) := E\xi_k^2 1\{|\xi_k| \le t\},$$
$$M_k(t) := tE|\xi_k|1\{|\xi_k| > t\},$$
$$G_k(t) = t^2/\big(H_k(t) + M_k(t)\big), \qquad t > 0.$$

By definition of the random variables $\{\xi_k\}$, we have

$$H_k(t) = \begin{cases} 0, & 0 \le t < \exp(\exp(k^2)), \\ c_\beta k^{2\beta}, & \exp(\exp(k^2)) \le t < \exp\big(\exp((k+1)^2)\big), \\ c_\beta\left(k^{2\beta} + \sum_{j=k+1}^{m} j^{-2}\right), & \exp(\exp(m^2)) \le t < \exp\big(\exp((m+1)^2)\big), \\ & m = k+1, k+2, \dots . \end{cases}$$

This implies in particular,

$$H_k(t) \ge 4^{-\beta} c_\beta (LLt)^\beta, \qquad \exp(\exp(k^2)) \le t \le \exp\big(\exp((k+1)^2)\big),\ k \ge 1,$$

from which we can infer

$$\inf_{\|f\|\le 1} G_f(t) \le \inf_k G_k(t) \le 4^\beta c_\beta^{-1} t^2/(LLt)^\beta, \qquad t \ge e^e.$$

Since we have, for any functional $f$ in $H^*$ with $\|f\| \le 1$,

$$G_f\big(\tilde{K}(n)\big) \ge G_f\big(K_f(n)\big) = n,$$

we obtain for large enough $n$,

$$n \le 4^{\beta} c_{\beta}^{-1} \tilde{K}(n)^2 / \big(LL\tilde{K}(n)\big)^{\beta}.$$

The last inequality immediately implies (6.3). □

PROOF OF (6.4). Let $y = \sum_{k=1}^{\infty} y_k e_k$ be a vector in $H$ such that $\|y\|^2 = \sum_{k=1}^{\infty} y_k^2 \le 1$.

Then we have

$$\begin{aligned} &E(X,y)^2 1\{|(X,y)| \le t\} \\ &\quad \le \sum_{k:\, y_k \ne 0} y_k^2 E\xi_k^2 1\{|\xi_k| \le t/|y_k|\} \\ &\quad \le \sum_{k \in K} c_{\beta} y_k^2 LL(t/|y_k|)^{\beta} + \sum_{l+1}^{\infty} t^2 P\{\xi_k \ne 0\} := T_1(t) + T_2(t), \end{aligned}$$

where $K := \{k \in \{1, \dots, l\}\colon y_k \ne 0\}$ and $l := [t] + 1$.

Using the fact that the function $(LLu)^{\beta}$, $u \ge u_{\beta}$, is concave, where $u_{\beta} \ge e^e$ is a constant, and recalling that $|y_k| \le 1$, we get for $t \ge u_{\beta}$,

$$T_1(t) \le c_{\beta} LL\left( \sum_{k=1}^{l} |y_k| t \right)^{\beta} \le c_{\beta} LL(t^2 + t)^{\beta} \le c'_{\beta} (LLt)^{\beta}.$$

From the definition of the random variables $\xi_k$ it follows that

$$\begin{aligned} P\{\xi_k \ne 0\} &= P\big\{|\xi_k| \ge \exp(\exp(k^2))\big\} \\ &\le c''_{\beta} k^{2\beta} \exp(-2\exp(k^2)), \qquad k \ge 1. \end{aligned}$$

It is easy now to see that

$$\begin{aligned} T_2(t) &\le c''_{\beta} t^2 \sum_{k=l+1}^{\infty} k^{2\beta} \exp(-2\exp(k^2)) \\ &\le c'''_{\beta} t^{2\beta+2} \exp(-2\exp(t^2)) \le \rho_3, \end{aligned}$$

where $\rho_3$ is a positive constant depending on $\beta$ only.

Combining the two above bounds for $T_1(t)$ and $T_2(t)$, we find that for some constant $\rho_4 > 0$,

$$\sup_{\|f\| \le 1} H_f(t) \le \rho_4 (LLt)^{\beta}. \tag{6.11}$$

Next note that

$$\begin{aligned} E|\xi_k| &= c_{\beta} k^{2\beta} \exp(-\exp(k^2)) + c_{\beta} \sum_{k+1}^{\infty} m^{-2} \exp(-\exp(m^2)) \\ &\le \rho_5 k^{2\beta} \exp(-\exp(k^2)), \end{aligned}$$

and also observe that for $m \ge k$,

$$E|\xi_k| 1\big\{|\xi_k| > \exp(\exp(m^2))\big\} \le \rho_6 m^{-2} \exp\Big(-\exp\big((m+1)^2\big)\Big).$$

This immediately implies

$$E|\xi_k|1\{|\xi_k| > t\} \le \rho_7(LLt)^{-1}t^{-1}, \qquad t \ge \exp(\exp(k^2)), \tag{6.12}$$

and also

$$E|\xi_k|1\{|\xi_k| > t\} \le \rho_8(LLt)^{\beta}t^{-1}, \qquad t < \exp(\exp(k^2)). \tag{6.13}$$

Let now $y = \sum_{k=1}^{\infty} y_k e_k$ be a vector in $H$ such that $\|y\| \le 1$. We then have

$$\begin{aligned} &E|(X,y)|1\{|(X,y)| > t\} \\ &\quad \le 3E|(X,x_t)|1\left\{|(X,x_t)| > \frac{t}{3}\right\} + 3E|\xi_{m_t}|1\left\{|\xi_{m_t}| > \frac{t}{3}\right\} \\ &\qquad + 3E|(X,z_t)|1\left\{|(X,z_t)| > \frac{t}{3}\right\} =: T_3(t) + T_4(t) + T_5(t), \end{aligned}$$

where $x_t := \sum_1^{m_t-1} y_j e_j$, $z_t = \sum_{m_t+1}^{\infty} y_j e_j$, and $m_t$ will be specified later.

We now derive upper bounds for the functions $T_i(t)$, $i = 3,4,5$. Setting $\eta_t := \max_{1 \le j < m_t} \xi_j^2$, it is easy to see that

$$\begin{aligned} T_3(t) &\le 3m_t^{1/2} E\eta_t^{1/2} 1\{\eta_t > t^2/9m_t\} \\ &\le 3m_t^{1/2} \sum_{j=1}^{m_t-1} E|\xi_j|1\{|\xi_j| > t/3m_t^{1/2}\}. \end{aligned}$$

Letting $m_t = \min\{m\colon 3\sqrt{m}\exp(\exp(m^2)) \ge t\} \le (LLt)^{1/2}$ and recalling (6.12), we obtain

$$T_3(t) \le \rho_9 m_t^2 t^{-1}(LLt)^{-1} \le \rho_9 t^{-1}. \tag{6.14}$$

Further note that on account of (6.12) and (6.13),

$$T_4(t) \le \rho_{10}(LLt)^{\beta}t^{-1}. \tag{6.15}$$

Finally observe that

$$\begin{aligned} T_5(t) &\le \sum_{m_t+1}^{\infty} E|\xi_j| \\ &\le \rho_5 \sum_{m_t+1}^{\infty} j^{2\beta}\exp(-\exp(j^2)) \\ &\le \rho_{11} m_t^{2\beta}\exp\big(-\exp\big((m_t+1)^2\big)\big) \\ &\le \rho_{11} m_t^{2\beta}\exp\big(-\exp(m_t^2)\big)^2 \\ &\le 9\rho_{11} m_t^{2\beta+1}t^{-2} \le 9\rho_{11}(LLt)^{\beta+1/2}t^{-2} \end{aligned}$$

by definition of $m_t$.

It is now clear that

$$\sup_{\|f\| \le 1} M_f(t) \le \rho_{12}(LLt)^{\beta}. \tag{6.16}$$

Combining (6.11) and (6.16), we find that, for some positive constant $\rho_{13}$,

$$\inf_{\|f\| \le 1} G_f(t) \ge \rho_{13}t^2/(LLt)^{\beta}. \tag{6.17}$$

Using a similar argument as in the proof of (6.3), we obtain (6.4) from (6.17). □

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## REFERENCES

BERGER, E. (1991). Majorization, exponential inequalities and almost sure behavior of vector-valued random variables. *Ann. Probab.* **19** 1206–1226.

EINMAHL, U. (1988). Strong approximations for partial sums of i.i.d. $B$-valued r.v.'s in the domain of attraction of a Gaussian law. *Probab. Theory Related Fields* **77** 65–85.

EINMAHL, U. (1989). Stability results and strong invariance principles for partial sums of Banach space valued random variables. *Ann. Probab.* **17** 333–352.

EINMAHL, U. (1991). On the almost sure behavior of sums of i.i.d. random variables in Hilbert space. *Ann. Probab.* **19** 1227–1263.

FELLER, W. (1968). An extension of the law of the iterated logarithm to variables without variance. *J. Math. Mech.* **18** 343–355.

HOFFMANN-JØRGENSEN, J. (1974). Sums of independent Banach space valued random variables. *Studia Math.* **52** 159–186.

KESTEN, H. (1972). Sums of independent random variables without moment conditions. *Ann. Math. Statist.* **43** 701–732.

KLASS, M. (1976). Toward a universal law of the iterated logarithm, I. *Z. Wahrsch. Verw. Gebiete* **36** 165–178.

KLASS, M. (1977). Toward a universal law of the iterated logarithm, II. *Z. Wahrsch. Verw. Gebiete* **39** 151–165.

KLASS, M. (1980). Precision bounds for the relative error in approximation of $E|S_n|$ and extensions. *Ann. Probab.* **8** 350–367.

KLASS, M. (1984). The finite mean LIL bounds are sharp. *Ann. Probab.* **12** 907–911.

KUELBS, J. (1985). The LIL when $X$ is in the domain of attraction of a Gaussian law. *Ann. Probab.* **13** 825–859.

KUELBS, J. and ZINN, J. (1983). Some results on LIL behavior. *Ann. Probab.* **11** 506–557.

LEDOUX, M. and TALAGRAND, M. (1988). Characterization of the law of the iterated logarithm in Banach space. *Ann. Probab.* **16** 1242–1264.

LEDOUX, M. and TALAGRAND, M. (1989). Comparison theorems, random geometry and some limit theorems for empirical processes. *Ann. Probab.* **17** 596–631.

LEDOUX, M. and TALAGRAND, M. (1990). Some applications of isoperimetric methods to strong limit theorems for sums of independent random variables. *Ann. Probab.* **18** 754–789.

LEDOUX, M. and TALAGRAND, M. (1991). *Probability in Banach Spaces*. Springer, Berlin.

PETROV, V. V. (1975). *Sums of Independent Random Variables*. Springer, Berlin.

PRUITT, W. (1981). General one-sided laws of the iterated logarithm. *Ann. Probab.* **9** 1–48.

TALAGRAND, M. (1989). An isoperimetric theorem on the cube and the Khintchine-Kahane inequalities. *Proc. Amer. Math. Soc.* **104** 905–909.

YURINSKII, V. V. (1976). Exponential inequalities for sums of random vectors. *J. Multivariate Anal.* **6** 473–499.

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