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TRIANGLE CONDITION FOR ORIENTED PERCOLATION IN HIGH DIMENSIONS

By Bao Gia Nguyen[1] and Wei-Shih Yang[2]

Illinois Institute of Technology and Temple University

In this paper, we apply the Brydges–Spencer lace expansion and the Hara–Slade analysis to obtain the triangle condition for the nearest-neighbor oriented bond percolation in high dimensions and for the spread-out oriented bond percolation in $Z^d \times Z$, $d \geq 5$. Furthermore, we also establish the infrared bound in the subcritical region and the mean-field behavior for these models.

1. Motivation and main results.

1.1. *Motivation*. Hara and Slade (1990a) show that the triangle condition, which was introduced by Aizenman and Newman (1984), holds for nearest-neighbor unoriented percolation in sufficiently high dimensions and for a class of spread-out unoriented percolation processes in dimensions $d > 6$. They note that their technique may not be extended in a straightforward manner to reproduce the same type of result for oriented percolation. One reason for the breakdown of their proof in applying the technique to oriented percolation is the symmetry of the unoriented models; that is, if x is connected to y, then y is also connected to x. The symmetry property often plays a crucial role in differentiating the two models. In this paper we show that the Hara–Slade technique can be modified to produce the triangle condition for nearest-neighbor oriented bond percolation in high dimensions and for a class of spread-out oriented percolation processes in $Z^d \times Z$ with $d > 4$ despite the lack of symmetry of the model. The spread-out oriented percolation is believed to be in the same universal class of nearest-neighbor oriented percolation. Our results for the spread-out oriented percolation in dimensions $d + 1 > 4 + 1$ strongly support a prediction by Obukhov (1980) that the critical dimension decreases from 6 for unoriented percolation down to 4 + 1 for oriented percolation.

Our modification of the Hara–Slade technique is based on the lace expansion. Hara and Slade's lace expansion originated from the Brydges–Spencer lace expansion that has been used successfully in studying self-avoiding random walks, as we may see from Brydges and Spencer (1985), Slade (1987,

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1989) and Yang and Klein (1988). The lace expansion also has been applied to study lattice trees and lattice animals [Hara and Slade (1990b, 1992c)]. Our observation is that oriented percolation is analogous to self-avoiding random walk. As we shall see later, it looks like a type of "self-avoiding bubble walk." This observation has helped us to apply directly the Brydges–Spencer lace expansion together with the techniques in Slade (1987) and Hara and Slade (1990a). It turns out that by exploiting the Markov structure of the oriented model, we have come up with a proof for establishing the triangle condition in high dimensions that is much simpler than the proof given by Hara and Slade (1990b) in the unoriented case. Furthermore, there seems to be no difficulty in applying our method to the continuous-time version of oriented percolation: the contact process. We note here that this method has also been exploited to produce a Gaussian limit for the connectivity function of oriented percolation for $p \le p_c$ in our other works [see Yang and Nguyen (1991) and Nguyen and Yang (1992)].

1.2. Main results. In this paper we first consider independent Bernoulli nearest-neighbor oriented bond percolation defined on the $(d+1)$-dimensional lattice $Z^d \times Z$. From each $(x,n) \in Z^d \times Z$ there is an oriented bond to $(x \pm e_{(\mu)}, n+1)$, where $e_{(\mu)}$, $\mu = 1,\dots,d$, are the canonical unit vectors in Z^d. If $b = \{(y,n),(x,n+1)\}$ is a bond, then we refer to (y,n) and $(x,n+1)$ as the bottom and the top of the bond b, respectively. Let each oriented bond be independently open with probability p and closed with probability $1-p$. Let P_p and E_p denote the probability and the expectation of the model accordingly. Declare that (x,n) can be reached from (y,m) if there is an open path connecting (x,n) from (y,m); that is, there is a sequence of open bonds $\{b_i;\ i = 1,\dots,n-m\}$ such that the bottom of $b_1 = (y,m)$, the top of $b_{n-m} = (x,n)$ and the top of b_i = the bottom of b_{i+1} for $i = 1,\dots,n-m-1$. Denote this event by $\{(y,m) \to (x,n)\}$. We also declare that (x,n) is connected to itself. Furthermore, we say that a (possibly random) set $B \in R^d \times R$ can be reached from a (possibly random) set $A \in R^d \times R$ if some site (x,n) in the closure $\overline{B}$ can be reached from some site (y,m) in the closure $\overline{A}$. This event is denoted by $\{A \to B\}$.

Let $C_0 = \{(x,n)\colon\ (0,0) \to (x,n)\}$ and set $|C_0|$ equal to the number of sites in C_0. Consider the percolation probability as a function of p: $\theta(p) = P_p(|C_0| = \infty)$. Let $p_c = \inf\{p\colon\ \theta(p) > 0\}$. It is well known that $0 < p_c < 1$, $p_c = 1/2d + o(d^{-1})$ [see Cox and Durrett (1983)], $p_c = \sup\{p\colon\ E_p(|C_0|) < \infty\}$ [see Aizenman and Barsky (1987) or Menshikov, Molchanov and Sidorenko (1986)] and $\theta(p_c) = 0$ [see Bezuidenhout and Grimmett (1990)]. We are interested in understanding the critical behavior of oriented percolation by studying the critical exponents γ, β, δ and Δ_{t+1} defined as follows:

$$E_p(|C_0|) \sim (p_c - p)^{-\gamma} \qquad \text{as } p \uparrow p_c;$$

$$P_p(|C_0| = \infty) \sim (p - p_c)^{\beta} \qquad \text{as } p \downarrow p_c;$$

$$\sum_{1 \le n \le \infty} P_{p_c}(|C_0| = n)[1 - e^{-nh}] \sim h^{1/\delta} \qquad \text{as } h \downarrow 0;$$

$$E_p(|C_0|^{t+1})/E_p(|C_0|^t) \sim (p_c - p)^{-\Delta_{t+1}} \qquad \text{as } p \uparrow p_c.$$

[Here $E_p(|C_0|) \sim (p_c - p)^{-\gamma}$ as $p \uparrow p_c$ means that there are positive constants K_1, K_2 such that $K_1(p_c - p)^{-\gamma} \le E_p(|C_0|) \le K_2(p_c - p)^{-\gamma}$, similarly for the others.] It is believed that these critical exponents should take the mean-field values

$$(1) \qquad \gamma = \beta = 1, \quad \delta = 2, \quad \text{and} \quad \Delta_{t+1} = 2, \quad \text{for } t = 1, 2, \dots,$$

in more than four dimensions [see Obukhov (1980)]. One reason for believing this is that in high dimensions (more than four) oriented percolation should resemble percolation on the Bethe lattice for which it can be shown that these critical exponents have the values in (1). Another, stronger, reason for this belief is that in more than four dimensions the following triangle condition is expected to hold:

$$(2) \qquad \lim_{R \to \infty} \sup\{\nabla_{p_c}(x, n) : |(x, n)| \ge R\} = 0,$$

where $|(x, n)| = (|x|^2 + |n|^2)^{1/2}$ with $|x| = (\sum_{\mu=1}^{d} |x_\mu|^2)^{1/2}$ and

$$(3) \qquad \begin{aligned} \nabla_p(x, n) = \sum_{(u_1, n_1)} \sum_{(u_2, n_2)} & P_p((0,0) \to (u_1, n_1)) P_p((u_1, n_1) \to (u_2, n_2)) \\ & \times P_p((x, n) \to (u_2, n_2)). \end{aligned}$$

Condition (2) is a modified version of the following triangle condition:

$$(4) \qquad \sup_{(x, n) \in Z^d \times Z} \nabla_p(x, n) < \infty,$$

which is analogous to the one that was introduced by Aizenman and Newman (1984) in the context of unoriented percolation. The modification was introduced by Barsky and Aizenman (1991) in order to deal with the oriented (as well as unoriented) situation. For unoriented percolation, using the Riemann–Lebesgue lemma, one can show that the Aizenman–Newman triangle condition implies the Barsky–Aizenman triangle condition. For oriented percolation in dimensions $d > d_c = 4$, we can obtain the Barsky–Aizenman triangle condition from the infrared bound which is stated as follows: For some positive constants c_1 and c_2, uniformly with respect to $p \in (0, p_c)$,

$$(5) \qquad |\hat{\Psi}(k, t)| \le \frac{1}{c_1 |k|^2 + c_2 |t|},$$

where $\hat{\Psi}(k, t)$ is the Fourier transform of

$$\Psi(x, n) = \begin{cases} P_p(x, n), & \text{if } n \ge 1, \\ 0, & \text{if } n \le 0. \end{cases}$$

In order to be more precise on the Fourier transform and its related norms, we introduce the following notation.

NOTATION. The Fourier transform of a function f defined on $Z^d \times Z$ is

$$\hat{f}(k, t) = \sum_{x \in Z^d} \sum_{n \in Z} f(x, n) e^{ikx} e^{itn} \quad \text{with } (k, t) \in [-\pi, \pi]^d \times [-\pi, \pi].$$

The integrals $\int \hat{f}\,dk$ and $\int\int \hat{f}\,dk\,dt$ mean

$$\int_{k\in[-\pi,\pi]^d} \hat{f}\,dk \quad \text{and} \quad \int\int_{(k,t)\in[-\pi,\pi]^d\times[-\pi,\pi]} \hat{f}\,dk\,dt,$$

respectively. We also use the following norms:

$$\|f(\cdot,n)\|_q = \Big(\sum_{x\in Z^d} |f(x,n)|^q\Big)^{1/q};$$

$$|||\, f \,|||_q = \Big(\sum_{x\in Z^d}\sum_{n\in Z} |f(x,n)|^q\Big)^{1/q};$$

$$\|\hat{f}(\cdot,t)\|_q = (2\pi)^{-d}\Big(\int |\hat{f}(k,t)|^q\,dk\Big)^{1/q};$$

$$|||\, \hat{f} \,|||_q = (2\pi)^{-d-1}\Big(\int\int |\hat{f}(k,t)|^q\,dk\,dt\Big)^{1/q}.$$

To see how the infrared bound (5) implies the Barsky–Aizenman triangle condition (2), we argue as follows. Since $d > d_c = 4$, the RHS of (5) is $L_{3(1+\varepsilon)}$-integrable with respect to the Lebesgue measure $dk\,dt$ over $[-\pi,\pi]^d \times [-\pi,\pi]$ for some $\varepsilon > 0$. Thus $|||\,\hat{\Psi}^3\,|||_{1+\varepsilon}$ is uniformly bounded on $p \in (0, p_c)$. By the Hausdorff–Young inequality, $|||\,\Psi * \Psi * \Psi\|_{\varepsilon'}$ is uniformly bounded on $p \in (0, p_c)$, where $(1+\varepsilon)^{-1} + (\varepsilon')^{-1} = 1$. Thus $|||\,\nabla_{p_c}|||_{\varepsilon'} < \infty$ because $|||\,\nabla_p\|_{\varepsilon'}$ is left continuous as a function of p; hence, (2) follows.

The triangle condition (2) plays a very important role in showing the mean-field behavior of the (unoriented as well as oriented) percolation process, as we may see from Aizenman and Newman (1984) for $\gamma = 1$, Aizenman and Barsky (1987) for $\{\beta = 1, \delta = 2\}$ and Nguyen (1987) or Barsky and Aizenman (1991) for $\{\Delta_t = 2; t = 2, 3, \ldots\}$.

In this paper we establish the following result:

THEOREM 1. *For nearest-neighbor independent Bernoulli oriented bond percolation on* $Z^d \times Z$, *there exists a sufficiently large number* d_0 *such that for* $d \ge d_0$ *the triangle conditions* (2) *and* (4) *and the infrared bound* (5) *are satisfied. Furthermore, the critical exponents take mean-field values as in* (1).

Our method of calculating d_0 is tedious; hence we have made no attempt to compute its best possible value. Even though it seems that we have an efficient expansion for oriented percolation, a new idea would be needed to obtain the triangle condition right down to the predicted critical dimension of four. The problem of achieving the mean-field behavior right down to the critical dimension $d_c = 6$ remains unsolved for unoriented percolation. Hara and Slade (1992a, b) show that for self-avoiding random walk the mean-field behavior may be obtained right down to the critical dimension four. It would be an important problem to combine their work and our method to obtain the mean-field behavior for oriented percolation down to the critical dimension.

In addition to the nearest-neighbor situation, we also apply our technique to obtain the triangle conditions (2) and (4) and the infrared bound (5) for the spread-out oriented percolation process in $Z^d \times Z$, $d > 4$. In this model each oriented bond $b = \{(y, n), (x, n+1)\}$ is independently open with probability $p_{0, x-y}$ and closed with probability $(1 - p_{0, x-y})$, where

$$p_{0,x} \equiv p_{0x} = pg(x/L)L^{-d}$$

for some nonnegative function g defined on R^d such that the following hold:

1. $\int_{R^d} g(x)\,dx = 1$;
2. $g(x) \le \text{const.} \exp(-\varepsilon|x|)$ for some $\varepsilon > 0$;
3. g is invariant under rotations by $\pi/2$ and reflections in the coordinate hyperplanes;
4. $\partial^I g(x)$ is piecewise continuous and $\int_{R^d} |\partial^I g(x)|\,dx < \infty$;
5. $g(x)$ and $\partial^I g(x)$ are continuous at 0 and $g(0) > 0$.

Here ∂^I means $\prod_{\mu \in I} \partial_\mu$, for $I \in \{1, 2, \dots, d\}$, and is interpreted as a distribution. This sort of spread-out distribution was introduced by Hara and Slade (1990a) to study unoriented percolation in dimensions $d > 6$. It is commonly believed that the spread-out percolation model belongs to the same universal class as the nearest-neighbor percolation model.

The notions of cluster, connectedness, and so forth, of the spread-out model can be defined similarly to the nearest-neighbor case. The following theorem is an analogue of Theorem 1.

THEOREM 2. *For the spread-out bond percolation model defined on* $Z^d \times Z$ *with* $d > 4$, *if* L_0 *is sufficiently large, then whenever* $L > L_0$ *the triangle conditions* (2) *and* (4) *and the infrared bound* (5) *hold*; *moreover*, *the critical exponents take mean-field values as in* (1).

The remainder of our paper consists of five sections. In Section 2 we discuss the lace expansion method for decomposing $\hat{\Psi}(k, t)$ into a sum of the lace parts. In Section 3 we describe the Feynman-type diagrams that play an important role in the proofs. In Section 4 we show that the lace parts can be estimated in terms of the Feynman diagrams. The proofs of Theorems 1 and 2 are given in Sections 5 and 6, respectively.

2. Lace expansion. Given a configuration η, we say that a bond b is pivotal for the connection $(0,0) \to (x, n)$, $n \ge 1$, if every open path in η reaching (x, n) from $(0,0)$ must use b. Observe that if there are no pivotal bonds for $(0,0) \to (x,n)$, then there are at least two disjoint open paths reaching (x,n) from $(0,0)$. In this case we say that $(0,0)$ is doubly connected to (x,n), denoted by $(0,0) \Rightarrow (x,n)$. On the other hand, if $(0,0)$ is not doubly connected to (x,n), then there are exactly m, $m \ge 1$, pivotal bonds $b_1, b_2, \dots, b_m$ for $(0,0) \to (x,n)$. We may arrange these pivotal bonds in natural order in the direction from $(0,0)$ to (x,n) and write $b_1 < b_2 < \cdots < b_m$. Thus by varying η in $\{(0,0) \to (x,n)\}$ we can decompose this event into the union of $\{(0,0) \Rightarrow (x,n)\}$ and

$$\bigcup_{m=1}^{\infty} \bigcup \{b_1, b_2, \dots, b_m \text{ are exactly } m \text{ pivotal bonds for } (0,0) \to (x,n)\},$$

where the second union is over the collection of m bonds $b_1 < b_2 < \cdots < b_m$ in the lattice $Z^d \times Z$. Note that this decomposition is disjoint. Our next step is to write

$$I\{b_1, b_2, \dots, b_m \text{ are exactly } m \text{ pivotal bonds for } (0,0) \to (x,n)\},$$

where I denotes the indicator function, as a product of $2m+1$ independent random variables and a Gibbs factor describing the interaction of the oriented percolation process. To do so we set

$$B_0 = \{(u,t) \in Z^d \times Z: (0,0) \to (u,t) \to \text{bottom of } b_1\},$$

$$B_i = \{(u,t) \in Z^d \times Z: \text{top of } b_i \to (u,t) \to \text{bottom of } b_{i+1}\}, \quad \text{for } i = 1, 2, \dots, m-1,$$

$$B_m = \{(u,t) \in Z^d \times Z: \text{top of } b_m \to (u,t) \to (x,n)\}.$$

We declare that, for $i = 1, \dots, m-1$,

$$B_i \text{ bubbles iff the top of } b_i \Rightarrow \text{the bottom of } b_{i+1}$$

and

$$B_0 \text{ bubbles iff } (0,0) \Rightarrow \text{the bottom of } b_1,$$

$$B_m \text{ bubbles iff the top of } b_m \Rightarrow (x,m).$$

We then write

$$I\{(0,0) \to (x,n) \text{ with exactly } m \text{ pivotal bonds } b_1 < b_2 < \cdots < b_m\}$$
$$= \prod_{i=1}^{m} I(b_i \text{ open}) \prod_{i=0}^{m} I(B_i \text{ bubbles})(\text{Gibbs factor})$$

with the Gibbs factor defined by $\prod_{0 \le i < j \le m}(1 + U_{ij})$, where

$$U_{ij} = -I(B_i \to B_j \text{ without using } b_{i+1}, \dots, b_j),$$

since all the reference bonds $b_1, \dots, b_m$ are not pivotal unless the Gibbs factor $\prod_{0 \le i < j \le m}(1 + U_{ij}) = 1$.

Expand the Gibbs product as $\sum_G \prod_{ij \in G} U_{ij}$, where G runs over all subsets of $\mathscr{B}_m = \{ij: 0 \le i < j \le m\}$. Each G can be represented by a graph on the set of vertices $\{0, 1, \dots, m\}$ with bonds $ij \in G$. A graph G is called connected if

$\bigcup_{ij \in G}[i, j] = [0, m]$. Define, for $n \geq 1$,

$$\Psi(x, n; b_1, \dots, b_m; G) = E_p\left(\prod_{i=1}^{m} I(b_i \text{ open}) \prod_{i=0}^{m} I(B_i \text{ bubbles}) \prod_{ij \in G} U_{ij} \right),$$

$$\Psi(x, n) = \begin{cases} P_p((0,0) \Rightarrow (x, n)) \\ \quad + \sum_{m=1}^{\infty} \sum_{b_1 < \cdots < b_m} \sum_{G \in \mathscr{B}_m} \Psi(x, n; b_1 \dots b_m; G), & \text{if } n \leq 1, \\ 0, & \text{if } n \leq 0, \end{cases}$$

$$\Psi_c(x, n) = \begin{cases} P_p((0,0) \Rightarrow (x, n)) \\ \quad + \sum_{m=1}^{\infty} \sum_{b_1 < \cdots < b_m} \sum_{G \text{ connected}} \Psi(x, n; b_1 \dots b_m; G), & \text{if } n \geq 1, \\ 0, & \text{if } n \leq 0. \end{cases}$$

Note that the function Ψ is still the same as the one defined earlier. Using the same argument as in Brydges and Spencer (1985) and exploiting the Markov property of the oriented percolation model, we can show the following.

THEOREM 3. *For $p < p_c$, we have the following renewal equations*:

$$(6) \qquad 1 + \hat{\Psi}(k, t) = \frac{1 + \hat{\Psi}_c(k, t)}{1 - 2dpe^{it}\hat{D}(k)\big(1 + \hat{\Psi}_c(k, t)\big)},$$

where $\hat{D}(k) = (1/d)\sum_{i=1}^{d} \cos k_i$, for the nearest-neighbor model, and

$$(7) \qquad 1 + \hat{\Psi}(k, t) = \frac{1 + \hat{\Psi}_c(k, t)}{1 - pp_L^{-1}e^{it}\hat{D}_L(k)\big(1 + \hat{\Psi}_c(k, t)\big)},$$

where $p_L^{-1} = \sum_x g(x/L)L^{-d}$ and $\hat{D}_L(k) = p_L \sum_x g(x/L)L^{-d}e^{ikx}$, for the spread-out model.

Note that for $p < p_c$ the function $\sum_x \Psi(x, n)$ decays exponentially as $n \to \infty$ with a positive rate which is known as the inverse correlation length [see Aizenman and Newman (1984)]. Thus $\hat{\Psi}(k, t)$ is well defined for $(k, t) \in [-\pi, \pi]^d \times [-\pi, \pi]$ if $p < p_c$. Notice also that, even though the connected part $\Psi_c(x, n)$ may be well defined for unoriented percolation, the renewal equations in Theorem 3 may not hold in this case due to the lack of the Markov property (this is the reason the Hara–Slade expansion for unoriented percolation is more complicated to deal with, in our opinion).

Further, we can decompose the connected part Ψ_c into the lace parts as follows. From any connected graph G defined on $\{0, 1, \dots, m\}$ we set inductively $s_1 = 0$, $t_1 = \max\{t\colon 0t \in G\}$, $t_{i+1} = \max\{t\colon st \in G \text{ for some } s \leq t_i\}$, $s_{i+1} = \min\{s\colon st_{i+1} \in G\}$. This defines a lace $\mathscr{L}(G)$ which is a graph consisting of all the bonds $\{L_i = s_i t_i\}$ obtained from G in this fashion. The number of bonds in $\mathscr{L}(G)$ is called the order of the lace. Given a lace $L = \{L_1, \dots, L_l\}$, we denote by $\mathscr{C}(L)$ the set of bonds $st \notin L$ that are compatible with L in such a way that $\mathscr{L}(L \cup \{st\}) = L$, and we let $A(x, n; b_1, \dots, b_m; L)$ be an event such that its indicator function is equal to

$$\prod_{i=1}^{m} I(b_i \text{ open}) \prod_{i=0}^{m} I(B_i \text{ bubbles}) \prod_{ij \in L} |U_{ij}| \prod_{ij \in C(L)} (1 + U_{ij}).$$

Define

$$\Psi_0(x,n) = P_p((0,0) \Rightarrow (x,n))$$

$$\Psi_l(x,n) = \sum_{m=1}^{\infty} \sum_{b_1 < \cdots < b_m} \sum_{L} E_p(A(x,n;b_1,\dots,b_m;L)),$$

where L runs over laces of order l defined on $\{0,1,\dots,m\}$. Then, following Brydges and Spencer (1985), we can obtain, for $p < p_c$,

$$\hat{\Psi}_c(k,t) = \sum_{l=0}^{\infty} (-1)^l \hat{\Psi}_l(k,t). \tag{8}$$

3. Description of Feynman-type diagrams. In this section we describe the Feynman-type diagrams that will be used later in the paper. We use the same Feynman diagram notation as in Hara and Slade (1990a) with modification for orientation since our model is oriented:

(y,m) (x,n) represents $P_p((y,m) \to (x,n))$

(y,m) b represents $P_p((y,m) \to \text{bottom of } b)P_p(b \text{ open})$.

The following are examples of Feynman diagrams that will play an important role in the proof of our main results.

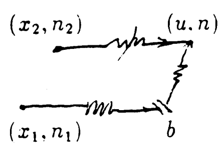
EXAMPLE 1. Given a bond b and sites (y,m), (x,n) and (u,n), define the triangle $T[((x_1,n_1),(x_2,n_2)),(b,(u,n))]$ as

$$P_p((x_1,n_1) \to \text{bottom of } b)P_p(b \text{ open})P_p(\text{top of } b \to (u,n))$$
$$\times \Psi(u - x_2, n - n_2).$$

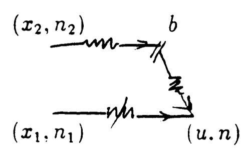
The diagram $T[((x_1,n_1),(x_2,n_2)),(b,(u,n))]$ with $(x_1,n_1) = (x_2,n_2) = (0,0)$ is denoted by $T[(0,0),(b,(u,n))]$. Furthermore, whenever site (x_i,n_i) is the top of b_i we write b_i instead of "top of b_i." Also, we set

$$T[((x_1,n_1),(x_2,n_2)),((u,n),b)] = T[((x_2,n_2),(x_1,n_1)),(b,(u,n))].$$

Their defined diagrams are as follows [note that the slashed oriented wavy lines connecting (x_i,n_i) and (u,n) represent $\Psi(u - x_i, n - n_i)$]:



$T[((x_1,n_1),(x_2,n_2)),(b,(u,n))]$



$T[((x_1,n_1),(x_2,n_2)),((u,n),b)]$

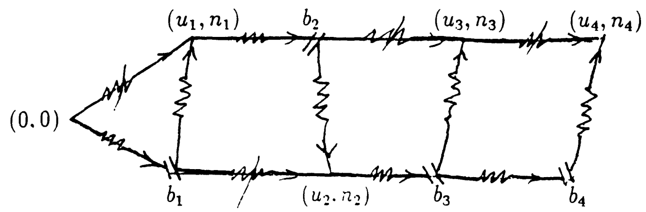
EXAMPLE 2. Given l pairs of sites and bonds $\{(b_i,(u_i,n_i)),\ i=1,2,\dots,l\}$, set $\sigma_i(b_i,(u_i,n_i)) = (b_i,(u_i,n_i))$ or $((u_i,n_i),b_i)$ depending on whether $\sigma_i = id$ or a permutation of sites and bonds. Let $\sigma = (\sigma_1,\dots,\sigma_l)$. The diagram

$$D_l\big[(b_i,(u_i,n_i)),\ i=1,2,\dots,l;\sigma\big]$$

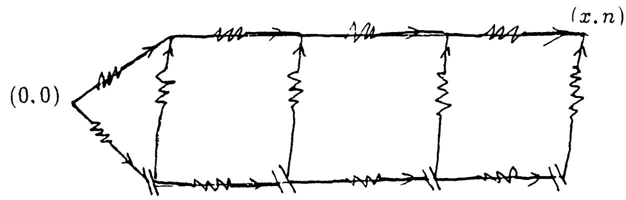
is defined by

$$T\big[(0,0),\sigma_1(b_1,(u_1,n_1))\big]\prod_{i=2,\dots,l} T\big[\sigma_{i-1}(b_{i-1},(u_{i-1},n_{i-1})),\sigma_i(b_i,(u_i,n_i))\big].$$

For $l=4$ and $\sigma_1,\sigma_3,\sigma_4 = id$ and σ_2 a permutation, the diagram D_l is represented by



EXAMPLE 3. The diagram $D_l(x,n)$ is defined as the sum of $D_l[(b_i,(u_i,n_i)),\ i=1,2,\dots,l;\sigma]$ over $\{(b_i,(u_i,n_i)),\ i=1,\dots,l\}$ and σ such that $\sigma_l = id$ and $(u_l,n_l) = (x,n)$. This diagram is represented by



Here we use the convention that labelled vertices and bonds are fixed and unlabelled vertices and bonds are summed over the lattice $Z^d \times Z$. Note that $(l-1)$ vertical wavy lines that are not oriented represent the sum over σ such that $\sigma_l = id$.

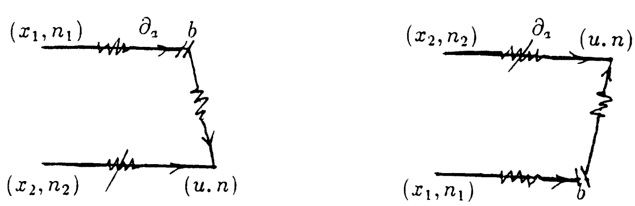
EXAMPLE 4. For $a = (a(1), \dots, a(d), a(d+1))$ we let $\partial_n = \prod_{\mu=1}^{d} \partial_\mu^{a(\mu)} \partial_t^{a(d+1)}$, where $\partial_\mu^{a(\mu)}$ is the $a(\mu)$th partial derivative with respect to the μ-coordinate of k and $\partial_t^{a(d+1)}$ is the $a(d+1)$th partial derivative with respect to the last coordinate t. We set

$$\begin{aligned} T_a&[((x_1,n_1),(x_2,n_2)),(b,(u,n))] \\ &= P_p((x_1,n_1) \to \text{bottom of } b) P_p(b \text{ open}) \\ &\quad \times \prod_{\mu=1}^{d+1} \left|(\text{top of } b)_{(\mu)} - (x_1,n_1)_{(\mu)}\right|^{a(\mu)} \\ &\quad \times P_p(\text{top of } b \to (u,n)) \Psi(u - x_2, n - n_2), \end{aligned}$$

where $(\text{top of } b)_{(\mu)}$ and $(x_1,n_1)_{(\mu)}$ are the μ-coordinates of the top of b and (x_1,n_1), repsectively. Similarly, we set

$$\begin{aligned} T^a&[((x_1,n_1),(x_2,n_2)),(b,(u,n))] \\ &= P_p((x_1,n_1) \to \text{bottom of } b) P_p(b \text{ open}) \prod_{\mu=1}^{d+1} \left|((u,n) - (x_2; n_2))_{(\mu)}\right|^{a(\mu)} \\ &\quad \times P_p(\text{top of } b \to (u,n)) \Psi(u - x_2, n - n_2). \end{aligned}$$

The corresponding diagrams are as follows:



EXAMPLE 5. We also define the diagram $\delta_{ia} D_l[(b_j,(u_j,n_j)),\ j = 1,2,\dots,l;\sigma]$ as the diagram $D_l[(b_j,(u_j,n_j)),\ j = 1,2,\dots,l;\sigma]$ in which the triangle

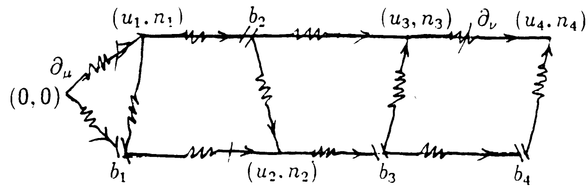
$$T[\sigma_{i-1}(b_{i-1},(u_{i-1},n_{i-1})),\sigma_i(b_i,(u_i,n_i))]$$

is replaced by

$$T^a[\sigma_{i-1}(b_{i-1},(u_{i-i},n_{i-1})),\sigma_i(b_i,(u_i,n_i))].$$

[If $i = 1$, we set $\sigma_{i-1}(b_{i-1},(u_{i-1},n_{i-1})) = (0,0)$.] This naturally induces the diagram $\delta_{ia} D_l(x,n)$ by summing $\delta_{ia} D_l[(b_j,(u_j,n_j)),\ j = 1,2,\dots,l;\sigma]$ over $\{(b_j,(u_j,n_j)),\ j = 1,2,\dots,l\}$ and σ in the same way as before. This process may be iterated to produce the diagram $\delta_{i\mu}\delta_{i'\nu} D_l[(b_j,(u_j,n_j)),\ j = 1,2,\dots,l;\sigma]$ for $i \neq i'$. If $i = i'$, then $\delta_{ia}\delta_{ia'}$ is interpreted as $\delta_{i(a+a')}$. For $l = 4$, σ_2 a permutation and $\sigma_1, \sigma_3, \sigma_4 = id$, the diagram $\delta_{1\mu}\delta_{4\nu} D_l[(b_j,(u_j,n_j)),$

$j = 1, 2, \ldots, l; \sigma]$ is represented by the following:



We make the following remarks concerning these Feynman diagrams.

REMARK 1. Let

$$\overline{T}_0 = \sup_{(x_1, n_1), (x_2, n_2)} \sum_{b, (u, n)} \{T[((x_1, n_1), (x_2, n_2)), (b, (u, n))] + T[((x_2, n_2), (x_1, n_1)), (b, (u, n))]\}.$$

Then we observe that $\sum_{(x, n)} D_l(x, n) \le \overline{T}_0^l$.

REMARK 2. By writing $(x, n)_{(\mu)}$ as a telescoping sum $\sum_{i=1}^l ((x_i, n_i) - (x_{i-1}, n_{i-1}))_{(\mu)}$, where $(x_0, n_0) = (0, 0)$, $(x_l, n_l) = (x, n)$ and (x_i, n_i) are in the upper corners of the diagram $D_l(x, n)$, we can easily show that

$$D_l(x, n) |(x, n)_{(\mu)}| \le \sum_{i=1}^{l} \delta_{i\mu}[D_l(x, n)].$$

REMARK 3. By a similar argument we can obtain

$$D_l(x, n) |(x, n)_{(\mu)}| \, |(x, n)_{(\nu)}| \le \sum_{i, j=1}^{l} \delta_{i\mu} \delta_{j\nu}[D_l(x, n)].$$

REMARK 4. For $a = (a(1), \ldots, a(d), a(d+1))$, we let

$$\overline{T}_a = \sup_{(x_1, n_1), (x_2, n_2)} \sum_{b(u, n)} \{T_a[((x_1, n_1), (x_2, n_2)), (b, (u, n))] + T^a[((x_1, n_1), (x_2, n_2)), (b, (u, n))]\}.$$

Then, using proof by induction on $l \ge 2$, we can improve the preceding inequalities as follows:

$$\sum_{(x, n)} \delta_{ia}[D_l(x, n)] \le \overline{T}_a \overline{T}_0^{l-1},$$

$$\sum_{(x, n)} \delta_{ia} \delta_{ja'}[D_l(x, n)] \le \begin{cases} \overline{T}_a \overline{T}_{a'} \overline{T}_0^{l-2}, & \text{if } i \ne j, \\ \overline{T}_{aa'} \overline{T}_0^{l-1}, & \text{if } i = j. \end{cases}$$

4. Estimates for $\Psi_l(x,n)$. Having introduced the defined diagrams, we now want to prove the following theorems concerning the estimates for the lace parts $\Psi_l(x,n)$.

THEOREM 4. $\Psi_0(x,n) \le \Psi^2(x,n) \le \Sigma_b T[(0,0),(b,(x,n))]$.

PROOF. By the well-known van den Berg–Kesten inequality [van den Berg and Kesten (1985)], we have

$$\begin{aligned}\Psi_0(x,n) &\le \Psi^2(x,n)\\ &\le \Psi(x,n) \sum_{b:\text{ top of } b=(x,n)} P_p\{(0,0)\to \text{bottom of } b\} P_p(b \text{ open})\\ &\le \sum_b T[(0,0),(b,(x,n))].\end{aligned}$$ □

THEOREM 5. $\Psi_1(x,n) \le \Sigma_b T[(0,0),(b,(x,n))]$.

PROOF. Consider the lace event $A(x,n;b_1,\dots,b_m;L)$ with $L = 0m$. Observe from the definition of the lace event that

$$(0,0) \to \text{bottom of } b_m \text{ with exactly } m \text{ pivotal bonds } b_1 < \cdots < b_{m-1}.$$

Moreover, we can find, in addition to the open bond b_m, three disjoint paths in $\{(0,0)\to(x,n)\}$, $\{(0,0)\to \text{bottom of } b_m \text{ via } b_1,\dots,b_{m-1}\}$ and $\{\text{top of } b_m \to (x,n)\}$. Thus, by the van den Berg–Kesten inequality [van den Berg and Kesten (1985)],

$$\begin{aligned}\Psi_1(x,n) &= \sum_{m=1}^{\infty} \sum_{b_1<\cdots<b_m} E_p(A(x,n;b_1,\dots,b_m;L=0m))\\ &\le \sum_{b_m} \Psi(x,n) P_p(b_m \text{ open})\ P_p\ (\text{top of } b_m \to (x,n))\\ &\quad\times \sum_{b_1,\dots,b_{m-1}} E_p((0,0)\to \text{bottom of } b_m \text{ with}\\ &\qquad\qquad \text{exactly } m \text{ pivotal bonds } b_1<\cdots<b_{m-1})\\ &= \sum_b T[(0,0),(b,(x,n))].\end{aligned}$$

THEOREM 6. *For $l \ge 2$ we have $\Psi_l(x,n) \le D_l(x,n)$.*

PROOF. Consider the lace event $A(x,n;b_1,\dots,b_m;L)$ with $L = \{L_1,\dots,L_l\}$, where $L_i = s_i t_i$, $s_1 = 0$, $t_l = m$. We say that the event $A(x,n;b_1,\dots,b_m;L)$ is compatible with $\{\bar b_i,\ i=1,\dots,l\}$ if $\bar b_{i-1} = b_{s_i}$, $i = 2,\dots,l$, and $\bar b_l = b_m$. It is clear from the definition of lace that the family of lace events $A(x,n;b_1,\dots,b_m;L)$ indexed by L for fixed $\{(x,n);b_1,\dots,b_m\}$ are mutually disjoint. Thus the family of lace events $A(x,n;b_1,\dots,b_m;L)$, indexed by $b_1,\dots,b_m;L$, that are compatible with $\{\bar b_i,\ i=1,\dots,l\}$ is a mutually disjoint family of events since the bond b is pivotal to $\{\text{top of } \bar b_i \to \text{bottom of } \bar b_{i+1}\}$ if and only if $b \in \{b_1,\dots,b_m\}$ and b is between these two bonds (same reason for $\{(0,0)\to \text{bottom of } \bar b_1\}$). To prove the theorem, it is enough to show that each configuration $\eta \in A(x, n; b_1, \dots, b_m; L)$ that is compatible with $\{\bar{b}_i, i = 1, \dots, l\}$ belongs to the event which corresponds to the Feynman diagram $D_l(x, n; \sigma)$ with fixed $\{\bar{b}_i, i = 1, \dots, l\}$ and σ. Next choose arbitrarily from each event $U_{s_i t_i}$, $i = 1, \dots, l$, an open path $\mathscr{P}_i$ that determines this event. To be more precise, we want $\mathscr{P}_i$ to be a path connecting the top of B_{t_i} (which is the bottom of b_{t_1+1}) from the bottom of B_{s_i} (which is the top of b_{s_1}) without using any bond $\bar{b}_j$ between B_{s_i} and B_{t_i}. There are two cases:

Case 1 ($\mathscr{P}_i$ hits $\mathscr{P}_{i+1}$). Let (u_i, n_i) be the first site where $\mathscr{P}_i$ hits $\mathscr{P}_{i+1}$ (e.g., see bubble B_2 in Figure 1a, bubble B_1 in Figure 1b).

Case 2 ($\mathscr{P}_i$ does not hit $\mathscr{P}_{i+1}$). Let (u'_i, n'_i) be the first site where $\mathscr{P}_i$ hits B_{t_i}. There are two subcases:

(a) If we can find a path in $b_{s_{i+1}}$ = top of $\bar{b}_i \to (u'_i, n'_i)$ that is disjoint from $\mathscr{P}_i, \mathscr{P}_{i+1}$, then we let $(u_i, n_i) = (u'_i, n'_i)$ (e.g., bubble B_4 in Figure 1a).

(b) Otherwise, we let (u_i, n_i) = the top of B_{t_i} (e.g., bubble B_2 in Figure 1b).

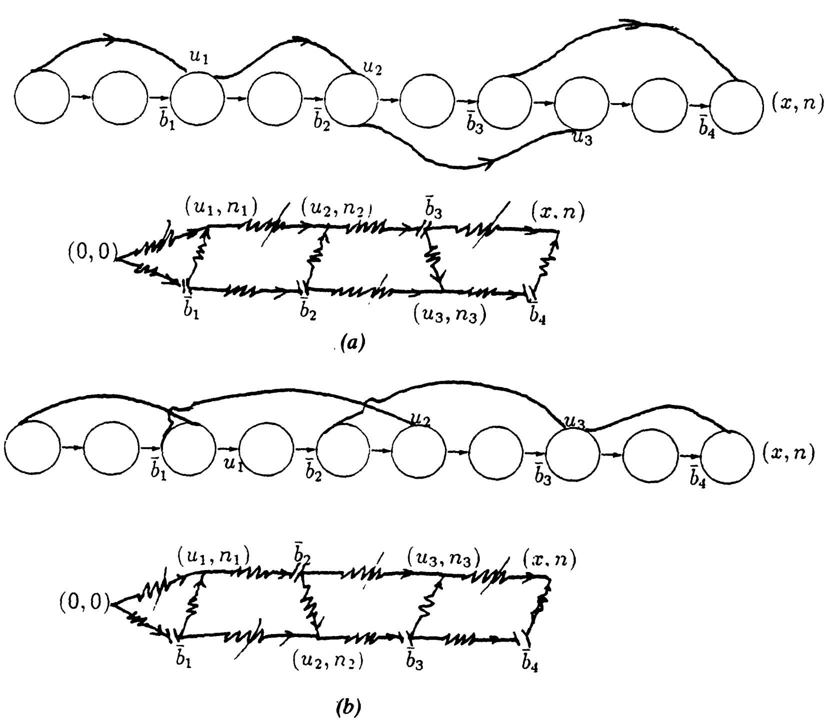


Fig. 1. *Two lace events of order 4 and their corresponding Feynman diagrams.*

Let us assume the claim that one can find mutually disjoint paths such as

$$\left[(0,0) \to (u_1,n_1)\right],\ \left[(0,0) \to \text{bottom of } \bar{b}_1\right],\ \left[\text{top of } \bar{b}_1 \to (u_1,n_1)\right]$$

and either

$$\begin{aligned}\text{(T.1)} \quad & \left[\text{top of } \bar{b}_i \to \text{bottom of } \bar{b}_{i+1}\right],\ \left[(u_i,u_i) \to (u_{i+1},n_{i+1})\right], \\ & \left[\text{top of } \bar{b}_{i+1} \to (u_{i+1},n_{i+1})\right]\end{aligned}$$

or

$$\begin{aligned}\text{(T.2)} \quad & \left[\text{top of } \bar{b}_i \to (u_{i+1},n_{i+1})\right],\ \left[(u_i,u_i) \to \text{bottom of } \bar{b}_{i+1}\right], \\ & \left[\text{top of } \bar{b}_{i+1} \to (u_{i+1},n_{i+1})\right]\end{aligned}$$

for $i = 1,\dots,l-1$ and $(u_l,n_l) = (x,n)$; see Figure 1a, b. Note that all the connections corresponding to the paths $\mathscr{P}_i$ are of length at least 1. Thus (T.1) and (T.2) correspond respectively to

$$T\left[\left(\bar{b}_i,(u_i,n_i)\right),\left(\bar{b}_{i+1},(u_{i+1},n_{i+1})\right)\right]$$

and

$$T\left[\left((u_i,n_i),\bar{b}_i\right),\left(\bar{b}_{i+1},(u_{i+1},n_{i+1})\right)\right].$$

Let $\sigma_i(\bar{b}_i,(u_i,n_i)) = (\bar{b}_i,(u_i,n_i))$ if there are an even number of (T.2) types in the last $(l-i)$ triangles, and let $\sigma_i(\bar{b}_i,(u_i,n_i)) = ((u_i,n_i),\bar{b}_i)$ otherwise. By the van den Berg–Kesten inequality [van den Berg and Kesten (1985)], the sum over the $A(x,n;b_1,\dots,b_m;L)$ that are compatible with $\{\bar{b}_i,(u_i,n_i),\ i = 1,\dots,l\}$ and σ is less than the diagram $D_l(x,n)$ with these fixed bonds, sites and permutations σ. To get the theorem, we simply sum both sides over $\{\bar{b}_i,(u_i,n_i),\ i = 1,\dots,l\}$ and σ.

It is left to justify the claim. We wish to point out that if $\mathscr{P}_i$ hits $\mathscr{P}_{i+1}$, then they must hit in bubble B_{t_i}; hence $(u_i,n_i) \in B_{t_i}$. Thus we can find an open path from $\bar{b}_i$ to $\bar{b}_{i+1}$ that is disjoint from $\mathscr{P}_i$ and $\mathscr{P}_{i+1}$. On the other hand, suppose that $\mathscr{P}_i$ does not hit $\mathscr{P}_{i+1}$. If $s_{i+1} \neq t_i$, then we can always find a path from $b_{s_{i+1}}$ to (u'_i,n'_i) that is disjoint from $\mathscr{P}_i$ and $\mathscr{P}_{i+1}$. If $s_{i+1} = t_i$, then such a path may not exist. However, there exists a path connecting the bottom of $B_{s_{i+1}}$ to (u_i,n_i), which is the top of $B_{s_{i+1}}$, and is disjoint from $\mathscr{P}_i$ and $\mathscr{P}_{i+1}$. With these observations we can find the disjoint paths as described in the claim above. □

5. Proof of Theorem 1. Note that since we are not attempting to achieve the best possible value for d_0, we prove Theorem 1 for $d_0 \geq 9$ even though it is possible to prove Theorem 1 for $d_0 \geq 5$ with a more complicated argument as in Section 6 of our paper (see also Remark 5).

One essential ingredient of our proof of Theorem 1 is the following assumption.

ASSUMPTION (P_K). There exist K-constants K_0, K_t, K_μ and $K_{\mu\nu}$, $\mu,\nu = 1,2,\dots,d$, such that

$$\overline{T}_0 \leq K_0 d^{-1}, \qquad \overline{T}_t \leq K_t d^{-1}, \qquad \overline{T}_\mu \leq K_\mu d^{-3/2}, \qquad \overline{T}_{\mu\mu} \leq K_{\mu\mu} d^{-3/2}$$

and

$$\overline{T}_{\mu\nu} \le K_{\mu\nu} d^{-5/2} \quad \text{if } \mu \ne \nu.$$

This assumption is then improved by a bootstrapping argument "(P_{3K}) implies (P_{2K})" which has been used successfully for many mathematical physics models [Slade (1987) and Hara and Slade (1990a, b, 1992a)] in high dimensions. For our proof of Theorem 1, we first verify, in Section 5.1, that the assumption (P_K) holds for some K^G-constants $\{K_0^G, K_t^G, K_\mu^G, K_{\mu\mu}^G, K_{\mu\nu}^G\}$ in the case of $p \le 1/2d$; then, in Section 5.2, we bootstrap Assumption (P_K) for the case $1/2d \le p \le p_c$ by proving the following theorem.

THEOREM 7 (Bootstrap theorem). *There exists a d_0 which is sufficiently large that, whenever $d \ge d_0$, we can find some K'-constants for which $(P_{3K'})$ implies $(P_{2K'})$ for every $p \in (0, p_c)$.*

This theorem creates a gap in the quantities appearing on the LHS of Assumption (P_K) for $d \ge d_0$. Furthermore, these quantities are continuous on $[0, p_c)$ and are left continuous at p_c [for the proof of this one may follow the proof of Lemma 4.2 in Hara and Slade (1990a)]; therefore, they must be bounded above by the gap for $p \le p_c$. This shows the Aizenman–Newman triangle condition for $d \ge d_0$. The infrared bound (5) will be shown in the course of doing this analysis. The Barsky–Aizenman triangle condition then follows from the infrared bound.

5.1. *Case $p \le 1/2d$, $d \ge 9$.* Lemmas 1–3 are needed to estimate the triangles in terms of the related Fourier transforms for every $p \in (0, p_c)$.

LEMMA 1. *Let*

$$\Phi(x) = \sum_{n=1}^{\infty} \Psi(x, n),$$

$$T_0(x) = \sup_{n \in Z} \sum_{b,(y,m)} T[((0,0),(x,n)),(b,(y,m))].$$

Then $T_0(x) \le \Phi * \Phi(x) + 4dpD * \Phi * \Phi(x) + 2dpD * \Phi * \Phi * \Phi(x)$.

PROOF. We have

$$\begin{aligned} T_0(x) &= \sup_{n \in Z} \sum_{b,(y,m)} T[((0,0),(x,n)),(b,(y,m))] \\ &= \sup_{n} \sum_{(y,m)} \sum_{j=1}^{4} \sum_{b}^{j} T[((0,0),(x,n)),(b,(y,m))], \end{aligned}$$

where the following hold:

(a) Σ_b^1 is over bonds b such that the bottom of $b = (0,0)$ and the top of $b = (y,m)$;

(b) Σ_b^2 is over bonds b such that the bottom of $b = (0,0)$ and the top of $b \neq (y,m)$;

(c) Σ_b^3 is over bonds b such that the bottom of $b \neq (0,0)$ and the top of $b = (y,m)$;

(d) Σ_b^4 is over bonds b such that the bottom of $b \neq (0,0)$ and the top of $b \neq (y,m)$.

This divides $T_0(x)$ into four terms accordingly. The first term is equal to

$$\sum_{n \in Z} \sum_{(y,1):\, |y|=1} \Psi(y,1)\Psi(y-x, 1-n) \le \sum_y \Phi(y)\Phi(y-x) = \Phi * \Phi(x).$$

The second term is equal to

$$\sup_{n \in Z} \sum_{u:\, |u|=1} \sum_{(y,m)} \Psi(u,1)\Psi(y-u, m-1)\Psi(y-x, m-n)$$

$$\le \sum_y \sum_{u:\, |u|=1} \Psi(u,1)\Phi(y-u)\Phi(y-x) = 2dpD * \Phi * \Phi(x).$$

The third term is equal to

$$\sup_{n \in Z} \sum_{(y,m)} \sum_{u:\, |y-u|=1} \Psi(u, m-1)\Psi(y-u,1)\Psi(y-x, m-n)$$

$$\le \sum_y \sum_{u:\, |y-u|=1} \Psi(y-u,1)\Phi(u)\Phi(y-x) = 2dpD * \Phi * \Phi(x).$$

The last term is

$$\sup_{n \in Z} \sum_{(y,m)} \sum_{(u_1,n_1)} \sum_{u_2:\, |u_1-u_2|=1} \Psi(u_1,n_1)\Psi(u_2-u_1,1)\Psi(y-u_2, m-u_1-1) \times \Psi(y-x, m-n)$$

$$\le \sum_y \sum_{u_1} \sum_{u_2:\, |u_1-u_2|=1} \Psi(u_1)\Psi(u_2-u_1,1)\Phi(y-u_2)\Phi(y-x)$$

$$= 2dpD * \Phi * \Phi * \Phi(x).$$

□

Similarly, with the same proof we can show the following.

LEMMA 2. *If we let, for* $a = (a(1), \dots, a(d), a(d+1))$,

$$\Phi_{\partial_a}(x) \equiv \sum_{n=1}^{\infty} \Psi(x,n) \prod_{\mu=1}^{d} |x_{(\mu)}|^{a(\mu)} |n|^{a(d+1)},$$

$$T_a(x) \equiv \sup_{n \in Z} \sum_{b,(y,m)} T_a[((0,0),(x,n)),(b,(y,m))],$$

$$T^a(x) \equiv \sup_{n \in Z} \sum_{b,(y,m)} T^a[((0,0),(x,n)),(b,(y,m))],$$

then both $T_a(x)$ *and* $T^a(x)$ *are less than*

$$\Phi * \Phi_{\partial_a}(x) + 4dpD * \Phi * \Phi_{\partial_a}(x) + 2dpD * \Phi * \Phi * \Phi_{\partial_a}(x).$$

It is easy to see that

$$\|\hat{\Phi}\|_2 = \|\hat{\Psi}(\cdot,0)\|_2, \qquad \|\hat{\Phi}^2\|_2 = \|\hat{\Psi}^2(\cdot,0)\|_2, \qquad \|\hat{\Phi}_{\partial_a}\|_2 = \|\partial_a\hat{\Psi}(\cdot,0)\|_2,$$

where $\partial_a\hat{\Psi}(k,0) = \partial_a\hat{\Psi}(k,t)$ evaluated at $t = 0$. Then using the Cauchy–Schwarz inequality and the Fourier inversion formula we can translate Lemma 2 into the following lemma.

LEMMA 3.

$$\overline{T}_0 \le \{(1+4dp)\|\hat{\Psi}(\cdot,0)\|_2 + 2dp\|\Psi^2(\cdot,0)\|_2\}\|\hat{\Psi}(\cdot,0)\|_2,$$

both $\overline{T}_a$ *and* $\overline{T}^a \le \{(1+4dp)\|\hat{\Psi}(\cdot,0)\|_2 + 2dp\|\hat{\Psi}^2(\cdot,0)\|_2\}\|\partial_a\hat{\Psi}(\cdot,0)\|_2.$

Observe that, for $p \le 1/2d$, by counting over all the paths connecting $(0,0)$ to (x,n), we can show that $\Psi(x,n) \le \text{Prob}(S_n = x)$, where $\{S_n;\ n = 1,2,\dots\}$ is the d-dimensional simple random walk; that is, $S_n = X_1 + X_2 + \cdots + X_n$ and $\{X_i\}$ are i.i.d. with $\text{Prob}(X_1 = x)$ equal to

$$D(x) = \begin{cases} 1/2d, & \text{if } x = \pm e_{(\mu)} \text{ for } \mu = 1,\dots,d, \\ 0, & \text{otherwise}. \end{cases}$$

Define $\hat{G}(k,t) = \sum_{n=1}^{\infty}\hat{D}(k)^n e^{int} = e^{it}\hat{D}(k)(1 - e^{it}\hat{D}(k))^{-1}$. Then we have the following lemma.

LEMMA 4. *For* $p \le 1/2d$,

$$\overline{T}_0 \le \{3\|\hat{G}(\cdot,0)\|_2 + \|\hat{G}^2(\cdot,0)\|_2\}\|\hat{G}(\cdot,0)\|_2,$$

both $\overline{T}_a$ *and* $\overline{T}^a \le \{3\|\hat{G}(\cdot,0)\|_2 + \|\hat{G}^2(\cdot,0)\|_2\}\|\partial_a\hat{G}(\cdot,0)\|_2.$

REMARK 5. As we shall see later, only $\partial_a\hat{G}(\cdot,0)$ with $a = \mu, \mu\nu, \mu\mu, t$ are needed in our proof. Note that the quantities in the RHS of the above inequalities (with $a = \mu, \mu\nu, \mu\mu, t$) are finite only if the dimension $d > 8$. This is the reason our proof works for $d \geq 9$. To improve this bound to $d \geq 5$, we need to use the same argument as in Section 6.

To obtain the upper bounds for the RHS of the inequalities in Lemma 3, we invoke the following lemma [whose proof can be found in Slade (1987)].

LEMMA 5. *Let S_n be the d-dimensional simple random walk. Then*

$$\sum_{n=1}^{\infty} n^3 \operatorname{Prob}(S_{2n} = 0) = O(d^{-1}).$$

This result implies further that

$$\left\| \hat{D}(1-\hat{D})^{-1} \right\|_2 \leq c_0 d^{-1/2}, \qquad \left\| \hat{D}(1-\hat{D})^{-2} \right\|_2 \leq c_0' d^{-1/2},$$

$$\left\| \partial_\mu \hat{D}(1-\hat{D})^{-2} \right\|_2 \leq c_\mu d^{-1}, \qquad \left\| \partial_\mu \partial_\mu \hat{D}(1-\hat{D})^{-2} \right\|_2 \leq c_{\mu\mu}' d^{-1}$$

and, for $\mu \neq \nu$,

$$\left\| \partial_\mu \hat{D}\, \partial_\nu \hat{D}(1-\hat{D})^{-3} \right\|_2 \leq c_{\mu\nu}' d^{-2}, \qquad \left\| \partial_\mu \partial_\nu \hat{D}(1-\hat{D})^{-2} \right\|_2 \leq c_{\mu\nu} d^{-2},$$

and

$$\left\| (1-\hat{D})^{-m} \right\|_1 \leq O(1), \quad \text{if } m = 2, 3, \dots, [(d-1)/2].$$

Applying Lemma 5 to Lemma 3, we can verify that (P_K) holds for some K^G-constants depending only on $\operatorname{Prob}(S_n = x)$. For instance,

$$\begin{aligned} \left\| \partial_t \hat{G}(\cdot, 0) \right\|_2 &= \left\| \partial_t \bigl(1 - e^{it}\hat{D}(k)\bigr)\bigl(1 - e^{it}\hat{D}(k)\bigr)^{-2} \right\|_2 \quad (\text{with } t = 0) \\ &= \left\| \hat{D}(k)\bigl(1 - \hat{D}(k)\bigr)^{-2} \right\|_2 \leq c_0' d^{-1/2}. \end{aligned}$$

The others, which can be estimated easily in the same way, are left to the reader.

5.2. *Bootstrapping argument.* We now want to show Theorem 7 and the infrared bound (5). The next three lemmas estimate the lace parts and their derivatives in terms of the corresponding triangles for $p < p_c$.

LEMMA 6. *For laces of order* 0 *and* 1, $a = (a(1), \dots, a(d), a(d+1))$, *we have*

$$\sum_{l=0,1} \left| \hat{\Psi}_l(k,t) \right| \leq \overline{T}_0,$$

$$\sum_{l=0,1} \left| \partial_a \hat{\Psi}_l(k,t) \right| \leq \overline{T}_a.$$

PROOF. Proofs of these are trivial.

LEMMA 7. *For laces of order $l \geq 2$, we have the following*:

$$\left|\hat{\Psi}_l(k,t)\right| \leq \overline{T}_0^l;$$

$$\left|\partial_t \hat{\Psi}_l(k,t)\right| \leq l\overline{T}_t \overline{T}_l^{l-1};$$

$$\left|\partial_\mu \hat{\Psi}_l(k,t)\right| \leq l\overline{T}_\mu \overline{T}_0^{l-1};$$

$$\left|\partial_{\mu\nu} \hat{\Psi}_l(k,t)\right| \leq l\overline{T}_{\mu\nu} \overline{T}_0^{l-1} + l(l-1)\overline{T}_\mu \overline{T}_\mu \overline{T}_0^{l-2}.$$

PROOF. The first three inequalities are trivial. We only give a proof for the last inequality. From Remarks 1–4 we have

$$\begin{aligned}\left|\partial_{\mu\nu} \hat{\Psi}_l(k,t)\right| &\leq \sum_{(x,n)} D_l(x,n)|x_{(\mu)}x_{(\nu)}| \\ &\leq \sum_{i,j=1}^{l} \delta_{i\mu}\delta_{j\nu}[D_l(x,n)] \\ &\leq l\overline{T}_{\mu\nu}\overline{T}_0^{l-1} + l(l-1)\overline{T}_\mu \overline{T}_\nu \overline{T}_0^{l-2}.\end{aligned}$$

Taking the results in Lemma 7 and summing over l, we obtain estimates for the connected part and their derivatives as in Lemma 8. □

LEMMA 8. *For $\overline{T}_0 < 1$, we have*

$$\left|\hat{\Psi}_c(k,t)\right| \leq \sum_{l=1}^{\infty} \overline{T}_0^l = \overline{T}_0\left(1-\overline{T}_0\right)^{-1},$$

$$\left|\partial_t \hat{\Psi}_c(k,t)\right| \leq \sum_{l=1}^{\infty} l\overline{T}_t \overline{T}_0^{l-1} = \overline{T}_t\left(1-\overline{T}_0\right)^{-2},$$

$$\left|\partial_\mu \hat{\Psi}_c(k,t)\right| \leq \sum_{l=1}^{\infty} l\overline{T}_\mu \overline{T}_0^{l-1} = \overline{T}_\mu\left(1-\overline{T}_0\right)^{-2},$$

$$\begin{aligned}\left|\partial_{\mu\nu} \hat{\Psi}_c(k,t)\right| &\leq \sum_{l=1}^{\infty} l\overline{T}_{\mu\nu}\overline{T}_0^{l-1} + \sum_{l=2}^{\infty} l(l-1)\overline{T}_\mu \overline{T}_\nu \overline{T}_0^{l-2} \\ &= \overline{T}_{\mu\nu}\left(1-\overline{T}_0\right)^{-2} + 2\overline{T}_\mu \overline{T}_\nu\left(1-\overline{T}_0\right)^{-3}.\end{aligned}$$

From this point we assume $p < p_c$, with $2dp_c \leq 2$ [which is possible if d is large enough, due to a result of Cox and Durrett (1983)], together with Assumption (P_K). It is easy to see from Lemma 8 that (P_K) implies the following.

COROLLARY 1.

$$\left|\hat{\Psi}_c(k,t)\right| \le 2K_0 d^{-1},$$

$$\left|\partial_t\hat{\Psi}_c(k,t)\right| \le 4K_t d^{-1},$$

$$\left|\partial_\mu\hat{\Psi}_c(k,t)\right| \le 4K_\mu d^{-3/2},$$

$$\left|\partial_{\mu\mu}\hat{\Psi}_c(k,t)\right| \le 4K_{\mu\mu} d^{-3/2} + 2^4K_\mu^2 d^{-3},$$

$$\left|\partial_{\mu\nu}\hat{\Psi}_c(k,t)\right| \le 4K_{\mu\nu} d^{-5/2} + 2^4K_\mu^2 d^{-3}, \quad \textit{if } \mu \ne \nu.$$

COROLLARY 2. *Let* $F(k,t) = 1 - 2dpe^{it}\hat{D}(k)(1 + \hat{\Psi}_c(k,t))$. *Then*

$$|\partial_t F| \le \left(2^2 + 2^3K_t d^{-1}\right)|\hat{D}|,$$

$$|\partial_\mu F| \le 2^2|\partial_\mu\hat{D}| + 2^3K_\mu d^{-3/2}|\hat{D}|,$$

$$|\partial_{\mu\mu} F| \le 2^2|\partial_{\mu\mu}\hat{D}| + 2^4K_\mu d^{-3/2}|\partial_\mu\hat{D}| + \left(2^3K_{\mu\mu}d^{-3/2} + 2^5K_\mu^2 d^{-3}\right)|\hat{D}|,$$

$$|\partial_\mu\partial_\nu F| \le 2^2|\partial_{\mu\nu}\hat{D}| + 2^3K_\mu d^{-3/2}|\partial_\nu\hat{D}| + 2^3K_\nu d^{-3/2}|\partial_\mu\hat{D}| + \left(2^3K_{\mu\nu}d^{-5/2} + 2^5K_\mu^2 d^{-3}\right)|\hat{D}|, \quad \textit{for } \mu \ne \nu.$$

PROOF. For the first inequality, we have

$$|\partial_t F| \le 2dp|\hat{D}|\,|1 + \hat{\Psi}_c| + 2dp|\hat{D}|\,|\partial_t\hat{\Psi}_c| \le 2^2|\hat{D}| + 2^3K_t d^{-1}|\hat{D}|.$$

To see the second inequality, we have

$$\begin{aligned}|\partial_\mu F| &= 2dp\left|\partial_\mu\hat{D}\left(1 + \hat{\Psi}_c\right) + \hat{D}\,\partial_\mu\hat{\Psi}_c\right| \\ &\le 2dp\left|\left(1 + \hat{\Psi}_c\right)\right|\,|\partial_\mu\hat{D}| + 2dp|\partial_\mu\hat{\Psi}_c|\,|\hat{D}| \\ &\le 2^2|\partial_\mu\hat{D}| + 2^3K_\mu d^{-3/2}|\hat{D}|.\end{aligned}$$

Proof for the third is similar:

$$\begin{aligned}|\partial_{\mu\mu} F| &= 2dp\left|\partial_{\mu\mu}\hat{D}\left(1 + \hat{\Psi}_c\right) + 2\,\partial_\mu\hat{D}\,\partial_\mu\hat{\Psi}_c + \hat{D}\,\partial_{\mu\mu}\hat{\Psi}_c\right| \\ &\le 2dp|\partial_{\mu\mu}\hat{D}|\,|1 + \hat{\Psi}_c| + 2|\partial_\mu\hat{D}|\,|\partial_\mu\hat{\Psi}_c| + |\hat{D}|\,|\partial_{\mu\mu}\hat{\Psi}_c| \\ &\le 2^2|\partial_{\mu\mu}\hat{D}| + 2^4K_\mu d^{-3/2}|\partial_\mu\hat{D}| + \left(2^3K_{\mu\mu}d^{-3/2} + 2^5K_\mu^2 d^{-3}\right)|\hat{D}|.\end{aligned}$$

For the fourth inequality, with $\mu \neq \nu$, we have

$$|\partial_{\mu\nu}F| = 2dp\left|\partial_{\mu\nu}\hat{D}\left(1+\hat{\Psi}_c\right) + \partial_\mu \hat{D}\,\partial_\nu \hat{\Psi}_c + \partial_\nu \hat{D}\,\partial_\mu \hat{\Psi}_c + \hat{D}\,\partial_{\mu\nu}\hat{\Psi}_c\right|$$

$$\leq 2dp\left\{|\partial_{\mu\nu}\hat{D}|\,|1+\hat{\Psi}_c| + |\partial_\mu \hat{D}|\,|\partial_\nu \hat{\Psi}_c| + |\partial_\nu \hat{D}|\,|\partial_\mu \hat{\Psi}_c| + |\hat{D}|\,|\partial_{\mu\nu}\hat{\Psi}_c|\right\}$$

$$\leq 2^2|\partial_{\mu\nu}\hat{D}| + 2^3 K_\mu d^{-3/2}|\partial_\nu \hat{D}| + 2^3 K_\nu d^{-3/2}|\partial_\mu \hat{D}|$$

$$+\left(2^3 K_{\mu\nu} d^{-5/2} + 2^5 K_\mu^2 d^{-3}\right)|\hat{D}|. \qquad \square$$

COROLLARY 3.

$$|\hat{\Psi}_c(k,t) - \hat{\Psi}_c(0,t)| = O(d^{-3/2})|k|^2,$$

$$|\hat{\Psi}_c(k,t) - \hat{\Psi}_c(k,0)| = O(d^{-1})|t|.$$

PROOF. Interpolating between 0 and k by sk, for $s \in [0,1]$, we have

$$\left|\hat{\Psi}_c(k,t) - \hat{\Psi}_c(0,t)\right| = \left|\int_0^1 ds(1-s)\frac{d^2}{ds^2}\hat{\Psi}_c(sk,t)\right|$$

$$\leq (1/2)\sup_{0\leq s\leq 1}\left|\sum_{\mu\nu=1}^{d} k_\mu k_\nu \partial_{\mu\nu}\hat{\Psi}_c(sk,t)\right|$$

$$\leq (1/2)\sum_{\mu,\nu=1}^{d} \overline{T}_{\mu\nu}\left(1-\overline{T}_0\right)^{-2}|k_\mu k_\nu|$$

$$+ \sum_{\mu,\nu=1}^{d} \overline{T}_\mu \overline{T}_\nu\left(1-\overline{T}_0\right)^{-3}|k_\mu k_\nu|$$

$$\leq (1/2)\left(1-\overline{T}_0\right)^{-2}\left[\sum_{\mu,\nu=1}^{d}\overline{T}_{\mu\nu}^2\right]^{1/2}\left[\sum_{\mu,\nu=1}^{d} k_\mu^2 k_\nu^2\right]^{1/2}$$

$$+\left(1-\overline{T}_0\right)^{-3}\left[\sum_{\mu,\nu=1}^{d}\overline{T}_\mu^2\overline{T}_\nu^2\right]^{1/2}\left[\sum_{\mu,\nu=1}^{d} k_\mu^2 k_\nu^2\right]^{1/2}$$

$$= (1/2)\left(1-\overline{T}_0\right)^{-2}|k|^2\left[\sum_{\mu,\nu=1}^{d}\overline{T}_{\mu\nu}^2\right]^{1/2} + d\left(1-\overline{T}_0\right)^{-2}|k|^2\overline{T}_\mu^2$$

$$= O(d^{-3/2})|k|^2.$$

Similarly, interpolating between 0 and t by st, for $s \in [0,1]$, we have

$$\begin{aligned}\left|\hat{\Psi}_c(k,t) - \hat{\Psi}_c(k,0)\right| &= \left|\int_0^1 \partial_s \hat{\Psi}_c(k,st)\,ds\right| \\ &\le |t| \int_0^1 \left|\partial_s \hat{\Psi}_c(k,st)\right| ds \\ &= O(d^{-1})|t|. \qquad \square\end{aligned}$$

COROLLARY 4. *For $1/2d \le p < p_c$ and d sufficiently large, we have*

$$|F(k,t)| \ge (1/2)\left|1 - e^{it}\hat{D}(k)\right| \ge c_3|k|^2 + c_4|t|, \tag{9}$$

for some positive constants $c_3, c_4 > 0$; in particular,

$$|F(k,0)| \ge (1/2)\left|1 - \hat{D}(k)\right| \ge c_3|k|^2. \tag{10}$$

PROOF. We write

$$\begin{aligned}F(k,t) &= F(0,0) + 2dp\left(1 - e^{it}\hat{D}(k)\right) \\ &\quad + 2dp\left(\hat{\Psi}_c(0,0) - e^{it}\hat{D}(k)\hat{\Psi}_c(k,t)\right).\end{aligned} \tag{11}$$

However, the modulus of the third term is bounded above by $2dp$ times

$$\begin{aligned}&\left|\hat{\Psi}_c(0,0) - \hat{\Psi}_c(0,t)\right| + \left|\hat{\Psi}_c(0,t) - \hat{\Psi}_c(k,t)\right| + \left|\hat{\Psi}_c(k,t)\left(1 - e^{it}\hat{D}(k)\right)\right| \\ &\quad \le O(d^{-1})|t| + O(d^{-3/2})|k|^2 + O(d^{-1})\left|1 - e^{it}\hat{D}(k)\right|.\end{aligned} \tag{12}$$

On the other hand, simple algebra shows from (6) of Theorem 3 that

$$1 + \hat{\Psi}_c(0,0) = \left[1 + \hat{\Psi}(0,0)\right]\left[1 + 2dp\hat{\Psi}(0,0)\right]^{-1},$$

so both $[1 + \hat{\Psi}(0,0)]$ and $[1 + \hat{\Psi}_c(0,0)]$ are nonnegative since the first one clearly is. This shows from (6) that $F(0,0) = 1 - 2dp\hat{\Psi}(0,0)$ is nonnegative. Thus, since the real part of $(1 - e^{it}\hat{D}(k))$ is nonnegative,

$$\left|F(0,0) + 2dp\left(1 - e^{it}\hat{D}(k)\right)\right| \ge \left|2dp\left(1 - e^{it}\hat{D}(k)\right)\right|.$$

Combining (11) and (12) and the above result, we obtain

$$\begin{aligned}|F(k,t)| &\ge \left|F(0,0) + 2dp\left(1 - e^{it}\hat{D}(k)\right)\right| - 2dpO(d^{-1})\left|\left(1 - e^{it}\hat{D}(k)\right)\right| \\ &\quad - O(d^{-3/2})|k|^2 - O(d^{-1})|t| \\ &\ge 2dp\left(1 - O(d^{-1})\right)\left|\left(1 - e^{it}\hat{D}(k)\right)\right| - O(d^{-3/2})|k|^2 - O(d^{-1})|t|.\end{aligned}$$

Furthermore, it is easy to see that for $\{(k,t): |(k,t)| \le \varepsilon\}$, for a small enough $\varepsilon > 0$,

$$\left|1 - e^{it}\hat{D}(k)\right| \ge \text{const.}\,|t| + \text{const.}\,d^{-1}|k|^2. \tag{13}$$

However, $1 - e^{it}\hat{D}(k)$ is bounded below by a positive constant for

$$\left\{(k,t) \in [-\pi,\pi]^d \times [-\pi,\pi]: |(k,t)| > \varepsilon\right\},$$

which implies that, for $d \geq d_0$ large enough,

$$|F(k,t)| \geq (1/2)\left|\left(1 - e^{it}\hat{D}(k)\right)\right| \geq c_3|k|^2 + c_4|t|.$$

In particular, we have $|F(k,0)| \geq (1/2)|1 - \hat{D}(k)| \geq c_3|k|^2$. □

We now apply Corollaries 1–4 to estimate the L_2- and $L_4([\pi,\pi]^d)$-norms of $\hat{\Psi}(\cdot,0)$ and its derivatives.

COROLLARY 5.

$$\|\hat{\Psi}(\cdot,0)\|_2 \leq 2^3 c_0 d^{-1/2} + O(d^{-1}).$$

PROOF. We have

$$\begin{aligned}\|\hat{\Psi}(\cdot,0)\|_2 &= \left\|\frac{\hat{\Psi}_c(\cdot,0) + 2dp\hat{D}\left(1 + \hat{\Psi}_c(\cdot,0)\right)}{1 - 2dp\hat{D}\left(1 + \hat{\Psi}_c(\cdot,0)\right)}\right\|_2 \\ &\leq \|\hat{\Psi}_c(\cdot,0)\|_\infty \|F(\cdot,0)^{-1}\|_2 + 2dp\|1 + \hat{\Psi}_c(\cdot,0)\|_\infty \|\hat{D}F(\cdot,0)^{-1}\|_2 \\ &\leq \|\hat{\Psi}_c(\cdot,0)\|_\infty \left\|2(1-\hat{D})^{-1}\right\|_2 + 2\|1 + \hat{\Psi}_c(\cdot,0)\|_\infty \left\|2\hat{D}(1-\hat{D})^{-1}\right\|_2 \\ &= O(d^{-1}) + 8c_0 d^{-1/2}.\end{aligned}$$

□

COROLLARY 6.

$$\|\hat{\Psi}^2(\cdot,0)\|_2 \leq 2^6 c_0' d^{-1/2} + O(d^{-3/2}).$$

PROOF. We have

$$\begin{aligned}\|\hat{\Psi}^2(\cdot,0)\|_2 &= \left\|\frac{\left[\hat{\Psi}_c(\cdot,0) + 2dp\hat{D}\left(1 + \hat{\Psi}_c(\cdot,0)\right)\right]^2}{\left[1 - 2dp\hat{D}\left(1 + \hat{\Psi}_c(\cdot,0)\right)\right]^2}\right\|_2 \\ &\leq \|\hat{\Psi}_c(\cdot,0)\|_\infty^2 \|F(\cdot,0)^{-2}\|_2 \\ &\quad + 4dp\left\|\hat{\Psi}_c(\cdot,0)\left(1 + \hat{\Psi}_c(\cdot,0)\right)\right\|_\infty \|\hat{D}F(\cdot,0)^{-2}\|_2 \\ &\quad + \left\|2dp\left(1 + \hat{\Psi}_c(\cdot,0)\right)\right\|_\infty^2 \|\hat{D}^2 F(\cdot,0)^{-2}\|_2 \\ &\leq \|\hat{\Psi}_c(\cdot,0)\|_\infty^2 \left\|4(1-\hat{D})^{-2}\right\|_2 \\ &\quad + 4dp\left\|\hat{\Psi}_c(\cdot,0)\left(1 + \hat{\Psi}_c(\cdot,0)\right)\right\|_\infty \left\|4\hat{D}(1-\hat{D})^{-2}\right\|_2 \\ &\quad + \left\|2dp\left(1 + \hat{\Psi}_c(\cdot,0)\right)\right\|_\infty^2 \left\|4\hat{D}(1-\hat{D})^{-2}\right\|_2 \\ &= O(d^{-2}) + O(d^{-1}d^{-1/2}) + 2^6 c_0' d^{-1/2}.\end{aligned}$$

□

COROLLARY 7.

$$\left\|\partial_\mu\hat{\Psi}(\cdot,0)\right\|_2 \le 2^5 c_\mu d^{-1} + O(d^{-3/2}).$$

PROOF. We have

$$\begin{aligned}\left\|\partial_\mu\hat{\Psi}(\cdot,0)\right\|_2 &= \left\|\partial_\mu\hat{\Psi}_c(\cdot,0)F(\cdot,0)^{-1} - \left(1+\hat{\Psi}_c(\cdot,0)\right)\partial_\mu F(\cdot,0)F(\cdot,0)^{-2}\right\|_2\\ &\le \left\|\partial_\mu\hat{\Psi}_c(\cdot,0)\right\|_\infty \left\|2(1-\hat{D})^{-1}\right\|_2\\ &\quad + \left\|1+\hat{\Psi}_c(\cdot,0)\right\|_\infty \left\|\left(2^2\partial_\mu\hat{D} + 2^3K_\mu d^{-3/2}\hat{D}\right)2^2(1-\hat{D})^{-2}\right\|_2\\ &\le O(d^{-3/2}) + 2^5\left\|\partial_\mu\hat{D}(1-\hat{D})^{-2}\right\|_2 + O(d^{-3/2})\left\|\hat{D}(1-\hat{D})^{-2}\right\|_2\\ &\le O(d^{-3/2}) + 2^5 c_\mu d^{-1}. \qquad \square\end{aligned}$$

COROLLARY 8.

$$\left\|\partial_{\mu\mu}\hat{\Psi}(\cdot,0)\right\|_2 \le 2^3 c_{\mu\mu} d^{-1} + O(d^{-2}).$$

PROOF. We have

$$\begin{aligned}\left\|\partial_{\mu\mu}\hat{\Psi}(\cdot,0)\right\|_2 &= \left\|\partial_{\mu\mu}\left[\left(1+\hat{\Psi}_c(\cdot,0)\right)F(\cdot,0)^{-1}\right]\right\|_2\\ &\le \left\|\partial_{\mu\mu}\hat{\Psi}_c(\cdot,0)F(\cdot,0)^{-1}\right\|_2 + 2\left\|\partial_\mu\hat{\Psi}_c(\cdot,0)\,\partial_\mu F(\cdot,0)^{-1}\right\|_2\\ &\quad + 2\left\|\left(1+\hat{\Psi}_c(\cdot,0)\right)\partial_\mu F(\cdot,0)\,\partial_\mu F(\cdot,0)F(\cdot,0)^{-3}\right\|_2\\ &\quad + \left\|\left(1+\hat{\Psi}_c(\cdot,0)\right)\partial_{\mu\mu}F(\cdot,0)F(\cdot,0)^{-2}\right\|_2.\end{aligned}$$

The first term is $O(d^{-3/2})$, the second is $O(d^{-3/2}d^{-1})$, the third term is of the same order as

$$\begin{aligned}&\left\|\left(2^2|\partial_\mu\hat{D}| + 2^3K_\mu d^{-3/2}|\hat{D}|\right)^2(1-\hat{D})^{-3}\right\|_2\\ &\quad\le \text{const.}\left\|\left(\partial_\mu\hat{D}\right)^2(1-\hat{D})^{-3}\right\|_2 + O(d^{-3/2})\left\|2\left(\partial_\mu\hat{D}\right)\hat{D}(1-\hat{D})^{-3}\right\|_2\\ &\qquad + O(d^{-3})\left\|\hat{D}^2(1-\hat{D})^{-3}\right\|_2,\end{aligned}$$

which is of order $O(d^{-2})$, and the last term is bounded by 2 times

$$\begin{aligned}&2^2\left\|\partial_{\mu\mu}\hat{D}(1-\hat{D})^{-2}\right\|_2 + \left\|\partial_\mu\hat{D}(1-\hat{D})^{-2}\right\|_2 O(d^{-3/2}) + \left\|\hat{D}(1-\hat{D})^{-2}\right\|_2 O(d^{-3/2})\\ &\quad\le 2^3 c_{\mu\mu} d^{-1} + O(d^{-2}). \qquad \square\end{aligned}$$

COROLLARY 9.

$$\left\|\partial_{\mu\nu}\hat{\Psi}(\cdot,0)\right\|_2 \le \left(2^4 c'_{\mu\nu} + 2^3 c_{\mu\nu}\right)d^{-2} + O(d^{-5/2}), \quad \textit{for } \mu \ne \nu.$$

PROOF. For $\mu \neq \nu$, we have

$$\begin{aligned}\left\|\partial_{\mu\nu}\hat{\Psi}(\cdot,0)\right\|_2 &= \left\|\partial_{\mu\nu}\left[(1+\hat{\Psi}_c(\cdot,0))F(\cdot,0)^{-1}\right]\right\|_2 \\ &\le \left\|\partial_{\mu\nu}\hat{\Psi}_c(\cdot,0)F(\cdot,0)^{-1}\right\|_2 \\ &\quad+\left\|\partial_{\mu}\hat{\Psi}_c(\cdot,0)\,\partial_{\nu}F(\cdot,0)^{-1}\right\|_2+\left\|\partial_{\nu}\hat{\Psi}_c(\cdot,0)\,\partial_{\mu}F(\cdot,0)^{-1}\right\|_2 \\ &\quad+2\left\|(1+\hat{\Psi}_c(\cdot,0))\,\partial_{\mu}F(\cdot,0)\,\partial_{\nu}F(\cdot,0)F(\cdot,0)^{-3}\right\|_2 \\ &\quad+\left\|(1+\hat{\Psi}_c(\cdot,0))\,\partial_{\mu\nu}F(\cdot,0)F(\cdot,0)^{-2}\right\|_2.\end{aligned}$$

The first three terms are of order $O(d^{-5/2})$, the fourth is bounded by 2^2 times

$$\begin{aligned}&\left\|\left(2^2|\partial_{\mu}\hat{D}|+2^3K_{\mu}d^{-3/2}|\hat{D}|\right)\left(2^2|\partial_{\nu}\hat{D}|+2^3K_{\nu}d^{-3/2}|\hat{D}|\right)(1-\hat{D})^{-3}\right\|_2 \\ &\quad\le 2^4\left\|\partial_{\mu}\hat{D}\partial_{\nu}\hat{D}(1-\hat{D})^{-3}\right\|_2+O(d^{-3/2})\left\|\partial_{\mu}\hat{D}\hat{D}(1-\hat{D})^{-3}\right\|_2 \\ &\quad\quad+O(d^{-3})\left\|\hat{D}^2(1-\hat{D})^{-3}\right\|_2,\end{aligned}$$

which is dominated by $c'_{\mu\nu}d^{-2}+O(d^{-5/2})$, and the last term is bounded by 2 times

$$\begin{aligned}&2^2\left\|\partial_{\mu\nu}\hat{D}(1-\hat{D})^{-2}\right\|_2+\left\|\partial_{\mu}\hat{D}(1-\hat{D})^{-2}\right\|_2 O(d^{-3/2})+\left\|\hat{D}(1-\hat{D})^{-2}\right\|_2 O(D^{-5/2}) \\ &\quad\le 2^3c_{\mu\nu}d^{-2}+O(d^{-5/2}).\end{aligned}$$ □

COROLLARY 10.

$$\left\|\partial_t\hat{\Psi}(\cdot,0)\right\|_2\le 2^5c'_t d^{-1/2}+O(d^{-1}).$$

PROOF. We have

$$\begin{aligned}\left\|\partial_t\hat{\Psi}(\cdot,0)\right\|_2 &= \left\|\partial_t\hat{\Psi}_c(\cdot,0)F(\cdot,0)^{-1}-(1+\hat{\Psi})\,\partial_tF(\cdot,0)F(\cdot,0)^{-2}\right\|_2 \\ &\le \left\|\partial_t\hat{\Psi}_c(\cdot,0)\right\|_\infty\left\|2(1-\hat{D}(k))^{-1}\right\|_2 \\ &\quad+\left\|1+\hat{\Psi}_c(\cdot,0)\right\|_\infty\left\|(2^2+2^3K_td^{-1})\hat{D}2^2(1-\hat{D})^{-2}\right\|_2 \\ &\le 2^2K_td^{-1}O(1)+2^5c'_0d^{-1/2}+O(d^{-1}).\end{aligned}$$

We note that the constants c_0, c_0', c_μ, $c_{\mu\mu}$, $c_{\mu\nu}$ and $c_{\mu\nu}'$ are independent of the constants in (P_K). Applying Corollaries 5–10 to Lemma 3, we obtain

$$\begin{aligned}\overline{T}_t &\le \{(1+4dp)2^3c_0d^{-1/2}+O(d^{-1})+2dp2^6c_0'd^{-1/2}+O(d^{-1})\}\\ &\quad\times\{2^5c_0'd^{-1/2}+O(d^{-1})\}\\ &= K_t'd^{-1}+O(d^{-3/2}),\end{aligned}$$

and similarly for the others:

$$\begin{aligned}\overline{T}_0 &\le K_0'd^{-1}+O(d^{-3/2}),\\ \overline{T}_\mu &\le K_\mu'd^{-3/2}+O(d^{-2}),\\ \overline{T}_{\mu\mu} &\le K_{\mu\mu}'d^{-3/2}+O(d^{-2}),\\ \overline{T}_{\mu\nu} &\le K_{\mu\nu}'d^{-5/2}+O(d^{-3}),\quad \text{for } \mu\ne\nu.\end{aligned}$$

For K-constants which are 3 times the K'-constants and for d large enough, we obtain the bootstrap theorem. Thus the numerator $1+\hat{\Psi}_c(k,t)$ of $\hat{\Psi}(k,t)$ is bounded above by 2 and, from Corollary 3, the denominator $F(k,t)$ of $\Psi(k,t)$ is bounded below by $(1/2)(1-e^{it}\hat{D}(k))^{-1}$; so the infrared bound (5) holds. This proves Theorem 1. □

6. Proof of infrared bound for spread-out model. Our proof of Theorem 2 for the spread-out model follows the same idea as in the proof of Theorem 1 in Section 5 and it is quite analogous to the proof in the last section of Hara and Slade (1990a). The only differences are in the following:

(i) estimates involving the random walk S_n;
(ii) estimates involving $\hat{\Psi}(k,t)$, $\hat{\Psi}_c(k,t)$ and their derivatives.

In the previous section the quantities in item (ii) are estimated in terms of L_2- and $L_4([-\pi,\pi]^d)$-norms. They need to be replaced by L_2- or $L_3([-\pi,\pi]^d\times[-\pi,\pi])$-estimates with respect to the Lebesgue measure $(2\pi)^{-(d+1)}\,dk\,dt$.

Recall that $p_{0x}=pg(x/L)L^{-d}$ and $p_L^{-1}=\Sigma_x g(x/L)L^{-d}$, where g satisfies the conditions described in Section 1. We also use $p_{0x}^{(L)}$ to denote $p_L g(x/L)L^{-d}$, the normalized version of p_{0x}. The renewal equation (6) in Theorem 3 takes the form

$$1+\hat{\Psi}(k,t)=\frac{1+\hat{\Psi}_c(k,t)}{1-pp_L^{-1}e^{it}\hat{D}_L(k)\big(1+\hat{\Psi}_c(k,t)\big)},\tag{14}$$

where $\hat{D}_L(k) = p_L \Sigma_x g(x/L) L^{-d} e^{ikx}$. Later we shall see that all the estimates of $\hat{\Psi}$, $\hat{\Psi}_c$ and their derivatives can be expressed in terms of L_2- or $L_3([-\pi, \pi]^d \times [-\pi, \pi])$-norms. Note that the asymptotic behavior of

$$\hat{C}_L(k,t) \equiv \{1 - e^{it}\hat{D}_L(k)\}^{-1}$$

is $(c_1|k|^2L^2 + c_2|t|)^{-1}$ for small k and t. Thus $\int\int |\hat{C}_L(k,t)|^3\, dk\, dt < \infty$ for $d > 4$. This gives a heuristic argument why the infrared bound (5) is satisfied for oriented percolation in $d + 1 > 5$ (unlike $d > 6$ for unoriented percolation).

Estimates of $\hat{\Psi}_c(k,t)$ and its derivatives can be bounded using Feynman diagrams as in Section 4. However, to improve $L_4([-\pi,\pi]^d)$-estimates to $L_3([-\pi,\pi]^d \times [-\pi,\pi])$-estimates, we need to improve the estimates of $\{\Psi_l(x,n);\ l \ge 0\}$ as follows:

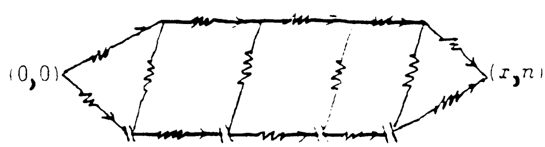
$$\Psi_l(x,n) \le E_l(x,n), \quad \text{for } l \ge 1, \tag{15}$$

$$\Psi_0(x,n) \le \Psi^2(x,n), \tag{16}$$

where

$$E_l(x,n) = \sum D_l[(b_i,(u_i,n_i));\ i = 1,2,\dots,l,\sigma]$$
$$\times P_p(\text{top of } b_l \to (x,n)) P_p((u_l,n_l) \to (x,n)),$$

with the summation taken over the set $\{\{b_i,(u_i,n_i);\ i = 1,2,\dots,l\},\sigma\}$. For instance, the Feynman diagram for $E_4(x,n)$ is given by



The proof of (16) is contained in the proof of Theorem 4, and the proof of (15) can proceed as in the proof of Theorem 6 with only a change in the estimate on the last bubble [see also Yang and Nguyen (1991) for more details].

To estimate $\partial_{\mu\nu}\hat{\Psi}_l(k,t)$, we use the method described in Section 3.2 of Hara and Slade (1990a). Let

$$W = \sum_{(x,n)} |x|^2 P_p^2((0,0) \to (x,n)),$$

$$W_{ym} = \sum_{(x,n)} |x|^2 P_p((0,0) \to (x,n))$$
$$\times \{P_p((y,m) \to (x,n)) + P_p((x,n) \to (y,m))\},$$

$$W'_{ym} = \sum_{(x,n)} \sum_u p_{0u}|x|^2 P_p((u,1) \to (x,n)) P_p((0,0) \to (x+y, n+m)),$$

$$\overline{W} = \sup_{(y,m)} W_{ym},$$

$$\overline{W}' = \sup_{(y,m)} W'_{ym},$$

$$\overline{Q} = \sup_b \sum_{(x,n)} P_p(\text{top of } b \to (x,n)) P_p((0,0) \to (x,n)).$$

Similarly, let U, U_{ym}, U'_{ym}, $\overline{U}$ and $\overline{U}'$ be defined in the same way as W, W_{ym}, W'_{ym}, $\overline{W}$ and $\overline{W}'$ but with $|x|^2$ replaced by n. Using the same argument as that in Hara and Slade [(1990a), page 364], together with (15) and (16), we get, for $l \geq 1$,

$$(17) \qquad \left|\partial_{\mu\nu}\hat{\Psi}_l(k,t)\right| \leq \overline{W}\overline{T}_0^l + l^2\overline{W}'\overline{T}_0^{l-1}\overline{Q} + pp_L^{-1}\overline{W}\overline{T}_0^{l-1}(1+\overline{T}_0),$$

$$(18) \qquad \left|\partial_t\hat{\Psi}_l(k,t)\right| \leq \overline{U}\overline{T}_0^l + (l-1)\overline{U}'\overline{T}_0^{l-1}\overline{Q} + pp_L^{-1}\overline{U}\overline{T}_0^{l-1}(1+\overline{T});$$

also, we can easily see that

$$(19) \qquad \left|\partial_{\mu\nu}\hat{\Psi}_0(k,t)\right| \leq \overline{W},$$

$$(20) \qquad \left|\partial_t\hat{\Psi}_0(k,t)\right| \leq \overline{U},$$

$$(21) \qquad \left|\partial_\mu\hat{\Psi}_l(k,t)\right| \leq \left|\partial_{\mu\nu}\hat{\Psi}_l(k,t)\right|, \quad \text{for } l = 0,1,2,\dots.$$

The undifferentiated $\hat{\Psi}_l(k,t)$ can be estimated by using Lemmas 6 and 7 of Section 5.2:

$$(22) \qquad \sum_{l=0,1} \left|\hat{\Psi}_l(k,t)\right| \leq \overline{T}_0,$$

$$(23) \qquad \left|\hat{\Psi}_l(k,t)\right| \leq \overline{T}_0^l, \quad \text{for } l \geq 2.$$

Next we want to invoke the following estimates from Lemma 5.1 of Hara and Slade (1990a).

LEMMA 9. *Given $d > 4$, we have*

$$\sup_x p_{0x}^{(L)} \sum_x \left(p_{0x}^{(L)}\right)^2 = O(L^{-d}),$$

$$\sup_x p_{0x}^{(L)}|x|^2 = O(L^{-d+2}),$$

$$\sum_x \left(p_{0x}^{(L)}\right)^2 |x|^4 = O(L^{-d+4}).$$

Note that Lemma 5.1 of Hara and Slade (1990a) states that $d > 6$. However, the above estimates hold for $d > 4$.

Let $M > 0$ be such that $\int_{x \in R^d:\ |x| \le M} g(x)\,dx > 0$, $\delta_1 = \pi/LM$ and $\delta = 3\pi \int_{R^d} \partial_1 g(x)\,dx/2L$. Since g is continuous at 0 and $g(0) > 0$, we can choose M so small that $(3/2)\int_{R^d} \partial_1 g(x)\,dx < 1/M$. Hence $\delta_1 \ge \delta$.

LEMMA 10. *There exist constants $c = c(M)$ and L_0 such that, for all $L \ge L_0$,*

$$(24)\quad \left|1 - e^{it}\hat{D}_L(k)\right|^2 \ge \tfrac{1}{16}\Big(\left|1 - \hat{D}_L(k)\right|^2 + |1 - \cos t|\Big), \quad \text{if } \hat{D}_L(k) \ge 0,$$

$$(25)\quad \left|1 - e^{it}\hat{D}_L(k)\right|^2 \ge \tfrac{1}{16}\Big(\left|1 + \hat{D}_L(k)\right|^2 + |1 + \cos t|\Big), \quad \text{if } \hat{D}_L(k) \le 0,$$

$$(26)\quad \left|1 - e^{it}\hat{D}_L(k)\right| \ge c(k^2L^2 + |t|), \quad \text{for } |k| \le \delta_1,$$

$$(27)\quad \left|\hat{D}_L(k)\right| \le \tfrac{2}{3}, \quad \text{for } |k| \ge \delta.$$

PROOF. For the first two inequalities, (24) and (25),

$$\begin{aligned}\left|1 - e^{it}\hat{D}_L(k)\right|^2 &= \left(1 - \hat{D}_L(k)\right)^2 + 2\hat{D}_L(k)(1 - \cos t)\\ &\ge \tfrac{1}{16}\Big(\left|1 - \hat{D}_L(k)\right|^2 + (1 - \cos t)\Big), \quad \text{if } \hat{D}_L(k) \ge 0,\\ \left|1 - e^{it}\hat{D}_L(k)\right|^2 &= \left(1 + \hat{D}_L(k)\right)^2 - 2\hat{D}_L(k)(1 + \cos t)\\ &\ge \tfrac{1}{16}\Big(\left|1 + \hat{D}_L(k)\right|^2 + (1 + \cos t)\Big), \quad \text{if } \hat{D}_L(k) \le 0.\end{aligned}$$

If $\hat{D}_L(k) \ge 0$, (26) follows from (24) and Lemma 5.5 of Hara and Slade (1990a). For $|k| \le \delta_1$,

$$\begin{aligned}1 + \hat{D}_L(k) &= \sum_x L^{-d} g(x/L)(1 + \cos kx)\\ &\ge \sum_{x:\,|kx| \le \pi/2} L^{-d} g(x/L)\\ &\ge \sum_{x:\,|x| \le LM/2} L^{-d} g(x/L)\\ &\sim \int_{|x| \le M/2} g(x)\,dx\\ &\equiv \alpha > 0\end{aligned}$$

if L is sufficiently large. By (25), we then have

$$\left|1 - e^{it}\hat{D}_L(k)\right| \ge \text{const.} \ge c\big(|k|^2L^2 + |t|\big),$$

for $|k| \le \delta_1$ and some sufficiently small constant c. Inequality (27) follows from Lemma 5.5 of Hara and Slade (1990a).

The estimates of $\hat{D}_L(k)$ and its derivatives for $|k| \ge \delta$ can be obtained using (5.34) of Hara and Slade (1990a) and the argument in the proof of Lemma 5.7

of Hara and Slade (1990a). Let $I \subset \{1, 2, \dots, d\}$, $I \neq \emptyset$, and let

$$R_I = \{k \in R^d \colon \delta < |k_\nu| \le \pi \text{ for all } \nu \in I, |k_\mu| \le \delta \quad \text{for all } \mu \notin I\}.$$

Then we have, for $s = 0, 1, 2$,

$$(28) \qquad (2\pi)^{-d} \int_{R_I} dk \left|\partial_\mu^s \hat{D}_L(k)\right|^N = \begin{cases} O(L^{sN-d}), & \text{if } N \ge 2 \\ O(L^{sN-d}|\ln L|^d), & \text{if } N \ge 1. \end{cases}$$

Here $\partial_\mu^s = \partial_\mu$ if $s = 1$, and $\partial_\mu^s = \partial_{\mu\mu}$ if $s = 2$. For small k, we use (26) to get

$$(29) \qquad \begin{aligned} &(2\pi)^{-(d+1)} \int dt \int_{|k| \le \delta_1} dk \left|1 - e^{it}\hat{D}_L(k)\right|^{-N} \\ &\quad = O\left(L^{-d}(d - 2(N-1))^{-1}\right), \end{aligned}$$

for $d > 2(N-1)$, $N = 1, 2, \dots$.

Furthermore, consider $\{p_n(x, y);\ n = 0, 1, 2, \dots;\ x, y \in Z^d\}$, the n-step transition functions of the random walk with one-step transition probabilities $p_{0x}^{(L)}$. Let

$$T_L = \sup_{(x_1, n_1), (x_2, n_2)} \sum p_{n-n_1}(x_1, u) p_{m-n}(u, v) p_{m-n_2}(x_2, v),$$

$$U_L = \sum_{n \ge 0, x} p_n^2(0, x) n,$$

$$W_L = \sum_{n \ge 1, x} p_n^2(0, x) |x|^2,$$

$$Q_L = \sup_{(y, m)} \sum_{(x, n)} [p_n(0, x) p_{n-m}(y, x) + p_n(0, x) p_{m-n}(x, y)],$$

$$U_{L, y, m} = \sum_{n \ge 0, x} p_n(0, x) n [p_{m-n}(x, y) + p_{n-m}(y, x)],$$

$$W_{L, y, m} = \sum_{n \ge 1, x} p_n(0, x) |x|^2 [p_{m-n}(x, y) + p_{n-m}(y, x)],$$

where the summation Σ in T_L is over $\{m, n, u, v \colon n - n_1 \ge 1,\ m - n_2 \ge 0\}$. The following lemma gives the estimates of these quantities in terms of L.

LEMMA 11. *Given $d > 4$ and $\varepsilon > 0$, there exists L_0 such that, for all $L \ge L_0$, the following hold*:

$$T_L \le O\left(|\ln L|^d L^{-d}/(d-4)\right),$$

$$W_L \le O(L^{-d+2}/(d-4)),$$

$$Q_L \le O(L^{-d}/(d-2)) + 4,$$

$$U_L \le O\left(L^{-d}|\ln L|^d/(d-2)\right),$$

$$W_{L, ym} \le O\left(L^{-d+2}|\ln L|^d/(d-4)\right),$$

$$U_{L, ym} \le O\left(L^{-d}|\ln L|^d\right).$$

PROOF. We have

$$T_L \le (2\pi)^{-d-1} \iint dk\, dt \left|\hat{D}_L(k)\right| \left|1 - e^{it}\hat{D}_L(k)\right|^{-3}$$

$$\le 3^3 (2\pi)^{-d-1} \int dt \int_{|k| \ge \delta} dk \left|\hat{D}_L(k)\right|$$

$$+ c(2\pi)^{-d-1} \int dt \int_{|k|<\delta} dk \left(|k|^2 L^2 + |t|\right)^{-3}.$$

By (28) and (29),

$$T_L = O(L^{-d} |\ln L|^d) + O(L^{-d}/(d-4)).$$

Similarly,

$$W_L = \sum_{\mu=1}^{d} (2\pi)^{-d-1} \iint dk\, dt \left|\partial_\mu \hat{D}_L(k)\right|^2 \left|1 - e^{it}\hat{D}_L(k)\right|^{-4}.$$

By (5.35) in Hara and Slade (1990a),

$$\left|\partial_\mu \hat{D}_L(k)\right| \le 2L^2 |k_\mu| \int_{R^d} \left|x_\mu^2 g(x)\right| dx, \tag{30}$$

for all k. We then split the last integral into two regions: $|k| \ge \delta$ and $|k| < \delta$. By (28), the integral over $|k| \ge \delta$ is equal to $O(L^{2-d})$. By Lemma 10, the integral over $|k| < \delta$ is bounded by

$$\text{const.}\ L^2 \int dt \int_{|k|<\delta} dk \left(|k|^2 L^2 + |t|\right)^{-4} |k^2|,$$

which is of order $O(L^d/(d-4))$. Hence, the total bound for W_L is of order $O(L^{d-2}/(d-4))$, for $d > 4$. Again using (28) and (29), we have for $d > 2(N-1)$,

$$(2\pi)^{-d-1} \iint dk\, dt \left|\hat{D}_L(k)\right|^N \left|1 - e^{it}\hat{D}_L(k)\right|^{-N}$$

$$= \begin{cases} O(L^{-d}/(d-2N+2)), & \text{for } N \ge 2 \\ O(L^{-d}/(d-2N+2)) + O(L^{-d}|\ln L|^d), & \text{for } N = 1. \end{cases}$$

Furthermore, for $d > 2$,

$$Q_L \le 2 \left\|\left| 1 + e^{it}\hat{D}_L(k)\left(1 - e^{it}\hat{D}_L(k)\right)^{-1} \right\|\right|_2^2$$

$$\le 4\left\{ \|\|1\|\|_2^2 + \left\| \hat{D}_L(k)\left(1 + \hat{D}_L(k)\right)^{-1} \right\|_2^2 \right\}$$

$$\le 4 + O(L^{-d}/(d-2))$$

and both U_L and $U_{L,ym}$ are bounded by

$$\left\| \left(1 - e^{it}\hat{D}_L(k)\right)^{-1} \partial_t \left(1 - e^{it}\hat{D}_L(k)\right)^{-1} \right\|_1$$

$$\le (2\pi)^{-d-1} \iint dk\, dt \left|\hat{D}_L(k)\right| \left|1 - e^{it}\hat{D}_L(k)\right|^{-2}$$

$$\le O\left(L^{-d}|\ln L|^d/(d-2)\right),$$

by (28) and (29). To bound $W_{L,ym}$, we use

$$|W_{L,ym}| \le (2\pi)^{-d-1} \iint dk\, dt \sum_{\mu=1}^{d} \left| \left(1 - e^{it}\hat{D}_L(k)\right)^{-1} \partial_{\mu\mu} \left(1 - e^{it}\hat{D}_L(k)\right)^{-1} \right|$$

$$\le (2\pi)^{-d-1} \iint dk\, dt \sum_{\mu=1}^{d} \left| \left(1 - e^{it}\hat{D}_L(k)\right)^{-3} \partial_{\mu\mu} \hat{D}_L(k) \right|$$

$$+ 2(2\pi)^{-d-1} \iint dk\, dt \sum_{\mu=1}^{d} \left| \left(1 - e^{it}\hat{D}_L(k)\right)^{-4} \left[\partial_\mu \hat{D}_L(k)\right]^2 \right|.$$

Thus the first term is bounded by $O(L^{2-d}|\ln L|^d)$. The second term is equal to $2W_L$, which is also of order $O(L^{2-d}/(d-4))$. □

REMARK 6. Durrett (1985) indicates that one can extend the work of Aizenman and Newman (1984) to show that if $p < p_c$, then $P_p((0,0) \to (y,m))$ decays exponentially as $(y,m) \to \infty$ for the nearest-neighbor oriented percolation model. Notice that because of the complication due to the long-range interactions of the spread-out oriented percolation, some modification is needed to show this same result. One can apply the same modification shown in Appendix A of Hara (1990), even though the proof there is given for the spread-out unoriented percolation. However, any change from the unoriented situation to the oriented case of this result is straightforward, so we omit the proof. Thus, for any $L > 0$, there exists a constant $M_1 = M_1(p, L)$ such that $W_{ym} \le L^{-d}$ and $U_{ym} \le L^{-d}$ for all $|(y,m)| \ge M_1(p, L)$.

Now let $d > 4$ be fixed. Let (P_K) be the statement that there exists a constant K such that for all $p_L \le p < p_c$, the following hold:

$$pp_L^{-1} \le K,$$

$$\overline{T}_0, U \le KL^{-d}|\ln L|^d,$$

$$W \le KL^{2-d},$$

$$W_{ym} \le KL^{2-d}|\ln L|^d, \quad \text{for every } |(y,m)| \le M_1(p,L),$$

$$U_{ym} \le KL^{-d}|\ln L|^d, \quad \text{for every } |(y,m)| \le M_1(p,L).$$

LEMMA 12. *Assume (P_K) holds. Then there exist constants L_0 and K' depending on K and independent of p such that, for all $L \geq L_0$,*

$$\begin{aligned}
\overline{W} &\leq K'L^{2-d}|\ln L|^d,\\
\overline{Q} &\leq K'L^{2-d} + 4,\\
\overline{W}' &\leq K'L^{2-d}|\ln L|^d,\\
W' &\leq K'L^{2-d},\\
\overline{U} &\leq K'L^{-d}|\ln L|^d,\\
\overline{U}' &\leq K'L^{-(d-2)/2},\\
U' &\leq K'L^{-(d-2)/2}.
\end{aligned}$$

PROOF. The first estimate follows from Assumption (P_K). Let $B = \sum_{(x,n)}\Psi^2(x,n)$. Then $B \leq W \leq KL^{2-d}$. Using the same argument as in the proof of Lemma 3.3 of Hara and Slade (1990a), we also get the same type of estimates for W':

$$W' \leq W\sum_y p_{0y} + (WB)^{1/2}\sum_y p_{0y}|y| + W^{1/2}\Big(\sum_y p_{0y}^2|y|^2\Big)^{1/2}, \tag{31}$$

$$W'_{al} \leq 2\overline{W}\sum_y p_{0y} + 2B\sum_y p_{0y}|y|^2 + 4B^{1/2}\Big\{\sum_{y\neq a} p_{0y}^2|y|^4\Big\}^{1/2} + 4p_{0a}|a|^2. \tag{32}$$

Then by Assumption (P_K) and symmetry of p_{0y}, W' is bounded by const. L^{2-d}; also, W'_{al} is bounded by const. $L^{-d+2}|\ln L|^d$. The bound on $\overline{Q}$ is as follows:

$$\overline{Q} \leq 2|||1 + \hat{\Psi}|||_2^2 \leq 4 + 4|||\hat{\Psi}|||_2^2$$
$$= 4 + 4B \leq 4 + \text{const. } L^{-d}.$$

The estimate for $\overline{U}$ follows from the same argument as that for $\overline{W}$. For estimating U' and U'_{al}, we need estimates similar to (31) and (32). Let $\tau_n(x) = P_p((0,0) \to (x,n))$. Then for $a \in Z^d$, $l = 0, 1, 2, \dots$,

$$\begin{aligned}
U'_{al} &= \sum_y p_{0y} \sum_{x,n\geq 1} \tau_{n-1}(x-y)\tau_{l+n}(x+a)n\\
&= \sum_y p_{0y}\tau_{l+1}(y+a) + \sum_y p_{0y}\sum_{x,n\geq 2} \tau_{n-1}(x-y)\tau_{l+n}(x+a)n\\
&\leq \Big(\sum_y p_{0y}^2\Big)^{1/2}\overline{W}^{1/2} + \sum_y p_{0y}\sum_{x,n\geq 1}\tau_n(x)\tau_{l+n+1}(x+y+a)(n+1)\\
&\leq \Big(\sum_y p_{0y}^2\Big)^{1/2}\overline{W}^{1/2} + 2\sum_y p_{0y}\sum_{x,n\geq 1}\tau_n(x)\tau_{l+n+1}(x+y+a)n\\
&\leq \Big(\sum_y p_{0y}^2\Big)^{1/2}\overline{W}^{1/2} + 2\sum_y p_{0y}\overline{U}\\
&\leq \text{const.}\left[L^{-(d-2)/2} + L^{-d}(\ln L)^d\right]\\
&\leq K'L^{-(d-2)/2}.
\end{aligned}$$

□

LEMMA 13. Assume (P_K). *Then there exists a universal constant K'' and a constant L_0 depending on K such that*

$$|F(k,t)| \geq K''\left|1 - e^{it}\hat{D}_L(k)\right|. \tag{33}$$

PROOF. Write $F(k,t) = X + Y + Z$, where

$$X = 1 - pp_L^{-1}\big(\hat{D}_L(0) + \hat{\Psi}_c(0,0)\big),$$

$$Y = pp_L^{-1}\big(1 - e^{it}\hat{D}_L(k)\big),$$

$$Z = pp_L^{-1}\big(\hat{\Psi}_c(0,0) - e^{it}\hat{D}_L(k)\hat{\Psi}_c(k,t)\big).$$

If $p < p_c$, then $X \geq 0$. Since $\mathrm{Re}(Y) \geq 0$ we have $|X+Y| \geq |Y|$. Therefore, $|F| \geq |X+Y| - |Z| \geq |Y| - |Z|$. Applying (P_K) we have, for L sufficiently large,

$$\begin{aligned}
\left|\hat{\Psi}_c(k,t) - \hat{\Psi}_c(0,t)\right| &= \left|\int_0^1 ds(1-s)\frac{d^2}{ds^2}\hat{\Psi}_c(sk,t)\right| \\
&\leq (1/2)\sup_{0\leq s\leq 1}\left|\sum_{\mu\nu=1}^{d} k_\mu k_\nu\,\partial_{\mu\nu}\hat{\Psi}_c(sk,t)\right| \\
&\leq \text{const.}\ L^{-d+2}|\ln L|^d\left(\sum_{\mu=1}^{d} k_\mu\right)^2,
\end{aligned}$$

$$\begin{aligned}
\left|\hat{\Psi}_c(0,t) - \hat{\Psi}_c(0,0)\right| &\leq \int_0^1 ds\left|\frac{d}{ds}\hat{\Psi}_c(0,st)\right| \\
&\leq \text{const.}\ L^{-(d-2)/2}|\ln L|^d|t|
\end{aligned}$$

and

$$\begin{aligned}
|Z| \leq pp_L\Big\{&\left|\hat{\Psi}_c(0,0) - e^{it}\hat{\Psi}_c(0,0)\right| + \left|e^{it}\hat{\Psi}_c(0,0) - e^{it}\hat{\Psi}_c(0,t)\right| \\
&+ \left|e^{it}\hat{\Psi}_c(0,t) - e^{it}\hat{\Psi}_c(0,t)\hat{D}_L(k)\right| \\
&+ \left|e^{it}\hat{\Psi}_c(0,t)\hat{D}_L(k) - e^{it}\hat{\Psi}_c(k,t)\hat{D}_L(k)\right|\Big\}.
\end{aligned}$$

Combining these estimates we get, for any $\varepsilon > 0$, $|Z| \leq \varepsilon(|k|^2 + |t|)$ if L is sufficiently large. From Lemma 10, $|Y| \geq$ const. $(|k|^2L^2 + |t|) \geq \frac{1}{3}$, for $|k| \leq \delta$, and $|Y| \geq \frac{1}{3}$, for $|k| \geq \delta$. Therefore, there exist constants K'' and $L_0(K)$ such that $|F| \geq K''|1 - e^{it}\hat{D}_L(k)|$, for all $p_L \leq p < p_c$ and all $L \geq L_0$. □

LEMMA 14. *There exist a constant K' and a constant L_0 such that for all $p_L \leq p < p_c$, $L \geq L_0$, the assumption $(P_{3K'})$ implies $(P_{2K'})$.*

PROOF. To estimate pp_L^{-1}, we note that for $p < p_c$, $1 - pp_L(\hat{D}_L(0) + \hat{\Psi}_c(0,0)) \geq 0$. However, $\hat{\Psi}_c(0,0) = O(L^{-d}|\ln L|^d)$, so $pp_L^{-1} \leq 3$ if L is sufficiently large. To obtain the estimate for $\overline{T}_0$, we use the Fourier inversion formula:

$$\overline{T}_0 \leq (2\pi)^{-d-1} \iint dk\, dt\, pp_L^{-1} |\hat{D}_L(k)|\, |\hat{\Psi}(k,t)|^3.$$

By Lemma 13, the r.h.s. is bounded by

$$\text{const.}\ (2\pi)^{-d-1} \iint dk\, dt\, pp_L^{-1} |\hat{D}_L(k)|\, |1 + \hat{\Psi}(k)|^3 |1 - e^{it}\hat{D}_L(k)|^{-3}$$

$$\leq 2\,\text{const.}\ (2\pi)^{-d-1} \iint dk\, dt\, pp_L^{-1} |\hat{D}_L(k)|\, |1 - e^{it}\hat{D}_L(k)|^{-3}$$

if L is large enough, by (22) and assumption $(P_{3K'})$. Using the same argument as in estimating $\overline{T}_L$ of Lemma 11, $\overline{T}_0$ can be bounded by $2K'L^{2-d}|\ln L|^d$. The other estimates for W, U, W_{Ym} and U_{ym} can be handled similarly; the only change is that $\hat{\Psi}_c(k,t)$ must be replaced by $\partial_{\mu\nu}\hat{\Psi}_c(k,t)$ and $\partial_t\hat{\Psi}_c(k,t)$, and their $L_3([-\pi,\pi]^d \times [-\pi,\pi])$-norms can be estimated using (28) and (29).

Finally, we want to prove that the infrared bound stated in Theorem 2 holds. When $p \in (0, p_L)$, $\overline{T}$, W, U and so on can be estimated by the estimates based on the random walk with transition function p_{0x}^L; therefore, (P_{K^g}) holds for some constant K^g. Furthermore, since they are continuous functions of p for $p < p_c$, by Lemma 14 $(P_{2K'})$ holds for $p < p_c$ if L is large enough. This then gives the desired infrared bound stated in Theorem 2. □

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DEPARTMENT OF MATHEMATICS
ILLINOIS INSTITUTE OF TECHNOLOGY
CHICAGO, ILLINOIS 60616

DEPARTMENT OF MATHEMATICS
TEMPLE UNIVERSITY
PHILADELPHIA, PENNSYLVANIA 19122