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CONSERVATION OF LOCAL EQUILIBRIUM FOR ATTRACTIVE PARTICLE SYSTEMS ON Z^d

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We prove conservation of local equilibrium for attractive particle systems. Our method applies as well to gradient asymmetric processes with mean drift 0 under diffusive (N^2) rescaling. The hydrodynamical behavior is proved for bounded continuous initial profiles under Euler (N) rescaling and for bounded a.s. continuous profiles under diffusive rescaling. We prove that, for attractive systems, the conservation of local equilibrium follows from a law of large numbers for the density field.

Introduction. This paper is devoted to the study of conservation of local equilibrium for attractive particle systems. To explain the meaning of conservation of local equilibrium, consider a gas or particles evolving on a d-dimensional volume V. Suppose that all the equilibrium states of the systems are characterized by a macroscopic parameter p (the density, the temperature, etc.) in some set P. If the system is not in equilibrium, in small neigborhoods of each macroscopic point x of the volume V we expect the process to be near equilibrium due to the great number of shocks between particles in small amounts of time. This local equilibrium is characterized by a parameter $p(x)$, possibly different at each point x. Observing the evolution of the systems, the local situation changes and at time t the equilibrium around x is characterized by $p(t, x) \in P$. We expect the parameter $p(t, x)$ to change in a smooth way in time and space according to a differential equation, called the hydrodynamic equation.

To give a precise formulation of this phenomenon, in this article we consider interacting particle processes where the equilibrium states are characterized by one parameter, the density of particles. To fix ideas and to keep notation simple, we state all results for the zero range process. This Markov process can be informally described as follows. Consider indistinguishable particles moving on the d-dimensional integers $\mathbf{Z}^d$. Let g: $\mathbf{N} \to \mathbf{R}$ be a nonnegative function with $g(0) = 0$ and $P(k, j)$ transition probabilities on $\mathbf{Z}^d$. Suppose that there are n particles on a site k of $\mathbf{Z}^d$. These particles, independently of particles on other sites, wait a mean $1/g(n)$-exponential time at the end of which one of them jumps to j with probability $P(k, j)$. This process has an infinite family of extremal invariant measures ν_ρ characterized by the density $\rho \in P = \mathbf{R}_+$.

In the sequel, for $k \in \mathbf{Z}^d$, we denote by τ_k the translation by k in the space of configurations $X_d = \mathbf{N}^{\mathbf{Z}^d}$ and extend them to the functions and to the

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measures in the natural way. Hence, for $\eta \in X_d$ [$\eta(i)$ represents the number of particles at site $i \in \mathbf{Z}^d$ for the configuration η], $(\tau_k \eta)(j) = \eta(k+j)$, $j \in \mathbf{Z}^d$. In the same way, for $f\colon X_d \to \mathbf{R}$, $(\tau_k f)(\eta) = f(\tau_k \eta)$ and for a probability measure μ on X_d, $\int f\, d(\tau_k \mu) = \int (\tau_k f)\, d\mu$.

Let $\{\mu_N,\ N \ge 1\}$ be a sequence of probability measures on X_d. We shall say that the sequence (μ_N) satisfies the local equilibrium property for a profile $u\colon \mathbf{R}^d \to P$ if

$$\lim_{N\to\infty} \tau_{[xN]}\mu_N = \nu_{u(x)} \quad \text{for every continuity point } x \text{ of } u,$$

where $[r]$ denotes the integer part of r and the limit, as all measure limits in this paper, is taken in the weak* sense.

Let S_t be the semigroup of the Markov process. We shall say that there is conservation of local equilibrium if there exists a time renormalization $T(N)$ and a function $u\colon \mathbf{R}_+ \times \mathbf{R}^d \to P$ such that for all $t \ge 0$,

$$\lim_{N\to\infty} \tau_{[xN]} S_{T(N)t}\mu_N = \nu_{u(t,x)} \quad \text{for every continuity point } x \text{ of } u(t,\cdot).$$

We expect $u(t,x)$ to be the solution of some P.D.E. with initial condition given by $u(x)$. This partial differential equation is called the hydrodynamic equation of the process. The time renormalization $T(N)$ is N^2 in the case where the mean drift of each particle is equal to 0 and N otherwise.

This problem has been considered before by several authors. Rost [20] and Galves, Kipnis, Marchioro, and Presutti [7] gave in 1981 the first contribution to the field. In [6] and [21] one can find a list of references.

In this paper, we prove conservation of local equilibrium for a large class of initial profiles for both asymmetric and symmetric cases from results obtained by Yau in [23], by Rezakhanlou in [19] and by the author in [12].

The conservation of local equilibrium is a corollary of the main theorem in this paper which states that it is possible to prove conservation of local equilibrium for attractive processes once we have a law of large numbers for the density field and if the solution of the hydrodynamic equation has some regularity properties (cf. Theorem 1). A law of large numbers for the density field can be proved by entropy arguments ([8], [22]), superexponential estimates ([9], [13]) or with Young measures and Rezakhanlou's method ([22], [19]). In this way, we can prove conservation of local equilibrium for a wide class of particle systems.

1. Notation and results. In this section we establish the notation and state the main results of the article.

This article is devoted to the study of conservation of local equilibrium for attractive particle systems. To fix ideas, we consider throughout the paper the zero range process. The reader will see below that the unique important assumptions are the attractiveness and some regularity of the hydrodynamic equations. In the diffusive rescaling, we also need the process to be gradient. Therefore the method presented applies to a wide class of particle systems.

Before defining the process, we introduce some notation. Throughout this paper, we consider $\mathbf{R}^d$ with the norm

$$|x| = \max\{|x_i|, 1 \le i \le d\}.$$

For a in $\mathbf{R}$ and $x = (x_1, \dots, x_d)$ in $\mathbf{R}^d$, we define $x + a$ and $[x]$ as

$$x + a = (x_1 + a, \dots, x_d + a), \qquad [x] = ([x_1], \dots, [x_d]),$$

where for $r \in \mathbf{R}$, $[r]$ denotes the integer part of r. For x, y in $\mathbf{R}^d$, we write $x \le y$ when $x_i \le y_i$ for $1 \le i \le d$.

The state space of the process $\mathbf{N}^{\mathbf{Z}^d}$ is denoted by X_d and the configurations by Greek letters η, ξ and χ. In this way, for $k \in \mathbf{Z}^d$, $\eta(k) \in \mathbf{N}$ represents the number of particles in site k for the configuration η.

The zero-range process $(\eta_t)_{t \ge 0}$, informally described in the introduction, is the Markov process on X_d whose generator acts on functions that depend only on a finite number of coordinates as follows:

$$(L_N f)(\eta) = \theta(N) \sum_{k, j \in \mathbf{Z}^d} g(\eta(k)) P(k, j) \left[f(\eta^{k,j}) - f(\eta) \right], \tag{1.1}$$

where, for configurations η such that $\eta(k) \ge 1$,

$$\eta^{k,j}(i) = \begin{cases} \eta(i), & \text{if } i \ne k, j, \\ \eta(k) - 1, & \text{if } i = k, \\ \eta(j) + 1, & \text{if } i = j. \end{cases} \tag{1.2}$$

The functions which depend only on a finite number of coordinates are called cylinder functions. We now state the main hypotheses on the process. Throughout this paper we assume the following:

ASSUMPTION 1. P is an irreducible translation invariant transition probability on $\mathbf{Z}^d$ with finite range,

$$P(k, j) = P(0, j - k) = p(j - k),$$

and there exists

$$A \in \mathbf{N} \quad \text{such that} \quad p(k) = 0 \text{ if } |k| \ge A.$$

ASSUMPTION 2. g is nondecreasing, $0 = g(0) < g(1)$,

$$G = \sup_n \{g(n + 1) - g(n)\} < \infty.$$

Let γ denote the mean drift of the particles:

$$\gamma = \sum_{k \in \mathbf{Z}^d} k p(k) \in \mathbf{R}^d. \tag{1.3}$$

In the case where $\gamma = 0$ let σ_{ij}, $1 \le i, j \le d$, be the diffusion coefficient,

$$\sigma_{ij} = \sum_{k \in \mathbf{Z}^d} k_i k_j p(k) \in \mathbf{R} \quad \text{if } k = (k_1, \dots, k_d). \tag{1.4}$$

It is easy to see that $\sigma = (\sigma_{ij})$ is a symmetric nonnegative definite matrix. To avoid degeneracy of the hydrodynamic equation, we assume (σ_{ij}) to be positive definite:

ASSUMPTION 3. There exist $\kappa > 0$ such that,

$$\sum_{i,j} \sigma_{ij} x_i x_j \ge \kappa \sum_i x_i^2 \quad \text{for every } x \in \mathbf{R}^d.$$

Let $\theta(N)$ be a renormalizing factor equal to N when $\gamma \ne 0$ and N^2 otherwise:

$$\theta(N) = \begin{cases} N^2, & \text{if } \gamma = 0, \\ N, & \text{otherwise}. \end{cases} \tag{1.5}$$

The reader should notice that in the diffusive rescaling we do not assume the transition probabilities to be symmetric.

The existence of this Markov process is proved in [1]. Before proceeding, we introduce some notation. Let S_t^N denote the semigroup of the Markov process with generator (1.1) and let $\mathscr{I}$ be the set of probability measures invariant under (S_t^N). Let $\{\tau_k,\ k \in \mathbf{Z}^d\}$ be the group of shifts on X_d: $\tau_k \eta$ is the configuration of X_d such that $(\tau_k \eta)(j) = \eta(k+j)$ for every $j \in \mathbf{Z}^d$. We extend the shift to the functions and to the measures in the natural way: $\tau_k f(\eta) = f(\tau_k \eta)$ and $\int f d(\tau_k \mu) = \int (\tau_k f)\, d\mu$. Let $\mathscr{S}$ denote the set of shift-invariant probability measures on X_d. It follows from Assumption 1 that S_t^N and τ_k commute.

We introduce in X_d the partial order defined by $\eta \le \chi$ if $\eta(k) \le \chi(k)$ for all $k \in \mathbf{Z}^d$. A continuous function f is said to be monotone if $f(\eta) \le f(\chi)$ whenever $\eta \le \chi$. We denote by $\mathscr{M}$ the set of monotone functions and we extend the partial order to the measures on X_d in the natural way: $\mu \le \nu$ if $\int f d\mu \le \int f d\nu$ for every monotone function f. A Feller process is said to be attractive if its semigroup S_t preserves the partial order: $\mu \le \nu \Rightarrow \mu S_t \le \nu S_t$ for every $t > 0$. It is proved in [1] that the monotonocity of g assumed in Assumption 2 implies the attractiveness of the zero-range process.

We now describe the invariant state of the process.

For $0 \le \varphi < \sup_k g(k)$, let ν_φ be the product measure on X_d with marginals given by

$$\nu_\varphi\{\eta; \eta(k) = j\} = \begin{cases} \dfrac{1}{Z(\varphi)} \dfrac{\varphi^j}{g(1) \cdots g(j)}, & \text{if } j \ge 1, \\ \dfrac{1}{Z(\varphi)}, & \text{if } j = 0, \end{cases}$$

where $Z(\varphi)$ is a normalizing constant. It is easy to see that this family of measures is continuous and increasing with φ:

$$\varphi_1 \le \varphi_2 \Rightarrow \nu_{\varphi_1} \le \nu_{\varphi_2},$$

$$\lim_{n\to\infty} \varphi_n = \varphi_0 \Rightarrow \lim_{n\to\infty} \nu_{\varphi_n} = \nu_{\varphi_0},$$

where all limits of measure in this article are to be understood in the weak* sense. Andjel [1] proved that this family is the extremal set of $(\mathscr{I} \cap \mathscr{S})$.

Let $\rho(\varphi)$ be the density of particles of the measure ν_φ,

$$\rho(\varphi) = \nu_\varphi[\eta(0)].$$

It is easy to see that ρ: $[0, \sup_k g(k)) \to [0, \infty)$ is a smooth strictly increasing bijection. Since $\rho(\varphi)$ has a physical meaning as the density of particles, instead of parametrizing this family of measures by φ, we use the density ρ as the parameter and we write,

$$\nu_\rho = \nu_{\varphi(\rho)}, \qquad \rho \ge 0. \tag{1.6}$$

With this convention, it is easy to see that

$$\varphi(\rho) = \nu_\rho[g(\eta(0))], \qquad \rho \ge 0. \tag{1.7}$$

Moreover, $\varphi(\rho) \le G\rho$, φ is in $C^\infty(\mathbf{R}_+)$ and φ' is bounded below by a positive constant on each compact subset of $\mathbf{R}_+$.

For every cylinder function Ψ on X_d, we define $\tilde{\Psi}$: $\mathbf{R}_+ \to \mathbf{R}$ by

$$\tilde{\Psi}(\rho) = \nu_\rho[\Psi(\eta)], \tag{1.8}$$

where ν_ρ is the product measure defined in (1.6).

Finally, we introduce the differential operator which describes the macroscopic evolution of the system.

Throughout this paper, $\mathscr{L}$ denotes a differential operator. Under Euler rescaling, $\mathscr{L}$ is a quasilinear first-order partial differential operator,

$$\mathscr{L}\rho = \frac{\partial \rho}{\partial t} + \sum_{j=1}^{d} \gamma_j \frac{\partial}{\partial x_j}\varphi(\rho) \tag{1.9}$$

and in the case where $\gamma = 0$ so that the process is rescaled by N^2, it is a quasilinear second-order partial differential operator,

$$\mathscr{L}\rho = \frac{\partial \rho}{\partial t} - \sum_{i,j=1}^{d} \sigma_{ij} \frac{\partial^2}{\partial x_i\, \partial x_j}\varphi(\rho), \tag{1.10}$$

where γ and σ were defined in (1.3) and (1.4). In both cases φ is smooth and increasing. Associated with the operators $\mathscr{L}$, we have differential equations

$$\begin{aligned} \mathscr{L}\rho &= 0, \\ \rho(0, \cdot) &= \rho_0(\cdot), \end{aligned} \tag{1.11}$$

with $\mathscr{L}$ as in (1.9) or (1.10). In the Appendix we fix the terminology of weak solutions to (1.9), (1.11) and (1.10), (1.11).

We now define approximating sequences for an initial profile ρ_0. For each $\rho_0 \in L^\infty(\mathbf{R}^d)$, let $(\rho_N)_{N \ge 1}$ be a sequence in $L^\infty(\mathbf{R}^d)$ such that:

(S1) ρ_N is nonnegative and constant in $[k/N, (k+1)/N))$ for every $k \in \mathbf{Z}^d$;

(S2) $\limsup_{N\to\infty} \|\rho_N\|_\infty < \infty$;

(S3) (i) $\rho_N \to \rho_0$ in $L^\infty_{\text{loc}}(\mathbf{R}^d)$ or (ii) $\rho_N(x) = \int_{[[xN]/N, ([xN]+1)/N]} N^d \rho_0(z)\,dz$, $x \in \mathbf{R}^d$, where for $x \le y \in \mathbf{R}^d$,

$$[x, y] = \{z \in \mathbf{R}^d;\ x_i \le z_i \le y_i,\ 1 \le i \le d\}. \tag{1.12}$$

In Section 3, we replace hypothesis (S3) by weaker ones. Sequences $(\rho_N)_{N\ge 1}$ satisfying assumptions (S1), (S2) and converging to ρ_0 in some topology are called approximating sequences of ρ_0.

Given an approximating sequence $(\rho_N)_{N\ge1}$ of ρ_0, we define a sequence $(\mu_N)_{N\ge1}$ of product probability measures on X_d corresponding to the initial profile ρ_0. We let μ_N be the product measure on X_d with marginals given by

$$\mu_N\{\eta;\ \eta(k) = j\} = \nu_{\rho_N(k/N)}\{\eta;\ \eta(0) = j\}, \qquad j \ge 0,\ k \in \mathbf{Z}^d, \tag{1.13}$$

where, for $\rho \ge 0$, ν_ρ is the product measure defined in (1.6).

Throughout this paper $C_K(\mathbf{R}^d)$ denotes the real continuous functions on $\mathbf{R}^d$ with compact support and $C_b(\mathbf{R}^d)$ the bounded real continuous functions on $\mathbf{R}^d$. We are now ready to state the theorems.

Theorem 1, the main theorem in this paper, states that conservation of local equilibrium follows from a law of large numbers for the density field and regularity assumptions on the hydrodynamic equation. In Theorems 2–5, from this result, we prove conservation of local equilibrium for some attractive interacting particle systems.

To state Theorem 1, we introduce a general setup.

For a profile ρ_0 in $L^\infty(\mathbf{R}^d)$ and for $\epsilon > 0$, define $\rho_0^{\epsilon,+}$ and $\rho_0^{\epsilon,-}$ as

$$\begin{aligned} \rho_0^{\epsilon,+}(x) &= \sup_{x \in [x-\epsilon, x+\epsilon]} \rho_0(z), \\ \rho_0^{\epsilon,-}(x) &= \inf_{z \in [x-\epsilon, x+\epsilon]} \rho_0(z), \end{aligned} \tag{1.14}$$

where, for x, y in$\mathbf{R}^d$, $[x, y]$ is defined in (1.12). In what follows, we sometimes denote ρ_0 by $\rho_0^{0,+}$ or by $\rho_0^{0,-}$.

For a fixed attractive interacting particle system (η_t) with generator L_N and for a partial differential operator $\mathscr{L}$, we consider functional spaces $\mathscr{F} = \mathscr{F}((L_N), \mathscr{L})$. In Theorem 1, we prove that if $\mathscr{F}$ satisfies the assumptions listed below then there is conservation of local equilibrium for every profile ρ_0 in $\mathscr{F}$. To fix ideas, the reader may think of L_N and $\mathscr{L}$ as defined by (1.1) and (1.9) and $\mathscr{F}$ as the space of continuous functions with compact support.

For a subset P of $\mathbf{R}_+$, let $\{m_\rho;\ \rho \in P\}$ be an ordered ($\rho_1 \le \rho_2 \Rightarrow m_{\rho_1} \le m_{\rho_2}$) and continuous family of invariant measures for the process with generator L_N. For a cylinder function Ψ, denote $m_\rho(\Psi)$ by $\tilde{\Psi}(\rho)$.

The assumptions are the following:

HYPOTHESIS 1. *For every ρ_0 in $\mathscr{F}$ and every $0 \le \epsilon \le 1$, there is a unique weak solution of $\mathscr{L}\rho = 0$ with initial data $\rho_0^{\epsilon, +}$ and $\rho_0^{\epsilon, -}$.*

The concept of weak solution will change from case to case. For instance, in the context of first order quasilinear hyperbolic equations it will mean entropic weak solution.

HYPOTHESIS 2. *For every ρ_0 in $\mathscr{F}$, let ρ ($\rho^{\epsilon, +}$) be the unique weak solution of $\mathscr{L}\rho = 0$ with initial data ρ_0 ($\rho_0^{\epsilon, +}$). There exists a version $\bar{\rho}^{\epsilon, +}$ of $\rho^{\epsilon, +}$ ($\bar{\rho}^{\epsilon, +} = \rho^{\epsilon, +}$ a.s.) such that for every $t > 0$ and every continuity point x of $\rho(t, \cdot)$,*

$$\lim_{\epsilon \to 0} \sup_{|z - x| \le \epsilon} |\bar{\rho}^{\epsilon, +}(t, z) - \rho(t, x)| = 0.$$

The same property is required for $\rho^{\epsilon, -}$ the unique weak solution of $\mathscr{L}\rho = 0$ with initial data $\rho_0^{\epsilon, -}$.

HYPOTHESIS 3. *For every $0 < \epsilon \le 1$ and every approximating sequence $(\rho_N)_{N \ge 1}$ of ρ_0 satisfying* (S1), (S2) *and* (S3), *there exists an approximating sequence $\rho_N^{\epsilon, +}$ ($\rho_N^{\epsilon, -}$) of $\rho_0^{\epsilon, +}$ ($\rho_0^{\epsilon, -}$) such that*:

(i) *For every bounded cylinder function Ψ, x, y in $\mathbf{R}^d$, $x \le y$ and $t \ge 0$,*

$$\lim_{N \to \infty} E_N^{\epsilon, \pm}\left[\frac{1}{N^d} \sum_{k \in \mathbf{Z}^d} \mathbf{1}\{[x, y]\}\left(\frac{k}{N}\right) \tau_k \Psi(\eta_t)\right] = \int_{[x, y]} \tilde{\Psi}(\rho^{\epsilon, \pm}(t, z))\, dz,$$

where $\tilde{\Psi}(\rho) = m_\rho[\Psi]$, $\rho^{\epsilon, +}$ ($\rho^{\epsilon, -}$) is the unique weak solution of (1.11) *with initial data $\rho_0^{\epsilon, +}$ ($\rho_0^{\epsilon, -}$), and $E_N^{\epsilon, +}$ ($E_N^{\epsilon, -}$) is the expectation on the path space $D([0, \infty), X_d)$ for the Markov process with generator L_N and with initial state distributed according to the product measure $\mu_N^{\epsilon, +}$ ($\mu_N^{\epsilon, -}$) defined from the approximating sequence $\rho_N^{\epsilon, +}$ ($\rho_N^{\epsilon, -}$) by* (1.13).

(ii) *There exists $\delta(\epsilon) > 0$ such that*

$$\rho_N^{\epsilon, -}(x) \le \rho_N(x + a) \le \rho_N^{\epsilon, +}(x) \quad \text{for every } |a| \le \delta,\ N \ge 1.$$

THEOREM 1. *Let (η_t) be an attractive particle process for which there exists a functional subspace $\mathscr{F}$ of $L^\infty(\mathbf{R}^d)$ satisfying Hypotheses 1–3. Then for every ρ_0 in $\mathscr{F}$ and every approximating sequence ρ_N of ρ_0 with properties* (S1), (S2) *and* (S3),

$$\lim_{N\to\infty} \mu_N S_t^N \tau_{[xN]} = m_{\rho(t,x)}$$

for every $t > 0$ and every continuity point x of $\rho(t,\cdot)$,

where μ_N is the product measure defined by (1.13) *and ρ the solution of* (1.11).

We now state the results concerning conservation of local equilibrium.

THEOREM 2. *Assume that $\gamma = 0$ and Assumptions* 1–3 *hold. Let ρ_0: $\mathbf{R}^d \to \mathbf{R}_+$ be an almost surely continuous function for which there exist constants $c < \infty$ and $\alpha > d$ such that*

$$\rho_0(x) \le \frac{c}{1 + |x|^\alpha}, \qquad x \in \mathbf{R}^d.$$

Let $(\rho_N)_{N\ge 1}$ be an approximating sequence satisfying (S1), (S2) *and* (S3). *Let μ_N be the product measure defined in* (1.13). *Then*

$$\lim_{N\to\infty} \mu_N S_t^N \tau_{[xN]} = \nu_{\rho(t,x)} \quad \text{for every } t > 0,\ x \in \mathbf{R}^d,$$

where $\rho(t,x)$ is the unique weak solution of (1.10), (1.11) *such that $\|\rho(t,\cdot)\|_1 \le \|\rho_0\|_1$ for every $t \ge 0$.*

THEOREM 3. *Assume that $\theta(N) = N$ and Assumptions* 1 *and* 2 *hold. Let $\rho_0 \in C_b(\mathbf{R}^d)$, $(\rho_N)_{N\ge 1}$ be an approximating sequence satisfying* (S1), (S2) *and* (S3) *and μ_N be the product measure defined in* (1.13). *Suppose that φ is strictly concave or convex in the range of ρ_0. Then*

$$\lim_{N\to\infty} \mu_N S_t^N \tau_{[xN]} = \nu_{\rho(t,x)}$$

for every $t > 0$ and every continuity point x of $\rho(t,\cdot)$, where $\rho(t,x)$ is the unique entropic weak solution of (1.9) *and* (1.11).

In dimension 1, we can prove an even stronger result for the Euler rescaling.

REMARK 1.1. In Theorem 3, in dimension 1, we may assume that the initial profile ρ_0 is bounded and almost surely continuous.

From Theorem 3 we consider the cases not covered by Theorem 1 of [12]. Define a cone $\mathscr{H}$ by

$$\mathscr{H} = \{x \in \mathbf{R}^d;\ x_i \ge 0,\ 1 \le i \le d\}.$$

COROLLARY. *For $\alpha, \beta \ge 0$, define the product measure $\mu_{\alpha,\beta}$ on X_d by*

$$\mu_{\alpha,\beta}\{\eta;\ \eta(k) = n\} = \begin{cases} \nu_\beta\{\eta;\ \eta(k) = n\}, & \text{if } k \in \mathscr{H}, \\ \nu_\alpha\{\eta;\ \eta(k) = n\}, & \text{otherwise}. \end{cases}$$

Suppose that the mean drift γ *defined in* (1.3) *does not belong to* $\mathscr{H} \cup (-\mathscr{H})$. *Then*

$$\lim_{N\to\infty} \mu_{\alpha,\beta} S_t^N \tau_{[xN]} = \nu_{\rho(t,x)}$$

for every $t > 0$ *and every continuity point* x *of* $\rho(t,\cdot)$, *where* $\rho(t,x)$ *is the unique entropic weak solution of* (1.9) *and* (1.11) *with initial data* ρ_0 *defined by*

$$\rho_0(x) = \begin{cases} \beta, & \text{if } x \in \mathscr{H}, \\ \alpha, & \text{otherwise}. \end{cases}$$

The easier case where γ belongs to $\mathscr{H} \cup (-\mathscr{H})$ is studied in [12].

REMARK 1.2. Theorem 3 extends to "misanthrope" processes [4]. On the other hand, to prove Theorem 2, we need the process to be gradient. Therefore our proof of Theorem 2 applies only to gradient "misanthrope" processes. We consider, for instance, the symmetric simple exclusion process in Theorem 5.

In the next theorem we present a partial answer to an open problem stated in ([15], Problem VIII.7.10) for the asymmetric zero-range process with mean drift $\gamma = 0$.

In dimension 1, let $m_{\alpha,\beta}$ be the product measure whose marginals are given by

$$m_{\alpha,\beta}\{\eta;\eta(k) = n\} = \begin{cases} \nu_\alpha\{\eta;\eta(k) = n\}, & \text{if } k < 0, \\ \nu_\beta\{\eta;\eta(k) = n\}, & \text{if } k \ge 0. \end{cases}$$

THEOREM 4. *Assume that* $\gamma = 0$ *and Assumptions* 1 *and* 2 *hold. Then*

$$\lim_{N\to\infty} m_{\alpha,\beta} S_t^N = \nu_{\rho(t)},$$

where $\rho(t) = \rho(t,0)$ *and* $\rho(t,x)$ *is the unique weak solution of* (1.10) *and* (1.11) *with initial data* ρ_0 *defined by*

$$\rho_0(\cdot) = \alpha \mathbf{1}\{(-\infty,0)\}(\cdot) + \beta \mathbf{1}\{[0,\infty)\}(\cdot)$$

such that

(a) $\rho(t,\cdot) - \rho_0(\cdot) \in L^1(\mathbf{R}^d)$ *for every* $t \ge 0$,
(b) $\lim_{t\to 0} \|\rho(t,\cdot) - \rho_0(\cdot)\|_1 = 0$.

Finally, we consider the symmetric simple exclusion process. This is the Markov process on the state space $\{0,1\}^{\mathbf{Z}^d}$ whose generator acts on cylinder functions as

$$(1.15)\quad (L_N f)(\eta) = N^2 \sum_{k,j\in\mathbf{Z}^d} \eta(k)[1-\eta(j)]p(j-k)[f(\eta^{k,j}) - f(\eta)],$$

where $\eta^{k,j}$ is defined in (1.2).

In addition to Assumptions 1 and 3, we assume that the transition probabilities are symmetric. In this way the process is gradient.

ASSUMPTION 4. $p(k) = p(-k)$ for every $k \in \mathbf{Z}^d$.

To state the next theorem, we introduce some notation. For $0 \le \rho \le 1$, let $\bar{\nu}_\rho$ be the translation invariant product measure on $\{0,1\}^{\mathbf{Z}^d}$ with marginals given by

$$\bar{\nu}_\rho\{\eta;\, \eta(k) = 1\} = \rho. \tag{1.16}$$

The macroscopic behavior of the symmetric simple exclusion is described by a linear partial differential operator $\mathscr{L}$,

$$\mathscr{L} = \frac{\partial}{\partial t} - \sum_{i,j=1}^{d} \sigma_{ij} \frac{\partial^2}{\partial x_i\, \partial x_j}. \tag{1.17}$$

In the next theorem, we apply Theorem 1 to symmetric simple exclusion processes. In this way, we obtain another proof of the hydrodynamical behaviour of this process (cf. [7], [5]).

THEOREM 5. *Consider the symmetric simple exclusion process. Assume Assumptions* 1, 3 *and* 4. *Let* $\rho_0\colon \mathbf{R}^d \to [0,1]$ *be an almost surely continuous function and let* $(\rho_N)_{N\ge 1}$ *be an approximating sequence satisfying* (S1), (S2) *and* (S3). *Let* μ_N *be the product measure defined in* (1.13). *Then*

$$\lim_{N\to\infty} \mu_N S_t^N \tau_{[xN]} = \bar{\nu}_{\rho(t,x)} \quad \text{for every } t > 0,\ x \in \mathbf{R}^d,$$

where $\rho(t,x)$ *is the unique weak solution of* (1.17) *and* (1.11).

The paper is divided as follows. In Section 2, we prove Theorem 1. In Section 3, we prove a law of large numbers for the density field in the diffusive regime. This result is based on a recent work of Yau and was obtained by Rezakhanlou for processes in the Euler regime. In Section 4 we prove Theorems 2–5.

2. Proof of Theorem 1.

Fix ρ_0 in $\mathscr{F}$ and ρ_N an approximating sequence of ρ_0 with properties (S1), (S2) and (S3). Let μ_N be the product measure defined by (1.13).

Let $\rho_N^{\epsilon,+}$ be the approximating sequence of $\rho_0^{\epsilon,+}$ given by Hypothesis 3 and let $\mu_N^{\epsilon,+}$ be the product measure defined from $\rho_N^{\epsilon,+}$ by (1.13). From Hypothesis 3(ii),

$$\mu_N \tau_k \le \mu_N^{\epsilon,+} \quad \text{for every } |k| \le \delta N.$$

Since the process is attractive,

$$\mu_N S_t^N \tau_k \le \mu_N^{\epsilon,+} S_t^N \quad \text{for every } |k| \le \delta N,\ t \ge 0.$$

Therefore, for Ψ a monotone bounded cylinder function and $x \in \mathbf{R}^d$,

$$\limsup_{N\to\infty} \mu_N S_t^N \tau_{[xN]}[\Psi] \le \lim_{N\to\infty} \frac{1}{(2[\delta N]+1)^d} \sum_{|k-[xN]|\le[\delta N]} \mu_N^{\epsilon,+} S_t^N \tau_k[\Psi]$$
$$= \frac{1}{(2\delta)^d} \int_{|z-x|\le\delta} \tilde{\Psi}(\rho^{\epsilon,+}(t,z))\,dz,$$

by Hypothesis 3(i).

Taking $\delta(\epsilon) \le \epsilon$ and x a continuity point of $\rho(t,\cdot)$, by Hypothesis 2 it follows that

$$\lim_{\varepsilon\to 0} \frac{1}{(2\delta(\epsilon))^d} \int_{|z-x|\le\delta(\epsilon)} \tilde{\Psi}(\rho^{\epsilon,+}(t,z))\,dz = \tilde{\Psi}(\rho(t,x)).$$

In conclusion, we proved that for every bounded monotone cylinder function Ψ, every $t \ge 0$ and every continuity point x of $\rho(t,\cdot)$,

$$\limsup_{N\to\infty} \mu_N S_t^N \tau_{[xN]}[\Psi] \le \tilde{\Psi}(\rho(t,x))$$
$$= m_{\rho(t,x)}[\Psi].$$

Considering $\rho^{\epsilon,-}$ and $\rho_N^{\epsilon,-}$ instead of $\rho^{\epsilon,+}$ and $\rho_N^{\epsilon,+}$, in the same way we prove the converse inequality, and this concludes the proof since every bounded cylinder function is the difference of two bounded monotone cylinder functions. □

3. A law of large numbers for the density field. Throughout this section we assume for the zero range process with generator (1.1) that $\gamma = 0$ and Assumptions 1–3 hold.

To state the main result of this section, we have to introduce some notation. We start by replacing assumption (S3) for approximating sequences $(\rho_N)_{N\ge 1}$ by weaker hypotheses:

(S4) $\rho_N \to \rho_0$ in $L^1_{\mathrm{loc}}(\mathbf{R}^d)$.

In some cases we need a slightly stronger hypothesis on (ρ_N):

(S5) $\rho_N \to \rho_0$ in $L^1(\mathbf{R}^d)$.

For a probability measure μ_N on X_d, denote by P_N the probability measure on the path space $D([0,\infty), X_d)$ corresponding to the Markov process with generator given by (1.1) and initial measure μ_N and by E_N the expectation with respect to P_N.

In this section, we prove the following theorem.

THEOREM 3.1. *Let $\rho_0 \in L^1(\mathbf{R}^d) \cap L^\infty(\mathbf{R}^d)$, let $(\rho_N)_{N\ge 1}$ be an approximating sequence satisfying* (S1), (S2) *and* (S4) *and let μ_N be the product measure* *defined in* (1.13). *For every* $J \in C_K(\mathbf{R}^d)$, *every bounded cylinder function* Ψ *and every* $t \geq 0$,

$$\lim_{N\to\infty} E_N\left[\frac{1}{N^d}\sum_{k\in Z^d} J\left(\frac{k}{N}\right)\tau_k\Psi(\eta_t)\right] = \int_{R^d} J(x)\tilde{\Psi}(\rho(t,x))\,dx,$$

where $\tilde{\Psi}$ *is defined in* (1.8) *and* $\rho(t,x)$ *is the unique weak solution of* (1.10) *and* (1.11) *such that* $\|\rho(t,\cdot)\|_1 \leq \|\rho_0\|_1$ *for every* $t \geq 0$.

The proof of Theorem 3.1. is based on recent work of Yau on relative entropy and on Rezakhanlou's method of replacing local quantities by functions of the microscopic density field. To state Yau's result we have to introduce some notation.

For a fixed $K \in N$, let $(S_N^K)^d$ be the d-dimensional torus with $2NK$ points,

$$(S_N^K)^d = \{-NK,\dots,NK-1\}^d$$

and let $(S^K)^d$ be the d-dimensional torus $[-K,K)^d$. To keep notation simple, since K and d are fixed, we omit them when no confusion arises.

Let (ξ_t) be the zero-range process with state space $Y_N = N^{S_N}$, that is, the Markov process on Y_N whose generator acts on functions $f\colon Y_N \to \mathbf{R}$ as

$$(3.1)\qquad (L_N f)(\xi) = N^2 \sum_{k,j\in S_N} g(\xi(k))p(j-k)\left[f(\xi^{k,j}) - f(\xi)\right],$$

where $\xi^{k,j}$ is defined in (1.2) and the sums are taken modulo $2KN$. Denote by $(\tilde{S}_t^N)_{t\geq 0}$ the semigroup of this Markov process on Y_N.

Let $\rho_0 \in C^3(S)$ such that

$$(3.2)\qquad \epsilon = \inf_{x\in S^d}\rho_0(x) > 0,$$

and let $\rho(t,x)$ be the unique solution of the differential equation on the torus

$$(3.3)\qquad \begin{aligned} \partial_t\rho &= \sum_{i,j}\sigma_{ij}\partial^2_{x_i,x_j}\varphi(\rho),\\ \rho(0,\cdot) &= \rho_0(\cdot). \end{aligned}$$

From the maximum principle,

$$\epsilon \leq \rho(t,x) \leq \|\rho_0\|_\infty, \qquad t\geq 0,\ x\in S.$$

On the other hand, from Theorem 14 of [17], $\rho(t,\cdot) \in C^3(S^d)$.

For each $t \geq 0$, define ν_t^N as the product probability measure on Y_N with marginals defined by

$$(3.4)\qquad \nu_t^N\{\eta;\eta(k) = n\} = \nu_{\rho(t,k/N)}\{\eta;\eta(0) = n\},\qquad k\in S_N,\ n\geq 0,$$

where $\rho(t,x)$ is the solution of (3.3) and for $\rho \geq 0$, ν_ρ represents the product measure introduced in (1.6).

Let μ and ν be two probability measures on Y_N. Define the relative entropy of μ with respect to ν by

$$H(\mu|\nu) = \sup_{u \in C_b(Y_N)} \left\{ \int u(\eta)\mu(d\eta) - \log \int e^{u(\eta)}\nu(d\eta) \right\}.$$

It is well known that the entropy of μ with respect to ν is given by

$$H(\mu|\nu) = \int \frac{d\mu}{d\nu} \log \frac{d\mu}{d\nu}\, d\nu,$$

if μ is absolutely continuous with respect to ν, and $H(\mu|\nu) = \infty$ otherwise. In the above formula, $d\mu/d\nu$ denotes the Radon–Nikodym derivative of μ with respect to ν. Moreover, for every subset A of Y_N, from the definition of the entropy, we obtain that

$$\mu(A) \le \frac{\log 2 + H(\mu|\nu)}{\log\{1 + \nu(A)^{-1}\}}. \tag{3.5}$$

With these notations established, we may state the theorem.

THEOREM 3.2 (Yau). *Let $\rho(t,x)$ be the unique solution of* (3.3) *with initial data ρ_0 in $C^3(S)$ satisfying* (3.2). *For $t \ge 0$, define the probability measure ν_t^N by* (3.4) *and let μ_N be a probability measure on Y_N. Suppose that $H(\mu_N|\nu_0^N) = o(N^d)$. Then $H(\mu_N \tilde{S}_t^N|\nu_t^N) = o(N^d)$.*

We omit the proof since it is not difficult to adapt the one of [23] to the zero-range context. Nevertheless, we point out that this proof requires the process to be gradient.

For $l \in N$, $k \in \mathbf{Z}^d$ and $\eta \in Y_N$ or X_d, define $\eta^l(k)$ by

$$\eta^l(k) = \frac{1}{(2l+1)^d} \sum_{|j-k| \le l} \eta(j). \tag{3.6}$$

From Theorem 3.2, we prove the following corollary.

COROLLARY 3.3. *Let ρ_0 be in $C^3(S)$ with property* (3.2), *and let $(\rho_N)_{N \ge 1}$ be an approximating sequence of ρ_0 satisfying assumptions* (S1), (S2) *and* (S5) *with S replacing $\mathbf{R}^d$ and for which there exists $\delta > 0$,*

$$\delta \le \inf_{N,x} \rho_N(x).$$

Let μ_N be the product measure defined in (1.13). *Then, for every $J \in C(S)$, every $\tilde{\Psi} \in C_b^1(\mathbf{R}_+)$ and every $t \ge 0$,*

$$\lim_{l \to \infty} \lim_{N \to \infty} E_N \left[\frac{1}{N^d} \sum_{k \in S_N} J\left(\frac{k}{N}\right) \tilde{\Psi}(\eta_t^l(k)) \right] = \int_S J(x) \tilde{\Psi}(\rho(t,x))\, dx,$$

where ρ is the unique solution of (3.3).

PROOF. Fix $J \in C(S)$, $\tilde{\Psi} \in C_b^1(\mathbf{R}_+)$ and $t \geq 0$. For each $\delta > 0$, define $A_\delta \subset Y_N$ by

$$A_\delta = \left\{\eta;\left|\frac{1}{N^d}\sum_k J\left(\frac{k}{N}\right)\left\{\tilde{\Psi}(\eta^l(k)) - \tilde{\Psi}\left(\rho\left(t,\frac{k}{N}\right)\right)\right\}\right| > \delta\right\}.$$

Since J and $\tilde{\Psi}$ are bounded and

$$\lim_{N\to\infty}\frac{1}{N^d}\sum_k J\left(\frac{k}{N}\right)\tilde{\Psi}\left(\rho\left(t,\frac{k}{N}\right)\right) = \int_S J(x)\tilde{\Psi}(\rho(t,x))\,dx,$$

to prove the corollary it is enough to show that for every $\delta > 0$,

$$\limsup_{l\to\infty}\limsup_{N\to\infty}\mu_N(t)[A_\delta] = 0, \tag{3.7}$$

where

$$\mu_N(t) = \mu_N \tilde{S}_t^N.$$

Since $\rho_N \to \rho_0$ in $L^1(S)$ and ρ_N is bounded below by a positive constant and above by a finite constant, a simple computation shows that

$$H(\mu_N|\nu_0^N) = o(N^d)$$

for ν_0^N defined by (3.4). Therefore, from Theorem 3.2, $H(\mu_N(t)|\nu_t^N) = o(N^d)$. From the entropy inequality (3.5), we obtain

$$\mu_N(t)[A_\delta] \leq \frac{\log 2 + o(N^d)}{\log\left(1 + \nu_t^N(A_\delta)^{-1}\right)}.$$

Therefore to conclude the proof of (3.7), we have to show that for every $\delta > 0$, there exists a positive constant $C(\delta)$ such that

$$\limsup_{l\to\infty}\limsup_{N\to\infty}\frac{1}{N^d}\log\nu_t^N[A_\delta] \leq -C(\delta). \tag{3.8}$$

By Chebycheff's exponential inequality, $\nu_t^N[A_\delta]$ is bounded above by

$$\exp\{-\delta\theta N^d\}E_{\nu_t^N}\left[\exp\left\{\theta\sum_k G(k/N)\left|\tilde{\Psi}(\eta^l(k)) - \tilde{\Psi}(\rho(t,k/N))\right|\right\}\right],$$

for every $\theta > 0$, where $G(x) = |J(x)|$. Since $\eta^l(j)$ and $\eta^l(k)$ are independent for the measure ν_t^N if $|j-k| > 2l+1$, by Hölder's inequality, the logarithm of the expectation in the last expression is bounded above by

$$\frac{1}{(2l+1)^d}\sum_k\log\left\{E_{\nu_t^N}\left[\exp\left\{\theta(2l+1)^d G\left(\frac{k}{N}\right)\left|\tilde{\Psi}(\eta^l(k)) - \tilde{\Psi}\left(\rho\left(t,\frac{k}{N}\right)\right)\right|\right\}\right]\right\}$$
$$+O(l^d).$$

Therefore,

$$\limsup_{N\to\infty} \frac{1}{N^d} \log \nu_t^N[A_\delta]$$
$$\le -\theta\delta + \int dx \frac{1}{(2l+1)^d}$$
$$\log E_{\nu_{\rho(t,x)}}\Big[\exp\big\{\theta(2l+1)^d G(x)\big|\tilde{\Psi}(\eta^l(0)) - \tilde{\Psi}(\rho(t,x))\big|\big\}\Big],$$

where $\nu_{\rho(t,x)}$ is the product measure defined in (1.6). From the large deviation principle for i.i.d. random variables, it is easy to see that for every $\delta > 0$, there exists a positive $\theta(\delta)$ such that the limit of this last expression when $l \uparrow \infty$ is strictly negative. □

PROPOSITION 3.4. *With the assumptions of Theorem* 3.2, *for every* $t \ge 0$ *and every bounded cylinder function* Ψ,

$$\limsup_{l\to\infty} \limsup_{N\to\infty} E_N\left[\frac{1}{N^d}\sum_k \left|\frac{1}{(2l+1)^d}\sum_{|j-k|\le l} \tau_j\Psi(\eta_t) - \tilde{\Psi}\big(\eta_t^l(k)\big)\right|\right] = 0.$$

The proof of this proposition is omitted since it is similar to the one of Lemma 7.4 in [19].

Corollary 3.3 and Proposition 3.4 prove the following corollary.

COROLLARY 3.5. *With the assumptions of Corollary* 3.3, *for every* $J \in C(S)$, *bounded cylinder function* Ψ *and* $t \ge 0$,

$$\lim_{N\to\infty} E_N\left[\frac{1}{N^d}\sum_k J\left(\frac{k}{N}\right)\tau_k\Psi(\eta_t)\right] = \int_S J(x)\tilde{\Psi}(\rho(t,x))\,dx,$$

where $\rho(t,x)$ *is the solution of* (3.3).

To conclude the proof of Theorem 3.1, we extend the last corollary to more general initial profiles and to a larger class of approximating sequences (ρ_N). The main tool in this last step is coupling and attractiveness.

LEMMA 3.6. *Let* $\rho_0 \in C^3(S)$ *and* $(\rho_N)_{N\ge 1}$ *be an approximating sequence satisfying assumptions* (S1), (S2) *and* (S5). *The conclusions of Corollary* 3.5 *hold.*

PROOF. Let $(\lambda^\epsilon)_{\epsilon>0}$ be a sequence of functions in $C^3(S)$ such that (i) $\lambda_\epsilon \in C^3(S)$, $\epsilon > 0$ and $\lambda_\epsilon \downarrow \rho_0$ uniformly in S; (ii) there exists $M < \infty$ such that

$$\epsilon \le \lambda_\epsilon \le M, \qquad \sup_\epsilon \|\partial_{x_i}\lambda_\epsilon\|_\infty \le M, \qquad \sup_\epsilon \|\partial^2_{x_i,x_j}\lambda_\epsilon\|_\infty \le M \quad \text{for } 1 \le i, j \le d.$$

For a fixed ϵ, define λ_N^ϵ: $S \to \mathbf{R}_+$ by

$$\lambda_N^\epsilon(x) = \max\left\{\lambda^\epsilon\left(\frac{[xN]}{N}\right), \rho_N\left(\frac{[xN]}{N}\right)\right\}.$$

A simple computation shows that $\lambda_N^\epsilon \to \lambda^\epsilon$ in $L^1(S)$ and that $(\lambda_N^\epsilon)_{N\ge 1}$ satisfies the assumptions of Corollary 3.3. Let μ_N and μ_N^ϵ be the product measures defined in (1.13), respectively, associated to the approximating sequences (ρ_N) and (λ_N^ϵ).

We couple two copies (η_t) and (ξ_t) of the zero-range process in the following way. First we place η-particles in S_N according to μ_N. Then, we add ζ-particles so that $\eta+\zeta$ is distributed according to μ_N^ϵ. This is possible since $\lambda_N^\epsilon \ge \rho_N$. Let $\xi = \eta + \zeta$ and let $\bar\mu_N(d\eta, d\xi)$ be the coupling measure on the product space $Y_N \times Y_N$ obtained in this way (we omit the indices ϵ when no confusion arises to keep notation as simple as possible). We let the η and ξ-particles evolve according to the basic coupling which is the Markov process whose generator acts on function $f\colon Y_N\times Y_N \to \mathbf{R}$ as

$$
\begin{aligned}
(\bar L_N f)(\eta,\xi) = N^2 \sum_{k,j} &[g(\eta(k)) \wedge g(\xi(k))]p(j-k) \\
&\times [f(\eta^{k,j},\xi^{k,j}) - f(\eta,\xi)] \\
&+ N^2 \sum_{k,j} [g(\eta(k)) - g(\xi(k))]^+ p(j-k) \\
&\times [f(\eta^{k,j},\xi) - f(\eta,\xi)] \\
&+ N^2 \sum_{k,j} [g(\xi(k)) - g(\eta(k))]^+ p(j-k) \\
&\times [f(\eta,\xi^{k,j}) - f(\eta,\xi)].
\end{aligned} \tag{3.9}
$$

Let $\bar S_t^N$ be the semigroup of this Markov process and let $\bar\mu_N(t) = \bar\mu_N \bar S_t^N$. Since for the coupled process, the η- and ξ-particles evolve as zero-range processes with generator given by (1.1), the first marginal of $\bar\mu_N(t)$ is equal to $\mu_N(t) = \mu_N \tilde S_t^N$ and the second marginal is $\mu_N^\epsilon(t) = \mu_N^\epsilon \tilde S_t^N$. Denote by $\bar E_N^\epsilon$ expectations with respect to the coupled process with initial measure $\bar\mu_N^\epsilon$. With this notation we have that

$$
\begin{aligned}
E_{\mu N}&\left[\frac{1}{N^d}\sum_k J\left(\frac{k}{N}\right)\tau_k\Psi(\eta_t)\right] \\
&= E_{\mu_N^\epsilon}\left[\frac{1}{N^d}\sum_k J\left(\frac{k}{N}\right)\tau_k\Psi(\xi_t)\right] \\
&\quad + \bar E_N^\epsilon\left[\frac{1}{N^d}\sum_k J\left(\frac{k}{N}\right)\{\tau_k\Psi(\eta_t) - \tau_k\Psi(\xi_t)\}\right].
\end{aligned} \tag{3.10}
$$

Since $(\lambda_N^\epsilon)_{N\ge1}$ satisfies the assumptions of Corollary 3.5, the first expression in the r.h.s. of (3.10) converges to

$$\int_S J(x)\tilde\Psi(\lambda^\epsilon(t,x))\,dx,$$

where $\lambda^\epsilon(t,x)$ is the unique solution of (3.3) with initial condition λ^ϵ.

Let $R \in N$ such that supp $\Psi \subset [-R, R]^d$ and let $\zeta_t = \xi_t - \eta_t$. Because

$$|\Psi(\xi_t) - \Psi(\eta_t)| \le 2\|\Psi\|_\infty \sum_{|j|\le R} \zeta_t(j),$$

the absolute value of the second expression in (3.10) is bounded above by

$$C(J,\Psi)\bar{E}_N^\epsilon\left[\frac{1}{N^d}\sum_k \zeta_t(k)\right] = C(J,\Psi)\bar{E}_N^\epsilon\left[\frac{1}{N^d}\sum_k \{\xi_0(k) - \eta_0(k)\}\right]$$
$$= C(J,\Psi)\int_S |\lambda_N^\epsilon(x) - \rho_N(x)|\,dx.$$

This last expression converges to 0 when $N \to \infty$ and then $\epsilon \downarrow 0$. Therefore

$$\lim_{N\to\infty} E_N\left[\frac{1}{N^d}\sum_k J\left(\frac{k}{N}\right)\tau_k\Psi(\eta_t)\right] = \lim_{\varepsilon\to 0}\int_S J(x)\tilde{\Psi}(\lambda^\epsilon(t,x))\,dx$$
$$= \int_S J(x)\tilde{\Psi}(\rho(t,x))\,dx.$$

The last equality follows from assumptions (i) and (ii) on the sequence λ^ϵ and Theorem 12 and 14 of [17]. □

LEMMA 3.7. *Let $\rho_0 \in C_K^3(\mathbf{R}^d)$ and let $(\rho_N)_{N\ge 1}$ be an approximating sequence of ρ_0 satisfying* (S1), (S2) *and* (S5). *Let μ_N be the product measure defined in* (1.13). *Then the conclusions of Theorem* 3.1 *hold.*

PROOF. We first consider the case where

$$\rho_N(x) = 0 \quad \text{if} \int_{[[xN]/N,([xN]+1)/N]} \rho_0(x)\,dx = 0, \tag{3.11}$$

where, for x, y in $\mathbf{R}^d$, $[x, y]$ was defined in (1.12). Fix $K \in \mathbf{N}$ such that supp $J \cup$ supp $\rho_0 \subset (-K, K)^d$. We couple a zero-range process (η_t) evolving on X_d with a zero-range process (ξ_t) evolving on $(S_N^K)^d$.

Let μ_N^K be a product probability measure on Y_N with marginals given by

$$\mu_N^K\{\xi; \xi(k) = n\} = \nu_{\rho_N(k/N)}\{\eta; \eta(0) = n\} \quad \text{for } n \ge 0,\ k \in S_N.$$

Let (ξ_t) be a Markov process evolving on Y_N with generator defined in (3.1) and initial state distributed according to μ_N^K and let (η_t) be a Markov process evolving on X_d with generator defined by (1.1) and initial state distributed according to μ_N. Define $\bar{\mu}_N^K$ as the probability measure on the product space $X_d \times Y_N$ with first marginal equal to μ_N, second marginal equal to μ_N^K and such that

$$\bar{\mu}_N^K\{(\eta,\xi); \eta(k) = \xi(k),\ k \in \Lambda_N^K\} = 1,$$

where

$$\Lambda_N^K = \{-KN, \dots, KN - 1\}^d.$$

We label the η and ξ particles in such a way that at time zero particles with the same label are at the same site. We let the η- and ξ-particles with same label evolve together before reaching $\partial\Lambda_N^K$, the border of Λ_N^K. From this time on, they behave independently of one another and as second class particles with respect to those particles that have not reached $\partial\Lambda_N^K$ (see [2], Section 2 or Lemma 2.2 in [11] for the terminology of second class particles). Denote by η^1 and ξ^1 the particles that have not reached $\partial\Lambda_N^K$ and by η^2 and ξ^2 the others. From our coupling,

$$\eta_t^1 \equiv \xi_t^1 \quad \text{and} \quad |\eta_t^2| = |\xi_t^2| \quad \text{for } t \geq 0,$$

since by hypothesis (3.11), $\rho_N(x) = 0$ if $|x| \geq K$. Denote by $\overline{E}_N^K$ the expectation with respect to this coupling.

Since ξ_t evolves as a zero-range process on the torus and the approximating sequence ρ_N satisfies the hypotheses of Lemma 3.6,

$$\lim_{N\to\infty} \overline{E}_N^K\left[\frac{1}{N^d}\sum_k J\left(\frac{k}{N}\right)\tau_k\Psi(\xi_t)\right] = \int_S J(x)\tilde{\Psi}(\rho^K(t,x))\,dx, \tag{3.12}$$

where $\rho^K(t,x)$ is the solution of (3.3) on the torus $(S^K)^d$ with initial data ρ_0. For $x \in \mathbf{R}^d$, $t \geq 0$, let

$$\bar{\rho}^K(t,x) = \rho^K\left(t, x - 2\left[\frac{x+K}{2K}\right]K\right),$$

where $[r]$ denotes the integer part of $r \in \mathbf{R}^d$. Then $\bar{\rho}^K(t,x)$ is the solution of (1.10) and (1.11) with initial data $\rho_0^K \in C^3(\mathbf{R}^d)$ defined by

$$\rho_0^K(x) = \rho_0\left(x - 2\left[\frac{x+K}{2K}\right]K\right) \quad \text{for } x \in \mathbf{R}^d.$$

By the maximum principle, $\bar{\rho}^{2^nK}(t,x)$ decreases in n and therefore the sequence converges to some function $\rho(t,x)$. It is not difficult to show that ρ is a weak solution of (1.10) and (1.11), and that $\|\rho(t,\cdot)\|_1 \leq \|\rho_0\|_1$ for every $t \geq 0$. By the uniqueness theorem (Proposition 1 in [3]) and the existence theorem, Theorem 14, in [17], $\rho(t,x)$ is the unique classical solution of (1.10) and (1.11).

On the other hand,

$$\begin{aligned} &\overline{E}_N^K\left[\frac{1}{N^d}\sum_k J\left(\frac{k}{N}\right)\tau_k\Psi(\eta_t)\right] \\ &\quad = \overline{E}_N^K\left[\frac{1}{N^d}\sum_k J\left(\frac{k}{N}\right)\tau_k\Psi(\xi_t)\right] \\ &\qquad + \overline{E}_N^K\left[\frac{1}{N^d}\sum_k J\left(\frac{k}{N}\right)\{\tau_k\Psi(\eta_t) - \tau_k\Psi(\xi_t)\}\right]. \end{aligned} \tag{3.13}$$

Let $R \in N$ such that $\operatorname{supp}\Psi \subset [-R,R]^d$. Since $\eta_t^1 \equiv \xi_1^t$,

$$|\Psi(\eta_t) - \Psi(\xi_t)| \leq 2\|\Psi\|_\infty \sum_{|j|\leq R} \eta_t^2(j) + \xi_t^2(j).$$

Therefore, the absolute value of the second expression in the r.h.s. of (3.13) is bounded above by

$$C(J,\Psi)\bar{E}_N^K\left[\frac{1}{N^d}\sum_k \eta_t^2(k)\right],$$

for $|\xi_t^2| = |\eta_t^2|$. This last expression represents the number of η-particles that reached $\partial\Lambda_N^K$ before time t.

Let (X_t) be a random walk with mean $1/G$ exponential holding times and transition probabilities $p(k)$ defined in (A1) and (A2). Let $B \in \mathbf{R}$ such that $\operatorname{supp}\rho_0 \subset [-B, B]^d$. It is easy to see that this last expression is bounded above by

$$C(J,\Psi,\|\rho_N\|_\infty)P\left[\sup_{s\le t}|X_{sN^2}| > (K-B)N\right]$$

and this probability converges to 0 when $N\uparrow\infty$ and after $K\uparrow\infty$. Therefore, from (3.12) and (3.13),

$$\lim_{N\to\infty} E_N\left[\frac{1}{N^d}\sum_k J\left(\frac{k}{N}\right)\tau_k\Psi(\eta_t)\right] = \int_S J(x)\tilde{\Psi}(\rho(t,x))\,dx,$$

where $\rho(t,x)$ is the solution of (1.10) and (1.11) with initial data ρ_0. This concludes the proof of the lemma in the case where ρ_N satisfies (3.11). For the general case, let

$$\tilde{\rho}_N(x) = \rho_N(x)\mathbf{1}\left\{\int_{[[xN]/N,([xN]+1)/N]}\rho_0(x)\,dx > 0\right\}.$$

It is easy to see that $\tilde{\rho}_N$ has all the properties required by Lemma 3.7 and satisfies (3.11). Let μ_N and $\tilde{\mu}_N$ be the product measures associated with the approximating sequences ρ_N and $\tilde{\rho}_N$, respectively. Repeating the coupling arguments presented in Lemma 3.6, we conclude the proof of the lemma. □

We are now ready to prove Theorem 3.1.

PROOF OF THEOREM 3.1. Let $(\rho_0^\epsilon)_{\epsilon>0}$ be a sequence in $C_K^3(\mathbf{R}^d)$ converging to ρ_0 in $L^1(\mathbf{R}^d)$ and uniformly bounded. For each positive ϵ, let (ρ_N^ϵ) be an approximating sequence of ρ_0^ϵ satisfying (S1), (S2) and (S5), and let μ_N^ϵ be the product measure defined in (1.13) associated with the approximating sequence (ρ_N^ϵ).

For $\epsilon > 0$ and $N \ge 1$, define $\bar{\mu}_N^\epsilon$ as a measure on the product space $X_d \times X_d$ with first marginal equal to μ_N, second marginal equal to μ_N^ϵ and such that

(3.14) $\bar{\mu}_N^\epsilon\{(\eta,\xi);\ \eta(k)\le\xi(k)\} = 1$ if and only if $\rho_N(k/N) \le \rho_N^\epsilon(k/N)$.

Let (η_t, ξ_t) be the Markov process with generator given by (3.9) and initial state (η_0, ξ_0) distributed according to $\bar{\mu}_N^\epsilon$. With this notation,

$$(3.15)\qquad \begin{aligned} &E_N\left[\frac{1}{N^d}\sum_k J\left(\frac{k}{N}\right)\tau_k\Psi(\eta_t)\right] \\ &= E_N^\epsilon\left[\frac{1}{N^d}\sum_k J\left(\frac{k}{N}\right)\tau_k\Psi(\xi_t)\right] \\ &\quad + \bar{E}_N^\epsilon\left[\frac{1}{N^d}\sum_k J\left(\frac{k}{N}\right)\{\tau_k\Psi(\eta_t) - \tau_k\Psi(\xi_t)\}\right]. \end{aligned}$$

From Lemma 3.7, the first term on the r.h.s. of (3.15) converges to $\int_S J(x)\tilde{\Psi}(\rho^\epsilon(t,x))\,dx$, where ρ^ϵ is the solution of (1.10) and (1.11) with initial data ρ_0^ϵ. Since $\varphi(\rho)$ is smooth, $\varphi'(\rho)$ is bounded below by a positive constant on each compact set of $\mathbf{R}_+$, σ is by Assumption 3 a positive definite matrix, and the sequence ρ_0^ϵ is uniformly bounded, by Nash's theorem [16] on linear parabolic differential equations, the sequence ρ^ϵ is uniformly (in ϵ) Hölder continuous on each compact set of $(0,\infty)\times\mathbf{R}^d$. Since ρ_0^ϵ converges in $L^1(\mathbf{R}^d)$ to ρ_0, every limit point of the sequence (ρ^ϵ) is a weak solution of (1.10) and (1.11) with initial data ρ_0. On the other hand, it is not difficult to see that $\|\rho(t,\cdot)\|_1 \le \|\rho_0\|_1$ for every $t \ge 0$. Therefore, from the uniqueness of weak solutions of partial differential equation (1.10) and (1.11) (Proposition 1 of [3]), ρ^ϵ converges uniformly on each compact set of $(0,\infty)\times\mathbf{R}^d$ to ρ, the weak solution of (1.10) and (1.11) with initial data ρ_0. Therefore,

$$\lim_{\varepsilon\to 0}\lim_{N\to\infty} E_N^\epsilon\left[\frac{1}{N^d}\sum_k J\left(\frac{k}{N}\right)\tau_k\Psi(\xi_t)\right] = \int_S J(x)\tilde{\Psi}(\rho(t,x))\,dx,$$

where $\rho(t,x)$ is the weak solution of (1.10) and (1.11) with initial data ρ_0 such that $\|\rho(t,\cdot)\|_1 \le \|\rho_0\|_1$ for every $t \ge 0$.

To conclude the proof of Theorem 3.1, we first have to show that the second term of the r.h.s. of (3.15) converges to 0. This is done with arguments similar to the ones used in the proof of Lemmas 3.6 and 3.7. In the same way, we prove that the assumption (S5) may be replaced by (S4). □

REMARK 3.8. Theorem 3.1 holds if assumptions (S2) and (S4) on the approximating sequence (ρ_N) are replaced by (S5).

4. Proof of Theorems 2–5. In this section, we prove the theorems stated in section 1. We begin by recalling a law of large numbers for the density field under Euler rescaling proved by Rezakhanlou [19].

THEOREM 4.1 (Rezakhanlou). *Denote by* (η_t) *the Markov process with generator* (1.1), *where* $\theta(N) = N$. *Let* $\rho_0 \in L^\infty(\mathbf{R}^d)$, *let* $(\rho_N)_{N\ge 1}$ *be an approximating sequence satisfying* (S1), (S2) *and* (S4) *and let* μ_N *be the product measure defined in* (1.13). *For every* $J \in C_K(\mathbf{R}^d)$, *bounded cylinder function* Ψ

and $t \geq 0$,

$$\lim_{N\to\infty} E_N\left[\frac{1}{N^d}\sum_{k\in Z^d} J\left(\frac{k}{N}\right)\tau_k\Psi(\eta_t)\right] = \int_{\mathbf{R}^d} J(x)\tilde{\Psi}(\rho(t,x))\,dx,$$

where $\tilde{\Psi}$ *is defined in* (1.8) *and* $\rho(t,x)$ *is the unique entropic weak solution of* (1.9) *and* (1.11).

The proofs of Theorems 2, 3 and 5 will consist of proving that the processes considered satisfy Hypotheses 1–3 of Theorem 1. In Theorem 2 and 3, $P = \mathbf{R}_+$ and the family of measures m_ρ is defined by (1.6) and denoted by ν_ρ. In Theorem 5, $P = [0,1]$ and the family of measures m_ρ is defined by (1.16) and denoted by $\bar{\nu}_\rho$.

Proof of Theorem 2. In Hypotheses 1–3 let L_N be the generator defined in (1.1) with Assumptions 1–3, let $\mathscr{L}$ be the differential operator defined in (1.10) and let $\mathscr{F}$ be the space of bounded a.s. continuous functions ρ_0 for which there exist constants $c < \infty$ and $\alpha > d$ such that

$$\rho_0(x) \leq \frac{c}{1+|x|^\alpha}. \tag{4.1}$$

Since ρ_0 is bounded, from (4.1), it is easy to see that $\rho_0^{\epsilon,+}$ and $\rho_0^{\epsilon,-}$ defined by (1.14) are in $L^1(\mathbf{R}^d) \cap L^\infty(\mathbf{R}^d)$. Therefore, from Proposition 1 of [3], there exists a unique weak solution of (1.10) and (1.11) with initial conditions $\rho_0^{\epsilon,+}$ and $\rho_0^{\epsilon,-}$ such that $\|\rho^{\epsilon,+}(t,\cdot)\|_1 \leq \|\rho_0^{\epsilon,+}\|_1$ $[\|\rho^{\epsilon,-}(t,\cdot)\|_1 \leq \|\rho_0^{\epsilon,-}\|_1]$ for every $t \geq 0$. Thus Hypothesis 1 is satisfied.

Since φ is smooth, since $\varphi'(\rho)$ is bounded below by a strictly positive constant on every compact subset of $\mathbf{R}_+$, since σ is a positive definite matrix and since ρ_0, $\rho_0^{\epsilon,+}$ and $\rho_0^{\epsilon,-}$ are uniformly bounded, by Nash's theorem [16] on linear strictly parabolic differential equations, the solutions $\rho^{\epsilon,+}$ and $\rho^{\epsilon,-}$ of (1.10) and (1.11) are equicontinuous on every compact set of $(0,\infty) \times \mathbf{R}^d$. Since ρ_0 is a.s. continuous, $\rho_0^{\epsilon,+}$ and $\rho_0^{\epsilon,-}$ converge in $L^1_{\text{loc}}(\mathbf{R}^d)$ to ρ_0. It is therefore not difficult to see that every limit point $\lambda(t,x)$ of the sequence $\rho^{\epsilon,+}$ and $\rho^{\epsilon,-}$ is a weak solution of (1.10) and (1.11) with initial data ρ_0 such that $\|\lambda(t,\cdot)\|_1 \leq \|\rho_0\|_1$ for every $t \geq 0$. By the uniqueness theorem of weak solutions (Proposition 1 of [3]), $\rho^{\epsilon,+}$ $(\rho^{\epsilon,-})$ converges uniformly on every compact set of $(0,\infty) \times \mathbf{R}^d$ to ρ, the weak solution of (1.10) and (1.11) with initial data ρ_0 such that $\|\rho(t,\cdot)\|_1 \leq \|\rho_0\|_1$ for every $t \geq 0$. This proves Hypothesis 2.

Let $(\rho_N)_{N\geq 1}$ be an approximating sequence of ρ_0 with properties (S1), (S2) and (S3). Define $\rho_N^{\epsilon,+}$ as

$$\rho_N^{\epsilon,+}(x) = \max\left\{\sup_{z\in[([xN]-[\delta N])/N,([xN]+[\delta N])/N]} \rho_N(z),\right.$$
$$\left.\int_{[[xN]/N,([xN]+1)/N]} N^d \rho^{\epsilon,+}(z)\,dz\right\},$$

where, for x and y in $\mathbf{R}^d$, $[x, y]$ is defined in (1.12) and $\delta(\epsilon) = \epsilon/2$. Define $\rho_N^{\epsilon,-}$ in an analogous way. Obviously $\rho_N^{\epsilon,+}$ ($\rho_N^{\epsilon,-}$) have properties (S1) and (S2) and it is not difficult to see that these sequences converge to $\rho_0^{\epsilon,+}$ ($\rho_0^{\epsilon,-}$) in L^1_{loc}. Therefore, by Theorem 3.1, Hypothesis 3(i) is satisfied. On the other hand, by construction the sequences $\rho_N^{\epsilon,+}$ and $\rho_N^{\epsilon,-}$ satisfy Hypothesis 3(ii) with $\delta = \epsilon/2$. □

PROOF OF THEOREM 3. The proof is almost the same as the one of Theorem 2. We just point out the needed modifications. In Hypotheses 1–3, let L_N be the generator defined in (1.1) with Assumptions 1 and 2, let $\mathscr{L}$ be the differential operator defined in (1.9) and, in dimension 1, let $\mathscr{F}$ be the space of bounded a.s. continuous functions ρ_0 and in higher dimensions let $\mathscr{F}$ be the space of bounded continuous functions.

Since ρ_0 is bounded, $\rho_0^{\epsilon,+}$ and $\rho_0^{\epsilon,-}$ defined in (1.14) are in $L^\infty(\mathbf{R}^d)$. Therefore, from Theorem 2 of [10], there exists a unique entropic weak solution of (1.11) with initial condition $\rho_0^{\epsilon,+}$ and $\rho_0^{\epsilon,-}$. Thus Hypothesis 1 is satisfied.

In dimension 1, since ρ_0 is a.s. continuous, $\rho_0^{\epsilon,+}$ and $\rho_0^{\epsilon,-}$ converge in $L^1_{\mathrm{loc}}(\mathbf{R}^d)$ to ρ_0. Lax obtained ([14], Section 2) an explicit formula for the weak solution of (1.9) and (1.11) when φ is strictly concave or convex. Repeating the arguments of the proof of Theorem 2.2 in [14], we can show that Hypothesis 2 is satisfied.

In dimension $d > 1$ with a change of variables the differential operator (1.9) can be rewritten as

$$\mathscr{L}(\rho) = \frac{\partial \rho}{\partial t} + C(\gamma)\frac{\partial}{\partial x_1}\varphi(\rho)$$

for some $C(\gamma) \in \mathbf{R}$. Therefore, the differential equation is one-dimensional and the same arguments prove that Hypothesis 2 is satisfied since we assumed that the initial profile is bounded and continuous.

Let $(\rho_N)_{N\geq 1}$ be an approximating sequence of ρ_0 with properties (S1), (S2) and (S3). Define $\rho_N^{\epsilon,+}$ and $\rho_N^{\epsilon,-}$ as in the proof of Theorem 2. $\rho_N^{\epsilon,+}$ ($\rho_N^{\epsilon,-}$) have properties (S1) and (S2) and it is not difficult to see that these sequences converge to $\rho_0^{\epsilon,+}$ ($\rho_0^{\epsilon,-}$) in L^1_{loc}. Therefore, by Theorem 4.1, Hypothesis 3(i) is satisfied. On the other hand, by construction the sequences $\rho_N^{\epsilon,+}$ and $\rho_N^{\epsilon,-}$ have Hypothesis 3(ii) with $\delta = \epsilon/2$. □

PROOF OF THE COROLLARY OF THEOREM 3. Let $\gamma \in \mathbf{R}^d$ be the mean drift defined in (1.3) and for $x \in \mathbf{R}^d$, let $l(x) = \{x + r\gamma;\ r \in \mathbf{R}\}$. In the proof of Theorem 3, to apply Lax's result [14] to show that Hypothesis 2 is satisfied, we only have to suppose that for every $x \in \mathbf{R}^d$, almost all points of $l(x)$ are continuity points of the initial data ρ_0. Since this is the case if $\gamma \notin \mathscr{H} \cup (-\mathscr{H})$, the proof of Theorem 3 easily extends to its corollary. □

The proof of Theorem 4 needs two preliminary lemmas. To fix ideas, we suppose that

$$\alpha \leq \beta.$$

To state the first lemma, for positive integers k, define ρ^k as the unique weak solution of (1.10) and (1.11) with initial data ρ_0^k given by

$$\rho_0^k(x) = \alpha \mathbf{1}\{[-k,0)\}(x) + \beta \mathbf{1}\{[0,k]\}(x).$$

LEMMA 4.2. *The sequence ρ^k converges uniformly on the compact sets of $(0,\infty) \times \mathbf{R}$ to the unique weak solution ρ of* (1.10) *and* (1.11) *with initial data*

$$\rho_0(x) = \alpha \mathbf{1}\{[-\infty,0)\}(x) + \beta \mathbf{1}\{[0,\infty)\}(x) \tag{4.2}$$

such that (a) $\rho(t,\cdot) - \rho_0(\cdot) \in L^1(\mathbf{R})$ *for every* $t > 0$, *and* (b) $\lim_{t\to 0}\|\rho(t,\cdot) - \rho_0(\cdot)\|_1 = 0$.

PROOF. From Theorem 1 of [3], there is a unique weak solution of (1.10) and (1.11) with initial data ρ_0 defined by (4.2) and satisfying (a) and (b). Therefore, to prove the lemma it is enough to show that the sequence (ρ^k) is equicontinuous and that every limit point of it is a weak solution of (1.10) and (1.11) satisfying the properties (a) and (b).

From Nash's theorem on weak solutions of linear partial differential equations [16], the sequence ρ^k is equicontinuous on each compact subset of $(0,\infty) \times \mathbf{R}^d$.

Let $\lambda(t,x)$ be a limit point. It is easy to see that λ is a weak solution of (1.10) and (1.11) with initial data ρ_0. To conclude the proof of the lemma, we have to show that λ has properties (a) and (b).

Since ρ^k is bounded below by α and above by β, for every $t > 0$,

$$\int_{\mathbf{R}} |\lambda(t,x) - \rho_0(x)|\,dx = \int_{-\infty}^{0} [\lambda(t,x) - \alpha]\,dx + \int_0^\infty [\beta - \lambda(t,x)]\,dx.$$

To keep notation simple, we suppose that ρ^k converges uniformly to λ on the compact sets of $(0,\infty) \times \mathbf{R}$. In the sequel, we prove that the second integral in the right-hand side of the last expression is bounded by a finite function $C(t)$ which converges to 0 when $t \downarrow 0$. The same argument applies to the first integral.

Since ρ^k converges uniformly to λ on the compact sets of $(0,\infty) \times \mathbf{R}$,

$$\int_0^\infty [\beta - \lambda(t,x)]\,dx = \lim_{b\to\infty}\lim_{k\to\infty}\int_0^b [\beta - \rho^k(t,x)]\,dx. \tag{4.3}$$

Define μ_N^k as the product measure on X_d with marginals given by

$$\mu_N^k\{\eta;\eta(j) = n\} = \begin{cases} \nu_\alpha\{\eta;\eta(j) = n\}, & \text{if } -kN \le j < 0,\\ \nu_\beta\{\eta;\eta(j) = n\}, & \text{if } 0 \le j \le kN,\\ \nu_0\{\eta;\eta(j) = n\}, & \text{otherwise,}\end{cases}$$

for every $n \in \mathbf{N}$. We define a product measure $\bar{\mu}_N^k$ on the space $X_1 \times X_1$ which has the first marginal equal to μ_N^k, second marginal equal to ν_β and such that $\bar{\mu}_N^k\{(\eta,\xi);\ \eta \le \xi\} = 1$. This measure is constructed in the following way. First, we place η-particles distributed according to μ_N^k. Then, we add ζ-particles for $\xi = \eta + \zeta$ being distributed according to ν_β. Let (η,ξ) evolve according to the Markov process with generator given by (3.9). From Theorem 3.1,

$$\begin{aligned}\int_0^b [\beta - \rho^k(t,x)]\,dx &= \lim_{N\to\infty} E_N^k\left[\frac{1}{N}\sum_{j=0}^{[bN]}(\beta - \eta_t(j))\right]\\ &= \lim_{N\to\infty}\left\{\overline{E}_N^k\left[\frac{1}{N}\sum_{j=0}^{[bN]}(\beta - \xi_t(j))\right]\right.\\ &\qquad\left.+\overline{E}_N^k\left[\frac{1}{N}\sum_{j=0}^{[bN]}\zeta_t(j)\right]\right\}.\end{aligned} \tag{4.4}$$

Since ν_β is an invariant measure,

$$\overline{E}_N^k\left[\frac{1}{N}\sum_{j=0}^{[bN]}(\beta - \xi_t(j))\right] = 0.$$

For each $j \in \mathbf{Z}$, let $Z(j)$ be the number of ζ-particles on site j at time 0. We label the ζ-particles on j with superscripts: $\zeta^{j,i}$, $1 \le i \le Z(j)$, $j \in \mathbf{Z}$. With this notation, we have

$$\overline{E}_N^k\left[\frac{1}{N}\sum_{j=0}^{[bN]}\zeta_t(j)\right] = \frac{1}{N}\sum_{\substack{j\ge kN\\ j\le 0}}\overline{E}_N^k\left[\sum_{i=1}^{Z(j)}\mathbf{1}\{\zeta_t^{j,i}\in[0,bN]\}\right]. \tag{4.5}$$

Let (X_t) be a random walk with mean $1/G$ exponential holding times and transition probabilities $p(k)$ defined in Assumptions 1 and 2. Since for every $j \in \mathbf{Z}$,

$$\overline{P}_N^k\left[\zeta_t^{j,i}\in[0,bN]\right] \le P\left[\sup_{s\le t}|X_{sN^2}| \ge |j| \wedge |j - bN|\right],$$

the left-hand side of (4.5) is bounded above by

$$\begin{aligned}&\frac{\beta}{N}\sum_{j\ge 0}P\left[\sup_{s\le t}|X_{sN^2}| \ge j + (k-b)N\right] + \frac{\beta}{N}\sum_{j\ge 0}P\left[\sup_{s\le t}|X_{sN^2}|\ge j\right]\\ &\quad= \beta\left\{E\left[\left(\sup_{s\le t}\frac{|X_{sN^2}|}{N} - (k-b)\right)^+\right] + E\left[\sup_{s\le t}\frac{|X_{sN^2}|}{N}\right]\right\}.\end{aligned}$$

Therefore, from (4.4),

$$\limsup_{k\to\infty}\int_0^b[\beta - \rho^k(t,x)]\,dx \le \beta E\left[\sup_{s\le t}\left|\sqrt{\sigma}B_{sG}\right|\right],$$

where (B_s) is a Brownian motion and G and σ are defined in Assumptions 1 and 3. We define the right-hand side as $C(t)$. It is easy to see that $C(t)$ is finite and converges to 0 as $t \downarrow 0$. Since from (4.3),

$$\int_0^\infty [\beta - \lambda(t,x)]\, dx \le C(t),$$

the proof is complete. □

LEMMA 4.3. *With the assumptions of Theorem 4, for every $J \in C_K(\mathbf{R}^d)$, bounded cylinder function Ψ and $t \ge 0$,*

$$\lim_{N\to\infty} m_{\alpha,\beta} S_t^N \left[\frac{1}{N} \sum_k J(k/N) \tau_k \Psi(\eta) \right] = \int J(x) \tilde{\Psi}(\rho(t,x))\, dx.$$

PROOF. For $k \in N$, let μ_N^k be as defined in Lemma 4.3. Distribute η-particles on X_1 according to μ_N^k. Add ζ-particles for $\xi = \eta + \zeta$ being distributed according to $m_{\alpha,\beta}$. Denote by $\bar{\mu}_N^k(\eta,\xi)$ the product measure on $X_1 \times X_1$ obtained in this way. Let (η_t, ξ_t) evolve according to the Markov process with generator defined in (3.9). From Theorem 3.1, Lemma 4.3 and coupling arguments similar to the ones used in the proof of Lemmas 3.6 and 3.7, we conclude the proof of the lemma. □

PROOF OF THEOREM 4. Let Ψ be a monotone bounded cylinder function. Since $m_{\alpha,\beta} \le \tau_1 m_{\alpha,\beta}$, and the process is attractive, for every $\epsilon > 0$,

$$m_{\alpha,\beta} S_t^N[\Psi(\eta)] \le m_{\alpha,\beta}\left[\frac{1}{[\epsilon N]+1} \sum_{j=0}^{[\epsilon N]} \tau_j \Psi(\eta_t) \right].$$

Therefore, from Lemma 4.3 and since $\rho(t,\cdot\,)$ is continuous for every $t > 0$,

$$\begin{aligned}\lim_{N\to\infty} m_{\alpha,\beta} S_t^N[\Psi(\eta)] &\le \lim_{\varepsilon\to 0} \frac{1}{\epsilon} \int_0^\epsilon \tilde{\Psi}(\rho(t,x))\, dx \\ &= \tilde{\Psi}(\rho(t,0)) = \nu_{\rho(t,0)}[\Psi].\end{aligned}$$

In the same way we prove the reverse inequality. Since every bounded cylinder function is the difference of two monotone bounded cylinder functions, the theorem is proved. □

PROOF OF THEOREM 5. The proof is similar to that of Theorem 2. We therefore point out only the needed modifications. In the present context L_N is the generator defined by (1.15) and the differential operator $\mathscr{L}$ defined by (1.17). Therefore Hypotheses 1 and 2 of Theorem 1 are satisfied, since ρ_0 is almost surely continuous and σ is a positive definite matrix by Assumption 3 (see [18] for an uniqueness theorem).

On the other hand, we can prove Theorem 3.1 for the symmetric simple exclusion process since it is gradient. With one step further, using coupling similar to the ones used in the proof of Theorem 3.1, we can extend the statement of Theorem 3.1 to initial profiles ρ_0 in $L^\infty(\mathbf{R}^d)$. Therefore, defining $\rho_N^{\epsilon,+}$ and $\rho_N^{\epsilon,-}$ as in the proof of Theorem 2, Hypothesis 3 is satisfied. □

APPENDIX

In this Appendix, we fix the terminology of weak solutions of quasilinear partial differential equations.

A bounded function ρ is an entropic weak solution of (1.9) and (1.11) if for every $c \in \mathbf{R}$,

$$\partial_t|\rho - c| + \sum_{1 \le j \le d} \gamma_j \partial_{x_j} |\phi(\rho) - \phi(c)| \le 0$$

in the sense of distributions on $(0, \infty) \times \mathbf{R}^d$, and if for every compact subset K of $\mathbf{R}^d$,

$$\lim_{t \to 0} \int_K |\rho(t, z) - \rho_0(z)|\, dz = 0.$$

Kružkov [10] proved the existence of an unique entropic weak solution of (1.9) and (1.11) for each $\rho_0 \in L^\infty(\mathbf{R}^d)$.

A bounded function $\rho(t, x)$ is a weak solution of (1.10) and (1.11) if

$$\int_0^\infty dt \int_{R^d} dx \left[\rho \partial_t J + \phi(\rho) \sum_{\substack{1 \le i \le d \\ 1 \le j \le d}} \sigma_{ij} \partial_{x_i, x_j} J \right] + \int_{\mathbf{R}^d} dx\, J(0, x) \rho_0(x) = 0$$

for every $J \in C_K^\infty(\mathbf{R}_+ \times \mathbf{R}^d)$. Brezis and Crandall [3] [cf. Proposition 1 and (1.22)] proved the uniqueness of weak solutions of (1.10) and (1.11) such that

$$\int_0^T \|\rho(t, \cdot)\|_1\, dt < \infty$$

for every $T < \infty$. They also proved (cf. Theorem 1 in [3]) that there is a unique weak solution of (1.10), (1.11) such that (a) $\|\rho(t, \cdot) - \rho_0(\cdot)\|_1 < \infty$ for every $t > 0$, and (b) $\lim_{t \to 0} \|\rho(t, \cdot) - \rho_0(\cdot)\|_1 = 0$.

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