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A SHAPE THEOREM FOR EPIDEMICS AND FOREST FIRES WITH FINITE RANGE INTERACTIONS

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Cox and Durrett proved the shape theorem of epidemics and forest fires with nearest neighbor interactions. They also conjectured that the shape theorem is still true with finite range interactions. In this paper, we answer this conjecture affirmatively. The method we develop allows us also to improve the result of Zhang and Zhang for all dimension.

1. Introduction and statement of results. We consider the model of the spread of an epidemic or a forest fire with finite range interactions (see [3]). Each site $x \in Z^2$ can be in one of three states 1, 2, or 0 and the state of the process is represented by a function ξ_t: $Z^2 \to \{1, 2, 0\}$, where $\xi_t(x)$ is the function giving the state of x at time t. In the epidemic interpretation, 1 = healthy, 2 = infected, 0 = immune; while for a forest fire, 1 = alive, 2 = on fire and 0 = burnt. An infected individual emits germs according to a Poisson process with rate α. A germ emitted from x goes to a point y in

$$N_x = \{y \in Z^2: \|y - x\|_\infty \le M\}$$

for some finite number M at rate $\alpha g(y - x)$, where g is a function from $Z^2 \to [0, 1)$ such that

$$g(z)\begin{cases} = 0, & \text{if } z \notin N_0, \\ > 0, & \text{otherwise,} \end{cases} \tag{1.1}$$

$$g(z) = g(-z) \tag{1.2}$$

and

$$\sum_{z \in N_0} g(z) = 1. \tag{1.3}$$

Let T_x, $x \in Z^2$, be independent random variables with distribution F. We assume that F is concentrated on the nonnegative half line and is not the unit point mass at zero. Let $e(x, y)$, for all x, $y \in Z^2$ and $\|x - y\|_\infty \le M$, be independent random variables with

$$P(e(x, y) > t) = \exp(-t\alpha g(y - x)). \tag{1.4}$$

T_x is the amount of time x will stay infected and $e(x, y)$ is the time lag from

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the infection of x until the first germ from x is sent to y. We let

$$(1.5) \qquad \tau(x,y) = \begin{cases} e(x,y), & \text{if } T_x > e(x,y), \\ \infty, & \text{otherwise.} \end{cases}$$

We say the oriented edge $\{x, y\}$ is 1 (open) if $\tau(x, y) < \infty$ and 0 (closed) otherwise, that is,

$$(1.6) \qquad P(\{x,y\} = 1) = 1 - \int_0^\infty \exp(-t\alpha g(y-x))\, dF(t).$$

Given the definition of T_x and $e(x, y)$ it should be clear that edge $\{x, y\}$ is open if x tries to infect y during its lifetime and $\tau(x, y)$ gives the time lag from the infection of x until it tries to infect y, with $\tau(x, y) = \infty$ if this never happens. We also denote by $Z^2(M)$ all the oriented edges $\{x, y\}$ with $\|x - y\|_\infty \le M$. Then $\{x, y\} \in Z^2(M)$ is open or closed independently of all other edges in $Z^2(M)$ by our definition of $\tau(x, y)$. As sample space, we take $\Gamma = \{0, 1\}^{Z^2(M)}$, points of which are represented as $w = \{w_{\{x,y\}}\colon \{x, y\} \in Z^2(M)\}$ and called configurations; the value $w_{\{x,y\}} = 0$ or $= 1$ corresponds to $\{x, y\}$ being closed or open. For fixed g and F, the corresponding probability measure on the configurations is denoted by P_α. Let

$$C_0 = \{x\colon x \text{ can be reached from 0 by a path of open edges}\}$$

and

$$\alpha_c(g, F) = \inf\{\alpha\colon P_\alpha(C_0 \text{ is infinite}) > 0\}.$$

In percolation language, C_0 is the open cluster containing the origin 0. It follows from the argument in [3] that

$$0 < \alpha_c(g, F) < \infty.$$

We abbreviate $\alpha_c(g, F)$ to α_c for some fixed g and F. We write

$$(1.7) \qquad t(x,y) = \inf\left\{ \sum_{i=1}^{m} \tau(x_{i-1}, x_i)\colon x_0, \dots, x_m \text{ is a path from } x \text{ to } y \text{ with } \|x_i - x_{i+1}\|_\infty \le M \text{ for } i = 0, 1, \dots, m-1 \right\}$$

for the first passage time from x to y. Cox and Durrett (see [3]) proved the shape theorem with nearest neighbor interactions (see Theorem 1 below with $M = 1$), and they conjectured the shape theorem still holds with finite range interactions. In this paper we shall prove their conjecture.

THEOREM 1. *We define B_t as the set of immune sites at time t and Q_t as the set of infected sites at time t, that is,*

$$(1.8) \qquad B_t = \{x\colon \xi_t(x) = 0\} \text{ and } Q_t = \{x\colon \xi_t(x) = 2\}.$$

We assume that initially the origin is infected and all other sites healthy. We *also assume that*

$$\int_0^\infty x^2\, dF(x) < \infty \tag{1.9}$$

and $\alpha > \alpha_c(g, F)$. *Then there is a convex set* A *such that for any* $\varepsilon > 0$ *as* $t \to \infty$,

$$P_\alpha(C_0 \cap t(1-\varepsilon)A \subset B_t \subset t(1+\varepsilon)A) \to 1 \tag{1.10}$$

and

$$P_\alpha(Q_t \subset t(1+\varepsilon)A \setminus t(1-\varepsilon)A) \to 1. \tag{1.11}$$

We also consider Z^d as a graph with edges connecting each pair of points with $\|x-y\| = 1$. For each edge e, there is an independent nonnegative random variable $X(e)$ with distribution F. For any vertices u, v, a path γ from u to v is an alternating sequence $(u_0, e_1, v_1, \dots, e_n, v_n)$ of vertices and edges in Z^d with $u_0 = u$, $u_n = v$. Define the passage time of γ as $t(\gamma) = \sum_{i=1}^n X(e_i)$. The minimum passage time from u to v is defined by

$$T(u, v) = \inf\{t(\gamma)\colon \gamma \text{ is a path from } u \text{ to } v\}. \tag{1.12}$$

Let

$$a_{mn} = T((m, 0, \dots, 0), (n, 0, \dots, 0)), \tag{1.13}$$

$$b_{mn} = \inf\{T((m, 0, \dots, 0)), (n, k_2, \dots, k_d)\colon k_2, \dots, k_a \in Z\}. \tag{1.14}$$

It is well known (see [7]) that

$$\lim_{n\to\infty} \frac{1}{n} a_{0n} = \lim_{n\to\infty} \frac{1}{n} b_{0n} = \mu \quad \text{a.s. and in } L^1. \tag{1.15}$$

We define, for $\theta = a$ or b,

$$N^\theta_{m,n} = \inf\{\text{the number of edges in } r\colon r \text{ is a path with } t(r) = \theta_{mn}\}. \tag{1.16}$$

It was shown (see [10]) that if $p_c(d) < F(0)$ and $d = 2$, then

$$\lim_n \frac{1}{n} N^\theta_n = \lambda \quad \text{a.s. and in } L_1, \tag{1.17}$$

where $p_c(d)$ is the critical probability for Bernoulli (bond) percolation on Z^d and λ is a nonrandom constant which is independent of a and b. In this paper we show their result is true for any d.

THEOREM 2. *If* $F(0) > p_c(d)$, *then*

$$\lim_{n\to\infty} \frac{1}{n} N^\theta_{0n} = \lambda \quad \textit{a.s. and in } L^1. \tag{1.18}$$

2. Proof of Theorem 1. We first review the proof of [3]. When $M = 1$ (the nearest neighbor case), Cox and Durrett denoted by $\kappa(z)$ the smallest k such that: (i) there are infinite open paths to and from the square $z + [-k, k]^2$, and (ii) there is an open circuit around $z + [-k, k]^2$ contained in $z + [-2k, 2k]^2$. Then they can show the existence of radial limits [see (2.16) below for $M = 1$]. The proof is based on the fact that

$$(2.1) \qquad P(\kappa(z) \geq n) \to 0 \quad \text{exponentially fast as } n \to 0$$

and the subadditive argument. After that they can show the shape theorem based on their main probability estimate

$$(2.2) \qquad P(t(0,z) \geq K\|z\|_\infty) = O(\|z\|_\infty^{-3}) \quad \text{as } \|z\|_\infty \to \infty$$

for some constant K.

When $M > 1$, the difficulty is that the open circuit method [see (ii) above] does not work since a path and a circuit do not have to be connected even if the path crosses the circuit. However, the recent percolation techniques (see, e.g., [6] and [8]) show that large sponge-crossing and sponge-connecting have a high probability at the supercritical state. It is natural to renormalize Z^2 into large sponge blocks to obtain some properties of connectedness. Indeed, partition Z^2 into some blocks $\{[in, (i+1)n] \times [jn, (j+1)n]\}$ for $i, j, n \in Z$. Each block $[in, (i+1)n] \times [jn, (j+1)n]$ is called the renormalized site (i, j). Denote by V_n all the renormalized sites. Therefore, V_n and the edges between (i, j) and $(i, j+1)$ or $(i+1, j)$ form a standard planar graph.

For each renormalized site (i, j), let $A_n(i, j)$ be the event that all three of the following hold:

1. Block $[in, (i+1)n] \times [jn, (j+2)n]$ is connected by bottom-top and top-bottom open paths.
2. If there exist any bottom-top or top-bottom open paths r_1 and left-right or right-left open paths r_2 of block $[in, (i+1)n] \times [jn, (j+2)n]$, then they are connected by some open paths from r_1 to r_2 and from r_2 to r_1 with edges in the block.
3. If there exist any two bottom-top or two top-bottom open paths r_1 and r_2 of $[in, (i+1)n] \times [jn, (j+1)n]$, they they are connected by some open paths from r_1 to r_2 and from r_2 to r_1 with the edges in $[in, (i+1)n] \times [jn, (j+1)n]$ (see Fig. 1).

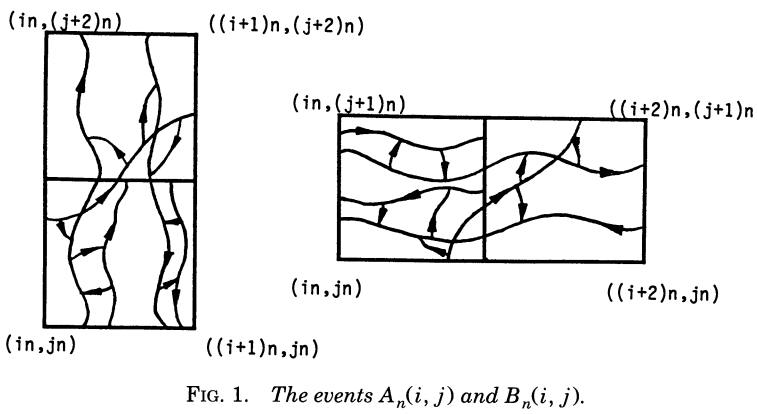

FIG. 1. *The events $A_n(i, j)$ and $B_n(i, j)$.*

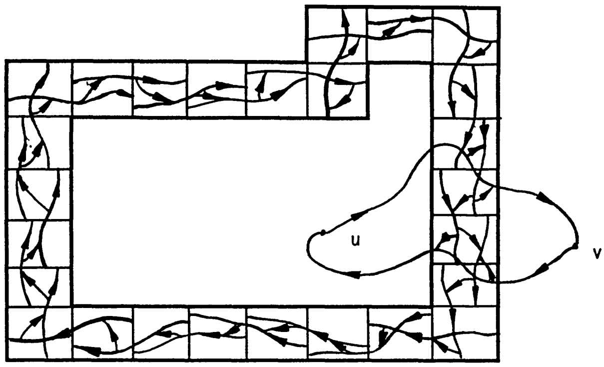

FIG. 2. *The event* $\Gamma(u, v)$.

Similarly, let $B_n(i, j)$ be the event that all three of the following hold:

1. Block $[in, (i+2)n] \times [jn, (j+1)n]$ is connected by left-right and right-left open paths.
2. If there exist any left-right or right-left open paths r_1 and bottom-top or top-bottom open paths r_2 of black $[in, (i+2)n] \times [jn, (j+1)n]$ they are connected by some open paths from r_1 to r_2 and form r_2 to r_1 with edges in the block.
3. If there exist any two left-right or two right-left of open paths r_1 and r_2 of $[in, (i+1)n] \times [jn, (j+1)n]$, they are connected by some open paths from r_1 to r_2 and from r_2 to r_1 with the edges in $[in, (i+1)n] \times [jn, (j+1)n]$ (see Fig. 1).

After that, renormalized site (i, j) is said to be occupied if $A_n(i, j) \cap B_n(i, j)$ occurs. With this occupied site in mind, next we construct an occupied circuit. Denote by $\Gamma(u, v)$ the event that there is an occupied circuit of the renormalized sites which surrounds u, separating it from v, for $u, v \in Z^2$ (see Fig. 2). By the definition of renormalized site we can see if $\Gamma(u, v)$ occurs, then (a) there is an open clockwise circuit C with edges in $Z^2(M)$ surrounding u, separating it from v and (b) each open path in $Z^2(M)$ from u to v or from v to u has to be connected to C by some open paths in both directions (see Fig. 2). Hence our renormalized circuit does not have the problem of connectedness described before.

To replace Cox and Durrett's circuit [see (ii) above] by our renormalized occupied circuit also requires a probability estimate corresponding to (2.1). Therefore, we need to show the following lemmas.

LEMMA 1. *If $\alpha > \alpha_c$, then for some suitably large n there is a positive constant $\kappa = \kappa(\alpha, n) > 0$ such that for each pair of sites $u, v \in Z^2$,*

$$P_\alpha(\Gamma(u, v)) \geq 1 - \exp[-\kappa \|u - v\|_\infty]. \tag{2.3}$$

The principal step in the proof of Lemma 1 is the following proposition, which is based on the methods of [6] and [8]. We shall first show how Lemma 1 follows from this proposition. Since the proof of the proposition is a little painful, we would rather put it in the Appendix. If some readers are not familiar with percolation language, they can skip the Appendix.

PROPOSITION. *Given* $\varepsilon > 0$, *there exists* N *such that*

$$P_\alpha((0,0) \text{ is occupied}) = P_\alpha(A_n(0,0) \cap B_n(0,0)) \geq 1 - \varepsilon \tag{2.4}$$

when $n \geq N$.

PROOF OF LEMMA 1 FROM THE PROPOSITION. We follow the proof of [1]. Let

$$r(a,b) = \{(i,j) \in V_n : 0 < i \leq a, 0 < j \leq b\}. \tag{2.5}$$

Denote by $R(a,b)$ and $I_i(b)$ the events of a left-right occupied crossing of $r(a,b)$ and a top-bottom open crossing of $r(a+2b,b)$ which lies in the left end or the right end square when $i=1$ or $i=2$ with renormalized sites. Define

$$C(a,b) = R(a+2b,b) \cap \left\{ \bigcap_{i=1}^{2} I_i(b) \right\}$$

(see Fig. 3). A standard argument (see, e.g., Lemma 4.12 and (4.61)–(4.65) in [1]) now shows that

$$P_\alpha(C(1,1)) \geq \tfrac{17}{18} \tag{2.6}$$

will imply (2.3). Therefore, Lemma 1 is implied by the above proposition and the FKG inequality. □

Let $B(l) = [-l, l]^2$. We denote by $d(l)\uparrow$ [or $d(l)\downarrow$] the event that there is an open path in $Z^2(M)$ from (or to) $\partial B(l)$ to (or from) ∞ in $Z^2 \setminus B(l)$ (see Fig. 4), where ∂A is the surface of the set A of vertices.

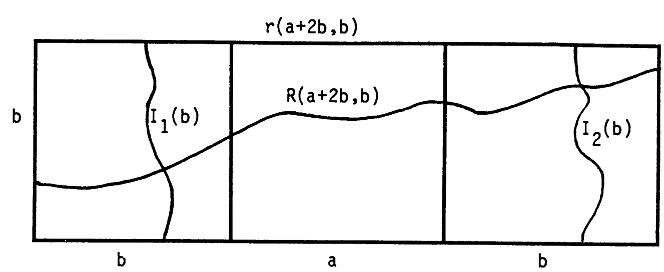

FIG. 3. *The events of* $C(a,b)$.

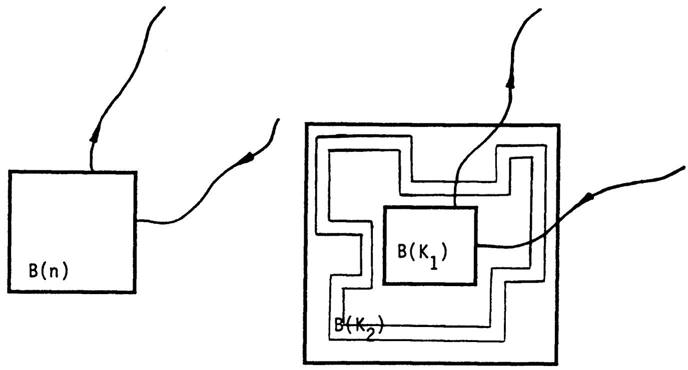

FIG. 4. *The event $d(l)\uparrow$ and $d(l)\downarrow$ and the definition of k_2.*

LEMMA 2. *If $\alpha > \alpha_c$, then there is a constant $c(\alpha)$ which only depends on α such that*

$$P_\alpha(d(l)\downarrow) = P_\alpha(d(l)\uparrow) \geq 1 - \exp\{-c(\alpha)l\}. \tag{2.7}$$

PROOF. By the symmetry of $g(z)$, the first equation is obvious. By the definition of $d(l)\uparrow$,

$$P_\alpha(\{d(l)\uparrow\}^c) \leq \sum_{k=l+1}^{\infty} 2k^2 \max_{y\in B(k)\setminus B(\max\{k-M,\, l\})} P_\alpha(a(k,y)), \tag{2.8}$$

where

$$a(m,y) = \{\text{There is an open path in } Z^2(M) \text{ from } \partial B(l) \text{ to the point } y \in \partial B(i) \text{ for } m - M \leq i \leq m \text{ and no open path in } Z^2(M) \text{ from } \partial B(l) \text{ to } \partial B(j) \text{ for } j > i\}. \tag{2.9}$$

If $a(m,y)$ occurs, there is no occupied circuit with renormalized sites in V_n which encircles the point y separating some point in $\partial B(l)$. Then

$$\begin{aligned} P_\alpha(d(l)\uparrow) &\leq \sum_{k\geq l+1}^{\infty} (2kl)^2 P_\alpha\left(\exists\, y \in Z^2,\ \|y\|_\infty > \frac{\max(k-M,l)}{n} \text{ such that } \{\Gamma(y,0)\}^c\right) \\ &\leq \sum_{k\geq l+1}^{\infty} (2kl)^2 \exp\left\{-\kappa(\alpha)\left(\frac{\max(k-M,l)}{n}\right)\right\} \quad \text{(by Lemma 1)} \\ &\leq \exp\{-c(\alpha)l\} \end{aligned} \tag{2.10}$$

for some constant $c(\alpha)$. This is the proof of Lemma 2. □

For any $z \in Z^2(M)$, we denote by $k_1(z)$ the smallest $k > 1$ such that there is an open path in $Z^2(M)$ from the boundary of $z + [-k, k]^2$ to ∞ and there is another open path in $Z^2(M)$ from ∞ to the boundary of $z + [-k, k]^2$. We also denote by $k_2(z)$ the smallest $m > k_1(z)$ such that there exists an occupied circuit with the renormalized sites of V_n in the annulus

$$\{z + [-m, m]^2\} \setminus \{z + [-k_1(z), k_1(z)]^2\}.$$

(See Fig. 4.) Then we have the following probability estimate:

LEMMA 3. *If $\alpha > \alpha_c$, there is a constant $c_1(\alpha)$ such that*

$$P_\alpha(k_2(0) > l) \le \exp(-c_1(\alpha) l). \tag{2.11}$$

PROOF. Clearly,

$$\begin{aligned} &P_\alpha(k_2(0) > l) \\ &\quad = P_\alpha\left(k_2(0) > l, k_1(0) < \frac{l}{2}\right) + P\left(k_2(0) > l, k_1(0) \ge \frac{l}{2}\right) \\ &\quad \le P_\alpha\left(k_2(0) > l, k_1(0) < \frac{l}{2}\right) + \exp\left(-c(\alpha)\frac{l}{2}\right) \quad \text{(by Lemma 2)}. \end{aligned} \tag{2.12}$$

Now by using the same proof of Lemma 2, we can show

$$\begin{aligned} P_\alpha\left(k_2(0) > l, k_1(0) < \frac{l}{2}\right) &\le P_\alpha\left(\begin{array}{l}\text{there is no occupied circuit with} \\ \text{the renormalized sites surrounding} \\ \left[-\frac{l}{2}, \frac{l}{2}\right]^2 \text{ in } [-l, l]^2 \setminus \left[-\frac{l}{2}, \frac{l}{2}\right]^2\end{array}\right) \\ &\le \exp(-c_2(\alpha) l). \end{aligned} \tag{2.13}$$

Therefore, Lemma 3 follows from (2.12) and (2.13). □

Lemma 3 prepares us to show the existence of radial limits. Let $\Delta(z)$ be the set $z + [-k_1(z), k_1(z)]^2$ and $\hat{t}(x, y)$ be the minimum passage time from a site of $\Delta(z)$ to a site of $\Delta(y)$. Let $u(z)$ be the sum of all $\tau\{x, y\} < \infty$ for $\{x, y\} \in z + [-k_2(z), k_2(z)]$.

If $t(x, y) < \infty$, then

$$\hat{t}(x, y) \le t(x, y) \le \hat{t}(x, y) + u(x) + u(y). \tag{2.14}$$

If we let $\xi(x, y) = \hat{t}(x, y) + u(y)$, then

$$\xi(x, z) \le \xi(x, y) + \xi(y, z)$$

by the definition of $k_1(z)$, $k_2(z)$ and (1.7). For any fixed $\theta \in Z^2$ let $\xi_{m,l} = \xi(m\theta, l\theta)$, $0 \le m < l < \infty$.

By Lemma 3, (1.9) and the argument of (3.6) in [4], it can be seen that $Eu^2(0) < \infty$ and $E(\hat{t}(x, y))^2 < \infty$. Hence $\xi_{m,l}$ is subadditive in the sense of

Kingman. The limit

$$\mu(\theta,\alpha) = \lim_{l\to\infty} \frac{\xi_{0,l}}{l} \quad \text{a.s. and in } L_1 \tag{2.15}$$

exists. Also

$$\left|\frac{\hat{t}(0,l\theta)}{l} - \frac{t(0,l\theta)}{l}\right| I\{l\theta \in C_0\} \le \frac{u(0)+u(l\theta)}{l}.$$

Since $Eu^2(l\theta) = Eu^2(0) < \infty$, the Chebyshev and Borel–Cantelli theorems imply that

$$\frac{u(l\theta)}{l} \to 0 \quad \text{a.s. and in } L_1.$$

Therefore, by (2.15),

$$\lim_l \frac{1}{l} t(0,l\theta) = \mu(\theta,\alpha) \quad \text{a.s. and in } L_1. \tag{2.16}$$

This is the proof of the existence of radial limits.

Now we begin to show the main probability estimate corresponding to (2.2):

$$P_\alpha(\hat{t}(0,z) \ge k\|z\|_\infty) = O\big(\|z\|_\infty^{-3}\big) \quad \text{as } z\to\infty \tag{2.17}$$

for some $k<\infty$ if $\alpha > \alpha_c$. Cox and Durrett [3] used the minimal circuit method to show (2.17) for the nearest neighbor case. However, this does not work for the finite range since we even cannot define the minimal circuit in $Z^2(M)$. Fortunately, we can define the minimal circuit with the renormalized sites since V_n is a planar graph. Let $c(z)$ be the minimal occupied circuit surrounding (i,j) with renormalized sites and $\bar{c}(z)$ be the union of $c(z)$, where $(i,j)\in V_n$ such that $z\in[in,(i+1)n)\times[jn,(j+1)n)$. Now we use this renormalized min-circuit instead of the min-circuit in [3]. It follows from the definition that our renormalized min-circuit also has the following properties described in [3]. First, if two renormalized min-circuits have a common site then they form a bigger circuit with the renormalized sites. Thus it corresponds to an open clockwise circuit with edges in $Z^2(M)$. In addition, $\{|\bar{c}(x)| = a\}$ and $\{|\bar{c}(y)| = b\}$ are independent if $\|x-y\|_\infty \ge a+b+4$ for $x,\ y\in V_n$. Therefore, (2.17) holds, when $\alpha>\alpha_c$, by Lemma 3 and the same argument in [3] (see pages 188–189 in [3] for details). Indeed, it is actually possible to show that $P_\alpha(\hat{t}(0,z)\ge k\|z\|_\infty)\to 0$ exponentially. We do not need this shaper result here.

Proof of Theorem 1. We follow the argument of [3] (see pages 189–190 in [3]). Let $\bar{A}_t = \{z: \hat{t}(0,z)\le t\}$. With the main estimate (2.17) one can derive (see [4]) that $\bar{A}_t$ contains a small ball with radius growing linearly in t. With the existence of radial limits, we can establish (see [4] for more details) for any fixed $\varepsilon>0$,

$$P\big((1-\varepsilon)A \subset t^{-1}\bar{A}_t \subset (1+\varepsilon)A \text{ for sufficiently large } t\big) = 1. \tag{2.18}$$

In addition to (2.18) we need:

(2.19) If $\varepsilon > 0$, then $P(u(z) > \varepsilon\|z\|_\infty \text{ i.o.}) = P(T_z > \varepsilon\|z\|_\infty \text{ i.o.}) = 0$.

This is a consequence of $Eu(z)^2 < \infty$ and $ET_z^2 < \infty$. We now wish to prove that the infected region does not sit far inside A. More precisely, we will prove

(2.20) if $\varepsilon > 0$, then $P(Q_t \cap (1-\varepsilon)tA = \varnothing \text{ for all sufficiently large } t) = 1$.

By (2.18) we know that a.s. for all large t, if $z \in (1-\varepsilon)tA$, then

$$\hat{t}(0,z) \le \left(1 - \frac{\varepsilon}{2}\right)t.$$

Add $u(0) + u(z) + T_z$ to both sides of this inequality and use (2.14) to obtain

$$t(0,z) + T_z \le \left(1 - \frac{\varepsilon}{2}\right)t + u(0) + T_z + u(z).$$

With $d = \sup_{x \in A}\|x\|_\infty$, we have from (2.19) that a.s. for all large t,

$$u(0) + u(z) + T_z \le \frac{\varepsilon}{3d}\|z\|_\infty \le \frac{\varepsilon}{3}(1-\varepsilon)t$$

[since $z \in (1-\varepsilon)tA$]. Combining this with the previous inequality gives us

$$t(0,z) + T_z \le \left(1 - \frac{\varepsilon}{6}\right)t,$$

and so z belongs to B_t, not Q_t. This proves (2.19) and

$$P(t(1-\varepsilon)A \cap C_0 \subset B_t \text{ for all sufficiently large } t) = 1.$$

On the other hand, if $z \in Q_t$ or $z \in B_t$, then $t(0,z) \le t$ so certainly $\hat{t}(0,z) \le t$, and by (2.18) $z \in (1+\varepsilon)tA$. That is,

$$P(Q_t \subset (1+\varepsilon)tA \text{ for all sufficiently large } t) = 1$$

and

$$P(B_t \subset (1+\varepsilon)tA \text{ for all sufficiently large } t) = 1. \qquad \square$$

3. Proof of Theorem 2. Let

$$W(e) = \begin{cases} 1, & \text{(closed) if } X(e) > 0, \\ 0, & \text{(open) if } X(e) = 0. \end{cases} \tag{3.1}$$

Then $W(e)$ has the Bernoulli distribution with the parameter $F(0)$. We call e open if $W(e) = 0$. If every bond on path r is open, r is called an open path. Since we assume $F(0) > p_c(d)$, there exists indeed an infinite open cluster C. Travel along bonds from C costs no time. The minimum passage time from $(0,\dots,0)$ to $(n,0,\dots,0)$ is therefore at most the time from $(0,\dots,0)$ to C plus the time from $(n,0\dots0)$ to C. If we can show $(0,\dots,0)$ is not far away from C, the argument of [9] might be applied. Now let us begin to show Theorem 2. We denote by $K_1(z)$ the smallest $K > 1$ such that there is an open path in Z^d from the boundary of $z + [-K,K]^d$ to ∞.

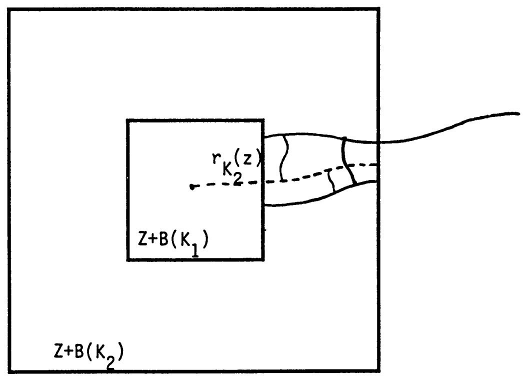

FIG. 5. *The definition of* K_1 *and* K_2. *The solid paths are open paths and the dashed path is the minimum passage path.*

Let $r_n(z)$ be one of the minimum passage paths from z to $\partial\{z + [-n, n]^d\}$. We denote by $K_2(z)$ the smallest $m > K_1(z)$ which satisfies the following:

(i) Any two open paths from boundary $z + [-K_1, K_1]^d$ to boundary $z + [-m, m]^d$ have to be connected by an open paths in $z + \{[-m, m]^d \setminus [-K_1, K_1]^d\}$, and

(ii) there exists a minimum passage path $r_m(z)$ such that any open paths from the boundary of $z + [-K_1(z), K_1(z)]^d$ to the boundary of $z + [-m, m]^d$ and $r_m(z)$ have to be connected by an open path in $z + \{[-m, m]^d \setminus [-K_1, K_1]^d\}$ (see Fig. 5).

We simply denote by $B(K_1(z))$ and $B(n)$ the sets $z + [-K_1, K_1]^d$ and $[-n, n]^d$. By the definition of K_1 and K_2 and the uniqueness of the infinite cluster (see [2]), we can see that there exists an open path from the boundary of $B(K_1(0, \dots, 0))$ to the boundary of $B(K_1(n, 0, \dots, 0))$ if $B(K_1(0, \dots, 0)) \cap B(K_1(n, 0, \dots, 0)) = \varnothing$. Let

$$\overline{N}_{mn} = \min\{|r|, r \text{ is an open path from } \partial B(K_1(m, \dots, 0)) \text{ to } \partial B(K_1(n, 0, \dots, 0))\}, \tag{3.2}$$

if $B(K_1(m, \dots, 0)) \cap B(K_1(n, 0, \dots, 0)) = \varnothing$, or $\overline{N}_{mn} = 0$ otherwise. Let

$$\xi_{mn} = \overline{N}_{mn}^a + 2^d K_2^d(n, 0, 0, \dots, 0). \tag{3.3}$$

By the definition of $K_2(z)$ and $\overline{N}_{0n}$ and the same argument as in [9],

$$\xi_{0(m+n)} \le \xi_{0m} + \xi_{mn}. \tag{3.4}$$

If we can show that

$$E(\overline{N}_{0n})^2 < \infty \tag{3.5}$$

and

$$EK_2^{2d}(n,0,0,\dots,0) = EK_2^{2d}(0,0,0,\dots,0) < \infty, \tag{3.6}$$

then $\{\xi_{0n}\}$ is a subadditive process in the sense of Kingman. Therefore

$$\frac{\xi_{0n}}{n} \to \lambda \quad \text{a.s. and in } L_1. \tag{3.7}$$

Also by the definition of $\overline{N}_{mn}$ and K_2,

$$N_{0n}^a \le 2^d K_2^d(0,\dots,0) + 2^d K_2^d(n,0,\dots,0) + \overline{N}_{0n}, \tag{3.8}$$

$$\overline{N}_{0n} \le 2^d K_2^d(0,\dots,0) + 2^d K_2^d(n,0,\dots,0) + N_{0n}^a. \tag{3.9}$$

Hence, by (3.3), (3.8) and (3.9),

$$\left|\frac{\xi_{0n} - N_{0n}^a}{n}\right| \le \frac{2(2^d K_2(0,0,\dots,0) + 2^d K_2^d(n,0,0,\dots,0))}{n}. \tag{3.10}$$

By Chebyshev, and Borel–Cantelli arguments and (3.6),

$$\frac{K_2^d(n,0,0,\dots,0)}{n} \to 0 \text{ a.s. in } L^1. \tag{3.11}$$

Hence

$$\lim_{n\to\infty} \frac{N_{0n}^a}{n} = \lambda \quad \text{a.s. and in } L^1 \tag{3.12}$$

from (3.7), (3.10) and (3.11). Now we show (3.5) and (3.6). Equation (3.5) holds by virtue of (8.32) in [9]. Equation (3.6) holds by the following argument. We simply denote $(0,\dots,0) = 0$. By $F(0) > p_c$, Theorem 6.126 in [5] and Theorem A in [6],

$$P(K_1(0) > n) \le L\exp(-c(F(0))n) \tag{3.13}$$

for some constants $c(F(0))$ and L. In addition,

$$\begin{aligned} &P(K_2(0) > n) \\ &= P\left(K_1(0) > \frac{n}{2}, K_2(0) > n\right) + P\left(\left(K_1(0) \le \frac{n}{2}, K_2(0) > n\right)\right) \\ &\le L\exp\left(-c(F(0))\frac{n}{2}\right) + P\left(K_1(0) \le \frac{n}{2}, K_2(0) > n\right). \end{aligned} \tag{3.14}$$

Now we need to estimate the second term of (3.14). We say that the vertex $u \in Z^d$ has the property Π_{n_1,n_2} if there exist two paths r_1 and r_2 from the boundary $\{u + B(n_1)\}$ to the boundary $\{u + B(n_2)\}$ which are not connected by an occupied path in $\{u + B(n_2) \setminus B(n_1)\}$. If $F(0) > p_c(d)$, it follows from (3.5) and (3.8) in [8] that

$$P(0 \text{ has the property } \Pi_{k_1 k_2}) < (6k_2)^{2d} \exp\{-c_1(F(0))|k_2 - k_1|\} \tag{3.15}$$

for some constant $c_1(F(0))$. It is also known from [7] that

$$P\big(t(r_n(0)) \geq \sqrt{n}\,\big) \leq c_4(F(0))\exp\{-c_2(F(0))\sqrt{n}\} \tag{3.16}$$

when $F(0) > p_c$, where $c_2(F(0))$ and $c_4(F(0))$ are constants. However, if $\{t(r_n(0)) < \sqrt{n}\}$, there exists an open path $r \subset r_n(0)$ from the boundary of $B((i+1)\sqrt{n}/3 + K_1)$ to the boundary of $B(i(\sqrt{n}/3) + K_1)$ in

$$B\left((i+1)\frac{\sqrt{n}}{3} + K_1\right) \setminus B\left(i\frac{\sqrt{n}}{3} + K_1\right)$$

for some $2\sqrt{n} \leq i \leq 3\sqrt{n}$. By the definition of $K_2(0)$,

$$\begin{aligned} P&\left(K_1(0) \leq \frac{n}{2}, K_2(0) \geq n, t(r_n(0)) \leq \sqrt{n}\right) \\ &\leq \sum_{i=0}^{n} P\big(0 \text{ has the property } \Pi_{i, i+\sqrt{n}/3}\big) \\ &\quad + P\big(0 \text{ has the property } \Pi_{n/2, n}\big) \\ &\leq (12n)^{2d} \exp\left(-c_1(F(0))\frac{\sqrt{n}}{3}\right). \end{aligned} \tag{3.17}$$

It follows from (3.14), (3.16) and (3.17) that

$$EK_2^{2d}(0) < \infty. \tag{3.18}$$

Hence (3.6) is proved.

Now we discuss

$$\lim_{n\to\infty} \frac{N_{0n}^b}{n} = \lambda \quad \text{a.s. and } L^1. \tag{3.19}$$

Here we only give a sketch of the proof. First we define

$$Y_{mn} = \inf\{T((m,0,\dots,0),(n,k_2,0,\dots,0)) : k_2 \in Z\}$$

and

$$N_{mn}^Y = \inf\{\text{the number of edges in } r, t(r) = Y_{nm}\}.$$

Then by using the result of [6] we define renormalized sites in the k-slab lattice (see definition (4.40) in [1]) for some k with $F(0) > p_c(k)$, where $p_c(k)$ is the critical point for the k-slab lattice. After renormalization, we can define the min-circuit with the renormalized sites as we did in last section, and replace the smallest circuit $\Delta(z)$ in [10] by the min-circuit as we just mentioned. Then we follow the exact proof in ([10], pp. 1074–1076) to show

$$\lim_{n\to\infty} \frac{N_{0n}^Y}{n} = \lambda \quad \text{a.s. and } L^1. \tag{3.20}$$

After that, we can discuss

$$Y^2_{mn} = \inf\{T((m,0,\dots,0),(n,k_2,k_3,0,\dots,0)) \colon k_2,k_3 \in Z\},$$
$$\vdots$$
$$Y^{d-1}_{mn} = \inf\{T((m,0,\dots,0),(n,k_2,\dots,k_d)) \colon k_2,\dots,k_d \in Z\}.$$

The argument of (3.20) and a simple induction then show

$$\lim_n \frac{N^{Y^{d-1}}_{0n}}{n} = \cdots = \lim_n \frac{N^{Y}_{0n}}{n} = \lim_n \frac{N^{b}_{0n}}{n} = \lambda \quad \text{a.s. and in } L^1. \tag{3.21}$$ □

APPENDIX

Proof of proposition. To show the proposition we only need to show that the large sponge-crossing and the large sponge-connection have a high probability at the supercritical case. By the FKG inequality, we can show them separately.

Estimation of the large sponge-crossing. We define

$$\begin{aligned} C_z &= \{x \colon x \text{ can be reached from } z \text{ by an open path in } Z^2(M)\},\\ C_z\downarrow &= \{x \colon z \text{ can be reached from } x \text{ by an open path in } Z^2(M)\},\\ \theta(\alpha) &= P_\alpha(|C_0| = \infty). \end{aligned} \tag{A.1}$$

By the symmetric property of $g(z)$,

$$\theta(\alpha) = P_\alpha(|C_0\downarrow| = \infty). \tag{A.2}$$

Let N be a positive integer. For any configuration $w \in \Gamma = \{0,1\}^{Z^2(M)}$, we define the sphere (see [1]) with radius N and center at w by

$$S^n(w) = \left\{w' \in \Gamma \colon \sum |w_{\{x,y\}} - w'_{\{x,y\}}| \le N\right\}. \tag{A.3}$$

$S^N(w)$ is the collection of configurations which differ from w on at most N edges. If $A \subset \Gamma$ is an event, we define the interior and the exterior of A by

$$I^NA = \{n \in \Gamma \colon S^N(n) \subset A\} \quad \text{and} \quad E^NA = \{n \in \Gamma \colon S^N(n) \cap A \ne \varnothing\}. \tag{A.4}$$

If $\alpha' \le \alpha$, then by (1.6) and the definition of $g(z)$,

$$0 \le P_{\alpha'}(\{x,y\} = 1) \le P_\alpha(\{x,y\} = 1). \tag{A.5}$$

We denote

$$\begin{aligned} m(\alpha) &= \min_{\|y\|_\infty \le M} \{(P_\alpha(\{0,y\}) = 1)\},\\ m(\alpha',\alpha) &= \min_{\|y\|_\infty \le M} \{P_\alpha(\{0,y\} = 1) - P_{\alpha'}(\{0,y\} = 1)\}. \end{aligned} \tag{A.6}$$

By the definition of F and (1.6), we can see that $m(\alpha) > 0$ and $m(\alpha,\alpha') > 0$ if $\alpha > \alpha' > \alpha_c$. With the definitions above, the following lemma is a straightforward adaptation of Theorem 2.45 in [5] or Lemma 4.2 in [1].

LEMMA A.1. *For any positive event A and $\alpha' < \alpha$,*

$$P_\alpha(A) \geq m(\alpha', \alpha)^N P_{\alpha'}(E^N A), \tag{A.7}$$

$$P_\alpha(I^N A) \geq 1 - \frac{1 - P_{\alpha'}(A)}{m(\alpha', \alpha)^N}. \tag{A.8}$$

Next we introduce some sets and events necessary for construction of the renormalized site lattice. If S, F and T are three sets, we denote by $S \to T$ in F the event that there is an open path from S to T with edges in F and $S \leftrightarrow T$ in F the event that $S \to T$ in F and $T \to S$ in F. We also denote by $S \nrightarrow T$ in F the event that there is no open path from S to T in F. A box in Z^2 is defined by

$$B(n_1, n_2, n_3, n_4) = [n_1, n_2] \times [n_3, n_4].$$

The top, bottom, left and right boundary of the above box are defined by

$$T(B(n_1, n_2, n_3, n_4)) = [n_1, n_2] \times \{n_4\},$$

$$U(B(n_1, n_2, n_3, n_4)) = [n_1, n_2] \times \{n_3\},$$

$$L(B(n_1, n_2, n_3, n_4)) = \{n_1\} \times [n_3, n_4],$$

$$R(B(n_1, n_2, n_3, n_4)) = \{n_2\} \times [n_3, n_4].$$

Finally, we define

$$T^L(B(n_1, n_2, n_3, n_4)) = \left[n_1, \frac{n_1 + n_2}{2}\right] \times \{n_4\},$$

$$T^R(B(n_1, n_2, n_3, n_4)) = \left[\frac{n_1 + n_2}{2}, n_2\right] \times \{n_4\}.$$

(See Fig. 6).

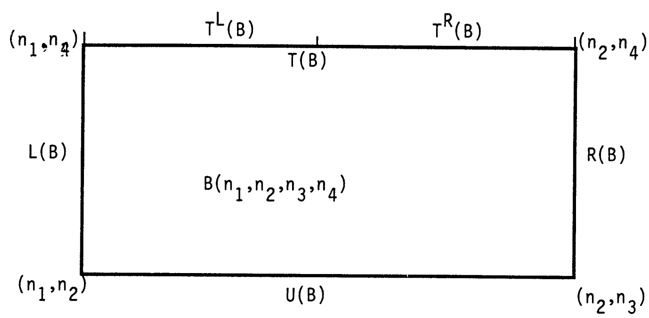

FIG. 6. *The definitions of the box, top, bottom, left and right boundary of box, and $T^L(B)$ and $T^R(B)$.*

LEMMA A.2. *Assume $\alpha > \alpha_c$. Then given $\varepsilon > 0$, there exists N such that for $n > 2N + M$,*

$$P_\alpha\left(U\left(B\left(0,n,0,\frac{n-N}{2}\right)\right) \to T\left(B\left(0,n,0,\frac{n-N}{2}\right)\right) \text{ in } B\left(0,n,0,\frac{n-N}{2}\right)\right) \geq 1-\varepsilon. \tag{A.9}$$

PROOF. Let $\alpha_c < \alpha_1 < \alpha$. Since $\theta(\alpha_1) > 0$, by a standard ergodic theorem, we can take N large enough that

$$P_\alpha\left(\left[0,\frac{N}{2}\right] \leftarrow \infty \text{ in } Z^2(M)\right) \geq 1-\varepsilon.$$

Then it follows from the square root trick (see [3] for details), and the symmetry and the translation invariance of $Z^2(M)$ that (see Fig. 7)

$$P_{\alpha_1}\left(B(0,n,0,M) \to \left[\frac{n-N}{2},\frac{n+N}{2}\right] \times \left\{\frac{n-N}{2}\right\} \text{ in } B(0,n,0,n-N)\right) \geq 1-\varepsilon. \tag{A.10}$$

Consequently,

$$P_\alpha\left(B(0,n,0,M) \to B\left(0,n,\frac{n-N}{2}-M,\frac{n-N}{2}\right) \text{ in } B\left(0,n,0,\frac{n-N}{2}\right)\right) \geq 1-\varepsilon. \tag{A.11}$$

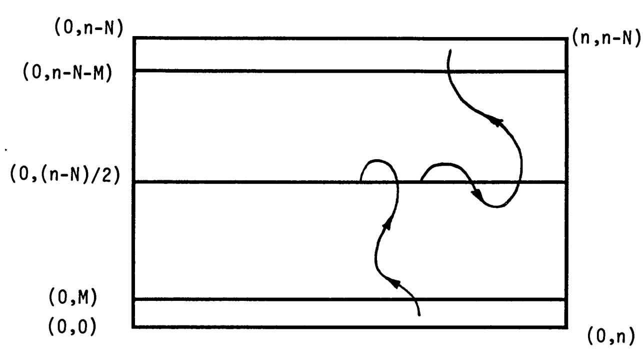

FIG. 7. *The event* $B(0,n,0,M) \to [(n-N)/2,(n-N)/2]$ *in* $B(0,n,0,n-N)$ *and the event* $[n/2,(n+N)/2] \times \{(n-N)/2\} \to B(n/2,n,n-N-M,n-N)$ *in* $B(0,n,n-N-M,n-N)$.

Then by Lemma A.1, for some large number k,

$$
\begin{aligned}
1 - &\frac{1 - P_{\alpha_1}(B(0,n,0,M) \to B(0,n,(n-N)/2 - M,(n-N)/2) \text{ in } B(0,n,0,(n-N)/2))}{m^k(\alpha,\alpha_1)} \\
&\le P_\alpha\left(I^k\left\{B(0,n,0,M) \to B\left(0,n,\frac{n-N}{2} - M,\frac{n-N}{2}\right) \text{ in } B\left(0,n,0,\frac{n-N}{2}\right)\right\}\right) \\
&\le P_\alpha\left(\exists\ k \text{ disjoint open paths from } B(0,n,0,M) \text{ to } B\left(0,n,\frac{n-N}{2} - M,\frac{n-M}{2}\right)\right) \\
&\le \sum_{0 \le i \le M} P_\alpha(H = i),
\end{aligned}
\tag{A.12}
$$

where H is the largest j, $0 \le j \le M$, such that there exist at least k/M open paths from $B(0,n,0,M)$ to $[0,n] \times \{(n-N)/2 - j\}$ in $[0,n] \times [0,((n-N)/2) - j]$. Since each such path can be connected to $T(B(0,n,0,(n-N)/2)$ by a single edge and $H = i$ only depends on the edges in $[0,n] \times [0,((n-N)/2) - i]$, we can pick k large such that at least one of these k/M open paths can be connected to $[0,n] \times \{(n-N)/2\}$ by a single open edge with a high probability. Note also that $H = i$ and $H = j$ are disjoint events if $i \ne j$. Combining this with (A.11) and (A.12) gives us that $P_\alpha(B(0,n,0,M) \to T(B(0,n,0,(n-N)/2)) \to 1$ by taking k, ε and N, respectively. Using this argument to connect some vertices, which belong to the open paths from $B(0,n,0,M)$ to $T(B(0,n,0,(n-N)/2))$ in $B(0,n,0,M)$, to $U(B(0,n,0,M)$ by a single open edge, we can see that Lemma A.2 holds. □

LEMMA A.3. *Suppose that* $\alpha > \alpha_c$ *and* $2N + M \le n$, *and let* $S \subset B(0,(3/2)n,0,n-N)$ *be a set with*

$$
\left| S \cap T^R\left(B\left(0,n,0,\frac{n-N}{2}\right)\right)\right| \ge k.
$$

For given $\varepsilon > 0$ *and integer* l, *if* $k < n$ *is big enough, then*

$$
P_\alpha\left(\exists \text{ at least } l \text{ disjoint open paths from } S \text{ to } T\left(B\left(\frac{n}{2},\frac{3}{2}n,0,n-N\right)\right) \text{ in } B\left(0,\frac{3}{2}n,0,n-N\right) \setminus S\right) \ge 1 - \varepsilon.
\tag{A.13}
$$

Note that a path in $B(0,(3/2)n,0,n-N) \setminus S$ means that each edge of the path is in $B(0,(3/2)n,0,n-N) \setminus S$, but some vertices of the path may still belong to S.

PROOF OF LEMMA A.3. By a standard ergodic theorem, the square root trick, the symmetry and the translation invariance of $Z^2(M)$, we can take N large such that (see Fig. 7)

$$P_{\alpha_1}\left(\left[\frac{n}{2},\frac{n+N}{2}\right]\times\left\{\frac{n-N}{2}\right\}\to B\left(\frac{n}{2},n,n-N-M,n-N\right)\right.$$
$$\text{(A.14)}\qquad \left.\text{in } B(0,n,0,n-N)\right)\geq 1-\frac{1}{2}\varepsilon m^l(\alpha_1,\alpha)$$

for some $\alpha > \alpha_1 > \alpha_c$. Then in turn there exists $h = h(\alpha, M)$ such that

$$\text{(A.15)}\qquad \mu\left(\bigcup_{i=1}^{h}E_i\right) > 1-\varepsilon\frac{1}{2}m^l(\alpha_1,\alpha),$$

where $E_1,\dots,E_h$ are independent events in some probability space, each having μ-probability given by

$$\mu(E_i) = P_{\alpha_1}\left(\forall \text{ edge with vertices in } \left[\frac{n}{2},\frac{n+N}{2}\right]\times\left\{\frac{n-N}{2}\right\} \text{ is open}\right).$$

In fact it is sufficient that

$$\left(1-m(\alpha)^{MN}\right)^h \leq \tfrac{1}{2}\varepsilon m^l(\alpha_1,\alpha),$$

but we do not require the precise condition. Now we can partition $[n/2,n]\times\{(n-N)/2\}$ into disjoint n/N segments $\{A_i\}$ with length $N/2$. Then by (A.14) and (A.15), $k > Nh$ implies

$$\text{(A.16)}\qquad 1-m(\alpha_1,\alpha)^l\varepsilon \leq P_{\alpha_1}(A),$$

where A is the event that there is an open path from A_i to $B(n/2,(3/2)n, n-N-M, n-N)$ for some i, every edge in A_i is open, and $A_i \cap S \neq \varnothing$ (see Fig. 8). It can be seen that there exists an open path from S to

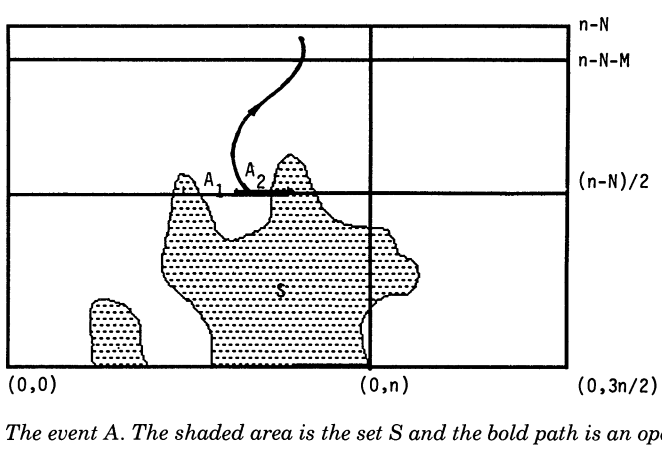

FIG. 8. *The event A. The shaded area is the set S and the bold path is an open path.*

$B(n/2,(3/2)n,n-N-M,n-N)$ if A occurs. By using the argument of Lemma A.2 to extend a single open edge in these open paths,

$$1-m(\alpha_1,\alpha)^l\varepsilon \le P_{\alpha_1}(\bar{A}) \tag{A.17}$$

for large N, where $\bar{A}$ is the event that there exists an open path from S to $[n/2,(3/2)n]\times\{n-N\}$. Then, by Lemma A.1,

$$\begin{aligned}
1-\varepsilon \le 1-\frac{1-P_{\alpha_1}(\bar{A})}{m^l(\alpha_1,\alpha)} &\le P_\alpha(I^l(\bar{A}))\\
&\le P_\alpha\Bigg(\exists \text{ at least } l \text{ disjoint paths in } B\left(0,\frac{3}{2}n,0,n-N\right)\\
&\qquad \text{from } S\cap T^R\left(B\left(0,n,0,\frac{n-N}{2}\right)\right)\\
&\qquad \text{to } T\left(B\left(\frac{n}{2},\frac{3}{2}n,0,n-N\right)\right)\Bigg)\\
&\le P_\alpha\Bigg(\exists\; l \text{ disjoint paths in } B\left(0,\frac{3}{2}n;0,n-N\right)\setminus S \text{ from } S\\
&\qquad \text{to } T\left(B\left(\frac{n}{2},\frac{3}{2}n,0,n-N\right)\right)\Bigg).
\end{aligned}\tag{A.18}$$

Lemma A.3 is proved. □

With these lemmas in mind, we begin to estimate the probability of the large sponge-crossing.

LEMMA A.4. *Given $\varepsilon>0$ and integer i, there exists N such that, when $\alpha>\alpha_c$ and $n\ge 2N+M$,*

$$P_\alpha(U(B(-n,2n,0,in))\to T(B(-n,2n,0,in)))\ge 1-\varepsilon. \tag{A.19}$$

PROOF. By Lemma A.2 and the symmetry property,

$$P_{\alpha_1}\left(U\left(B\left(0,n,0,\frac{n-N}{2}\right)\right)\to T^R\left(B\left(0,n,0,\frac{n-N}{2}\right)\right)\right)\ge 1-\varepsilon \tag{A.20}$$

for $\alpha>\alpha_1>\alpha_c$ and some large N. We write $C_z(n)$ for the open cluster containing z in $B(0,n,0,(n-N)/2)$ for some

$$z\in U\left(B\left(0,n,0,\frac{n-N}{2}\right)\right).$$

We denote by V the unit of all clusters $C_z(n)$ in $B(0,n,0,(n-N)/2)$. Let W be the edge set of the boundary of these clusters in $B(0,n,0,(n-N)/2+M)$, and let (V_0,W_0) be a possible pair of clusters for the random pair (V,W) (see Fig. 9). Note that each edge in $W\cap B(0,n,0,(n-N)/2)$ has to be closed. We

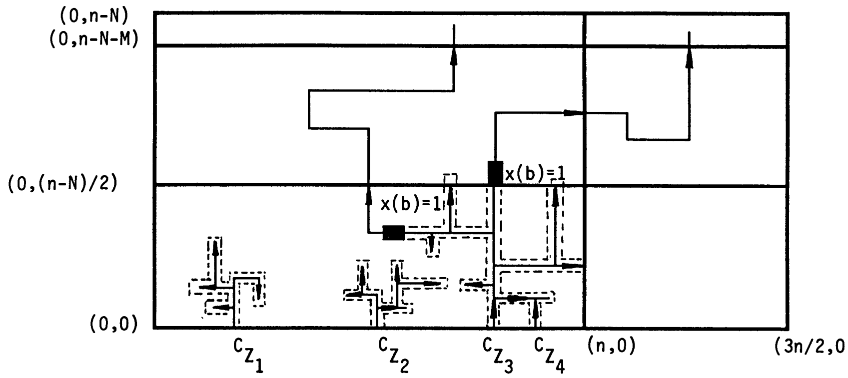

FIG. 9. *A configuration of* (V, W). *The dot lines are the set* W_0 *and the solid lines are open clusters and open paths. The solid small boxes are the edges* b *with* $X(b) = 1$.

also define event $Q(k)$ by

$$Q(k) = \left\{ \left| V \cap T^R\left(B\left(0, n, 0, \frac{n-N}{2}\right)\right)\right| \geq k \right\}$$

Then by Lemma A.1,

$$\begin{aligned} 1 - \frac{1 - P_{\alpha_1}(U(B(0,n,0,(n-N)/2)) \to T^R(B(0,n,0,(n-N)/2)))}{m(\alpha_2,\alpha_1)^k} \\ \leq P_{\alpha_2}\left(I^k\left(U\left(B\left(0,n,0,\frac{n-N}{2}\right)\right)\right) \to T^R\left(B\left(0,n,0,\frac{n-N}{2}\right)\right)\right) \\ \leq P_{\alpha_2}(Q(k)). \end{aligned} \tag{A.21}$$

Then if we take N large, by (A.20),

$$1 - \varepsilon \leq P_{\alpha_2}(Q(k)) \tag{A.22}$$

for some $\alpha_1 < \alpha_2 < \alpha$. Then by the definition of (V_0, W_0),

$$\begin{aligned} 1 - \varepsilon &\leq P_{\alpha_2}(Q(k)) \\ &= \sum_{V_0, W_0} P_{\alpha_2}((V,W) = (V_0,W_0) \cap Q(k)), \end{aligned} \tag{A.23}$$

where the summation is taken over all possible of pairs (V_0, W_0) such that $|V_0 \cap T^R(B(0,n,0,(n-N)/2))| \geq k$. For a fixed pair (V_0, W_0) and any integer l, let (see Fig. 9)

$$\begin{aligned} J(l) = \Big\{ &\exists \text{ at least } l \text{ disjoint open paths from } W_0 \text{ to} \\ &T\left(B\left(\frac{n}{2}, \frac{3}{2}n, 0, (n-N)\right)\right) \\ &\text{in } B\left(0, \frac{3n}{2}, 0, (n-N)\right) \setminus (V_0, W_0) \Big\}. \end{aligned} \tag{A.24}$$

It can be seen that $\{J(l)\}$ is independent of $\{(V,W)=(V_0,W_0)\}$ since $J(l)$ only depends on the edges outside of (V_0,W_0). Since

$$\left|V_0 \cap T^R\left(B\left(0,n,0,\frac{n-N}{2}\right)\right)\right| \geq k,$$

if $Q(k)$ occurs, we can take k large and use Lemma A.3 to obtain

$$\begin{aligned}
\sum_{V_0,W_0} & P_{\alpha_2}((V,W)=(V_0,W_0)\cap Q(k)) \\
&= \sum_{V_0,W_0} P_{\alpha_2}((V,W)=(V_0,W_0)\cap Q(k)\cap J(l)) \\
&\quad + \sum_{V_0,W_0} P_{\alpha_2}\left((V,W)=(V_0,W_0)\cap Q(k)\cap\{J(l)\}^C\right) \\
&\leq \sum_{V_0,W_0} P_{\alpha_2}((V,W)=(V_0,W_0)\cap Q(k)\cap J(l)) \\
&\quad + \sum_{V_0,W_0} P_{\alpha_2}((V,W)=(V_0,W_0))P_{\alpha_2}\left(\{J(l)\}^C\right) \\
&\leq \sum_{V_0,W_0} P_{\alpha_2}((V,W)=(V_0,W_0)\cap Q(k)\cap J(l)) \\
&\quad + \varepsilon \quad \text{(by Lemma A.3 and taking } k \text{ big and then } N \text{ big)}.
\end{aligned} \tag{A.25}$$

Let $S \subset W_0$ be the set such that, for each edge $b \in S$, there is an open path from one site of b to $T(B(n/2,(3/2)n,0,(n-N))$ in $B(0,3n/2,0,(n-N))\setminus\{W_0 \cup V_0\}$. Then $|S| \geq l$ if $J(l)$ occurs. We also define another random variable $X(b)$ for each edge b in $Z^2(M)$ as follows: If b is open, then $X(b)$ is 1. If b is closed, then $X(b)=1$ with probability $\delta/P_{\alpha_2}(b$ is closed) or $X(b)=0$ with probability $(P_{\alpha_2}(b$ is closed$)-\delta)/P_{\alpha_2}(b$ is closed). We say that b is $X(b)$ open (closed) if $X(b)=1$ $(X(b)=0)$. The determinations are made independently over the edges. It can be seen that the distribution of $X(b)$ is Bernoulli with parameter $P_{\alpha_2}(b$ is open$)+\delta$. We say a path is an $X(b)$ open path if all edges in the path are $X(b)$ open. Then, by (A.25),

$$\begin{aligned}
\sum_{V_0,W_0} & P_{\alpha_2}((V,W)=(V_0,W_0)\cap Q(k)\cap J(l)) \\
&= \sum_{V_0,W_0}\sum_{S_0} P_{\alpha_2}((V,W)=(V_0,W_0)\cap Q(k)\cap J(l),\ S=S_0) \\
&= \sum_{V_0,W_0}\sum_{S_0} P_{\alpha_2}((V,W)=(V_0,W_0)\cap Q(k)\cap J(l),\ S=S_0, \\
&\qquad\qquad \exists\, b\in S_0 \text{ with } X(b)=1) \\
&\quad + \sum_{V_0,W_0}\sum_{S_0} P_{\alpha_2}\Big((V,W)=(V_0,W_0)\cap Q(k)\cap J(l),\ S=S_0, \\
&\qquad\qquad \{\exists\, b\in S_0 \text{ with } X(b)=1\}^C\Big) \\
&= \mathrm{I}+\mathrm{II},
\end{aligned} \tag{A.26}$$

where the second summation is taken over all S_0 in W_0 such that, for each $b \in S_0$, there is an open path from one site of b to $T(B(n/2, (3/2)n, 0, (n - N))$ in $B(0, 3n/2, 0, (n - N)) \setminus \{W_0 \cup V_0\}$. Clearly,

$$I \le P_{\alpha_2}\Big(\exists \text{ an } X(b) \text{ open path from } U(B(0,n,0,n-N)) \text{ to } T\Big(B\Big(\frac{n}{2}, \frac{3}{2}n, 0, n-N\Big) \text{ in } B\Big(0, \frac{3n}{2}, 0, n-N\Big)\Big)\Big).$$

Next we estimate II. If $b \in W_0 \cap B(0, n, 0, (n - N)/2)$, it has to be closed. Then b is $X(b)$ closed with probability $1 - (\delta / P_{\alpha_2}(b \text{ is closed}))$. If $b \in W_0 \setminus \{W_0 \cap B(0, n, 0, (n - N)/2)\}$, $X(b)$ is closed with probability $1 - P_{\alpha_2}(b \text{ is open}) - \delta$. Then we can pick δ small enough that

$$P_{\alpha_2}(\text{none of the bonds in } S_0 \text{ is } X(b) \text{ open} | S = S_0) \le \left(1 - \frac{\delta}{1 - m(\alpha_2)}\right)^{|S_0|}. \tag{A.27}$$

Then given $\varepsilon > 0$, by taking l big, we have

$$\begin{aligned} \text{II} &\le \sum_{V_0, W_0} \sum_{S_0} P_{\alpha_2}((V,W) = (V_0,W_0) \cap I(k) \cap J(l), S = S_0) \\ &\quad \times P_{\alpha_2}(\text{none of the bonds in } S_0 \text{ is } X(b) \text{ open} | S = S_0) \\ &\le \varepsilon. \end{aligned} \tag{A.28}$$

By (A.23), (A.25), (A.26), (A.28), and choosing δ such that $P_{\alpha_2}(X(b) = 1) \le P_{\alpha_3}(b \text{ is open})$ for each bond in $Z^2(M)$ and some $\alpha_2 < \alpha_3 < \alpha$, then

$$\begin{aligned} 1 - 3\varepsilon &\le P_{\alpha_2}\Big(\exists \text{ an } X(b) \text{ open path from } U(B(0,n,0,n-N)) \\ &\qquad \text{to } T\Big(B\Big(\frac{n}{2}, \frac{3}{2}n, 0, n-N\Big) \text{ in } B\Big(0, \frac{3n}{2}, 0, n-N\Big)\Big)\Big) \\ &\le P_{\alpha_3}\Big(U\Big(B(0,n,0,n-N) \to T\Big(B\Big(\frac{n}{2}, \frac{3}{2}n, 0, n-N\Big)\Big) \\ &\qquad \text{in } B\Big(0, \frac{3n}{2}, 0, n-N\Big)\Big)\Big) \end{aligned} \tag{A.29}$$

(see Fig. 9). By the same argument and the symmetry property, we show that

$$\begin{aligned} &P_{\alpha_4}\Big(U(B(0,n,0,(n-N))) \to T\Big(B\Big(0,n,0,\frac{3}{2}(n-N)\Big) \\ &\qquad \text{in } B\Big(-\frac{n}{2}, 2n, 0, \frac{3}{2}(n-N)\Big)\Big)\Big) \ge 1 - 6\varepsilon \end{aligned} \tag{A.30}$$

for some $\alpha_3 \le \alpha_4 < \alpha$ and some large N and $n \ge 2N + M$. If we continue doing these steps $4i + 1$ times with large n depending on i, and use the argument of extending a single open edge in these open paths from $U(B(-n, 2n, 0, in))$ to $B(-n, 2n, in - M, in)$ (see Lemma A.2), then Lemma A.4 holds. □

Next we use Lemma A.4 to extend an open circuit in an annular region with positive probability. More precisely, let

$$C_\delta(n) = B(-(1+\delta)n, (1+\delta)n, -(1+\delta)n, (1+\delta)n) \setminus B(-n, n, -n, n)$$

and (see Fig. 10)

$$\overline{C}_\delta(n) = \{\text{there exists an open circuit in } C_\delta(n)\}$$

for some $0 < \delta < 1$ and integer n.

LEMMA A.5. *If $\alpha > \alpha_c$ and $\delta > 0$, there exists N depending on δ and a constant $c > 0$ such that*

$$P_\alpha(\overline{C}_\delta(n)) > c \tag{A.31}$$

for all $n \geq N$.

PROOF. By Lemma A.4 and the FKG inequality, we can take n large such that the event $A_1 \cap A_2 \cap A_3 \cap A_4$ has probability larger than $1/2$ for some $\alpha_c < \alpha_1 < \alpha$, where A_1 is the event of a left-right open crossing of $[-n(1+\delta), n(1+\delta)] \times [-n(1+\delta), -n]$, A_2 is the event of a bottom-top open crossing of $[n, n(1+\delta)] \times [-n(1+\delta), n(1+\delta)]$, A_3 is the event of a right-left open crossing of $[-n(1+\delta), n(1+\delta)] \times [n, n(1+\delta)]$ and A_4 is the event of a

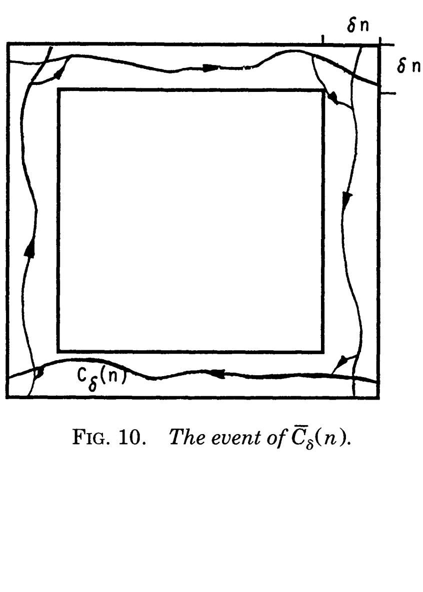

FIG. 10. *The event of $\overline{C}_\delta(n)$.*

top-bottom open crossing of $[-n(1+\delta), -n] \times [-n(1+\delta), n(1+\delta)]$ (see Fig. 10). By Lemma A.1,

$$\begin{aligned} P_\alpha(\overline{C}_\delta(n)) &\geq m^4(\alpha_1, \alpha) P_{\alpha_1}\big(E^4(\overline{C}_\delta(n))\big) \\ &\geq m^4(\alpha_1, \alpha) P_{\alpha_1}(A_1 \cap A_2 \cap A_3 \cap A_4) \geq \frac{m^4(\alpha_1, \alpha)}{2} \end{aligned} \tag{A.32}$$

for some $\alpha_c < \alpha_1 < \alpha$. Hence Lemma A.5 is proved. □

Estimation of the probability of large sponge-connection. We begin to estimate the connectedness of two open paths in a large box. Before giving some results, we need some definitions. Let $Z^+ = \{0, 1, 2, \dots, \}$. Define the following:

$$\begin{aligned} C_0(H) &= \{x\colon x \text{ can be reached from 0 by an open path} \\ &\qquad\qquad\qquad \text{in the half space } Z^+ \times Z\}; \\ \theta_H(\alpha) &= P_\alpha(|C_0(H)| = \infty), \\ \alpha_c(H) &= \inf\{\alpha\colon P_\alpha(|C_0(H)| = \infty) > 0\}. \end{aligned} \tag{A.33}$$

Now we need to show that $\alpha_c(H) = \alpha_c$. In [6], Grimmett and Marstrand have obtained a similar result for a general Z^d percolation. The key proof of their argument is called the sprinkling method. That is to add a small density of extra open vertices to create an open path which connects two large square boxes with a high probability (see Lemma 6 in [6]). However, this work in our case is done by our Lemma A.4. Therefore, by adapting the same argument in [6] (see the details of the proof of Theorem A in [6]), we have:

COROLLARY A.6. $\alpha_c(H) = \alpha_c$.

Now we start to estimate the probability of the large sponge connectedness.

LEMMA A.7. *Define (see Fig. 11) the following*:

$$\begin{aligned} D_n = \{&\exists \text{ two open paths } \gamma_1 \text{ and } \gamma_2 \text{ such that} \\ &U(B(0,2n,0,n)) \xrightarrow{\gamma_1} T(B(0,2n,0,n)), \\ &L(B(0,2n,0,n)) \xrightarrow{\gamma_2} R(B(0,2n,0,n)), \text{ but} \\ &\gamma_1 \not\leftrightarrow \gamma_2\}. \end{aligned} \tag{A.34}$$

Then given $\varepsilon > 0$, there exists N such that

$$P_\alpha(D_n) < \varepsilon \tag{A.35}$$

when $n > N$ and $\alpha > \alpha_c$.

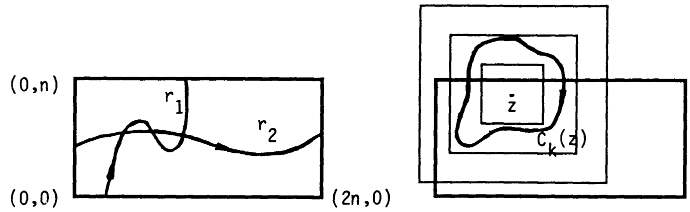

FIG. 11. *The left figure is the event D_n and the right one is the events $\overline{C}_k(z)$.*

PROOF. For any $0 < \delta < 1$, m and any vertex $z \in B(0, 2n, 0, n)$, we define the annulus surrounding z as follows (see Fig. 11):

$$C_0(z) = \{z + [-m, m]^2\},$$
$$C_1(z) = \{z + [-m(1+\delta), m(1+\delta)]\} \setminus \{z + [-m, m]\},$$
$$\vdots$$
$$C_k(z) = \left\{z + \left[-m(1+\delta)^k, m(1+\delta)^k\right]\right\} \setminus \left\{z + m(1+\delta)^{k-1}, m(1+\delta)^{k-1}\right\},$$

where k is the smallest number such that

$$[0, n]^2 \subset z + \left[-m(1+\delta)^{1+k}, m(1+\delta)^{1+k}\right],$$

if $z \in [0, n]^2$ or

$$[n, 2n] \times [0, n] \subset z + \left[-m(1+\delta)^{1+k}, m(1+\delta)^{1+k}\right]$$

if $z \in B(n, 2n, 0, n)$. By the definition of k, $k \geq (\log n - (\log 2m))/\log(1+\delta)$. Also we denote by $\partial^i C_j(z)$ and $\partial^o C_j(k)$ the inside boundary and outside boundary of $C_k(z)$. Define

$$\overline{C}_k(z) = \{\text{there is an open circuit in } C_k(z)\}.$$

Now we first show that there is a constant $c_1 > 0$ such that, by taking m large,

$$P_\alpha(v_1 \leftrightarrow v_2 \text{ in } C_k(z)) > c_1 \tag{A.36}$$

for any v_1, $v_2 \in \partial^i C_k(z)$. We follow the proof of Lemma 4.3 in [1]. For any $\delta > 0$ and $\alpha_c < \alpha_1 < \alpha$, we choose m such that

$$P_{\alpha_1}(\Gamma_k) \geq c\theta_H^2(\alpha_1)$$

by Lemma A.5, Corollary A.6 and the FKG inequality, where

$$\Gamma_k = \{\text{there are two open paths } r_1 \text{ and } r_2 \text{ from } v_1 \text{ and } v_2 \text{ to } \partial^o C_k(z) \cap \overline{C}_k(z)\}.$$

If Γ_k occurs, there are four single edges (not necessary open) in $C_k(z)$ from r_1 and r_2 to the open circuit and from the open circuit to r_1 and r_2. Specifically,

$r_1 \leftrightarrow r_2$ occurs if these four edges are open. Define the event

$$G_k = \{\Gamma_k \cap \{\text{these four edges are open}\}\}.$$

Then by Lemma A.1,

$$\begin{aligned} P_\alpha(v_1 \leftrightarrow v_2 \text{ in } C_k(z)) &\ge P_\alpha(G_k) \\ &\ge m^4(\alpha_1, \alpha) P_{\alpha_1}(E^4(G_k)) \\ &\ge m^4(\alpha_1, \alpha) P_{\alpha_1}(\Gamma_k) \\ &\ge c m^4(\alpha_1, \alpha) \theta_H^2(\alpha_1). \end{aligned} \tag{A.37}$$

Hence, (A.36) is proved. Now we estimate $P_\alpha(D_n)$ by the method of Proposition 1 in [8]. Partition $B(0, 2n, 0, n)$ into equally likely squares with side length m ($2m > M$). Since γ_1 and γ_2 are bottom-top and a left-right open crossings of $B(0, 2n, 0, n)$, they have to come to one of these squares. We denote by z the center vertex of this square. Then

$$P_\alpha(D_n) \le 32m\left(\frac{n}{m}\right)^2 \max_{x_1, x_2 \in \partial C_0(z)} \mathrm{P}_\alpha(\exists \text{ two open paths } \gamma_1, \gamma_2 \text{ from } x_1 \text{ and } x_2 \text{ of } \partial C_0(z) \text{ to } \partial^o C_k(z), \gamma_1 \not\leftrightarrow \gamma_2). \tag{A.38}$$

Let $C(x_1)$ and $C(x_2)$ be the open clusters in $z + [-m(1+\delta)^{k-1}, m(1+\delta)^{k-1}]$. Then $C(x_1)$ and $C(x_2)$ only depend on the edges in $z + [-m(1+\delta)^{k-1}, m(1+\delta)^{k-1}]$ denoted by E. Indeed, once the configuration in E is fixed, we merely pick any $x \in C(x_1)$ and $y \in C(x_2)$ in $z + [-m(1+\delta)^{k-1}, m(1+\delta)^{k-1}] \setminus \{z + [-m(1+\delta)^{k-1} + M, m(1+\delta)^{k-1} - M]\}$, which are connected by two open edges (not in E) to $C_k(z)$ and apply (A.36). Then we have a conditional probability at least $cm^2(\alpha) m^4(\alpha_1, \alpha) \theta_H^2(\alpha_1)$ of connecting x and y in $C_k(z)$. Iteration of (A.38) by using this argument on $z + [-m(1+\delta)^{k-1}, m(1+\delta)^{k-1}], \dots, z + [-m(1+\delta), m(1+\delta)]$ shows

$$P(D_n) \le \frac{32n^2}{m}\left(1 - cm^2(\alpha) m^4(\alpha_1, \alpha) \theta_H^2(\alpha_1)\right)^{(\log n - \log(2m))/\log(1+\delta)}. \tag{A.39}$$

Then simply take δ small, and m and n large to see that

$$P(D_n) \to 0 \quad \text{as } n \to \infty.$$

Then Lemma A.7 is proved. □

Define

$$\begin{aligned} K_n = \{&\exists \text{ two open paths } \gamma_1 \text{ and } \gamma_2 \text{ such that} \\ &U(B(0, n, 0, n)) \overset{\gamma_1}{\to} T(B(0, n, 0, n)), \\ &U(B(0, n, 0, n)) \overset{\gamma_2}{\to} T(B(0, n, 0, n)), \text{ but} \\ &\gamma_1 \not\leftrightarrow \gamma_2\}. \end{aligned} \tag{A.40}$$

If some vertices of γ_1 and γ_2 are not far apart, then we can use the argument of Lemma A.7 to show the following corollary. If each two vertices on γ_1, γ_2, respectively, are far apart, then we can add a left-right open crossing of $B(0, n, 0, n)$ and use the same argument of Lemma A.7 and Lemma A.4 to show the following corollary.

COROLLARY A.8. *Given* $\varepsilon > 0$, *there exists* N *such that*

$$P_\alpha(k_n) < \varepsilon \tag{A.41}$$

when $n \geq N$ *and* $\alpha > \alpha_c$.

Also by the argument of Lemma A.7, we have the following:

COROLLARY A.9. *If* $\alpha > \alpha_c$, *then the infinite cluster in* $Z^2(M)$ *is unique.*

PROOF OF THE PROPOSITION. By the FKG inequality, Lemma A.4, Lemma A.7 and Corollary A.8, there exists N such that

$$P_\alpha(A_n(0,0) \cap B_n(0,0)) \geq 1 - \varepsilon \tag{A.42}$$

when $\alpha > \alpha_c$ and $n \geq N$. □

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