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DECOMPOSITION OF DIRICHLET PROCESSES AND ITS APPLICATION

BY T. J. LYONS AND T. S. ZHANG

Edinburgh University

We extend the forward–backward martingale approach to Stratonovich integrals developed by Zheng and Lyons to the general context of Dirichlet spaces. From this perspective, it is clear that the Stratonovich integral of an L^2 1-form against a Dirichlet process is well defined, coordinate invariant, and obeys appropriate chain rules.

The paper continues by examining the tightness and continuity of the mapping from Dirichlet forms to probability measures on path space. Some positive results are obtained for a class of infinite-dimensional diffusions.

0. Introduction. Itô's stochastic calculus has been extremely well developed in the context of semimartingales. Any function on Euclidean space composed with Brownian motion gives rise to a semimartingale process provided it has two derivatives (or more precisely is in the domain of the Laplace operator into finite measures). However, from many perspectives one would like to understand the process one obtains when the function is in $W_{1,2}$ and has only one derivative; in this case the process will not in general be a semimartingale. This has directed attention to the stochastic calculus of Dirichlet processes, an extension of the notion of semimartingale. This study of the stochastic calculus of Dirichlet processes is also important in the study of diffusion processes corresponding to uniformly elliptic second-order differential operators in divergence form with measurable coefficients, for in this case the coordinate functionals are not always in the domain of the infinitesimal generator and so the processes will not in general be semimartingales.

The basic framework for understanding Dirichlet processes is the Dirichlet space developed by Fukushima [6]. In this context, let $\{X_t, P_x\}$ be a nice symmetric Markov process on a state space X which is a locally compact Hausdorff space with countable base, and let $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ be the associated Dirichlet space. For every $u \in \mathscr{D}(\mathscr{E})$ let $\tilde{u}$ be a quasicontinuous version of u. Fukushima [6] showed that $A_t^{[u]} = \tilde{u}(X_t) - \tilde{u}(X_0)$ is a Dirichlet process and that $A^{[u]}$ admits the decomposition

$$A_t^{[u]} = M_t^{[u]} + N_t^{[u]} \tag{0.1}$$

The process $M^{[u]}$ is a martingale additive functional (a.f.) of finite energy and the process $N^{[u]}$ is a continuous a.f. of zero energy. The process $N_t^{[u]}$ will only be of bounded variation if u is in some domain of the infinitesimal generator, and so generally $A_t^{[u]}$ is not a semimartingale. The theory of semimartingales

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cannot be used directly to define stochastic integrals against $A_t^{[u]}$. By using a particular linear operator and (0.1), Nakao [11] gave a definition of the Stratonovich integral in this context and obtained some applications. On the other hand, under the assumption that the Markov process X is conservative, Lyons and Zheng [10] obtained another expression for $A^{[u]}$. Let $P_m(\cdot) = \int P_x(\cdot)m(dx)$ be the measure on paths corresponding to the law of X conditional on X_0 having distribution m (not necessarily finite). Suppose m is the canonical invariant measure for the Dirichlet space. Then

$$(0.2) \qquad A_t^{[u]} = \tfrac{1}{2}M_t^{[u]} - \tfrac{1}{2}\Big[\overline{M}_T^{[u]} - \overline{M}_{T-t}^{[u]}\Big], \qquad 0 \le t \le T,\ P_m\text{-a.e.}$$

Here $M_t^{[u]}$ is an $(\mathscr{F}_t, P_m)$ martingale where $\mathscr{F}_t = \sigma(X_s, 0 \le s \le t)$, and $\overline{M}_t^{[u]}$ is an $(\overline{\mathscr{F}}_t, P_m)$ or backward martingale where $\overline{\mathscr{F}} = \sigma(X_{T-s}, s \le t)$. The decomposition (0.2) incorporates important cancellations pathwise and allows useful estimates to be deduced even in situations where the original process was a semimartingale (see Takeda [17]).

In this paper we explain in a general context how the martingales can be used to define the Stratonovich integral of a forward and backward predictable process against the Dirichlet process. This approach to constructing integrals first appeared in [10], but on that occasion the forms contracted with the paths of X_t were concretely chosen.

We prove that all the integrals studied in this paper fall into the class of Dirichlet processes. Furthermore, the integral can be constructed as a limit of Riemann sums, making the definition of a Stratonovich integral clear and easy to understand pathwise. In particular, we obtain the chain rule.

Another purpose of this paper is to give a criterion for convergence of certain diffusion processes on infinite-dimensional space. Assume H is a Hilbert space densely and continuously embedded in a Hilbert space X; X acts as the state space for the process associated with the diffusion form $\mathscr{E}(u, v) = \int \langle \nabla u, A(z)\nabla v\rangle_H \mu(dz)$ (see the details in [3]). Under a reasonable condition we obtain a tightness criterion for the measures on path space $C([0,\infty) \to X)$ associated with the Markov processes of a family of diffusion forms. In some special cases, we identify the limit process.

The paper is organized as follows: In Section 1 we state the martingale decomposition of the Dirichlet process and prove that the decomposition is unique in a certain sense. In Section 2 we develop the Stratonovich calculus for Dirichlet processes. The criterion for tightness of the diffusion processes on infinite-dimensional space is given in Section 3. Finally, we give another approach to the construction of diffusion processes on Hilbert space.

1. A martingale decomposition. Let X be a locally compact Hausdorff space with a countable base, and let m be a positive Radon measure on X such that supp$[m] = X$. Let $\mathscr{E}$ be a regular Dirichlet form on $L^2(X, m)$ with the local property, and let $\mathscr{D}(\mathscr{E})$ be the domain of $\mathscr{E}$ as described, for example, in [6]. By [6], we know that there is a diffusion process $\{\Omega, \mathscr{F}, X_t, P_x, x \in X\}$ associated with $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ so that if L is the unbounded self-adjoint operator on $L^2(X, m)$ defined by

$$\mathscr{E}(f, g) = (-Lf, g)_{L^2(X,m)}, \qquad f \in \mathscr{D}(L), g \in \mathscr{D}(\mathscr{E}),$$

then L is the generator of the transition semigroup of X on $L^2(X, m)$. For any additive functional A_t of the Markov process, the energy of A is defined by the formula

$$e(A) = \lim_{t \to 0} \frac{1}{2t} E_m(A_t^2).$$

Here $E_m(A_t^2)$ denotes the expectation of A_t^2 with respect to $P_m(\cdot)$. Let

$$\mathscr{M} = \{M;\ M \text{ is an a.f. such that, for each } t > 0,\ E_x[M_t^2] < +\infty \text{ and } E_x[M_t] = 0 \text{ q.e. } x \in X\}.$$

If $M \in \mathscr{M}$, then M_t is a square-integrable martingale additive functional. We say A_t is a continuous additive functional of zero energy if A_t is an element of the following family:

$$N_c = \{N\colon N \text{ is a continuous a.f., } e(N) = 0,\ E_x[|N_t|] < +\infty, \text{ q.e. for each } t\}.$$

For $f \in \mathscr{D}(\mathscr{E})$ let $\tilde{f}$ denote a quasicontinuous version of f and let $A_t^{[f]} = \tilde{f}(X_t) - \tilde{f}(X_0)$. Because $\tilde{f}$ is unique up to sets of capacity zero, $A_t^{[f]}$ is uniquely defined for all t P_x-almost surely for quasi-all x. Then it is known by Fukushima [6] that the additive function $A^{[f]}$ can be written as

$$(1.0) \qquad A_t^{[f]} = M_t^{[f]} + N_t^{[f]}, \qquad M_t^{[f]} \in \mathscr{M},\ N_t^{[f]} \in N_c,$$

extending the remark that if $f \in \mathscr{D}(L)$, then $A^{[f]}$ is a semimartingale. We now prove (in a slightly more general context) the forward–backward martingale composition for $A^{[u]}$ given in [10]. Consider the stochastic process on a fixed time parameter space $[0, 1]$. Set $\mathscr{F}_t = \sigma(X_s,\ s \le t)$, $\overline{\mathscr{F}}_t = \sigma(X_{1-s},\ s \le t)$. The $\mathscr{F}_t$-martingales and $\overline{\mathscr{F}}_t$-martingales are called forward martingales and backward martingales, respectively. For $f \in \mathscr{D}(\mathscr{E})$, the following decomposition holds.

THEOREM 1.1. *Assume that X is conservative. Then, under P_m and for any $f \in \mathscr{D}(\mathscr{E})$, there exists a continuous forward P_m-martingale M_t^f and a continuous backward P_m-martingale $\overline{M}_t^f$ which satisfy the following conditions*:

(i) $\tilde{f}(X_t) - \tilde{f}(X_0) = \frac{1}{2}M_t^f - \frac{1}{2}(\overline{M}_1^f - \overline{M}_{1-t}^f)$.

(ii) $N_t^f = M_t^f + (\overline{M}_1^f - \overline{M}_{1-t}^f)$ *is a continuous additive functional of zero energy*.

(iii) M_t^f *and* $\overline{M}_t^f$ *are square P_m-integrable*, $\overline{M}_0^f = M_0^f = 0$.

Conditions (i), (ii) *and* (iii) *uniquely determine the martingales*.

PROOF. *Existence*. If $f \in \mathscr{D}(L)$, it is well known that

$$(1.1) \qquad M_t^f = \tilde{f}(X_t) - \tilde{f}(X_0) - \int_0^t L\tilde{f}(X_s)\, ds, \qquad 0 \le t \le 1,$$

is an $(\mathscr{F}_t, P_m)$ square-integrable martingale. Since L is a self-adjoint operator on $L^2(X, m)$, $\{X_t\}$ and $\{X_{1-t}, 0 \le t \le 1\}$ are identical in law under P_m. So it follows from (1.1) that

$$(1.2) \qquad \overline{M}_t^f = \tilde{f}(X_{1-t}) - \int_0^t L\tilde{f}(X_{1-s})\, ds - \tilde{f}(X_1), \qquad 0 \le t \le 1,$$

is an $(\overline{\mathscr{F}}_t, P_m)$ square-integrable martingale. Combining (1.1) with (1.2), we see that

$$(1.3) \qquad \overline{M}_t^f = \tilde{f}(X_{1-t}) - \left(N_1^{[f]} - N_{1-t}^{[f]}\right) - \tilde{f}(X_1),$$

$$(1.4) \qquad M_t^f + \left(\overline{M}_1^f - \overline{M}_{1-t}^f\right) = -2N_t^{[f]},$$

$$(1.5) \qquad \tilde{f}(X_t) - \tilde{f}(X_0) = \tfrac{1}{2}M_t^f - \tfrac{1}{2}\left(\overline{M}_1^f - \overline{M}_{1-t}^f\right)$$

and, furthermore,

$$(1.6) \qquad \begin{aligned} &\int_\Omega \left[\langle M^f, M^f\rangle_1 - \langle M^f, M^f\rangle_0\right] dP_m \\ &= \int_\Omega \left[\langle \overline{M}^f, \overline{M}^f\rangle_1 - \langle \overline{M}^f, \overline{M}^f\rangle_0\right] dP_m \\ &= 2\mathscr{E}(f, f). \end{aligned}$$

If $f \in \mathscr{D}(\mathscr{E})$, one may approximate it by a sequence $f_n \in \mathscr{D}(L)$ so that $\mathscr{E}(f_n - f, f_n - f) + \int |f_n - f|^2\, dm \to 0$. By (1.6), we can define $\overline{M}_t^f = \lim_{n\to\infty} \overline{M}_t^{f_n}$, where the limit of $\overline{M}^{f_n}$ is taken in $L^2(\Omega, P_m)$ and similarly $M_t^f = \lim_{n\to\infty} M_t^{f_n}$. On the other hand, from Lemma 5.1.2 of [6], we know that there exists a subsequence $\{n_k\}$ such that $\lim_{k\to\infty} \tilde{f}_{n_k}(X_t) = \tilde{f}(X_t)$ uniformly on $[0,1]$, P_m-a.s. Therefore (1.5) holds for any $f \in \mathscr{D}(\mathscr{E})$. Since we use the same limit procedure to get M^f as Fukushima [6], it is obvious that $M^f = M^{[f]}$ [as introduced in (0.1)], and (1.4) holds for $f \in \mathscr{D}(\mathscr{E})$. So (ii) is satisfied and the proof of existence is finished.

Uniqueness. If there are two systems $\{{}^1M^f, {}^1\overline{M}^f\}, \{{}^2M^f, {}^2\overline{M}^f\}$ satisfying (i)–(iii), then

$$(1.7) \qquad \tilde{f}(X_t) - \tilde{f}(X_0) = {}^1M_t^f - \tfrac{1}{2}\left[{}^1M_t^f + \left({}^1\overline{M}_1^f - {}^1\overline{M}_{1-t}^f\right)\right],$$

$$(1.8) \qquad \tilde{f}(X_t) - \tilde{f}(X_0) = {}^2M_t^f - \tfrac{1}{2}\left[{}^2M_t^f + \left({}^2\overline{M}_1^f - {}^2\overline{M}_{1-t}^f\right)\right].$$

Thus, ${}^1M_t^f - {}^2M_t^f = 2({}^1N_t^f - {}^2N_t^f)$ is a continuous martingale of zero energy.

Consequently, ${}^1M_t^f = {}^2M_t^f$. Letting $t = 1$ in the equalities (1.7) and (1.8), we get

$${}^2\overline{M}_1^f = {}^1\overline{M}_1^f = {}^1M_1^f + 2\left[\tilde{f}(X_0) - \tilde{f}(X_1)\right].$$

Therefore, ${}^2\overline{M}_t^f = {}^1\overline{M}_t^f$, $0 \le t \le 1$. The uniqueness holds. □

REMARK 1.2. It is possible to decompose $\tilde{f}(X_t) - \tilde{f}(X_0)$ into a difference of martingales in a number of ways. Condition (ii) forces the uniqueness.

EXAMPLE 1.3. Let $X = R$,

$$\mathscr{E}(f,g) = \int_R \frac{df}{dx}\frac{dg}{dx}\,d\mu, \qquad \mu = e^{-x^2/2}\frac{1}{\sqrt{2\pi}}\,dx,$$

$$\mathscr{D}(\mathscr{E}) = \{\mathscr{E}(f,f) < +\infty,\ f \in L^2(R:\mu)\}.$$

In this case, as we know, $\{\Omega, \mathscr{F}, X_t, \theta_t, P_x, x \in R\}$ is the Ornstein–Uhlenbeck process, that is,

$$X_t = X_0 + B_t - \int_0^t X_s\,ds,$$

where B is standard Brownian motion on $\mathbb{R}$. Let $f(x) = x$; this allows one to decompose X. The forward martingale part M_t of X is $B_t = X_t - X_0 + \int_0^t X_s\,ds$. Using symmetry, the backward martingale is given by

$$\overline{M}_t = X_{1-t} - X_1 - \int_0^{1-t} X_s\,ds + \int_0^1 X_s\,ds.$$

One may readily check that $X_t - X_0 = \frac{1}{2}M_t - \frac{1}{2}(\overline{M}_1 - \overline{M}_{1-t})$, $0 \le t \le 1$.

2. Stratonovich calculus for Dirichlet processes

2.1. *The definition of Stratonovich integrals.* In this and the following sections, we fix a Dirichlet space $(\mathscr{E}, \mathscr{D}(\mathscr{E}), L^2(X, m))$. Let $\{\Omega, \mathscr{F}, X_t, \theta_t, P_m, P_x, x \in X\}$ be the associated diffusion process. Assume that X is conservative and that it satisfies the "champs de carrée." That is to say, there exists a linear subset C_1 of $\mathscr{D}(\mathscr{E}) \cap C_0(X)$ and a quadratic functional $f \to \tilde{f}$ from C_1 into the positive cone in $L^2(X, m)$ with the following properties:

(2.1.1)
(i) C_1 is a core of $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$;
(ii) $2\mathscr{E}(fh, f) - \mathscr{E}(h, f^2) = \int h\tilde{f}\,dm$, for any $h \in C_1$.

For $f, g \in C_1$, we define a map $\Gamma(f,g) = \frac{1}{4}(\widetilde{f+g} - \widetilde{(f-g)})$ from $C_1 \times C_1$ to $L^1(X, m)$. Then (see [4]),

$$\int_X |h|\Gamma(f,f)\,dm \le \|h\|_\infty \mathscr{E}(f,f), \qquad f, h \in C_1. \tag{2.1.2}$$

Let $h \in \mathscr{D}(\mathscr{E}) \cap L^\infty$. By Theorem 1.4.2(ii) in [6], one can choose a sequence $\{h_n\}$ so that

$$h_n \xrightarrow{\mathscr{E}_1} h, \qquad h_n \in C_1,\ \|h_n\|_\infty \le M.$$

It follows that fh_n converges to fh weakly in the Hilbert space $(\mathscr{E}_1, \mathscr{D}(\mathscr{E}_1))$ [where $\mathscr{E}_\alpha(f,g) = \mathscr{E}(f,g) + \alpha(f,g)_{L^2(X,m)}$], and h_n converges to h in m-measure. Replace h by h_n in (2.1.1) and let $n \to \infty$. Then (2.1.1) holds for any $h \in \mathscr{D}(\mathscr{E}) \cap L^\infty$. From (2.1.1) and (2.1.2), we know that the form Γ is uniformly continuous and so extends uniquely to a positive symmetric bilinear continuous form from $\mathscr{D}(\mathscr{E}_1) \times \mathscr{D}(\mathscr{E}_1)$ to $L^1(X, m)$.

EXAMPLE 2.1.1. Let $X = R^d$,

$$\mathscr{E}(f,g) = \sum_{i=1}^{d}\sum_{j=1}^{d}\int_{R^d} a_{ij}\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j}\,dx, \qquad f,g \in \mathscr{D}(\mathscr{E}) \supset C_0^\infty(R^d).$$

Whenever $\mathscr{E}$ defines a Dirichlet form on $L^2(R^d, dx)$ then

$$\Gamma(f,g) = 2\sum_{i,j} a_{ij}\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j}.$$

For $f \in \mathscr{D}(\mathscr{E})$, let M^f and $\overline{M}^f$ be, respectively, the forward martingale and backward martingales described in Theorem 1.1. We have remarked that the martingale M^f can be identified with the additive functional $M^{[f]}$ and so there is a smooth measure $\mu_{\langle f\rangle}$ corresponding to the bracket $\langle M^f\rangle$ of the additive functional M^f. A function f is a local Dirichlet function [i.e., $f \in \mathscr{D}_{\mathrm{loc}}(\mathscr{E})$] if one can find $f_n \in \mathscr{D}(\mathscr{E})$ and an exhaustion of X by open sets U_n such that $f_n = f$ on U_n.

We have the following proposition for $\mu_{\langle f\rangle}$.

PROPOSITION 2.1.1. *Let $f \in \mathscr{D}(\mathscr{E})_{\mathrm{loc}}$. Then $\mu_{\langle f\rangle}$ is absolutely continuous with respect to m; moreover,*

$$\frac{d\mu_{\langle f\rangle}}{dm} = \Gamma(f,f) \quad \textit{when } f \in \mathscr{D}(\mathscr{E}).$$

PROOF. By Theorem 5.2.3 in [6] and by (2.1.1), it is easy to see that, for $f \in \mathscr{D}(\mathscr{E}) \cap L^\infty$, $\mu_{\langle f\rangle} \ll m$ and $d\mu_{\langle f\rangle}/dm = \Gamma(f,f)$. However, for $h \in \mathscr{B}_b^+$ and $u, v \in \mathscr{D}(\mathscr{E})$, the following inequality holds:

$$\left(\sqrt{\int_X h\,d\mu_{\langle u\rangle}} - \sqrt{\int_X h\,d\mu_{\langle v\rangle}}\right)^2 \le 2\|h\|_\infty \mathscr{E}(u-v, u-v). \tag{2.1.3}$$

(See the proof of Lemma 5.4.6 in [6].)

Given $f \in \mathscr{D}(\mathscr{E})$, we take a sequence $\{f_n\} \subset \mathscr{D}(\mathscr{E}) \cap L^\infty$ such that $\mathscr{E}_1(f_n - f, f_n - f) \to 0$ as $n \to \infty$. Then (2.1.3) and the property of Γ yield

$$\int_X h\,d\mu_{\langle f\rangle} = \lim_{n\to\infty}\int_X h\,d\mu_{\langle f_n\rangle} = \lim_{n\to\infty}\int_X h\Gamma(f_n, f_n)\,dm = \int_X h\Gamma(f,f)\,dm.$$

So $\mu_{\langle f\rangle}$ is absolutely continuous with respect to m and $d\mu_{\langle f\rangle}/dm = \Gamma(f,f)$.

If $f \in \mathscr{D}(\mathscr{E})_{\mathrm{loc}}$ and $m(K) = 0$ (K is a compact set), we choose a relatively compact open set $U \supset K$ and $f_U \in \mathscr{D}(\mathscr{E})$ with property $f_U = f$ on U. Then $\mu_{\langle f\rangle}(K) = \mu_{\langle f_U\rangle}(K) = 0$. Therefore, $\mu_{\langle f\rangle} \ll m$. □

This yields the immediate corollary:

COROLLARY 2.1.2. *For $f \in \mathscr{D}(\mathscr{E})$, $\langle M^f\rangle_t = \int_0^t \Gamma(f,f)(X_s)\,ds$.*

Using the symmetry of time reversal, we also obtain

$$(2.1.4)\qquad \langle \overline{M}^f\rangle_t = \int_0^t \Gamma(f,f)(X_{1-s})\,ds.$$

Hereafter, we denote $\mu_{\langle f\rangle}$ by μ, M^f by M, $\overline{M}^f$ by $\overline{M}$, and so forth, unless confusion would arise.

Let u be a bounded measurable function on X and let U^n be open sets such that $U^n \uparrow X$ and $u|_{U^n} \in L^2(X,\mu)$.

Define $T_n = \inf\{t, X_t \notin U^n\}$ and $\overline{T}_n = \inf\{t, X_{1-t} \notin U^n\}$. We have

$$(2.1.5)\qquad \begin{aligned} E\left[\int_0^{T_n} u^2(X_s)\,d\langle M\rangle_s\right] &= E\left[\int_0^{T_n} u^2(X_s)\Gamma(f,f)(X_s)\,ds\right] \\ &\le \int_X u^2(X)\chi_{U^n}(x)\Gamma(f,f)\,dm < +\infty. \end{aligned}$$

So the Itô integral $\int_0^t u(X_s)\,dM_s$ is well defined; by symmetry, so is $\int_0^t u(X_{1-s})\,d\overline{M}_s$.

Definition. The Stratonovich integral of $u(X_s)$ with respect to the Dirichlet processes $f(X_s)$ is

$$(2.1.6)\qquad \int_0^t u(X_s)\circ df(X_s) \triangleq \tfrac{1}{2}\int_0^t u(X_s)\,dM_s - \tfrac{1}{2}\int_{1-t}^1 u(X_{1-s})\,d\overline{M}_s.$$

The use of Stratonovich's name is justified by Theorem 2.3.1. It is obvious that the definition can be extended to the case where u is a time-dependent measurable function satisfying $\int_0^1\int_X u^2(x,t)\,dt\,d\mu_{\langle f\rangle} < \infty$.

2.2. *The properties of the Stratonovich integral.* In this section we study the Stratonovich integral and obtain some useful properties.

Definition 2.2.1. An R^1-valued stochastic process A_t is said to be of 0-quadratic variation if

$$(2.2.1)\qquad \lim_{n\to\infty} \sum_{t_i\in\tau^n} \left(A_{t_{i+1}} - A_{t_i}\right)^2 = 0 \quad \text{in } P_m,$$

for any sequence $\{\tau^n\}$ of partitions of $[0,1]$ with $\delta(\tau^n) \to 0$.

Definition 2.2.2. We call the stochastic process Y_t a regular Dirichlet process if the following decomposition holds:

$$(2.2.2)\qquad Y_t = M_t + A_t.$$

Here M_t is a martingale and A_t is of 0-quadratic variation.

First we remark that the Stratonovich integral $\int_0^t u(X_s)\circ df(X_s)$ is bilinear with respect to u and f. The main result of this section is the following.

THEOREM 2.2.1. *If* $u \in L^2(X, d\mu_{\langle f\rangle})$, *then* $\int_0^t u(X_s)\circ df(X_s)$ *is a regular Dirichlet process and the martingale part is* $\int_0^t u(X_s)\,dM_s^f$.

PROOF. Since $\int_0^t u(X_s)\circ df(X_s) = \frac{1}{2}\int_0^t u(X_s)\,dM_s^f - \frac{1}{2}\int_{1-t}^1 u(X_{1-s})\,d\overline{M}_s^f$, it is sufficient that $N_t = \int_0^t u(X_s)\,dM_s^f + \int_{1-t}^1 u(X_{1-s})\,d\overline{M}_s^f$ is a process of zero quadratic variation. We divide the proof of this fact into four steps.

Step 1. Assume $u \in \mathscr{D}(\mathscr{E})$ and $f \in \mathscr{D}(L)$. By Theorem 1.1(ii),

$$\overline{M}_t^f = M_{1-t}^f - M_1^f - 2\int_{1-t}^1 Lf(X_s)\,ds.$$

Now, take a sequence $\{\tau^n = (t_i^n)\}$ of partitions of $[0,t]$ such that the mesh size $\delta(\tau^n) = \max(t_{i+1}^n - t_i^n)$ converges to 0. For simplicity of notation, we write t_i for t_i^n, M_t for M_t^f and $\overline{M}_t$ for $\overline{M}_t^f$, unless it is otherwise specified. Put $\hat{\tau}^n = (\hat{t}_0^n = 1 - t_{k_n}^n, \dots, \hat{t}_{k_n}^n = 1)$. Then $\{\hat{\tau}^n\}$ is a sequence of partitions of $[1-t, 1]$ and $\delta(\hat{\tau}^n) \to 0$.

Because $u \in \mathscr{D}(\mathscr{E})$ we may assume that it is quasicontinuous and, in consequence, $u(X_t)$ is continuous for all t with probability 1. We write

$$\begin{aligned}
&\sum_{\hat{\tau}^n} u(X_{1-\hat{t}_{i-1}})\big(\overline{M}_{\hat{t}_i} - \overline{M}_{\hat{t}_{i-1}}\big)\\
&\quad = \sum_{\hat{\tau}^n} u(X_{1-\hat{t}_{i-1}})\big(M_{1-\hat{t}_i} - M_{1-\hat{t}_{i-1}}\big)\\
&\qquad - 2\sum_{\hat{\tau}^n} u(X_{1-\hat{t}_{i-1}})\int_{1-\hat{t}_i}^{1-\hat{t}_{i-1}} Lf(X_s)\,ds\\
&\quad = -\sum_{\tau^n} u(X_{t_{i+1}})\big[M_{t_{i+1}} - M_{t_i}\big] - 2\sum_{\tau^n} u(X_{t_{i+1}})\int_{t_i}^{t_{i+1}} Lf(X_s)\,ds\\
&\quad = -\sum_{\tau^n} u(X_{t_i})\big[M_{t_{i+1}} - M_{t_i}\big] - 2\sum_{\tau^n} u(X_{t_{i+1}})\int_{t_i}^{t_{i+1}} Lf(X_s)\,ds\\
&\qquad - \sum_{\tau^n}\big(u(X_{t_{i+1}}) - u(X_{t_i})\big)\big(M_{t_{i+1}} - M_{t_i}\big).
\end{aligned} \tag{2.2.3}$$

Since $u \in \mathscr{D}(\mathscr{E})$, we have that

$$\lim_{n\to\infty}\sum_{\tau^n}\big(u(X_{t_{i+1}}) - u(X_{t_i})\big)\big(M_{t_{i+1}} - M_{t_i}\big) = \langle M^u, M^f\rangle_t \quad \text{in } P_m. \tag{2.2.4}$$

Hence, it follows that

$$\begin{aligned}
\tfrac{1}{2}\int_{1-t}^{1} u(X_{1-s})\,d\overline{M}_s &= \lim_{n\to\infty} \tfrac{1}{2}\sum_{\hat{\tau}^n} u(X_{1-\hat{t}_{i-1}})\left(\overline{M}_{\hat{t}_i} - \overline{M}_{\hat{t}_{i-1}}\right)\\
&= -\tfrac{1}{2}\lim_{n\to\infty}\sum_{\tau^n} u(X_{t_i})(M_{t_{i+1}} - M_{t_i})\\
&\quad - \lim_{n\to\infty}\sum_{\tau^n} u(X_{t_{i+1}})\int_{t_i}^{t_{i+1}} Lf(X_s)\,ds\\
&\quad - \tfrac{1}{2}\lim_{n\to\infty}\sum_{\tau^n}\left(u(X_{t_{i+1}}) - u(X_{t_i})\right)(M_{t_{i+1}} - M_{t_i})\\
&= -\tfrac{1}{2}\int_0^t u(X_s)\,dM_s - \int_0^t u(X_s)Lf(X_s)\,ds - \tfrac{1}{2}\langle M^u, M^f\rangle_t.
\end{aligned}$$

Consequently,

$$\begin{aligned}
\int_0^t u(X_s)\circ df(X_s) &= \int_0^t u(X_s)\,dM_s - \tfrac{1}{2}\left(\int_0^t u(X_s)\,dM_s + \int_{1-t}^{1} u(X_{1-s})\,d\overline{M}_s\right)\\
&= \int_0^t u(X_s)\,dM_s + \int_0^t u(X_s)Lf(X_s)\,ds + \tfrac{1}{2}\langle M^u, M^f\rangle_t
\end{aligned}$$

is a regular Dirichlet process.

Step 2. Assume $u \in \mathscr{D}(\mathscr{E}) \cap L^\infty$, $f \in \mathscr{D}(\mathscr{E})$. Choose $f_n \in \mathscr{D}(L)$ such that $\mathscr{E}_1(f_n - f, f_n - f) \to 0$. Consequently, $\Gamma(f_n, f_n) \to \Gamma(f, f)$ in $L^1(X; m)$. Put $M_s^n = M_s^{f_n}$, and so forth. Let

$$N_t = \tfrac{1}{2}\int_0^t u(X_s)\,dM_s + \tfrac{1}{2}\int_{1-t}^{1} u(X_{1-s})\,d\overline{M}_s,$$

$$N_t^n = \tfrac{1}{2}\int_0^t u(X_s)\,dM_s^n + \tfrac{1}{2}\int_{1-t}^{1} u(X_{1-s})\,d\overline{M}_s^n.$$

As before, we must show that N_t is of 0-quadratic variation. Let $\{\tau^m\}$ be a sequence of partitions of $[0, 1]$ such that $\delta(\tau^m) \to 0$ as $m \to \infty$. Then

$$\sum_{\tau^m}(N_{t_{i+1}} - N_{t_i})^2 \le 2\sum_{\tau^m}\left[(N_{t_{i+1}} - N_{t_i}) - (N_{t_{i+1}}^n - N_{t_i}^n)\right]^2 + 2\sum_{\tau^m}(N_{t_{i+1}}^n - N_{t_i}^n)^2.$$

However,

$$\begin{aligned}
&E\left[\sum_{\tau^m}\left[(N_{t_{i+1}} - N_{t_i}) - (N_{t_{i+1}}^n - N_{t_i}^n)\right]^2\right]\\
&\quad\le E\Bigg[\tfrac{1}{2}\sum_{\tau^m}\left(\int_{t_i}^{t_{i+1}} u(X_s)\,dM_s - \int_{t_i}^{t_{i+1}} u(X_s)\,dM_s^n\right)^2\\
&\qquad + \tfrac{1}{2}\sum_{\tau^m}\left(\int_{1-t_{i+1}}^{1-t_i} u(X_{1-s})\,d\overline{M}_s - \int_{1-t_{i+1}}^{1-t_i} u(X_{1-s})\,d\overline{M}_s^n\right)^2\Bigg]
\end{aligned}$$

$$\leq \tfrac{1}{2}\sum_{\tau^m} E\left(\int_{t_i}^{t_{i+1}} u(X_s)\, d\overline{M}_s^{f-f_n}\right)^2 + \tfrac{1}{2}\sum_{\tau^m} E\left(\int_{1-t_{i+1}}^{1-t_i} u(X_{1-s})\, d\overline{M}_s^{f-f_n}\right)^2$$

$$\leq \tfrac{1}{2}\sum_{\tau^m} E\left(\int_{t_i}^{t_{i+1}} u^2(X_s)\Gamma(f-f_n, f-f_n)(X_s)\, ds\right)$$

$$+ \tfrac{1}{2}\sum_{\tau^m} E\left(\int_{1-t_{i+1}}^{1-t_i} u^2(X_{1-s})\Gamma(f-f_n, f-f_n)(X_{1-s})\, ds\right)$$

$$\leq E\left(\int_0^1 u^2(X_s)\Gamma(f-f_n, f-f_n)(X_s)\, ds\right)$$

$$= \int_X u^2(x)\Gamma(f-f_n, f-f_n)(x)\, dm,$$

and by our choice of f_n this goes to zero.

So for $\varepsilon > 0$,

$$P\left(\sum_{\tau^m}(N_{t_{i+1}} - N_{t_i})^2 > \varepsilon\right)$$

$$\leq \frac{4}{\varepsilon}\left(\int_X u^2(x)\Gamma(f-f_n, f-f_n)(x)\, dm\right) \tag{2.2.5}$$

$$+ P\left(\sum_{\tau^m}(N^n_{t_{i+1}} - N^n_{t_i})^2 > \frac{\varepsilon}{4}\right).$$

Letting $m \to \infty$, and then $n \to \infty$ in (2.2.5), we get that N_t is of 0-quadratic variation, that is,

$$\int_0^t u(X_s) \circ df(X_s) \quad \text{is a regular Dirichlet process.}$$

Step 3. Fix $u \in \mathscr{B}_b(X) \cap L^2(X, m)$, $f \in \mathscr{D}(\mathscr{E})$. Then $P_t u \in \mathscr{D}(\mathscr{E})$ and $\lim_{t\to 0} P_t u = u$ in $L^2(X, m)$. This shows that a sequence $\{u_n\}$ can be chosen satisfying $u_n \in \mathscr{D}(\mathscr{E})$, $\|u_n\|_\infty \vee \|u\|_\infty \leq A$, $\lim_{t\to 0} u_n = u$ (m-a.e.). Hence,

$$\lim_{n\to\infty} \int_X (u_n - u)^2 \Gamma(f, f)\, dm = 0. \tag{2.2.6}$$

Put

$$N_t = \tfrac{1}{2}\int_0^t u(X_s)\, dM_s + \tfrac{1}{2}\int_{1-t}^1 u_n(X_{1-s})\, d\overline{M}_s,$$

$$N_t^n = \tfrac{1}{2}\int_0^t u_n(X_s)\, dM_s + \tfrac{1}{2}\int_{1-t}^1 u_n(X_{1-s})\, d\overline{M}_s.$$

Let $\{\tau^m\}$ be a sequence of partitions of $[0,1]$ with $\delta(\tau^m) \to 0$. Then

$$\begin{aligned}
&E\left[\sum_{\tau^m} \left[(N_{t_{i+1}} - N_{t_i}) - (N^n_{t_{i+1}} - N^n_{t_i})\right]^2\right] \\
&\quad \le \tfrac{1}{2}\sum_{\tau^m} E\left(\int_{t_i}^{t_{i+1}} (u(X_s) - u_n(X_s))\, dM_s\right)^2 \\
&\qquad + \tfrac{1}{2}\sum_{\tau^m} E\left(\int_{1-t_{i+1}}^{1-t_i} (u(X_{1-s}) - u_n(X_{1-s}))\, d\overline{M}_s\right)^2 \\
&\quad \le \tfrac{1}{2}\sum_{\tau^m} E\left(\int_{t_i}^{t_{i+1}} (u(X_s) - u_n(X_s))^2 \Gamma(f,f)(X_s)\, ds\right) \\
&\qquad + \tfrac{1}{2}\sum_{\tau^m} E\left(\int_{1-t_{i+1}}^{1-t_i} (u(X_{1-s}) - u_n(X_{1-s}))^2 \Gamma(f,f)(X_{1-s})\, ds\right) \\
&\quad \le E\left(\int_0^1 (u(X_s) - u_n(X_s))^2 \Gamma(f,f)(x_s)\, ds\right) \\
&\quad \le \int_X (u(x) - u_n(x))^2 \Gamma(f,f)(x)\, dm.
\end{aligned}$$

Thus, for $\varepsilon > 0$,

$$\begin{aligned}
&P\left(\sum_{\tau^m} (N_{t_{i+1}} - N_{t_i})^2 > \varepsilon\right) \\
&\qquad \le \frac{4}{\varepsilon}\int_X (u(x) - u_n(x))^2 \Gamma(f,f)(x)\, dm + P\left(\sum_{\tau^m} (N^n_{t_{i+1}} - N^n_{t_i})^2 > \frac{\varepsilon}{4}\right).
\end{aligned}$$

First let $m \to \infty$. It follows that

$$\limsup_{m\to\infty} P\left(\sum_{\tau^m} (N_{t_{i+1}} - N_{t_i})^2 > \varepsilon\right) \le \frac{4}{\varepsilon}\int_X (u(x) - u_n(x))^2 \Gamma(f,f)(x)\, dm.$$

Then, let $n \to \infty$. We have

$$\lim_{m\to\infty} P\left(\sum_{\tau^m} (N_{t_{i+1}} - N_{t_i})^2 > \varepsilon\right) = 0,$$

that is, $\int_0^t u(X_s) \circ df(X_s)$ is a regular Dirichlet process.

Step 4. Fix $u \in L^2(X, \mu_{\langle f\rangle})$, $f \in \mathscr{D}(\mathscr{E})$. Choose a sequence $\{F_n\}$ of compact sets such that $F_n \uparrow X$. Set $u_n = (((-n) \vee u) \wedge n)\chi_{F_n}$. Then $u_n \in \mathscr{B}_b(X) \cap L^2(X, m)$, and

$$\int_X (u(x) - u_n(x))^2 \Gamma(f,f)(x)\, dm \to 0.$$

Set

$$N_t = \tfrac{1}{2}\int_0^t u(X_s)\,dM_s + \tfrac{1}{2}\int_{1-t}^1 u(X_{1-s})\,d\overline{M}_s,$$

$$N_t^n = \tfrac{1}{2}\int_0^t u_n(X_s)\,dM_s + \tfrac{1}{2}\int_{1-t}^1 u_n(X_{1-s})\,d\overline{M}_s.$$

By the same method as in Step 3, we can conclude that N_t is of 0-quadratic variation and $\int_0^t u(X_s)\circ df(X_s)$ is a regular Dirichlet process. □

COROLLARY 2.2.2.

$$\left\langle \int_0^t u(X_s)\circ df(X_s), \int_0^t v(X_s)\circ dg(X_s)\right\rangle = \int_0^t u(X_s)v(X_s)\Gamma(f,g)(X_s)\,ds,$$

for $f, g \in \mathscr{D}(\mathscr{E})$, $u \in L^2(X\colon \mu_{\langle f\rangle})$ *and* $v \in L^2(X\colon \mu_{\langle g\rangle})$.

PROOF. Let

$$N_t^1 = \int_0^t u(X_s)\circ df(X_s) - \int_0^t u(X_s)\,dM_s^f,$$

$$N_t^2 = \int_0^t v(X_s)\circ dg(X_s) - \int_0^t v(X_s)\,dM_s^g.$$

Then N_t^1 and N_t^2 are of 0-quadratic variation and

$$\begin{aligned}
&\left\langle \int_0^t u(X_s)\circ df(X_s), \int_0^t v(X_s)\circ dg(X_s)\right\rangle \\
&\quad = \left\langle \int_0^t u(X_s)\,dM_s^f, \int_0^t v(X_s)\,dM_s^g\right\rangle + \left\langle \int_0^t u(X_s)\,dM_s^f, N_t^2\right\rangle \\
&\qquad + \left\langle N_t^1, \int_0^t v(X_s)\,dM_s^g\right\rangle + \langle N_t^1, N_t^2\rangle.
\end{aligned}$$

It is not difficult to show that

$$\left\langle \int_0^t u(X_s)\,dM_s^f, N_t^2\right\rangle = \left\langle N_t^1, \int_0^t v(X_s)\,dM_s^g\right\rangle = \langle N_t^1, N_t^2\rangle = 0, \qquad P_m\text{-a.e.}$$

For instance, to show that $\langle \int_0^t u(X_s)\,dM_s^f, N_t^2\rangle = 0$, let $\{\tau^n\}$ be a sequence of partitions of $[0,t]$ such that $\delta(\tau^n) \to 0$. Then,

$$\begin{aligned}
E\left[\sum_{\tau^n}\left(\int_{t_i}^{t_{i+1}} u(X_s)\,dM_s^f\right)^2\right] &= E\left(\int_0^t u^2(X_s)\Gamma(f,f)(X_s)\,ds\right) \\
&\le \int_X u^2(x)\Gamma(f,f)(x)\,dm < +\infty.
\end{aligned} \tag{2.2.7}$$

For any $\varepsilon > 0$, we have

$$
\begin{aligned}
P&\left(\left(\sum_{\tau^n}\left(\int_{t_i}^{t_{i+1}} u(X_s)\,dM_s^f\right)\left(N_{t_{i+1}}^2 - N_{t_i}^2\right)\right)^2 > \varepsilon\right)\\
&\le P\left(\sum_{\tau^n}\left(\int_{t_i}^{t_{i+1}} u(X_s)\,dM_s^f\right)^2 \sum_{\tau^m}\left(N_{t_{i+1}}^2 - N_{t_i}^2\right)^2 > \varepsilon\right)\\
&\le P\left(\sum_{\tau^n}\left(\int_{t_i}^{t_{i+1}} u(X_s)\,dM_s^f\right)^2 \sum_{\tau^n}\left(N_{t_{i+1}}^2 - N_{t_i}^2\right)^2 > \varepsilon,\right.\\
&\qquad\qquad\left.\sum_{\tau^n}\left(\int_{t_i}^{t_{i+1}} u(X_s)\,dM_s^f\right)^2 \le M\right)\\
&\quad + P\left(\sum_{\tau^n}\left(\int_{t_i}^{t_{i+1}} u(X_s)\,dM_s^f\right)^2 > M\right)\\
&\le P\left(\sum_{\tau^n}\left(N_{t_{i+1}}^2 - N_{t_i}^2\right)^2 > \frac{\varepsilon}{M}\right) + \frac{1}{M}\int_X u^2(x)\Gamma(f,f)(x)\,dm.
\end{aligned}
\tag{2.2.8}
$$

First let $n \to \infty$, then let $M \to \infty$ in (2.2.8). It follows that

$$
\left\langle \int_0^t u(X_s)\,dM_s^f, N_t^2 \right\rangle = \lim_{n\to\infty} \sum_{\tau^n} \int_{t_i}^{t_{i+1}} u(X_s)\,dM_s^f\left(N_{t_{i+1}}^2 - N_{t_i}^2\right) = 0. \qquad \square
$$

We have defined the Stratonovich integral of u with respect to $f(X_t)$. Moreover, the map can obviously be extended to the linear space H_0 of formal linear combinations $h = \sum u_j\,dg_j$, where $u_j \in L^\infty(X, m)$ and $g_j \in \mathscr{D}(\mathscr{E})$. It is an important but easy observation that the mapping $h \to \sum_j \int_0^1 u_j(X_t)\,dM_t^{g_j}$ is an isometry if we impose the inner product

$$
(h, h') \triangleq \int \sum_{j,j'} u_j u'_{j'} \Gamma(g_j, g_{j'})\,dm
$$

on H_0 and use the $L^2(\Omega, P_m)$ norm for the martingale. It follows that the integral and martingale decomposition

$$
\begin{aligned}
\int h(X_t)\circ dX_t &= \sum \int u_j \circ dg_j(X_t)\\
&= \tfrac{1}{2}\left(M_t - \left(\overline{M}_1 - \overline{M}_{1-t}\right)\right),
\end{aligned}
$$

have meaning for any element of H (the completion of H_0 in $\langle\ ,\ \rangle$). One can view H as the space of L^2 differential forms on X. Some caution is required because if ω is a differential form on a manifold M, then there are many ways to write it as a sum $\{\sum u_i\,dg_i\}$. However, one readily checks that distance between two such sums with respect to the inner product $\langle\ ,\ \rangle$ defined here will be zero if the Dirichlet space arose from a uniformly elliptic operator.

As a corollary, one learns, for example, that if X_t is a uniformly elliptic diffusion on $\mathbb{R}^2$, then the area integrals $\int(X\,dY - Y\,dX)$ exist from quasi-every starting point. To show that the area swept out about the initial point of the diffusion is also finite and well behaved, one observes that

$$\begin{aligned}\int(X - X_0)\,dY - \int(Y - Y_0)\,dX \\ = \left(\int_0^t X\,dY - \int_0^t Y\,dX\right) - X_0(Y_t - Y_0) + Y_0(X_t - X_0);\end{aligned} \tag{2.2.9}$$

the first right-hand expression makes sense for all t and almost all X_0, Y_0; the second right-hand expression makes sense for all X_0, Y_0 and all t. The theorem of this section cannot be used directly because the differential form ω integrated in the left-hand side of the expression is not backwards predictable.

2.3. *Pathwise approximation to Stratonovich integral.* In this section, we prove that the Stratonovich integral $\int_0^t u(X_s)\circ df(X_s)$ can be approximated pathwise. Let τ^n be a sequence of partitions on $[0,t]$ with $\delta(\tau^n)\to 0$. Assume $u\in L^2(X:\mu_{\langle f\rangle})\cap\mathscr{D}(\mathscr{E})$, $f\in\mathscr{D}(\mathscr{E})$. We have the following main result.

THEOREM 2.3.1.

$$\int_0^t u(X_s)\circ df(X_s) = \lim_{n\to\infty}\sum_{\tau^n}\frac{u(X_{t_{i+1}}) + u(X_{t_i})}{2}\big(f(X_{t_{i+1}}) - f(X_{t_i})\big)$$

in P_m.

To simplify notation, here and in what follows we leave the dependence of t_i on n implicit. Convergence is interpreted as convergence in P_m.

PROOF. Let $\overline{M}_t^u$ be the backward martingale part of $u(X_s)$. Denote M_t^f and $\overline{M}_t^f$, respectively, by M_t and $\overline{M}_t$. From the decomposition of $f(X_s)$, we have

$$\begin{aligned}&\sum_{\tau^n}\frac{u(X_{t_{i+1}}) + u(X_{t_i})}{2}\big(f(X_{t_{i+1}}) - f(X)_{t_i}\big)\\ &\quad= \frac{1}{2}\sum_{\tau^n}\frac{u(X_{t_{i+1}}) + u(X_{t_i})}{2}\big(M_{t_{i+1}} - M_{t_i}\big)\\ &\qquad - \frac{1}{2}\sum_{\tau^n}\frac{u(X_{t_{i+1}}) + u(X_{t_i})}{2}\big(\overline{M}_{1-t_i} - \overline{M}_{1-t_{i+1}}\big).\end{aligned}$$

Denote this expression to be $\frac{1}{2}I_1^n - \frac{1}{2}I_2^n$, where

$$\begin{aligned}I_1^n &= \sum_{\tau^n}\frac{u(X_{t_{i+1}}) + u(X_{t_i})}{2}\big(M_{t_{i+1}} - M_{t_i}\big)\\ &= \sum_{\tau^n}u(X_{t_i})(M_{t_{i+1}} - M_{t_i}) + \frac{1}{2}\sum_{\tau^n}\big(u(X_{t_{i+1}}) - u(X_{t_i})\big)(M_{t_{i+1}} - M_{t_i}).\end{aligned}$$

Since $u,\ f \in \mathscr{D}(\mathscr{E})$, we have

$$\begin{aligned}\lim_{n\to\infty} I_1^n &= \lim_{n\to\infty} \sum_{\tau^n} u(X_{t_i})(M_{t_{i+1}} - M_{t_i}) \\ &\quad + \tfrac{1}{2} \lim_{n\to\infty} \sum_{\tau^n} \big(u(X_{t_{i+1}}) - u(X_{t_i})\big)(M_{t_{i+1}} - M_{t_i}) \\ &= \int_0^t u(X_s)\, dM_s + \tfrac{1}{2}\langle M^u, M^f\rangle_t \\ &= \int_0^t u(X_s)\, dM_s + \tfrac{1}{2}\int_0^t \Gamma(u, f)(X_s)\, ds.\end{aligned}$$

By an essentially symmetric argument,

$$\begin{aligned}\lim_{n\to\infty} I_2^n &= \int_{1-t}^1 u(X_{1-s})\, d\overline{M}_s + \tfrac{1}{2}\Big[\langle \overline{M}^u, \overline{M}\rangle_1 - \langle \overline{M}^u, \overline{M}\rangle_{1-t}\Big] \\ &= \int_{1-t}^1 u(X_{1-s})\, d\overline{M}_s + \tfrac{1}{2}\int_0^t \Gamma(u, f)(X_s)\, ds.\end{aligned}$$

Consequently,

$$\begin{aligned}&\lim_{n\to\infty} \sum_{\tau^n} \frac{u(X_{t_{i+1}}) + u(X_{t_i})}{2}\big(f(X_{t_{i+1}}) - f(X_{t_i})\big) \\ &\quad = \lim_{n\to\infty} \frac{1}{2} I_1^n - \lim_{n\to\infty} \frac{1}{2} I_2^n \\ &\quad = \frac{1}{2}\int_0^t u(X_s)\, dM_s - \frac{1}{2}\int_{1-t}^1 u(X_{1-s})\, d\overline{M}_s \\ &\quad = \int_0^t u(X_s) \circ df(X_s).\end{aligned}$$

The theorem is proved. □

Set $\tau_G = \inf\{t > 0,\ X_t \in G^c\}$. For a compact set G, we have the following corollary.

COROLLARY 2.3.2. *If $u_1 = u_2$ and $f_1 = f_2$ on G, then*

$$\int_0^t u_1(X_s) \circ df_1(X_s) = \int_0^t u_2(X_s) \circ df_2(X_s), \qquad t < \tau_G, \quad P_m\text{-a.e.}$$

Suppose $u \in \mathscr{D}(\mathscr{E}) \cap L^2(X, \mu_{\langle f\rangle})$, $f \in \mathscr{D}(\mathscr{E})$. Then define

$$(2.3.1) \quad \int_0^t u(X_s) \circ dN_s^f \triangleq \int_0^t u(X_s) \circ df(X_s) - \int_0^t u(X_s)\, dM_s^f - \tfrac{1}{2}\langle M^u, M^f\rangle_t.$$

We call $\int_0^t u(X_s) \circ dN_s^f$ the Stratonovich integral of $u(X_s)$ against the zero energy process N^f.

COROLLARY 2.3.3.

$$(2.3.2) \quad \int_0^t u(X_s) \circ dN_s^f = \lim_{n\to\infty} \sum_{\tau^n} \frac{u(X_{t_{i+1}}) + u(X_{t_i})}{2}\big(N_{t_{i+1}}^f - N_{t_i}^f\big).$$

PROOF. Since

$$\begin{aligned}
&\sum_{\tau^n} \frac{u(X_{t_{i+1}}) + u(X_{t_i})}{2}\left(N^f_{t_{i+1}} - N^f_{t_i}\right)\\
(2.3.3)\qquad &= \sum_{\tau^n} \frac{u(X_{t_{i+1}}) + u(X_{t_i})}{2}\left(f(X_{t_{i+1}}) - f(X_{t_i})\right)\\
&- \sum_{\tau^n} \frac{u(X_{t_{i+1}}) + u(X_{t_i})}{2}\left(M_{t_{i+1}} - M_{t_i}\right),
\end{aligned}$$

by Theorem 2.3.1 and (2.3.3), it is not difficult to see that (2.3.2) holds true. □

REMARK 2.3.4. Note that the definition of the Stratonovich integral here and Nakao's definition in [11] are consistent. In fact, recall

$$(2.3.4)\qquad \int_0^t u(X_s)\circ df(X_s) = \tfrac{1}{2}\int_0^t u(X_s)\,dM_s^f - \tfrac{1}{2}\int_{1-t}^1 u(X_{1-s})\,d\overline{M}_s^f,$$

and Nakao's definition,

$$(2.3.5)\qquad \int_0^t u(X_s)\circ df(X_s) = u\cdot M_t^f + \Gamma(u\cdot M^f)_t.$$

Here Γ is a special functional; see the details in [11].

It is obvious that (2.3.4) and (2.3.5) are the same when $u \in \mathscr{D}(\mathscr{E})_b$ and $f \in \mathscr{D}(L)$. For $u \in \mathscr{D}(\mathscr{E}) \cap L^2(X, d\mu)$, $f \in \mathscr{D}(\mathscr{E})$, by a limit procedure and the properties of the two definitions, we can see that they are also equal.

2.4. *Itô's formula for Dirichlet processes.* In this section we give a chain rule for Dirichlet processes. Take $u_1, \dots, u_n \in \mathscr{D}(\mathscr{E})_b$. We can assume, by modification on sets of capacity zero, that they are quasicontinuous. If Φ is in $C^1(R^n)$, then it was shown in [10] and would easily follow in the more general context introduced here, that the usual change of variable formula holds, so that

$$\begin{aligned}
&\Phi(u_1(X_t), \dots, u_n(X_t)) - \Phi(u_1(X_0), \dots, u_n(X_0))\\
(2.4.1)\qquad &= \sum_{i=1}^n \int_0^t \frac{\partial \Phi}{\partial u_i}(X_s)\circ du_i(X_s).
\end{aligned}$$

From this, $\Phi(u_1(X_t), \dots, u_n(X_t))$ is a Dirichlet process. It is interesting to express its 0-quadratic variation part in terms of those for u_i. In the case where Φ is C^2, this can be done explicitly as the following theorem of Nakao [11] shows. We present two proofs here. The first shows that the result follows easily from (2.4.1). The other is a proof closer to the classical approach.

THEOREM 2.4.1. *If $\Phi \in C^2(R^n)$, then the following holds*:

$$\Phi(u_1(X_t),\dots,u_n(X_t)) - \Phi(u_1(X_0),\dots,u_n(X_0))$$
$$= \sum_1^n \int_0^t \Phi_i(u_1(X_s),\dots,u_n(X_s))\, dM_s^{u_i}$$
$$(2.4.2) \qquad + \sum_1^n \int_0^t \Phi_i(u_1(X_s),\dots,u_n(X_s)) \circ dN_s^{u_i}$$
$$+ \frac{1}{2}\sum_{i,j} \int_0^t \Phi_{ij}(u_1(X_s),\dots,u_n(X_s))\, d\langle M^{u_i}, M^{u_j}\rangle_s.$$

Here

$$\Phi_i(x_1,\dots,x_n) = \frac{\partial \Phi}{\partial x_i}(x_1,\dots,x_n) \quad \textit{and} \quad \Phi_{ij} = \frac{\partial^2 \Phi}{\partial x_i\, \partial x_j}(x_1,\dots,x_n).$$

FIRST PROOF. Let τ^m be a sequence of partitions on $[0,t]$ with $\delta(\tau^m) \to 0$ as $m \to +\infty$. From the assumption $\Phi \in C^2(R^n)$, it is known (see, e.g., [6]) that

$$\Phi_i(u_i(x),\dots,u_n(x)) \in \mathscr{D}(\mathscr{E}) \quad \text{for each } i \le n.$$

Using the fact

$$M_t^{\psi}(u_1,\dots,u_n) = \sum_{j=1}^n \int_0^t \frac{\partial \psi}{\partial u_j}(u_1(X_s),\dots,u_n(X_s))\, dM_s^{u_i}, \quad \text{for } \psi \in C^1(R^n)$$

(see [6]), together with Theorem 2.3.1 and (2.4.1), we have

$$\Phi(u_1(X_t),\dots,u_n(X_t)) - \Phi(u_1(X_0),\dots,u_n(X_0))$$
$$= \sum_{i=1}^n \lim_{m\to\infty} \sum_{\tau^m} \frac{\Phi_i\big(u_1(X_{t_{j+1}}),\dots,u_n(X_{t_{j+1}})\big) + \Phi_i\big(u_1(X_{t_j}),\dots,u_n(X_{t_j})\big)}{2}$$
$$\times \big(u_i(X_{t_{j+1}}) - u_i(X_{t_j})\big).$$

Splitting the sum, this equals

$$\sum_{i=1}^n \lim_{m\to\infty} \sum_{\tau^m} \frac{\Phi_i\big(u_1(X_{t_{j+1}}),\dots,u_n(X_{t_{j+1}})\big) + \Phi_i\big(u_1(X_{t_j}),\dots,u_n(X_{t_j})\big)}{2}$$
$$\times \big(M_{t_{j+1}}^{u_i} - M_{t_j}^{u_i}\big)$$
$$+ \sum_{i=1}^n \lim_{m\to\infty} \sum_{\tau^m} \frac{\Phi_i\big(u_1(X_{t_{j+1}}),\dots,u_n(X_{t_{j+1}})\big) + \Phi_i\big(u_1(X_{t_j}),\dots,u_n(X_{t_j})\big)}{2}$$
$$\times \big(N_{t_{j+1}}^{u_i} - N_{t_j}^{u_i}\big).$$

Using Corollary 2.3.3, this becomes

$$\sum_{i=1}^{n} \lim_{m\to\infty} \sum_{\tau^m} \Phi_i\big(u_1(X_{t_j}),\dots,u_n(X_{t_j})\big)\big(M_{t_{j+1}}^{u_i} - M_{t_j}^{u_i}\big)$$

$$+ \sum_{i=1}^{n} \lim_{m\to\infty} \tfrac{1}{2} \sum_{\tau^m} \Big(\Phi_i\big(u_1(X_{t_{j+1}}),\dots,u_n(X_{t_{j+1}})\big) - \Phi_i\big(u_1(X_{t_j}),\dots,u_n(X_{t_j})\big)\Big) \times \big(M_{t_{j+1}}^{u_i} - M_{t_j}^{u_i}\big)$$

$$+ \sum_{i=1}^{n} \int_0^t \Phi_i(u_1(X_s),\dots,u_n(X_s)) \circ dN_s^{u_i}.$$

Identifying the limits with the Itô integral and simplifying, we obtain,

$$\begin{aligned}
&\sum_{i=1}^{n} \int_0^t \Phi_i(u_1(X_s),\dots,u_n(X_s))\, dM_s^{u_i} \\
&\qquad + \tfrac{1}{2} \sum_{i=1}^{n} \langle M^{\Phi_i(u_1,\dots,u_n)}, M^{u_i}\rangle_t \\
&\qquad + \sum_{i=1}^{n} \int_0^t \Phi_i(u_1(X_s),\dots,u_n(X_s)) \circ dN_s^{u_i} \\
&= \sum_{i=1}^{n} \int_0^t \Phi_i(u_1(X_s),\dots,u_n(X_s))\, dM_s^{u_i} \\
&\qquad + \sum_{i=1}^{n} \int_0^t \Phi_i(u_1(X_s),\dots,u_n(X_s)) \circ dN_s^{u_i} \\
&\qquad + \tfrac{1}{2} \sum_{i,j} \int_0^t \Phi_{ij}(u_1(X_s),\dots,u_n(X_s))\, d\langle M^{u_i}, M^{u_j}\rangle_s.
\end{aligned}$$

This ends the first proof. □

Second proof. For simplicity, we only deal with the case $n = 1$. First of all, we note the fact that $u_i(X_t)$, $i = 1,\dots,n$, is a continuous process, because u_i is quasicontinuous and X_t is conservative. Since u_i, $i = 1,\dots,n$, are bounded, we can assume that $\Phi \in C_b^2(R)$. Take a sequence τ^n of subdivisions on $[0,t]$ with $\delta(\tau^n) \to 0$. By Taylor expansion,

$$\begin{aligned}
\Phi(u(X_t)) - \Phi(u(X_0)) &= \sum_{n} \big[\Phi\big(u(X_{t_{i+1}})\big) - \Phi\big(u(X_{t_i})\big)\big] \\
&= \sum_{\tau^n} \Phi_1\big(u(X_{t_i})\big)\big(u(X_{t_{i+1}}) - u(X_{t_i})\big) \qquad (2.4.3)\\
&\quad + \tfrac{1}{2} \sum_{\tau^n} \Phi_{11}\big(\overline{u(X_{t_i})}\big)\big(u(X_{t_{i+1}}) - u(X_{t_i})\big)^2 \\
&= I^n + II^n.
\end{aligned}$$

Here $\overline{u(X_{t_i})}$ is a random intermediate point between $u(X_{t_i})$ and $u(X_{t_{i+1}})$. Now we write $II^n = II_1^n + II_2^n$, with

$$II_1^n = \tfrac{1}{2}\sum_{\tau^n}\Phi_{11}(u(X_{t_i}))(u(X_{t_{i+1}}) - u(X_{t_i}))^2,$$

$$II_2^n = \tfrac{1}{2}\sum_{\tau^n}\left(\Phi_{11}(\overline{u(X_{t_i})}) - \Phi_{11}(u(X_{t_i}))\right)(u(X_{t_i}) - u(X_{t_i}))^2.$$

Let μ^n be the random measure

$$\tfrac{1}{2}\sum_{\tau^n}(u(X_{t_{i+1}}) - u(X_{t_i}))^2\delta_{t_i}$$

on $[0, 1]$, and let μ be the random measure on $[0, 1]$ generated by $\langle M^u, M^u\rangle_t$. Then for all $t \in [0, 1]$,

$$\mu^n[0,t] \to \mu[0,t], \tag{2.4.4}$$

$$\lim_{M\to\infty}\sup_n P_m(\mu^n[0,1] > M) = 0. \tag{2.4.5}$$

In fact, (2.4.4) follows by definition. For the proof of (2.4.5), it is sufficient to note the fact

$$\begin{aligned} E|\mu^n[0,1]| &= \sum_{\tau^n}E(u(X_{t_{i+1}}) - u(X_{t_i}))^2 \\ &\le \sum_{\tau^n}(t_{i+1} - t_i)\mathscr{E}(u,u) \le \mathscr{E}(u,u) < \infty. \end{aligned}$$

From (2.4.4), (2.4.5) and Lemma 2.2 in [12], we have that

$$\lim_{n\to\infty} II_1^n = \tfrac{1}{2}\int_0^t\Phi_{11}(u(X_s))\,d\langle M^u, M^u\rangle_s \quad \text{in } P_m. \tag{2.4.6}$$

On the other hand,

$$|II_2^n| \le \sup_{t_i}\left|\Phi_{11}(\overline{u(X_{t_i})}) - \Phi_{11}(u(X_{t_i}))\right|\sum_{\tau^n}(u(X_{t_{i+1}}) - u(X_{t_i}))^2.$$

Since $u(X_s)$ is uniformly continuous on $[0, 1]$ and

$$\lim_{n\to\infty}\sum_{\tau^n}(u(X_{t_{i+1}}) - u(X_{t_i}))^2 = \langle M^u, M^u\rangle_t,$$

it follows that $\lim_{n\to\infty} II_2^n = 0$.

Now, we look back at $I^n = I_1^n + I_2^n + I_3^n$, where

$$I_1^n = \sum_{\tau^n}\Phi_1(u(X_{t_i})(M_{t_{i+1}}^u - M_{t_i}^u)),$$

$$I_2^n = \sum_{\tau^n}\frac{\Phi_1(u(X_{t_i})) + \Phi_1(u(X_{t_{i+1}}))}{2}(N_{t_{i+1}}^u - N_{t_i}^u),$$

$$I_3^n = -\frac{1}{2}\sum_{\tau^n}(\Phi_1(u(X_{t_{i+1}})) - \Phi_1(u(X_{t_i})))(N_{t_{i+1}}^u - N_{t_i}^u).$$

Then,

$$\lim_{n\to\infty} I_1^n = \int_0^t \Phi_1(u(X_s))\, dM_s^u. \tag{2.4.7}$$

Since $\Phi_1(u) \in \mathscr{D}(\mathscr{E})_b$, it follows by Corollary 2.3.3 that

$$\lim_{n\to\infty} I_2^n = \int_0^t \Phi_1(u(X_s)) \circ dN_s^u. \tag{2.4.8}$$

However,

$$E|I_3^n| \le \tfrac{1}{2}\mathscr{E}(\Phi_1(u), \Phi_1(u))\left(E\sum_{\tau^n}\left(N_{t_{i+1}}^u - N_{t_i}^u\right)^2\right)^{1/2} \to 0.$$

So, letting $n \to \infty$ in (2.4.3), we complete the proof of the theorem. □

3. Application to diffusion processes on infinite-dimensional space.

3.1. *Tightness for Dirichlet processes on infinite-dimensional space.* In this section, we intend to use the martingale decomposition to establish the tightness of some simple classes of infinite-dimensional Dirichlet processes. Our methods are strongly influenced by Takeda's use of the decomposition in the finite-dimensional situation [17].

One should always be cautious about tightness results because there are very simple examples to show that the limiting process need not in general be the process associated with the limiting form.

In certain rather special situations it is relatively easy to establish the relevant continuity theorem. We outline two. First, we look at situations where some monotonicity is present. Second, we treat the case where the limit is a Markovian measure. If we restrict ourselves to a Gaussian environment, we can establish quite explicitly the Dirichlet form of the limit process as the limit of the sequence of the Dirichlet forms (Theorem 3.1.3).

In this subsection, our state space is a Hilbert space E. Let H be a Hilbert space such that H is densely embedded in E by a Hilbert–Schmidt map. Identifying H with its dual, we obtain that $E' \subset H \subset E$. Define the linear space of finitely based smooth functions

$$\mathscr{F}C_b^\infty(E) = \{u\colon E \to R \mid \exists l_1, \dots, l_m \in E',\ f \in C_b^\infty(R^m) \text{ such that } u(z) = f(l_1(z), \dots, l_m(z)),\ z \in E\}. \tag{3.1.1}$$

For $k \in E \setminus \{0\}$, $u \in \mathscr{F}C_b^\infty(E)$, define the following Gâteaux-type derivative (in the direction of k) by

$$\frac{\partial}{\partial k}u(z) = \left.\frac{du(z+sk)}{ds}\right|_{s=0}, \qquad z \in E. \tag{3.1.2}$$

Denote the unique element in H representing the continuous linear map $h \to \partial u(z)/\partial h$, $h \in H$, by $\nabla u(z)$. Let $L^\infty(H)$ be the family of all bounded linear operators on H. For each $n \ge 0$, let μ^n be a probability measure on $(E, \mathscr{B}(E))$ with $\operatorname{supp}[\mu^n] = E$ and let $A^n(z)\colon E \to L^\infty(H)$ be a strongly measurable map into the bounded positive self-adjoint operators on H for which the condition

$$\int_E \| A^n(z) \|_{L^\infty(H)} \mu^n(dz) < +\infty$$

holds, and for which the following symmetric form on $L^2(E: \mu^n)$ is closable:

$$(3.1.3)\qquad \begin{aligned} \bar{\mathscr{E}}_n(u, v) &= \int_E \langle A^n(z) \nabla u(z), \nabla v(z) \rangle_H \, d\mu^n, \\ \mathscr{D}(\bar{\mathscr{E}}_n) &= \mathscr{F}C_b^\infty(E). \end{aligned}$$

Denote the closure on $L^2(E, \mu^n)$ by $(\mathscr{E}_n, \mathscr{D}(\mathscr{E}_n))$. In this case, it is known from [3] that there exists a diffusion system $\{\Omega, \mathscr{F}, \mathscr{F}_t, X_t, \theta_t, P_x^n, \ x \in E\}$ whose Dirichlet form is $(\mathscr{E}_n, \mathscr{D}(\mathscr{E}_n))$, where $\Omega = C([0, \infty) \to E)$ and X is the canonical process with associated σ-fields $\mathscr{F} = \sigma(X_s, 0 \le s < +\infty)$ and $\mathscr{F}_t = \sigma(X_s, 0 \le s \le t)$. Let P_n be the probability measure on path space Ω defined by

$$(3.1.4)\qquad P_n(\cdot) = \int_E P_x^n(\cdot) \mu^n(dx).$$

Since $H \hookrightarrow E$ densely by a Hilbert–Schmidt map, we can find an orthonormal basis $\{e_n | n \in N\}$ of $(H, \langle\ \rangle_H)$ and $\lambda_n > 0$, $n \in N$, with $\sum_1^\infty \lambda_n^2 < +\infty$, which is such that $\{\bar{e}_n = e_n/\lambda_n,\ n \in N\}$ is an orthonormal basis of $(E, \langle\ \rangle_E)$ and

$$(3.1.5)\qquad \langle e_n, h \rangle_E = \lambda_n^2 \langle e_n, h \rangle_H \quad \text{for all } h \in H.$$

The following theorem is one of the main results in this subsection.

THEOREM 3.1.1. *Suppose that the following hold*:

$$\text{(i)}\qquad \sup_n \int_E \| A^n(z) \|_{L^\infty(H)}^2 \mu^n(dz) < +\infty;$$

$$\text{(ii)}\qquad \sup_n \int_E |z|_E^2 \, \mu^n(dz) < +\infty;$$

$$\text{(iii)}\qquad \{\mu^n, n \ge 0\} \text{ is tight on } (E, \mathscr{B}(E)).$$

Then $\{P_n, n \ge 0\}$ *is tight on* $C([0, \infty) \to E)$.

PROOF. For $l \in E'$, condition (ii) implies that

$$\int_E |l(z)|^2 \, d\mu^n \le \|l\|^2 \textstyle\int_E |z|^2 \, d\mu^n < +\infty.$$

From this and the result in [3] we conclude that $l(z) \in \mathscr{D}(\mathscr{E}_n)$. Consequently, it follows from Section 1 that, under P_n,

$$(3.1.6)\qquad \begin{aligned} &\langle \bar{e}_i, X_t \rangle_E - \langle \bar{e}_i, X_0 \rangle \\ &\quad = \tfrac{1}{2} M_t^{\bar{e}_i} - \tfrac{1}{2} \big[M_t^{\bar{e}_i}(\gamma_T) - M_{T-t}^{\bar{e}_i}(\gamma_T) \big], \qquad 0 \le t \le T. \end{aligned}$$

Here $M_t^{\bar{e}_i}$ is a square-integrable martingale with initial value zero and

$$\langle M_t^{\bar{e}_i} \rangle = \int_0^t \langle A^n(X_s) \nabla \bar{e}_i(X_s), \nabla \bar{e}_i(X_s) \rangle_H \, ds. \tag{3.1.7}$$

Here, γ_T is the backward shift operator, that is, $X_t(\gamma_T) = X_{T-t}$, $t \le T$.

We denote $M_t^i = M_t^{\bar{e}_i}$, $E_n = E_{P_n}$ for simplicity of notation. By (3.1.5), it holds that

$$\nabla \bar{e}_i = \lambda_i e_i \in H.$$

Therefore, for $0 \le s \le t \le T$,

$$\langle \bar{e}_i, X_t - X_s \rangle_E = \tfrac{1}{2}\left(M_t^i - M_s^i\right) - \tfrac{1}{2}\left(M_{T-s}^i(\gamma_T) - M_{T-t}^i(\gamma_T)\right) \tag{3.1.8}$$

and, by symmetry,

$$\begin{aligned}
E_n\left[\langle \bar{e}_i, X_t - X_s \rangle_E^4\right] &\le 2E_n\left[\left(M_t^i - M_s^i\right)^4\right] + 2E_n\left[\left(M_{T-s}^i - M_{T-t}^i\right)^4\right] \\
&\le CE_n\left[\left(\int_s^t \langle A^n(X_u) \nabla \bar{e}_i(X_u), \nabla \bar{e}_i(X_u) \rangle_H \, du\right)^2\right] \\
&\quad + CE_n\left[\left(\int_{T-t}^{T-s} \langle A^n(X_u) \nabla \bar{e}_i(X_u), \nabla \bar{e}_i(X_u) \rangle_H \, du\right)^2\right] \\
&\le CE_n\left[\left(\int_s^t \langle A^n(X_u) \lambda_i e_i, \lambda_i e_i \rangle_H \, du\right)^2\right] \\
&\quad + CE_n\left[\left(\int_{T-t}^{T-s} \langle A^n(X_u) \lambda_i e_i, \lambda_i e_i \rangle_H \, du\right)^2\right] \\
&\le CE_n\left(\int_s^t \| A^n(X_u) \|_{L^\infty(H)}^2 \, du\right)(t-s)\lambda_i^4 \\
&\quad + CE_n\left(\int_{T-t}^{T-s} \| A^n(X_u) \|_{L^\infty(H)}^2 \, du\right)(t-s)\lambda_i^4 \\
&\le 2C(t-s)^2 \lambda_i^4 \int_E \| A^n(z) \|_{L^\infty(H)}^2 \mu^n(dz).
\end{aligned}$$

Here we have used the fact that μ^n is an invariant measure of the diffusion system $\{X_t, \mathscr{F}_t, P_x^n, x \in E\}$. The quantity C is a constant.

Set $a_i := \langle \bar{e}_i, X_t - X_s \rangle_E^2$. Then

$$\begin{aligned}
E_n(a_i a_j) &\le \left(E_n\left(a_i^2\right)\right)^{1/2}\left(E_n\left(a_j^2\right)\right)^{1/2} \\
&\le 2C\lambda_i^2 \lambda_j^2 (t-s)^2 \int_E \| A^n(z) \|_{L^\infty(H)}^2 \mu^n(dz).
\end{aligned}$$

Thus,

$$E_n(\|X(t)-X(s)\|_E^4=E_n\left(\sum_1^\infty a_i\right)^2=\sum_{i,j}E_n(a_i,a_j)$$
$$\le\sum_{i,j}2C\lambda_i^2\lambda_j^2(t-s)^2\int_E\|A^n(z)\|_{L^\infty(H)}^2\mu^n(dz)$$
$$\le 2C\int_E\|A^n(z)\|_{L^\infty(H)}^2\mu^n(dz)\left(\sum_1^\infty\lambda_i^2\right)^2(t-s)^2.$$

Consequently,

$$\sup_n E_n\left(\|X(t)-X(s)\|_E^4\right)$$
$$\le 2C\sup_n\int_E\|A^n(z)\|_{L^\infty(H)}^2\mu^n(dz)\cdot\left(\sum_1^\infty\lambda_i^2\right)^2(t-s)^2 \tag{3.1.9}$$
$$\le M(t-s)^2,$$

where M is a constant independent of n, and therefore, by Kolmogorov's criteria,

$$\lim_{h\to 0}\sup_n P_n\left(\sup_{\substack{|t-s|\le h\\ t,s\le T}}|X(t)-X(s)|>\rho\right)=0\quad\text{for any }\rho>0. \tag{3.1.10}$$

Let $\mu_{t_0}^n$ denote the distribution of X_{t_0} under P_n. Then $\{\mu_{t_0}^n=\mu^n,\ n\ge 0\}$ is tight. Combining this fact with (3.1.10) we have that $\{P_n,\ n\ge 0\}$ is tight on $C([0,\infty)\to E)$. The proof is complete. □

Although the sequence $\{P_n,\ n\ge 0\}$ above is tight, it will not in general converge. In some special cases one can identify a limit. We first explain one of these. In addition to our previous assumptions, fix

$$A^n(z)=I_{dH},\qquad d\mu^n=\varphi_n^2\,d\mu, \tag{3.1.11}$$

where μ is a probability measure on $(E,\mathscr{B}(E))$ with supp$[\mu]=E$ and suppose

$$|z|_E\in L^P(E:\mu)\quad\text{for any }P\ge 1. \tag{3.1.12}$$

Assume the form $\bar{\mathscr{E}}(u,v)=\int_E\langle\nabla u(z),\nabla v(z)\rangle_H\,d\mu$, $\mathscr{D}(\bar{\mathscr{E}})=\mathscr{F}C_b^\infty(E)$ is closable on $L^2(E,\mu)$. Let $\{\Omega,\mathscr{F},\mathscr{F}_t,X_t,P_x,x\in E\}$ be the diffusion associated with its closure $(\mathscr{E},\mathscr{D}(\mathscr{E}))$ on $L^2(E,\mu)$. Define the probability measure P on Ω by

$$P(\cdot)=\int_E P_x(\cdot)\,d\mu. \tag{3.1.13}$$

We introduce the following conditions:

(I) $0<\varphi_1\le\varphi_2\le\cdots\le 1$, μ-a.e., $\varphi_n\to 1$ in $L^\infty(d\mu)$,

(II) $1\le\varphi_n$, μ-a.e., $\varphi_n\to 1$ in $L^\infty(d\mu)$.

We are going to prove that P_n converges to P provided (I) or (II) is fulfilled. First, if (I) is true, then following the proof of Theorem 7 in [1] exactly, the finite-dimensional distributions of P_n converge to those of P. Thus by Theorem 3.1.1, we have that P_n converges to P in the Prohorov topology.

Now assume (II) holds. Note that the gradient operator ∇ with domain $\mathscr{F}C_b^\infty(E)$ is closable on $L^2(E, \mu^n)$, where $\mu^n = \varphi_n^2 \mu$. We still use ∇ to denote the closure of ∇ on $L^2(E, \mu^n)$ for convenience. Set

$$\begin{aligned} h^n(f) &= \int_E \langle \nabla(f), \nabla(f) \rangle_H \varphi_n^2 \, d\mu, \\ \mathscr{D}(h^n) &= \{ f \in L^2(E, \mu); h^n(f) < +\infty \} \end{aligned} \tag{3.1.14}$$

and

$$\begin{aligned} h(f) &= \int_E \langle \nabla(f), \nabla(f) \rangle_H \, d\mu \\ \mathscr{D}(h) &= \{ f \in L^2(E, \mu); h(f) < +\infty \}. \end{aligned} \tag{3.1.15}$$

Then $(h^n(f), \mathscr{D}(h^n))$ and $(h(f), \mathscr{D}(h))$ are closed forms on $L^2(E, \mu)$. Denote the generators of $(h^n, \mathscr{D}(h^n))$ and $(h, \mathscr{D}(h))$ on $L^2(E, \mu)$, by H^n and H, respectively. We have the following theorem.

THEOREM 3.1.2. *H^n converges to H in the strong resolvent sense.*

PROOF. Set $\overline{\varphi}_n = \sup_{k \geq n} \varphi_k$. Define symmetric forms $(\overline{h}^n, \mathscr{D}(\overline{h}^n))$ as

$$\begin{aligned} \overline{h}^n(f) &= \int_E \langle \nabla(f), \nabla(f) \rangle_H \varphi_n^{-2} \, d\mu, \\ \mathscr{D}(\overline{h}^n) &= \{ f \in L^2(E, \mu) : \overline{h}^n(f) < +\infty \}. \end{aligned}$$

Then $(\overline{h}^n, \mathscr{D}(\overline{h}^n))$ is a densely defined closed form on $L^2(E, \mu)$. In fact, since $\overline{\varphi}_n = \sup_{k \geq n} \varphi_k \in L^2(E, \mu)$, we have that $\mathscr{F}C_b^\infty(E) \subset \mathscr{D}(\overline{h}^n)$. So $(\overline{h}^n, \mathscr{D}(\overline{h}^n))$ is densely defined. Clearly $(\overline{h}^n, \mathscr{D}(\overline{h}^n))$ is closed.

Let $\{\overline{R}_\alpha^n\}$ be the resolvent associated with $(\overline{h}^n, \mathscr{D}(\overline{h}^n))$. Denote by (R_α^n) and (R_α) the resolvents associated, respectively, with $(h^n, \mathscr{D}(h^n))$ and $(h, \mathscr{D}(h))$. Since $\overline{\varphi}_n \geq \varphi_n \geq 1$, we have that

$$\left(\overline{R}_\alpha^n f, f \right) \leq (R_\alpha^n f, f) \leq (R_\alpha f, f) \quad \text{for } f \in L^2(E, \mu). \tag{3.1.16}$$

On the other hand, from $\tilde{\varphi}_n \downarrow 1$ and Theorem 6 in [1] it follows that $\lim_{n \to \infty} (\overline{R}_\alpha^n f, f) = (R_\alpha f, f)$. Thus we deduce from (3.1.16) that $\lim_{n \to \infty} (R_\alpha^n f, f) = (R_\alpha f, f)$. By the polarization identity, we have that

$$\lim_{n \to \infty} (R_\alpha^n f, g) = (R_\alpha f, g) \quad \text{for } f, g \in L^2(E, \mu). \tag{3.1.17}$$

This implies the strong convergence of $R_\alpha^n f$ to $R_\alpha f$ and ends the proof. □

Once we have the above theorem, one can follow exactly the proof of Theorem 7 in [1] to show that the finite-dimensional distributions of P_n

converge to those of P. This together with our tightness criteria gives the convergence of P_n to P on Ω in Prohorov topology. Thus we finish the first part of the discussion.

The rest of this section is devoted to discussing the convergence problem in the situation where μ is a centered Gaussian measure on E with covariance space H, that is, (H, E, μ) is an abstract Wiener space, $\mu^n = \varphi_n^2\, d\mu$ and we also assume the following conditions hold:

$$(3.1.18)\qquad \begin{aligned}&\text{(i) } A^n(z) = I_{dH}, \varphi_n \to \varphi \text{ in } L^{2+\varepsilon}(E:\mu) \quad \text{for some } \varepsilon > 0;\\ &\text{(ii) there exists a constant } C > 0 \text{ such that } \varphi_n \geq C > 0 \quad \text{for } n \geq 1.\end{aligned}$$

Let $(\bar{\mathscr{E}}_\varphi, \mathscr{D}(\bar{\mathscr{E}}_\varphi))$ be the symmetric form on $L^2(E: \varphi^2\, d\mu)$ defined by

$$(3.1.19)\qquad \begin{aligned}\bar{\mathscr{E}}_\varphi(u,v) &= \int_E \langle \nabla u(z), \nabla v(z)\rangle_H \varphi^2\, d\mu,\\ \mathscr{D}\left(\bar{\mathscr{E}}_\varphi\right) &= \mathscr{F}C_b^\infty(E).\end{aligned}$$

By assumption (3.1.18)(ii), it follows that $(\bar{\mathscr{E}}_\varphi, \mathscr{D}(\bar{\mathscr{E}}_\varphi))$ is closable on $L^2(E: \varphi^2\, d\mu)$. Let $\{\Omega, \mathscr{F}, \mathscr{F}_t, X_t, \bar{P}_x,\ x \in E\}$ denote the diffusion associated with its closure $(\mathscr{E}_\varphi, \mathscr{D}(\mathscr{E}_\varphi))$. We introduce the probability measure P_φ on Ω by

$$(3.1.20)\qquad P_\varphi(\cdot) = \int \bar{P}_x(\cdot)\varphi^2\, d\mu.$$

Suppose that P_n converges to some probability P, where P is Markovian (i.e., X_t is a Markov process under P). Because the time reversibility obviously carries over from P_n to P, one expects to be able to show that P is Dirichlet. In fact, one can give an explicit characterization of the associated Dirichlet form $\mathscr{E}$ on the space $\mathscr{F}C_b^\infty(E)$. Furthermore, providing $\varphi \in D_2^1$ (the Sobolev space in the Malliavin sense, defined below), one can show that P is actually equal to P_φ, and therefore $P_n \to P_\varphi$. We explain the above description in detail in the rest of this section.

Now fix an arbitrary limit point P of the tight family $\{P_n,\ n \geq 0\}$, and assume that P is Markovian [i.e., $E_p[f(X_s)|\mathscr{F}_t] = E_p[f(X_s)|X_t]$ if $s > t$, for any $f \in \mathscr{B}_b(E)$]. By choosing a subsequence, we suppose that P is the limit of $\{P_n,\ n \geq 0)$.

First we prove that X has a stationary measure $\varphi^2\, d\mu$ under P. Since $P_n \to P$ as $n \to +\infty$, we have $E_n(f(X_t)) \to E(f(X_t))$ as $n \to \infty$ for any $f \in C_b(E)$, where $E(f(X_t))$ denotes the expectation of $f(X_t)$ with respect to P. However,

$$E_n(f(X_t)) = \int_E f(z)\varphi_n^2(z)\, d\mu \to \int_E f(z)\varphi^2(z)\, d\mu \quad \text{as } n \to +\infty;$$

therefore, $\{X_t\}$ has the marginal distribution $\varphi^2\, d\mu$ under P. Note that, for any $T > 0$, $\{X_t(\omega),\ 0 \leq t \leq T\}$ and $\{X_{T-t}(\omega),\ 0 \leq t \leq T\}$ have the same distribution under P_n, $n \geq 0$. This implies that $\{X_t(\omega)\}$ is also reversible under P.

Let P_t be the semigroup associated with $\{\Omega, \mathscr{F}, \mathscr{F}_t, X_t, P\}$ and let P_t^n be the semigroup associated with $\{\Omega, \mathscr{F}_t, X_t, P_x^n,\ x \in E\}$. Then from the preceding discussion, we see that P_t is a self-adjoint contraction semigroup on $L^2(E, \varphi^2\mu)$.

PROPOSITION 3.1.1. *For $g \in L^\infty(d\mu)$, $P_t^n g$ converges to $P_t g$ in $L^2(E, \varphi^2\, d\mu)$ as $n \to \infty$.*

PROOF. First we show that $P_t^n g$ converges to $P_t g$ weakly in $L^2(E, \varphi^2\, d\mu)$. Since $C_b(E)$ is dense in $L^2(E, \varphi^2\, d\mu)$, by the contractivity of P_t^n and P_t, it is sufficient to prove

$$\lim_{n\to\infty}\int_E (P_t^n f(x))g(x)\varphi^2(x)\, d\mu = \int_E (P_t f(x))g(x)\varphi^2(x)\, d\mu \qquad \text{for } f, g \in C_b(E).$$

In fact,

$$\begin{aligned}\lim_{n\to\infty}\int_E (P_t^n f(x))g(x)\varphi^2\, d\mu &= \lim_{n\to\infty}\int_E (P_t^n f(x))g(x)\varphi_n^2(x)\, d\mu\\ &= \lim_{n\to\infty} E_n[f(X_t)g(X_0)] = E[f(X_t)g(X_0)]\\ &= \int_E (P_t f(x))g(x)\varphi^2(x)\, d\mu.\end{aligned}$$

On the other hand,

$$\begin{aligned}\lim_{n\to\infty}\int_E (P_t^n g)^2\varphi^2\, d\mu &= \lim_{n\to\infty}\int_E (P_{2t}^n g)g\varphi_n^2\, d\mu\\ &= \lim_{n\to\infty}\int_E (P_{2t}^n g)g\varphi^2\, d\mu = \int_E (P_{2t} g)g\varphi^2\, d\mu\\ &= \int_E (P_t g)^2\varphi^2\, d\mu.\end{aligned}$$

Hence we conclude that $P_t^n g \to P_t g$ in $L^2(E\colon \varphi^2\, d\mu)$ as $n \to \infty$. This ends the proof. □

Let us denote by $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ the Dirichlet form on $L^2(E, \varphi^2\, d\mu)$ associated with the semigroup P_t. Set $R_\alpha^n = \int e^{-\alpha t}P_t^n\, dt$ and $R_\alpha = \int e^{-\alpha t}P_t\, dt$. Let D_2^1 be the completion of $\mathscr{F}C_b^\infty(E)$ under the norm $\|u\|_2^1 = [\int |\nabla u|_H^2\, d\mu + \int u^2\, d\mu]^{1/2}$. Recall the following definitions used in [9].

DEFINITION 3.1.1. A measurable function u defined on E is said to be Ray absolutely continuous (abbreviated RAC) if, for any $h \in H$, there exists a measurable function $\tilde{u}_h$ on E such that the following hold:

(i) $\tilde{u}_h(z) = u(z)$ μ-a.e., $z \in E$;
(ii) $\tilde{u}_h(z + th)$ is absolutely continuous in t for each $z \in E$.

DEFINITION 3.1.2. We say that a measurable function u on E is stochastically H Gâteaux differentiable (abbreviated by SGD) if there exists a measurable map $\nabla u \colon E \to H$ such that, for any $h \in H$,

$$\frac{1}{t}\left[u(z+th) - u(z) - t(\nabla u(z), h)_H\right] \to 0 \quad \text{in probability } \mu \text{ as } t \to 0.$$

The map $\nabla u(z)$ is called a stochastic Gâteaux derivative of u.

We have the following characterization result for $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$.

THEOREM 3.1.3. *Assume* (3.1.18) *holds. Then* $\mathscr{F}C_b^\infty(E) \subset \mathscr{D}(\mathscr{E})$ *and for any* $f \in \mathscr{B}_b(E)$, $R_1 f$ *is Ray absolutely continuous and stochastically Gâteaux differentiable. Moreover,*

$$\mathscr{E}(g, R_1 f) = \int_E \langle \nabla g, \nabla R_1 f\rangle_H \varphi^2 \, d\mu \quad \text{for } g \in \mathscr{F}C_b^\infty(E). \tag{3.1.21}$$

If φ *is also in* D_2^1, *then* $(\mathscr{E}, \mathscr{D}(\mathscr{E})) = (\mathscr{E}_\varphi, \mathscr{D}(\mathscr{E}_\varphi))$ *and hence* $P_n \to P_\varphi$ *as* $n \to +\infty$.

PROOF. We prove this theorem along the lines of [2]. Let $g \in \mathscr{F}C_b^\infty(E)$. Since $g \in \bigcap_n \mathscr{D}(\mathscr{E}_n)$ and $\liminf_{n\to\infty} \mathscr{E}_n(g, g) < +\infty$, it is easy to see that $\mathscr{F}C_b^\infty(E) \subset \mathscr{D}(\mathscr{E})$ by Proposition 3.1.1. Let $f \in \mathscr{B}_b(E)$. Since $R_1^n f \to R_1 f$ in μ-measure φ by Proposition 3.1.1 and $\|R_1^n f\|_\infty \le \|f\|_\infty < +\infty$, we have that $\lim_{n\to\infty} R_1^n f = R_1 f$ in $L^P(E\colon \mu)$, for any $P > 1$. The idea now is to use the derivatives of the functions $R_1^n f$ to construct a "derivative" for $R_1 f$.

Since

$$\begin{aligned}
\int_E \|\nabla R_1^n f\|_H^2 \varphi_n^2 \, d\mu &= \bar{\mathscr{E}}_n(R_1^n f, R_1^n f) \\
&= \bar{\mathscr{E}}_{n1}(R_1^n f, R_1^n f) - (R_1^n f, R_1^n f)_{L^2(\mu^n)} \\
&= (R_1^n f, f)_{L^2(\varphi_n^2 \, d\mu)} - (R_1^n f, R_1^n f)_{L^2(\varphi_n^2 \, d\mu)} \\
&\le 2\|f\|_\infty^2 \sup_n \int \varphi_n^2 \, d\mu < +\infty,
\end{aligned}$$

we can assume that $\varphi_n \nabla R_1^n f$ converges weakly to some function $K(z) \in L^2(E \to H, d\mu)$.

Since $1/\varphi_n$ is bounded and converges to $1/\varphi$ in L^2, one can easily establish that, for any $h \in L^\infty(E \to H, \mu)$,

$$\lim_{n\to\infty} \int_E \langle h, \nabla R_1^n f\rangle_H \, d\mu = \int_E \langle h, \varphi^{-1} K\rangle_H \, d\mu. \tag{3.1.22}$$

By (3.1.22) and Lemma 1.1 in [9], it follows that $R_1 f$ is RAC and SGD, and $\nabla R_1 f = \varphi^{-1} K(z) \in L^2(E \to H, \varphi^2 \, d\mu)$, which completes the proof of the first part of the theorem.

In the following, we will show that $\mathscr{E}(g, R_1 f) = \int_E \langle \nabla g, \nabla R_1 f \rangle_H \varphi^2 \, d\mu$ for $g \in \mathscr{F}C_b^\infty(E)$. We have shown $\lim \varphi_n \nabla R_1^n f = \varphi \nabla R_1 f$ weakly, so that

$$\int_E \langle \nabla g, \nabla R_1 f \rangle_H \varphi^2 \, d\mu = \lim_{n \to \infty} \int_E \langle \varphi \nabla g, \varphi_n \nabla R_1^n f \rangle_H \, d\mu.$$

However,

$$\left| \int_E \langle \varphi \nabla g, \varphi_n \nabla R_1^n f \rangle_H \, d\mu - \int_E \langle \varphi_n \nabla g, \varphi_n \nabla R_1^n f \rangle_H \, d\mu \right|$$

$$\leq \left(\int_E \|\nabla g\|_H^2 (\varphi_n - \varphi)^2 \, d\mu \right)^{1/2} \left(\int_E \varphi_n^2 \|\nabla R_1^n f\|_H^2 \, d\mu \right)^2$$

$$\leq M \left(\int_E (\varphi_n - \varphi)^2 \, d\mu \right)^{1/2} \to 0.$$

Hence,

$$\begin{aligned} \int_E \langle \nabla g, \nabla R^1 f \rangle_H \varphi^2 \, d\mu &= \lim_{n \to \infty} \int_E \langle \nabla g, \varphi_n^2 \nabla R_1^n f \rangle_H \, d\mu \\ &= \lim_{n \to \infty} \int_E \langle \nabla g, \nabla R_1^n f \rangle_H \varphi_n^2 \, d\mu = \lim_{n \to \infty} \mathscr{E}_n(R_1^n f, g) \\ &= \lim_{n \to \infty} \left[\mathscr{E}_{n1}(R_1^n f, g) - (R^n f, g)_{L^2(\phi_n^2 \, d\mu)} \right] \\ &= \lim_{n \to \infty} \left[(f, g)_{L^2(\varphi_n^2 \, d\mu)} - (R_1^n f, g)_{L^2(\varphi_n^2 \, d\mu)} \right]. \end{aligned}$$

It is obvious that

$$\lim_{n \to \infty} \int f g \varphi_n^2 \, d\mu = \int f g \varphi^2 \, d\mu,$$

$$\begin{aligned} \int_E g R_1^n f \varphi_n^2 \, d\mu - \int_E g R_1 f \varphi^2 \, d\mu &= \int_E g R_1^n f \varphi_n^2 \, d\mu - \int_E g R_1^n f \varphi^2 \, d\mu \\ &\quad + \int_E g R_1^n f \varphi^2 \, d\mu - \int_E g R_1 f \varphi^2 \, d\mu \to 0. \end{aligned}$$

Therefore,

$$\int_E \langle \nabla g, \nabla R_1 f \rangle_H \varphi^2 \, d\mu = (f, g)_{L^2(\varphi^2 \, d\mu)} - (R_1 f, g)_{L^2(\varphi^2 \, d\mu)} = \mathscr{E}(g, R_1 f).$$

We have identified the form $\mathscr{E}$ on the space $\mathscr{F}C_b^\infty(E)$ as claimed in the theorem.

Finally, if we have $\varphi \in D_2^1$, then by integration by parts we obtain

$$\mathscr{E}(g, R_1 f) = \int R_1 f \left(-Lg - \frac{\langle \nabla \varphi, \nabla g \rangle}{\varphi} \right) \varphi^2 \, d\mu, \tag{3.1.23}$$

where L is the Ornstein–Uhlenbeck operator on E. Note that $\{R_1 f,\ f \in \mathscr{B}_b(E)\} \subset \mathscr{D}(\mathscr{E})$ densely. It is clear by (3.1.23) that $g \in \mathscr{D}(\mathscr{L})$ and $\mathscr{L} g = Lg + \langle \nabla \varphi, \nabla g \rangle / \varphi$, where $\mathscr{L}$ denotes the generator of $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ on

$L^2(E, \varphi^2\,d\mu)$. Consequently, according to the uniqueness result in [13], we conclude that $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ must be $(\mathscr{E}_\varphi, \mathscr{D}(\mathscr{E}_\varphi))$. The theorem is proved. □

3.2. *Another approach to the construction of diffusion processes on Hilbert space.* In this section, diffusion processes on Hilbert space will be constructed by using the decomposition given in Section 3.1. We adopt the same state space and notation as in Section 3.1. Let m be a probability measure on $(E, \mathscr{B}(E))$ such that $\operatorname{supp}[m] = E$, $\int \|z\|_E^2 m(dz) < +\infty$.

Assume that the symmetric form

$$\mathscr{E}(u, v) = \int_E \langle \nabla u, \nabla v \rangle_H \, dm,$$

$$\mathscr{D}(\mathscr{E}) = \mathscr{F}C_b^\infty(E)$$

is densely defined and closable on $L^2(E\colon m)$. We use $\overline{\mathscr{E}}$ to denote the closure of $\mathscr{E}$.

THEOREM 3.2.1. *There exists a diffusion process on E such that its Dirichlet form is* $(\overline{\mathscr{E}}, \mathscr{D}(\overline{\mathscr{E}}))$.

REMARK. The result stated in Theorem 3.2.1 is not new. See [3] for details. We give another approach here.

PROOF. Let $\{e_n\} \subset E'$ be an orthonormal basis of H such that $\{e_n^* = e_n/\lambda_n\}$ is an orthonormal basis of $(E, \langle\ ,\ \rangle_E)$ and $\langle e_n^*, h\rangle_E = \lambda_n \langle e_n, h\rangle_H$, for $h \in H$, with

$$\sum_1^\infty \lambda_n^2 < +\infty. \tag{3.2.1}$$

Let $X = \prod_{k=1}^\infty \overline{R}_k$. Here $\overline{R}_k$ is the one-point compactification of R. Define the continuous injection $i\colon E \to X$ by

$$i(z) = (e_1(z), \dots, e_n(z), \dots).$$

Set $m = \nu \circ i^{-1}$. By i we obtain an image form $(\hat{\mathscr{E}}, \hat{\mathscr{D}}(\hat{\mathscr{E}}))$ of $(\overline{\mathscr{E}}, \mathscr{D}(\overline{\mathscr{E}}))$ on $L^2(X, m)$. Then $(\hat{\mathscr{E}}, \hat{\mathscr{D}}(\hat{\mathscr{E}}))$ is a regular and local Dirichlet form (see [14]). So by [6] there exists a diffusion process $M = \{\Omega, \mathscr{F}, \{X_t\}_{t \ge 0}, P_x, x \in X\}$ on X associated with $(\hat{\mathscr{E}}, \hat{\mathscr{D}}(\hat{\mathscr{E}}))$. We identify $i(E)$ with E. Then, $P_m(X_t \in E) = m(E) = 1$, for any $t > 0$. [Here, $P_m(\cdot) = \int P_x(\cdot) m(dx)$.] Let $\mathscr{F}_\infty^0 = \sigma(X_s, 0 \le s < +\infty)$ and $\mathscr{F}_t^0 = \sigma(X_s, 0 \le s \le t)$. Denote by $\mathscr{F}_\infty^m$ (resp., $\mathscr{F}_t^m$) the completion of $\mathscr{F}_\infty^0$ (resp., completion of $\mathscr{F}_t^0$ in $\mathscr{F}_\infty^m$) with respect to P_m. By Theorem 4.1.1 in [6], we know that M is a strong Markov diffusion process with respect to $\mathscr{F}_t^m$. On the other hand, from the definition of $\hat{\mathscr{E}}$ we can see that

$$e_n^* \in \mathscr{D}(\hat{\mathscr{E}}) \quad \text{and} \quad \hat{\mathscr{E}}(e_n^*, e_n^*) = \int \langle \nabla e_n^*, \nabla e_n^* \rangle (x)\, m(dx).$$

Consequently, under P_m, the decomposition of $e_n^*(X_t) - e_n^*(X_0)$ in Section 1 is as follows:

$$e_n^*(X_t) - e_n^*(X_0) = \tfrac{1}{2}M_t^{e_n^*} - \tfrac{1}{2}\left(M_T^{e_n^*}(\gamma_T) - M_{T-t}^{e_n^*}(\gamma_T)\right). \tag{3.2.2}$$

Using the same method as in Theorem 3.1.1, we get from (3.2.2) that

$$E_m\| X_t - X_s \|_E^4 \le M\left(\sum_1^\infty \lambda_n^2\right)^2 (t-s)^2, \qquad 0 \le s \le t \le T. \tag{3.2.3}$$

Here E_m means expectation with respect to P_m. From (3.2.3), it is well known that there exists a continuous process $\tilde{X}_t$ on E such that $P_m(X_t = \tilde{X}_t) = 1$.

Next we show $\widehat{\mathrm{Cap}}(X - E) = 0$. [Here $\widehat{\mathrm{Cap}}(X - E)$ denotes the capacity of $X - E$ with respect to the Dirichlet form $(\hat{\mathscr{E}}, \mathscr{D}(\hat{\mathscr{E}}))$.] To this end, we set

$$\sigma_{X-E} = \inf\{t > 0,\ X_t \in X - E\}.$$

Then we know that $\widehat{\mathrm{Cap}}(X - E) = 0$ if and only if

$$P_m(\sigma_{X-E} < +\infty) = 0.$$

Under P_m, since $\tilde{X}_t$ is a continuous process on E, it is automatically a continuous process with respect to the topology on X. However, we know that, for each $t > 0$, $\tilde{X}_t = X_t$ P_m-a.e. So there exists a set $N \in \mathscr{F}_\infty^m$ such that $P_m(N) = 0$ and, for $\omega \notin N$, $X_t(\omega) = \tilde{X}_t(\omega)$ for all $t \ge 0$. This indicates that

$$\{\sigma_{X-E} < +\infty\} \subset N.$$

Consequently,

$$P_m(\sigma_{X-E} < +\infty) = 0, \quad \text{that is,} \quad \widehat{\mathrm{Cap}}(X - E) = 0.$$

Once we have $\widehat{\mathrm{Cap}}(X - E) = 0$, following the same argument as in [9] we see that a diffusion process X_t does exist on E.

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DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF EDINBURGH
THE KING'S BUILDINGS
MAYFIELD ROAD
EDINBURGH EH9 3JZ
SCOTLAND