The Annals of Probability
1994, Vol. 22, No. 1, 334–371

SPATIAL PATTERNS WHEN PHASES SEPARATE IN AN INTERACTING PARTICLE SYSTEM[1]

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We consider a one-dimensional Glauber–Kawasaki process which gives rise in the hydrodynamical limit to a reaction diffusion equation with a double-well potential. We study the case when the process starts off from a product measure with zero averages, which, hydrodynamically, corresponds to a stationary unstable state. We prove that at times longer than the hydrodynamical ones the reaction diffusion equation no longer describes the behavior of the system, which in fact leaves the unstable equilibrium. The spatial patterns of the typical configurations when this happens are investigated.

1. Introduction. A gas suddenly cooled below the critical temperature becomes unstable whenever its density ρ lies inside the phase transition region. Liquid droplets appear and liquid and vapor separate. No pure phase corresponds to the initial density ρ, and the final state is a mixture of liquid, with density ρ_L, and vapor, with density ρ_V, the total density remaining equal to ρ. Mathematically, the final state is a linear combination of the Gibbs states with densities ρ_L and ρ_V.

Similar phenomena arise in several branches of science and technology and draw considerable interest both theoretically and in applications. Several phenomenological equations are used to study these effects. An example [see Fife (1979)] is provided by the reaction diffusion equation

$$\frac{\partial m}{\partial t} = \frac{1}{2}\Delta m - V'(m), \qquad m \in \mathbb{R}, \tag{1.1}$$

where Δ is the Laplacian and $V(m)$ is a double well potential. The reactive term $-V'(m)$ describes the drift toward the pure phases, here determined by the values of the parameter m corresponding to the two minima of $V(m)$. The homogenization phenomena are taken into account by the diffusive term Δm.

The purpose of this paper is to study a stochastic system of interacting particles, the Glauber–Kawasaki process, which models (1.1) and to analyze at the microscopic level, that is, at the particle level, how phases separate. We accomplish this by characterizing, in a one-dimensional model, the phase separation and its spatial pattern.

Received April 1991; revised October 1992.

[1]This work has been supported by CNPq, CNR-MMAIT and IBM of Brazil grants.

AMS 1991 *subject classifications*. Primary 60K35; secondary 60F05, 82A05.

Key words and phrases. Interacting particle systems, reaction diffusion equation, unstable equilibria.

In the next section we give the main definitions and results. In Section 3 we study the "magnetization fields" and their critical fluctuations, characterizing the early stage of the phase separation. In Section 4 we describe the final stage of the phase separation, while in Section 5 we prove the probability estimates used earlier.

2. The model and the main results. The model under study consists of a family $(\sigma^\varepsilon(t))_{t\geq 0}$ of Markov processes taking values on $X_\varepsilon = \{-1, 1\}^{\mathbb{Z}_\varepsilon}$, where $\mathbb{Z}_\varepsilon = \mathbb{Z}$ modulo $\varepsilon^{-1}|\ln \varepsilon|$ for $0 < \varepsilon \leq 1/2$ and such that $\varepsilon^{-1}|\ln \varepsilon| \in \mathbb{Z}$. Of course, we could take instead $\mathbb{Z}_\varepsilon = \mathbb{Z}$ modulo $[\varepsilon^{-1}|\ln \varepsilon|]$ with any $0 < \varepsilon \leq 1/2$ ($[\cdot]$ denoting integer part). Identifying X_ε with $\{-1, 1\}^{\Lambda_\varepsilon}$ where $\Lambda_\varepsilon = \{0, 1, \dots, \varepsilon^{-1}|\ln \varepsilon| - 1\}$, we write the generator of $\sigma^\varepsilon(t)$ as

$$L_\varepsilon = \varepsilon^{-2} L_0 + L_G, \tag{2.1}$$

where, for any $\sigma \in X_\varepsilon$ and any function f on $\{-1, 1\}^{\Lambda_\varepsilon}$:

$$L_0 f(\sigma) = \frac{1}{2} \sum_{x \in \Lambda_\varepsilon} \left[f(\sigma^{x, x+1}) - f(\sigma) \right], \tag{2.2a}$$

$$L_G f(\sigma) = \sum_{x \in \Lambda_\varepsilon} c(x, \sigma) \left[f(\sigma^x) - f(\sigma) \right], \tag{2.2b}$$

with

$$\sigma^{x, x+1}(z) = \begin{cases} \sigma(x), & \text{if } z = x + 1, \\ \sigma(x+1), & \text{if } z = x, \\ \sigma(z), & \text{otherwise} \end{cases}$$

(recall that we are identifying 0 and $\varepsilon^{-1}|\ln \varepsilon|$) and

$$\sigma^x(z), = \begin{cases} \sigma(z), & \text{if } z \neq x, \\ -\sigma(x), & \text{otherwise.} \end{cases}$$

The general assumptions concerning $c(\cdot, \cdot)$ are: (i) $c(x, \sigma) = c(0, \tau_x \sigma)$, that is, they are translationally invariant $[\tau_y \sigma(x) = \sigma(y + x)]$; (ii) $c(0, \cdot)$ is a cylinder function, that is, the interaction has a fixed finite range; (iii) $c(0, \cdot)$ is strictly positive. The process corresponding to the generator L_0 is called the symmetric simple exclusion process or the stirring process.

In De Masi, Ferrari and Lebowitz (1986) these dynamics have been studied in unbounded volumes; the proofs apply as well to the case we are considering. After fixing the basic notation we recall some known results on our model.

NOTATION. We denote by $\sigma(x)$, $x \in \Lambda_\varepsilon$, both the x-coordinate of $\sigma \in X_\varepsilon$ and the random variable on X_ε whose value at σ is the x-coordinate of σ. We then write $\mu(f)$ for the integral $\int f \, d\mu$, μ being a measure on X_ε or X. Finally, for any $n \geq 1$, M_n^ε is the set of all the n-tuples $\mathbf{x} = (x_1, \dots, x_n)$ of distinct sites in Λ_ε.

Let us now assume that the initial measure μ^ε is a product measure on X_ε with $\mu^\varepsilon(\sigma(x)) = m_\varepsilon(\varepsilon x)$, where m_ε is a C^3 function from the circle $[0, |\ln \varepsilon|]$ to $[-1, 1]$, with uniformly bounded derivatives, which converges uniformly on compact subsets to a limiting function $m(\cdot)$. Then let μ_t^ε denote the law of the process at time t. Under these assumptions it is proven in De Masi, Ferrari and Lebowitz (1986) that for any $r > 0$ and any positive integer n:

$$\lim_{\varepsilon \to 0} \sup_{\substack{\mathbf{x} \in M_n^\varepsilon \\ \forall i,\, |x_i| \le \varepsilon^{-1} r}} \left| \mu_t^\varepsilon\left(\prod_{i=1}^n \sigma(x_i) \right) - \prod_{i=1}^n m_\varepsilon(\varepsilon x_i, t) \right| = 0, \tag{2.3}$$

where $m_\varepsilon(\cdot, t)$ is the solution of the reaction–diffusion equation

$$\frac{\partial}{\partial t} m = \frac{1}{2} \frac{\partial^2}{\partial r^2} m + F(m), \qquad m(\cdot, 0) = m_\varepsilon(\cdot), \tag{2.4}$$

with periodic boundary conditions on $[0, |\ln \varepsilon|]$ and

$$F(m) = -2\nu_m(\sigma(0)c(0, \sigma)),$$

with ν_m denoting the Bernoulli measure on $X = \{-1, +1\}^{\mathbb{Z}}$ such that $\nu_m(\sigma(x)) = m$. As a consequence, for any $r \in \mathbb{R}$,

$$\lim_{\varepsilon \to 0} \mu_t^\varepsilon(\sigma([\varepsilon^{-1} r])) = m(r, t),$$

which solves (2.4) with initial condition $m(r, 0) = m(r)$.

Our goal is to study the long-term behavior of μ_t^ε when $F(m) = -V'(m)$ with V being a double-well potential. For this we may choose

$$c(0, \sigma) = 1 - \gamma\sigma(0)[\sigma(1) + \sigma(-1)] + \gamma^2 \sigma(1)\sigma(-1), \tag{2.5}$$

where $\gamma \in (1/2, 1]$.

When the intensities are given by (2.5), we have $F(m) = -V'(m)$ with

$$V(m) = \frac{\beta m^4}{4} - \frac{\alpha m^2}{2}, \qquad \alpha = 2(2\gamma - 1), \qquad \beta = 2\gamma^2. \tag{2.6}$$

In this case $m \equiv 0$ is an unstable stationary solution of (2.4) and for such an initial profile, $\mu_t^\varepsilon \to \nu_0$ as $\varepsilon \to 0$, for any fixed t. The problem we are concerned with is the behavior of μ_t^ε for t tending to ∞ as $\varepsilon \to 0$. The first questions in this direction are "When does the system escape from ν_0? How should $t \to \infty$ when $\varepsilon \to 0$?" Some heuristic arguments drawn from the "fluctuating hydrodynamic theory" [see Spohn (1991)] suggest that the deviations of the system from (2.4) are described by the stochastic differential equation

$$\begin{aligned} dm &= \left\{ \frac{1}{2} \frac{\partial^2 m}{\partial r^2} + F(m) \right\} dt + \sqrt{\varepsilon}\, dw, \\ m(r, 0) &= 0, \end{aligned} \tag{2.7}$$

where w is a white noise in space and time. Indeed, equations like (2.7) have been used to model phase separation phenomena and have their own interest; for us here, (2.7) only has the purpose of indicating the right time scale for the escape. If we linearize (2.7) around $m \equiv 0$, replacing $F(m)$ by $F'(0)m = \alpha m$, we readily see that the right scaling is $t = \tau|\ln \varepsilon|$. Indeed, for bounded volumes, that is, if we replace Λ_ε by $\{0, \dots, \varepsilon^{-1}L - 1\}$ with $L\varepsilon^{-1} \in \mathbb{N}$, we already know [see De Masi and Presutti (1991)] that

$$\lim_{\varepsilon \to 0} \mu^\varepsilon_{\tau|\ln \varepsilon|} = \begin{cases} \nu_0, & \text{for } \tau < 1/2\alpha, \\ \frac{1}{2}(\nu_{m^*} + \nu_{-m^*}), & \text{for } \tau > 1/2\alpha, \end{cases} \tag{2.8a}$$

where the limit is in the w^*-topology and $m^* = \sqrt{\alpha/\beta}$. Furthermore,

$$\lim_{\varepsilon \to 0} \mu^\varepsilon_{|\ln \varepsilon|/(2\alpha)+t} = \int_{-1}^{1} \lambda_t(dm)\nu_m, \tag{2.8b}$$

where $\lambda_t(\cdot)$ is absolutely continuous with respect to the Lebesgue measure. In fact, one has something more than w^*-convergence, since one can prove the convergence of the integrals of any fixed number of spins, uniformly on their location. We refer to Calderoni, Pellegrinotti, Presutti and Vares (1989) for a discussion of motivation and for the analysis of the case when the potential $V(m)$ has a quartic maximum at $m = 0$. The phenomenology in such a case is quite different [cf. also Vares (1990), where (2.7) is studied for the case when $V(\cdot)$ is exponentially flat at its maximum and w is a white noise only with respect to time, at each time being constant in space, so that (2.7) becomes an ordinary stochastic differential equation].

The case of a potential V with a quadratic maximum $[V''(0) < 0]$ is very special. In the proper scale the "escape time" becomes asymptotically deterministic; in our example, (2.6), it is equal to $1/(2\alpha)$, in the time scale whose unit is $|\ln \varepsilon|$. Otherwise, the escape time is stochastic, and we observe the so-called bimodality effects, that is, at each time, in the proper scale, the state of the system is approximated as $\varepsilon \to 0$ by a nontrivial convex combination of ν_0 and $(\nu_{m^*} + \nu_{-m^*})/2$, in agreement with the behavior of the solution of (2.7) [see Calderoni, Pellegrinotti, Presutti and Vares (1989) and Vares (1990)].

We extend here the analysis in De Masi and Presutti (1991) to unbounded volumes. For technical reasons it is simpler not to study the system in the whole of $\mathbb{Z}$, but only in Λ_ε, which is, however, large enough for exhibiting a nontrivial spatial structure, as shown in the following theorem. (For further comments on this point, see Section 6).

THEOREM 2.1. *Let μ^ε be the product measure on X_ε, with $\mu^\varepsilon(\sigma(x)) = 0$ for all $x \in \Lambda_\varepsilon$. Let also μ^ε_t be the law at time t of the process generated by L_ε with $c(\cdot, \cdot)$ given by (2.5) and initial measure μ^ε. We set $\alpha = 2(2\gamma - 1)$, $\beta = 2\gamma^2$, $m^* = \sqrt{\alpha/\beta}$, $t_f = |\ln \varepsilon|/(2\alpha) + |\ln \varepsilon|^{1/3}$ and, for $r > 0$, $r_\varepsilon = r\varepsilon^{-1}|\ln \varepsilon|$. We then have, for any $r > 0$ and any $n \geq 1$,*

$$\text{(i)} \quad \lim_{\varepsilon \to 0} \sup_{\substack{\mathbf{x} \in M^\varepsilon_n \\ \forall i,\, |x_i| \leq r_\varepsilon}} \left| \mu^\varepsilon_{\tau|\ln \varepsilon|}\left(\prod_{i=1}^n \sigma(x_i) \right) \right| = 0 \quad \textit{if } \tau \leq 1/(2\alpha), \tag{2.9a}$$

$$\text{(ii)} \quad \lim_{\varepsilon \to 0} \sup_{\substack{\mathbf{x} \in M^\varepsilon_n \\ \forall i,\, |x_i| \leq r_\varepsilon}} \left| \mu^\varepsilon_{t_f}\left(\prod_{i=1}^n \sigma(x_i) \right) - \tilde{E}\left(\prod_{i=1}^n \rho\left(\varepsilon|\ln \varepsilon|^{-1/2} x_i\right) \right) \right| = 0, \tag{2.9b}$$

where $\rho(r) = m^* \operatorname{sign} \tilde{X}(r)$ *and* $(\tilde{X}(r))_{r \in \mathbb{R}}$ *is a zero-average Gaussian process on some probability space* $(\tilde{\Omega}, \tilde{\mathscr{A}}, \tilde{P})$ *with* $\tilde{E}(\tilde{X}(r)\tilde{X}(r')) = e^{-\alpha(r-r')^2/2}$, *for* $r, r' \in \mathbb{R}$.

REMARKS. The convergence in (2.9a) is uniform in $\tau \le 1/(2\alpha)$, but we shall only prove the weaker statement (2.9a). Notice also the difference between the present result and the corresponding one for the bounded volume case; here the magnetization at time $|\ln \varepsilon|/(2\alpha)$ is still infinitesimal, while in (2.8b) it is already finite.

We shall prove Theorem 2.1 in the remaining part of the paper. The basic techniques we use involve: (a) sufficiently sharp estimates on some sort of truncated correlation functions, the v-functions introduced below; (b) separation of several time scales. To understand the meaning of such points, we recall the bounded volume case [cf. De Masi and Presutti (1991)]. Using the estimates mentioned in (a), one can study the stochastic fluctuations at the initial stage of the escape, when the typical magnetization grows from $\varepsilon^{1/2}$ to ε^{a} for any given $a \in (0, 1/2)$ and ε is small enough. Since the magnetization is still infinitesimal, we can safely use only the linear part of the drift and this procedure works up to times when the magnetization is "almost finite" in a sense to be made precise in Section 4; from then on, one exploits the convergence of the process to (2.4) to show that the magnetization reaches finite values. The first stage of this analysis is similar both in the bounded volume and in the present case, except for a few subtle technical points. The main difference, however, appears in the last part, since in the bounded volume case we could exploit the absence of spatial structures. The typical configurations were "flat," and, taking advantage of this, we could see that the deterministic evolution is essentially ruled by an ordinary differential equation. Now, in the unbounded case, the spatial structure makes the analysis truly infinite-dimensional. Another crucial point is that we always have regions of arbitrarily small magnetization, where the stochastic fluctuations are important; thus we must control their influence on any fixed region of the space. It is at this point that we use the time lag $|\ln \varepsilon|^{1/3}$ in (2.9b). Such a time interval is chosen so long that the magnetization can reach finite values in the "good regions," but also small enough for neglecting the influence of the "bad regions" where the magnetization at time $d/(2\alpha)|\ln \varepsilon|$ is too small. This is, we believe, only a technical difficulty, because the same statement (2.9b) should hold for $t_f \to \tau|\ln \varepsilon|$, with any $\tau > 1/(2\alpha)$. This is peculiar to one dimensional space, as mentioned in Section 6; in higher dimensional space, $d > 1$, the clusters (i.e., the regions with the same magnetization) are expected to move by curvature when τ increases past $d/(2\alpha)$, which is the time when the escape occurs in d dimensions. In one dimension we conjecture that the clusters will not move significantly in this same scale, but only after times which grow like some positive power of ε^{-1}.

Before defining the v-functions and stating the basic estimates, we introduce some notation and definition.

NOTATION. Let $\mathbb{P}_{\mu}^{\varepsilon}$ denote the law of $(\sigma^{\varepsilon}(t))_{t\geq 0}$ on the canonical space $D([0, +\infty), X_{\varepsilon}) = \{\sigma: [0, +\infty) \to X_{\varepsilon} | \sigma(\cdot)$ is right continuous and with left limits$\}$, when $\sigma^{\varepsilon}(0)$ is distributed according to μ, where μ is a probability on X_{ε}, and let $\mathbb{E}_{\mu}^{\varepsilon}$ denote the expectation with respect to the law $\mathbb{P}_{\mu}^{\varepsilon}$.

DEFINITION [The functions $m_{\varepsilon}(r, t; \lambda)$]. If λ is a product measure on X_{ε}, we denote by $m_{\varepsilon}(r, t; \lambda)$ the solution of (2.4) when the initial condition is periodic with period $|\ln \varepsilon|$ and its values for $0 \leq r < |\ln \varepsilon|$ are

$$m_{\varepsilon}(r, 0; \lambda) = \lambda(\sigma([\varepsilon^{-1} r])). \tag{2.10}$$

We are now ready for the definition of the v-functions, which, for our purposes here, are introduced in a way slightly different than usual [see, for instance, Chapter 9 of De Masi and Presutti (1991)].

DEFINITION (The v-functions). Let λ be any product measure on X_{ε}. Then for any $\varepsilon > 0$, $n \geq 1$, $\mathbf{x} \in M_n^{\varepsilon}$ and $t \geq 0$, we define

$$v_n^{\varepsilon}(\mathbf{x}, t; \lambda) = \mathbb{E}_{\lambda}^{\varepsilon}\left(\prod_{i=1}^{n} [\sigma(x_i, t) - m_{\varepsilon}(\varepsilon x_i, t; \lambda)] \right). \tag{2.11}$$

Notice that in particular λ may be any measure supported by a single configuration of X_{ε}. A basic bound on the v-function is proven in Section 7 of De Masi and Presutti (1991) and it is reported here without proof.

PROPOSITION 2.2 [Theorem 9.2.1 in De Masi and Presutti (1991)]. *There are a^*, δ^*, β^* positive such that for any $n \geq 1$ there is a c such that for any $\varepsilon > 0$ and any product measure λ on X_{ε},*

$$\sup_{\varepsilon^{\beta^*} \leq t \leq t_{a^*}} \sup_{\mathbf{x} \in M_n^{\varepsilon}} |v_n^{\varepsilon}(\mathbf{x}, t; \lambda)| \leq c \varepsilon^{\delta^* n}, \qquad t_{a^*} = a^* |\ln \varepsilon|. \tag{2.12}$$

REMARK. In Theorem 9.2.1 in De Masi and Presutti (1991), one considers the bounded volume case, that is, $\mathbb{Z}_{\varepsilon} = \mathbb{Z}/[\varepsilon^{-1}]$, but the estimate (2.12) follows exactly as in that proof.

In the next proposition we give an explicit expression for the quantity δ^* appearing in Proposition 2.2, under the assumption that the initial measure is the product probability measure on X_{ε} with zero spin averages. We have a "good" estimate for v_2 in this case, but we have not been able to prove that v_{2n} behaves as the nth power of v_2, as we expect. More precisely, let μ^{ε} be as in Theorem 2.1. In this case we write for $n \geq 1$ and $\mathbf{x} \in M_n^{\varepsilon}$,

$$v_n^{\varepsilon}(\mathbf{x}, t) = \mathbb{E}_{\mu^{\varepsilon}}^{\varepsilon}\left(\prod_{i=1}^{n} \sigma(x_i, t) \right). \tag{2.13}$$

Then we can prove the following proposition.

PROPOSITION 2.3. *Under the above conditions* $v_n^\varepsilon = 0$ *when* n *is odd. Otherwise, for any* $\eta_0 < 1/8$, $\tau < 1/(2\alpha)$ *and* $n \geq 1$, *there is* c *such that*

$$\sup_{\mathbf{x} \in M_{2n}^\varepsilon} |v_{2n}^\varepsilon(\mathbf{x}, t)| \leq c(\Gamma_t(\varepsilon))^{2n}, \qquad 0 \leq t \leq \tau|\ln \varepsilon|, \tag{2.14}$$

where

$$\Gamma_t(\varepsilon) = \sqrt{\varepsilon}\, e^{\alpha t} \frac{1}{(1+t)^{\eta_0}}. \tag{2.15}$$

Furthermore,

$$\sup_{\mathbf{x} \in M_2^\varepsilon} |v_2^\varepsilon(\mathbf{x}, t)| \leq c\varepsilon e^{2\alpha t} \frac{1}{(1+t)^{1/2}}, \qquad 0 \leq t \leq \tau|\ln \varepsilon|. \tag{2.16}$$

REMARKS. The statement that $v_n^\varepsilon = 0$ if n is odd holds trivially by the symmetry of the process under the transformation $\sigma \to -\sigma$ and the symmetry of the initial measure μ^ε under the same transformation; the statement when n is even is proven in Section 5.

Since $t \leq \tau|\ln \varepsilon|$ and $\tau < 1/(2\alpha)$, (2.16) tells us that at time t we still have a vanishing magnetization. It is then quite natural (since the law should still be close to being a product) to expect that v_{2n} should be bounded by the nth power of the r.h.s. of (2.16). We were not able to show this but only something slightly weaker, according to (2.14), which nevertheless suffices for our purposes. Notice that (2.14) already gives us (2.8) in Theorem 2.1, for the case $\tau < 1/(2\alpha)$.

3. The early stage of the escape. In this section we characterize "the early stage of the escape" from μ^ε, the product measure with zero spin average. As an introduction to such an analysis, we recall that according to Theorem 3 in De Masi, Ferrari and Lebowitz (1986), under suitable assumptions on the initial measure, the density fluctuation field

$$Z_t^\varepsilon(\phi) = \sqrt{\varepsilon} \sum_x \phi(\varepsilon x) \sigma^\varepsilon(x, t), \qquad \phi \in \mathscr{S}(\mathbb{R}), \qquad t \geq 0,$$

converges in law to a generalized Ornstein–Uhlenbeck process. In particular, if we fix t and ϕ, then the law of $Z_t^\varepsilon(\phi)$ is approximated, as $\varepsilon \to 0$, by a Gaussian law with zero average and finite variance $C_t(\phi)$, and so, for finite times, the typical values of the magnetization density $\varepsilon \sum_x \phi(\varepsilon x) \sigma^\varepsilon(x, t)$ are of the order of $\sqrt{\varepsilon}$. By "early stage of the escape" we mean their growth from $\sqrt{\varepsilon}$ to ε^a, for some $a \in (0, 1/2)$. Applied to our case, the results proven in De Masi, Ferrari and Lebowitz (1986) show that $C_t(\phi)$ grows exponentially. This, however, does not imply that the magnetization is also increasing exponentially. In fact, the result proven in the above paper is obtained when $\varepsilon \to 0$, with time restricted to a bounded interval $[0, T]$, while now we want to look at times $t_\varepsilon \to +\infty$ as $\varepsilon \to 0$. However, the result suggests that the right time scale for observing the escape is $t_\varepsilon = \tau|\ln \varepsilon|$, $\tau > 0$. We prove below that the Gaussian character of the suitably rescaled magnetization fields is preserved even at these longer times, provided $\tau < 1/(2\alpha)$. For this we introduce the following definition.

DEFINITION. For $\phi \in \mathscr{S}(\mathbb{R})$, $t \geq 0$ and $0 < \varepsilon \leq 1/2$, we define

$$X_\tau^\varepsilon(\phi) = Y_{\tau|\ln \varepsilon|}^\varepsilon(\phi), \qquad 0 \leq \tau < 1/(2\alpha), \tag{3.1}$$

where

$$Y_t^\varepsilon(\phi) = \delta^{1/2}\varepsilon^{-\alpha t}\sum_x \phi(\delta x)\sigma^\varepsilon(x,t), \qquad \delta = \frac{\varepsilon}{\sqrt{|\ln \varepsilon|}}. \tag{3.2}$$

We remark that the reason for the "anomalous" δ instead of ε in (3.2) is simply the longer time scale $t = \tau|\ln \varepsilon|$ to be used [cf. (3.1)]. The factor $e^{-\alpha t}$ is present to depress the exponential growth of the field, keeping it finite.

NOTATION. On the path space $\Omega = C([0,\infty), \mathscr{S}'(\mathbb{R}))$, or $\Omega = D([0,\infty), \mathscr{S}'(\mathbb{R}))$, we shall use $\{X_\tau(\phi),\ \phi \in \mathscr{S}(\mathbb{R})\}$ to denote the canonical (coordinate) process, that is, $X_\tau(\phi)(\omega) = \omega(\tau)(\phi)$, for all $\omega \in \Omega$ and $\tau \geq 0$.

THEOREM 3.1. *Let $\bar\tau \in (0, 1/(2\alpha))$ and let $\mathscr{P}^\varepsilon$ denote the law of the process $X_\tau^\varepsilon(\phi)$, $0 \leq \tau \leq \bar\tau$ (when the initial measure is μ^ε), on the space $D([0,\infty), \mathscr{S}'(\mathbb{R}))$. Let $\mathscr{P}$ be the probability on $C([0,\infty), \mathscr{S}'(\mathbb{R}))$ concentrated on the deterministic evolution satisfying*

$$X_\tau(\phi) = X_0(\phi_\tau), \tag{3.3a}$$

where

$$\phi_\tau(y) = \int dz\, \phi(z)\frac{1}{\sqrt{2\pi\tau}}\exp\left\{-\frac{(z-y)^2}{2\tau}\right\} \tag{3.3b}$$

and such that under $\mathscr{P}$, X_0 is Gaussian with

$$\mathscr{E}(X_0(\phi)) = 0, \qquad \mathscr{E}(X_0(\phi)X_0(\psi)) = \left(1 + \frac{2}{\alpha}\right)\int_{-\infty}^{\infty} dx\, \phi(x)\psi(x) \tag{3.4}$$

for all ϕ and ψ in $\mathscr{S}(\mathbb{R})$. Then, for each $\tau_0 \in (0, \bar\tau)$, $\mathscr{P}^\varepsilon$, restricted to $D([\tau_0, \bar\tau], \mathscr{S}'(\mathbb{R}))$, converges weakly to the restriction of $\mathscr{P}$ to $C([\tau_0, \bar\tau], \mathscr{S}'(\mathbb{R}))$.

REMARKS 3.2.

(a) The equal time covariances of X_τ under $\mathscr{P}$ are given by the kernel

$$C(r, r', \tau) = \left(1 + \frac{2}{\alpha}\right)\frac{1}{\sqrt{4\pi\tau}}\exp\left\{-\frac{(r-r')^2}{4\tau}\right\}. \tag{3.5}$$

(b) By the classical central limit theorem, the distribution of X_0^ε converges weakly on $\mathscr{S}'(\mathbb{R})$ to the standard white noise, that is, instead of $1 + 2/\alpha$, as in (3.4), we have simply 1. Thus the above theorem cannot be extended to $D([0, \bar\tau], \mathscr{S}'(\mathbb{R}))$. That is, our scaling $\tau|\ln \varepsilon|$ produces an "initial layer," a jump at time 0, and, after that, the evolution becomes essentially deterministic. This is completely consistent with the following intuitive picture: The magnetization field evolves according to a linear stochastic differential equation obtained by the addition of a $\sqrt{\varepsilon}$ noise to the linearized version of (2.4), namely,

$$dm = \left(\frac{1}{2} \frac{\partial^2}{\partial r^2} m + \alpha m \right) dt + \sqrt{\varepsilon}\, dw.$$

As soon as the magnetization becomes of the order of ε^a, $a < 1/2$, the noise is overcome by the effects of the deterministic drift caused by the linear instability αm. The larger this is, the shorter is the time it takes to overcome the noise, whose effect, in the meantime, is correspondingly smaller, hence the dependence on $1/\alpha$ in (3.5). However, for any finite $\alpha > 0$ the noise on the scale $\tau|\ln \varepsilon|$ is immediately "switched off."

PROOF OF THEOREM 3.1. Based on the general results of Holley and Stroock (1978) and Mitoma (1983), we shall prove: (i) for any fixed $0 < \tau_0 < \bar{\tau} < 1/(2\alpha)$, the family $\mathscr{P}^\varepsilon$ is tight on $D([\tau_0, \bar{\tau}], \mathscr{S}'(\mathbb{R}))$, and any sequence $\mathscr{P}^{\varepsilon_n}$, $\varepsilon_n \to 0$, has a subsequence which converges to a probability measure concentrated on $C([\tau_0, \bar{\tau}), \mathscr{S}'(\mathbb{R}))$; (ii) any possible weak limit point must be in fact concentrated on the deterministic evolution which solves the heat equation, that is, $X_\tau(\phi) = X_{\tau_0}(\phi_{\tau - \tau_0})$, where ϕ_τ is given by (3.3b). Finally, using the fact that this holds for τ_0 arbitrarily small, we will be able to prove the convergence of $X^\varepsilon_{\tau_0}$ which will complete the proof.

For (i) and (ii) we use the martingale characterization of the limiting process and of tightness. According to Mitoma (1983), the family $(\mathscr{P}^\varepsilon)_{\varepsilon \in (0,1]}$ is tight on $D([\tau_0, \bar{\tau}], \mathscr{S}'(\mathbb{R}))$ if and only if for each $\phi \in \mathscr{S}(\mathbb{R})$ the laws of the processes $(X^\varepsilon_\tau(\phi)\colon \tau_0 \le \tau \le \bar{\tau})$ with $\varepsilon \in (0, 1]$ form a tight family of probability measures on $D([\tau_0, \bar{\tau}], \mathbb{R})$. On the other hand, for each test function ϕ, the jumps of $X^\varepsilon_\cdot(\phi)$ on $[\tau_0, \bar{\tau}]$ are uniformly vanishing as $\varepsilon \to 0$; thus for the tightness we may just use the usual C-criterion and in the case of an affirmative answer any limit point will be supported by the set of continuous trajectories.

For this we define

$$\gamma_1^\varepsilon(\tau, \phi) = \mathscr{L}_\varepsilon X^\varepsilon_\tau(\phi) - \alpha X^\varepsilon_\tau(\phi), \tag{3.6a}$$

$$\gamma_2^\varepsilon(\tau, \phi) = \mathscr{L}_\varepsilon X^\varepsilon_\tau(\phi)^2 - 2 X^\varepsilon_\tau(\phi) \mathscr{L}_\varepsilon X^\varepsilon_\tau(\phi), \tag{3.6b}$$

with

$$\mathscr{L}_\varepsilon = |\ln \varepsilon| L_\varepsilon = |\ln \varepsilon| \left[\varepsilon^{-2} L_0 + L_G \right]. \tag{3.6c}$$

Notice that

$$M^\varepsilon_{\tau, \tau_0}(\phi) := X^\varepsilon_\tau(\phi) - X^\varepsilon_{\tau_0}(\phi) - \int_{\tau_0}^{\tau} ds\, \gamma_1^\varepsilon(s, \phi) \tag{3.7a}$$

and

$$\left(M^{\varepsilon}_{\tau,\tau_0}(\phi)\right)^2 - \int_{\tau_0}^{\tau} ds\, \gamma_2^{\varepsilon}(s,\phi) \tag{3.7b}$$

are martingales which vanish at τ_0.

Thus it suffices to show that for any $\phi \in \mathscr{S}(\mathbb{R})$ there is a constant c so that

$$\sup_{\tau_0 \le \tau \le \bar{\tau}} \mathbb{E}^{\varepsilon}_{\mu^{\varepsilon}}\left(\gamma_i^{\varepsilon}(\tau,\phi)\right)^2 \le c, \qquad i = 1, 2, \tag{3.7c}$$

$$\sup_{\tau_0 \le \tau \le \bar{\tau}} \mathbb{E}^{\varepsilon}_{\mu^{\varepsilon}}\left(X^{\varepsilon}_{\tau}(\phi)\right)^2 \le c. \tag{3.7d}$$

Indeed, with Doob's inequality, (3.7a)–(3.7c) give us the following. For each $\phi \in \mathscr{S}$, each η, δ positive, we may find $\zeta > 0$ so that

$$P\left(\sup_{\substack{\tau_0 \le \tau,\, \tau' \le \bar{\tau} \\ |\tau - \tau'| \le \zeta}} |X^{\varepsilon}_{\tau}(\phi) - X^{\varepsilon}_{\tau'}(\phi)| > \delta \right) \le \eta,$$

which together with (3.7d) gives the tightness.

We now check (3.7c) and (3.7d). The action of L_0 is very simple to compute. Recall that $\varepsilon^{-2}|\ln \varepsilon| = \delta^{-2}$. Then set $x_{\pm} = x \pm 1$ and

$$\Delta_{\delta}\phi(x) = \frac{1}{\delta^2}\left[\phi(x+\delta) + \phi(x-\delta) - 2\phi(x)\right]$$

(sometimes we drop the time from the argument of the spin variables below). We then have

$$\varepsilon^{-2}|\ln \varepsilon| L_0 X^{\varepsilon}_{\tau}(\phi) = X^{\varepsilon}_{\tau}\left(\frac{1}{2}\phi''\right) + R_0(\varepsilon,\tau,\phi), \tag{3.8a}$$

where

$$\begin{aligned} |R_0(\varepsilon,\tau,\phi)| &\le e^{-\alpha\tau|\ln \varepsilon|}\frac{\delta^{1/2}}{2}\sum_x |\phi''(\delta x) - \Delta_{\delta}\phi(\delta x)| \\ &\le e^{-\alpha\tau|\ln \varepsilon|}\frac{\delta^{1/2}}{2} C(\phi), \end{aligned} \tag{3.8b}$$

with $C(\phi) < \infty$, for any ϕ.

Similarly, we get

$$\begin{aligned} &\left|\varepsilon^{-2}|\ln \varepsilon|\left(L_0\left(X^{\varepsilon}_t(\phi)^2\right) - 2X^{\varepsilon}_t(\phi) L_0 X^{\varepsilon}_t(\phi)\right)\right| \\ &\quad \le e^{-2\alpha\tau|\ln \varepsilon|}\delta^{-2}\delta\sum_x |\phi(\delta x) - \phi(\delta x_+)|^2 \\ &\quad \le e^{-2\alpha\tau|\ln \varepsilon|} C_1(\phi), \end{aligned} \tag{3.8c}$$

with $C_1(\phi) < \infty$, for any ϕ.

To compute the action of L_G, we first rewrite the rate function $c(x,\sigma)$ as

$$c(x,\sigma) = (1-2\gamma) + \gamma[2 - \sigma(x)\sigma(x_+) - \sigma(x)\sigma(x_-)] + \gamma^2\sigma(x_-)\sigma(x_+) \tag{3.9a}$$

and recall that $\alpha = 2(2\gamma - 1)$. Thus we easily get

$$|\ln \varepsilon| L_G X_t^\varepsilon(\phi) = \alpha |\ln \varepsilon| X_t^\varepsilon(\phi) + R_1(\varepsilon,\tau,\phi) + R_2(\varepsilon,\tau,\phi), \tag{3.9b}$$

where

$$R_1(\varepsilon,\tau,\phi) = 2\gamma|\ln\varepsilon| e^{-\alpha\tau|\ln\varepsilon|}\delta^{1/2}\sum_x [2\sigma(x) - \sigma(x_+) - \sigma(x_-)]\phi(\delta x),$$

so that

$$|R_1(\varepsilon,\tau,\phi)| \le 2\gamma|\ln\varepsilon| e^{-\alpha\tau|\ln\varepsilon|}\sum_x \delta^{5/2}|\Delta_\delta\phi(\delta x)| \tag{3.9c}$$

yielding

$$\lim_{\varepsilon\to 0}\sup_{\tau_0\le\tau\le\bar\tau}|R_1(\varepsilon,\tau,\phi)| = 0 \tag{3.9d}$$

and

$$R_2(\varepsilon,\tau,\phi) = 2\gamma^2|\ln\varepsilon| e^{-\alpha\tau|\ln\varepsilon|}\delta^{1/2}\sum_x \phi(\delta x)\sigma(x)\sigma(x_+)\sigma(x_-).$$

For each $\phi \in \mathscr{S}(\mathbb{R})$ we can take $C_2(\phi) < \infty$ in such a way that ($\mathbf{x}$ below denotes any set of six different sites in Λ_ε)

$$\begin{aligned}\mathbb{E}^\varepsilon_{\mu^\varepsilon}\big(R_2(\varepsilon,\tau,\phi)^2\big) &\le 4\gamma^4|\ln\varepsilon|^2 e^{-2\alpha\tau|\ln\varepsilon|}C_2(\phi)\Big(1 + \delta^{-1}\sup_{\mathbf{x}}|v_6(\mathbf{x},\tau|\ln\varepsilon|)|\Big)\\ &\le 4\gamma^4|\ln\varepsilon|^2 e^{-2\alpha\tau|\ln\varepsilon|}C_2(\phi) + C_3(\phi)4\gamma^4|\ln\varepsilon|^{5/2}e^{4\alpha\tau|\ln\varepsilon|}\varepsilon^2,\end{aligned}$$

where we have used (2.14). Therefore,

$$\lim_{\varepsilon\to 0}\sup_{\tau_0\le\tau\le\bar\tau}\mathbb{E}^\varepsilon_{\mu^\varepsilon}\big(\big|R_2(\varepsilon,\tau,\phi)^2\big|\big) = 0. \tag{3.9e}$$

Finally, we compute the contribution to γ_2^ε coming from L_G:

$$\begin{aligned}&|\ln\varepsilon|\big|L_G X_t^\varepsilon(\phi)^2 - 2X_t^\varepsilon(\phi)L_G X_t^\varepsilon(\phi)\big|\\ &\quad= 4|\ln\varepsilon| e^{-2\alpha\tau|\ln\varepsilon|}\delta\sum_x \phi^2(\delta x)|c(x,\sigma)|\\ &\quad\le 4|\ln\varepsilon| e^{-2\alpha\tau|\ln\varepsilon|}C_3(\phi),\end{aligned} \tag{3.9f}$$

where $C_3(\phi) < \infty$, for any $\phi \in \mathscr{S}(\mathbb{R})$.

From (3.8) and (3.9) we have reduced the proof of (3.7c) and (3.7d), and so of tightness on $D([\tau_0,\bar\tau],\mathscr{S}'(\mathbb{R}))$, to showing that for any $\phi \in \mathscr{S}(\mathbb{R})$,

$$\sup_{\tau_0\le\tau\le\bar\tau}\mathbb{E}^\varepsilon_{\mu^\varepsilon}\big(X_\tau^\varepsilon(\phi)^2\big) < \infty. \tag{3.10}$$

But

$$\mathbb{E}^{\varepsilon}_{\mu^{\varepsilon}}\left(X^{\varepsilon}_{\tau}(\phi)^2\right) \le e^{-2\alpha\tau|\ln\varepsilon|}\left[\delta\sum_x \phi^2(\delta x) + \delta^{-1}C_4(\phi)\sup_{x_1\neq x_2\in\mathbb{Z}_\varepsilon}|v_2(\mathbf{x},\tau|\ln\varepsilon|)|\right]$$
$$\le e^{-2\alpha\tau|\ln\varepsilon|}\left[C_5(\phi) + \varepsilon^{-1}|\ln\varepsilon|^{1/2}c_2 e^{2\alpha\tau|\ln\varepsilon|}\varepsilon|\ln\varepsilon|^{-1/2}\right]$$

according to (2.16), and (3.10) follows.

We have just proven that if $[\tau_0, \bar\tau]$ is fixed as above, any $\mathscr{P}$ which is a weak limit of some sequence $\mathscr{P}^{\varepsilon_n}$, $\varepsilon_n \to 0$, must concentrate on $C([\tau_0,\bar\tau],\mathscr{S}'(\mathbb{R}))$. For the identification of the possible limit points of $\mathscr{P}^{\varepsilon}$, we need to look at the limiting behavior of $\int_0^t \gamma_i^{\varepsilon}(s,\phi)\,ds$, for $i = 1,2$, and then use the characterization of $\mathscr{P} = \lim_n \mathscr{P}^{\varepsilon_n}$ as a solution of a martingale problem [cf. Holley and Stroock (1978) and Rebolledo (1980)]. Here this is particularly simple since from (3.8) and (3.9) it follows that under any such $\mathscr{P}$ the canonical process $X_\tau(\cdot)$ satisfies the following:

(a) For any $\phi \in \mathscr{S}(\mathbb{R})$,

$$M_{\tau_0,\tau}(\phi) := X_\tau(\phi) - X_{\tau_0}(\phi) - \int_{\tau_0}^{\tau} ds\, X_s\left(\tfrac{1}{2}\phi''\right)$$

is a martingale vanishing at τ_0.

(b) $(M_{\tau_0,\tau}(\phi))^2$ is also a martingale.

As is well known, this implies that $M_{\tau_0,\tau}$ vanishes for all τ, with probability 1. This says that with probability 1 (with respect to $\mathscr{P}$),

$$X_\tau(\phi) = X_{\tau_0}\left(\phi_{\tau-\tau_0}\right)$$

for $\tau \in [\tau_0, \bar\tau]$ and for ϕ_τ defined as in (3.3b). It remains to prove convergence of $X^{\varepsilon}_{\tau_0}(\phi)$ to the Gaussian distribution indicated by (3.5) with $\tau = \tau_0$. Given any sequence $\varepsilon_n \to 0$, we may take by diagonalization a subsequence ε_n' such that the measures $\mathscr{P}^{\varepsilon_n'}$ converge on each $D([2^{-m},\tau],\mathscr{S}'(\mathbb{R}))$, for all $m \ge 1$ such that $2^{-m} < \bar\tau$, and by the above argument the limit must satisfy (a) and (b) for any $\tau_0 = 2^{-m}$.

On the other hand, from (3.7) and the Cauchy–Schwarz inequality, we get for all $s_0 \ge 0$,

$$\mathbb{E}^{\varepsilon}_{\mu_\varepsilon}\left(\left|X^{\varepsilon}_{\tau_0}(\phi) - X^{\varepsilon}_{s_0/|\ln\varepsilon|}(\phi)\right|^2\right) \le 2\tau_0 c + 2\int_{s_0/|\ln\varepsilon|}^{\tau_0} d\tau'\,\gamma_2(\varepsilon,\tau',\phi) \tag{3.11}$$
$$\le \bar c\left(\tau_0 + e^{-2\alpha s_0}\right).$$

We will eventually let $\tau_0 \to 0$ and $s_0 \to \infty$ after $\varepsilon \to 0$. But first consider the fluctuation field $Y^{\varepsilon}_t(\phi)$ defined by (3.2a) so that $X^{\varepsilon}_{s_0/|\ln\varepsilon|} = Y^{\varepsilon}_{s_0}$ and let

$$\bar\gamma^{\varepsilon}_1(t,\phi) = L_\varepsilon Y^{\varepsilon}_t(\phi) - \alpha Y^{\varepsilon}_t(\phi),$$
$$\bar\gamma^{\varepsilon}_2(t,\phi) = L_\varepsilon Y^{\varepsilon}_t(\phi)^2 - 2Y^{\varepsilon}_t(\phi) L_\varepsilon Y^{\varepsilon}_t(\phi).$$

Proceeding as before, we may prove the tightness of the laws of Y^{ε} on $D([0,T],\mathscr{S}'(\mathbb{R}))$. Indeed, just recall that the $\bar\gamma^{\varepsilon}_i$ are obtained from the γ^{ε}_i by changing $\tau|\ln \varepsilon|$ to t, $X_\tau^\varepsilon(\phi)$ to $Y_t^\varepsilon(\phi)$ and dividing everything by $|\ln \varepsilon|$. The verification of the conditions analogous to (3.7c) and (3.7d) is very simple. Now $\mathbb{E}_{\mu^\varepsilon}^\varepsilon(Y_0^\varepsilon(\phi)^2) < \infty$ trivially. To identify the possible limit points, we must identify the limiting behavior of $\int_0^t \bar{\gamma}_i^\varepsilon(s, \phi)\, ds$ for $i = 1, 2$. Usually, the hardest part comes from the drift term $\bar{\gamma}_1^\varepsilon$, which generally is not a function of the fluctuation field itself. In order to prove that

$$Y_t(\phi) - \int_0^t Y_s(A_s \phi)\, ds$$

is a martingale (for any limiting process Y), we need to find operators A_s so that

$$\lim_{\varepsilon} E_{\mu^\varepsilon}^\varepsilon \left| \int_0^t ds \left[\bar{\gamma}_1^\varepsilon(s, \phi) - Y_s^\varepsilon(A_s \phi) \right] \right|^2 = 0.$$

This is the so-called Boltzmann–Gibbs principle, and for the models under study it has been proven in De Masi, Ferrari and Lebowitz (1986) for the standard fluctuation fields. In the present situation notice first that due to (3.8a) and (3.8c) we see that the contribution of L_0 to $\bar{\gamma}_i^\varepsilon$ vanishes for $i = 1, 2$. The reason for this is that we scaled space by δ^{-1} and not by ε^{-1}. The contribution to $\bar{\gamma}_1^\varepsilon(t, \phi)$ coming from L_G, by (3.9b) and (3.9d), is given by

$$2\gamma^2 e^{-\alpha t} \delta^{1/2} \sum_x \phi(\delta x) \sigma(x) \sigma(x_+) \sigma(x_-) + \bar{R}_1(t, \varepsilon, \phi),$$

where

$$\lim_{\varepsilon \to 0} \sup_{0 \le t \le T} \mathbb{E}_{\mu^\varepsilon}^\varepsilon \left(\bar{R}_1(t, \varepsilon, \phi)^2 \right) = 0.$$

Adapting the proof of the Boltzmann–Gibbs principle in De Masi, Ferrari and Lebowitz (1986), Theorem 4, we can see that

$$\lim_{\varepsilon \to 0} \mathbb{E}_{\mu^\varepsilon}^\varepsilon \left(\left[\int_0^t ds\, e^{-\alpha s} \delta^{1/2} \sum_x \phi(\delta x) \sigma(x, s) \sigma(x_+, s) \sigma(x_-, s) \right]^2 \right) = 0.$$

The contribution of L_G to $\bar{\gamma}_2^\varepsilon$ is, by (3.9f),

$$\lambda_2^\varepsilon(t, \phi) = 4 e^{-2\alpha t} \delta \sum_x \phi^2(\delta x) c(x, \sigma_t)$$

and, using Proposition 2.3,

$$\lim_{\varepsilon \to 0} \mathbb{E}_{\mu^\varepsilon}^\varepsilon \left(\left[\lambda_2^\varepsilon(t, \phi) - 4 e^{-2\alpha t} \delta \sum_x \phi^2(\delta x) \right]^2 \right) = 0.$$

Therefore, using again the Holley and Stroock theory, we have that if $\mathscr{Q}$ is a probability on $C([0, T], \mathscr{S}'(\mathbb{R}))$, which is a limit point of the laws of the $Y_t^\varepsilon(\phi)$, then under $\mathscr{Q}$ the following hold (denote below by $\mathscr{E}$ the expectation with respect to $\mathscr{Q}$):

(a) $X_0(\phi)$ is a Gaussian field with

$$\mathscr{E}(X_0(\phi)) = 0, \qquad \mathscr{E}(X_0(\phi)X_0(\psi)) = \int_{-\infty}^{\infty} dr\,\phi(r)\psi(r)$$

for all ϕ and ψ in $\mathscr{S}(\mathbb{R})$.

(b) $X_t(\phi)$ is a martingale and

$$\left\{X_t(\phi)^2 - 4\int_0^t ds\,e^{-2\alpha s}\int_{\mathbb{R}} dr\,\phi(r)^2\right\} \text{ is a martingale.}$$

But this uniquely determines the process so that we have convergence of the Y^ε to the Gaussian process with law $\mathscr{Q}$. This has zero average and the equal time covariance kernel is

$$C(r, r', t) = \left[1 + \frac{2}{\alpha}(1 - e^{-2\alpha t})\right]\delta(r - r').$$

From this convergence result and (3.11) the conclusion of Theorem 3.1 follows. □

4. The final stage of the escape. In this section we characterize the final stage of the escape; this will complete the proof of Theorem 2.1. We first introduce some notation: We fix $a > 0$ in such a way that αa is sufficiently small; in particular, we require that $\alpha a < 1/8$ and that $3a < a^*$, a^* being as in Proposition 2.2. For each $\varepsilon > 0$, we denote by

$$t_c = \frac{1}{2\alpha}|\ln \varepsilon|$$

the critical time for the escape (cf. Theorem 2.1), and we set

$$(4.1) \qquad t^* = t_c - 2t_a, \qquad t_a \equiv a|\ln \varepsilon|, \qquad t_f = t_c + |\ln \varepsilon|^{1/3}.$$

By the final stage of the escape, we mean the evolution of the system from time t^* until time t_f, when the escape will be completed. Since $t_f - t^* \le 3t_a \le t_{a^*}$ (at least for ε sufficiently small), we can use Proposition 2.2 to study this final stage of the escape. Call $\mathscr{F}(t^*)$ the σ-algebra generated by $(\sigma^\varepsilon(t))_{t \le t^*}$ and σ^* the configuration at time t^*. Denote by $m_\varepsilon(r, t; \delta_{\sigma^*})$ the solution to (2.4) with potential given by (2.6) and initial condition periodic with period $|\ln \varepsilon|$ and such that

$$m_\varepsilon(r, 0; \delta_{\sigma^*}) = \sigma^*([\varepsilon^{-1}r]) \quad \text{for } 0 \le r < |\ln \varepsilon|.$$

Then the law of the process at time t_f conditioned on $\mathscr{F}(t^*)$ is approximated, in the sense of Theorem 2.1, by a product measure with averages $m_\varepsilon(\varepsilon x, t_f - t^*; \delta_{\sigma^*})$, $x \in \Lambda_\varepsilon$. Since the analogous property also holds for all $t^* + \varepsilon^{\beta^*} \le t \le t_f$, the final stage of the escape is determined by the behavior of the function $m_\varepsilon(\varepsilon x, t - t^*; \delta_{\sigma^*})$. The problem is then reduced to an analysis of (2.4) when the initial conditions are the typical configurations at time t^*. We first outline the main steps, then we give the proofs.

LEMMA 4.1. *For any positive b and u there is c so that for all ε,*

$$\mathbb{P}^{\varepsilon}_{\mu^\varepsilon}\big(\sigma(\cdot, t^*) \equiv \sigma^*: \big\| m_\varepsilon\big(\cdot, \varepsilon^{1/4}; \delta_{\sigma^*}\big)\big\| \le \varepsilon^{2\alpha a - b}\big) \ge 1 - c\varepsilon^u, \tag{4.2}$$

where $\|\cdot\|$ denotes the sup norm with respect to the spatial variable.

Lemma 4.1 will be proven by showing that until time $\varepsilon^{1/4}$ one can neglect the reactive term $F(m)$ in (2.4), with an error which becomes negligible when $\varepsilon \to 0$. As a consequence, m_ε is essentially the solution of a purely diffusive equation; hence it is a Gaussian average over σ^*. By (3.2) and (3.5) the typical size of the magnetization at time t^* is $\varepsilon^{2\alpha a}|\ln \varepsilon|^{-1/4}$. Then, using (2.14) and the Chebyshev inequality, we will take advantage of the factor ε^{-b} in (4.2) and prove the statement.

Notice that the bound in (4.2) must not be sharp. In fact, if $m_\varepsilon(r, \varepsilon^{1/4}; \delta_{\sigma^*}) \approx \varepsilon^{2\alpha a - b}$, then m_ε would already be finite at the end of the time interval $2t_a$, because of the exponential growth, and the escape would occur before t_c. We therefore need to improve the bound in (4.2). We first establish the following result.

LEMMA 4.2. *If $b < \alpha a$ and $\|m_\varepsilon(\cdot, \varepsilon^{1/4}; \delta_{\sigma^*})\| \le \varepsilon^{2\alpha a - b}$, then there is c so that for all ε,*

$$\big\| m_\varepsilon(\cdot, t_a; \delta_{\sigma^*})\big\| \le c\varepsilon^{\alpha a - b}, \tag{4.3a}$$

$$\big\| m_\varepsilon(\cdot, t_a; \delta_{\sigma^*}) - l_\varepsilon(\cdot, t_a; \delta_{\sigma^*})\big\| \le c\varepsilon^{3(\alpha a - b)}, \tag{4.3b}$$

where l_ε solves the linearization of (2.4), *namely,*

$$l_\varepsilon(r, t; \delta_{\sigma^*}) = \int dr'\, e^{\alpha t} G_t(r - r')\sigma^*([\varepsilon^{-1} r']) \tag{4.4}$$

and $G_t(r) = \exp\{-r^2/2t\}(2\pi t)^{-1/2}$ and $\sigma^*(x)$, $x \in \mathbb{Z}_\varepsilon$, is periodic with period $\varepsilon^{-1}|\ln \varepsilon|$.

We therefore know from Lemmas 4.1 and 4.2 that $m_\varepsilon(r, t_a; \delta_{\sigma^*})$ depends on σ^* approximately as the right-hand side of (4.4), with large probability. The important point is that this expression varies slowly so that it will be possible to introduce and prove bounds on its sup norm. For this purpose we recall the following classical inequality, easily proven using the Cauchy–Schwarz inequality, which expresses the variation of a function in terms of its H_2 norm.

LEMMA 4.3. *For any r and $L > 0$, let*

$$N_\varepsilon(r, L, t) = \sup_{|r - r'| \le L\sqrt{|\ln \varepsilon|}} \big| l_\varepsilon(r, t; \delta_{\sigma^*}) - l_\varepsilon(r', t; \delta_{\sigma^*})\big|^2 \tag{4.5}$$

(*we omit the dependence of* N_ε *on* σ^* *for notational simplicity*). *Then*

$$N_\varepsilon(r, L, t) \le 2L\sqrt{|\ln \varepsilon|} \int_{r-L\sqrt{|\ln \varepsilon|}}^{r+L\sqrt{|\ln \varepsilon|}} dr' \left(\frac{\partial}{\partial r'} l_\varepsilon(r', t; \delta_{\sigma^*}) \right)^2. \tag{4.6}$$

The right-hand side of (4.6) is a quadratic expression in σ^*. Using again the Chebyshev inequality, (2.14) and (2.16), we shall prove the following result.

LEMMA 4.4. *For any* $\eta < 1/8$ *and* $u > 0$ *there is* c *so that for any* r *and* ε,

$$\mathbb{P}^\varepsilon_{\mu^\varepsilon}\left(N_\varepsilon(r, 1/2, t_a) > \left\{ \frac{\varepsilon^{\alpha a}}{|\ln \varepsilon|^\eta} \right\}^2 \right) \le c|\ln \varepsilon|^{-u}. \tag{4.7}$$

Furthermore, for any $\zeta > 0$ *and* $d > 0$ *there is* $L > 0$ *so that for any* r *and* ε,

$$\mathbb{P}^\varepsilon_{\mu^\varepsilon}\left(N_\varepsilon(r, L, 2t_a) > d\left\{ \frac{1}{|\ln \varepsilon|^{1/4}} \right\}^2 \right) \le \zeta. \tag{4.8}$$

By using the estimate (2.14) and the Chebyshev inequality to bound the right-hand side of (4.4), we will prove the following result.

LEMMA 4.5. *For any* $\eta < 1/8$ *and* $u > 0$ *there is* c *so that for any* r *and* ε,

$$\mathbb{P}^\varepsilon_{\mu^\varepsilon}\left(|l_\varepsilon(r, t_a; \delta_{\sigma^*})| > \frac{\varepsilon^{\alpha a}}{|\ln \varepsilon|^\eta} \right) \le c|\ln \varepsilon|^{-u}, \tag{4.9}$$

where we recall $\sigma^* = \sigma(t^*)$ *with* t^* *defined by* (4.1).

Combing (4.7) and (4.9), we will obtain the following result.

LEMMA 4.6. *For any* $\eta < 1/8$ *and* $u > 0$ *there is* c *so that for all* $\varepsilon \in (0, 1)$,

$$\mathbb{P}^\varepsilon_{\mu^\varepsilon}\left(\| l_\varepsilon(\cdot, t_a; \delta_{\sigma^*}) \| > \frac{\varepsilon^{\alpha a}}{|\ln \varepsilon|^\eta} \right) \le c|\ln \varepsilon|^{-u}. \tag{4.10}$$

We will also show the following result.

LEMMA 4.7. *Assume that* (4.3b) *holds with* $b > 0$ *and* $3b < 2\alpha a$. *Assume also that*

$$\| l_\varepsilon(\cdot, t_a; \delta_{\sigma^*}) \| \le \frac{\varepsilon^{\alpha a}}{|\ln \varepsilon|^\eta}.$$

Then there is c *so that*

$$\begin{gathered} \| m_\varepsilon(\cdot, 2t_a; \delta_{\sigma^*}) \| \le c|\ln \varepsilon|^{-\eta}, \\ \| m_\varepsilon(\cdot, 2t_a; \delta_{\sigma^*}) - l_\varepsilon(\cdot, 2t_a; \delta_{\sigma^*}) \| \le c|\ln \varepsilon|^{-3\eta}. \end{gathered} \tag{4.11}$$

This together with (4.10) and Lemmas 4.1 and 4.2 will prove (2.9a), namely that $\mu^{\varepsilon}_{t_c}$ converges to the Bernoulli measure with zero average, and hence that the escape has not yet occurred at time t_c.

To prove that $m_\varepsilon(\cdot, t_f - t^*; \delta_{\sigma^*})$ is not infinitesimal, we need lower bounds on $|m_\varepsilon|$, hence on $|l_\varepsilon|$, at time $2t_a$. We first give a definition.

DEFINITION. Given two functions ψ and ψ' on $\mathbb{R}$ and $L > 0$, we write

$$\psi =_L \psi' \quad \text{if } \psi(r) = \psi'(r) \text{ for all } |r| \le L\sqrt{|\ln \varepsilon|}, \tag{4.12a}$$

$$\psi \ge_L \psi' \quad \text{if } \psi(r) \ge \psi'(r) \text{ for all } |r| \le L\sqrt{|\ln \varepsilon|}. \tag{4.12b}$$

LEMMA 4.8. *Denote by $m(r, t; \phi)$ the solution to* (2.4)–(2.6) *with initial condition ϕ. Then for any $L > 0$,*

$$\lim_{\varepsilon \to 0} \sup_{\substack{\psi =_L \psi' \\ \|\psi\|, \|\psi'\| \le 1}} \sup_{|r| \le L/2} \Big| m\big(r|\ln \varepsilon|, |\ln \varepsilon|^{1/3}; \psi\big) - m\big(r|\ln \varepsilon|, |\ln \varepsilon|^{1/3}; \psi'\big) \Big| = 0. \tag{4.13}$$

We fix n distinct sites, $x_1, \dots, x_n$, as in (2.9). Then, because of Lemma 4.8, as ε tends to 0, $m_\varepsilon(\varepsilon x_i, t_f - t^*; \delta_{\sigma^*})$ will be determined by $m_\varepsilon(r, 2t_a; \delta_{\sigma^*})$ with r such that $\{|r - \varepsilon x_i| \le L\sqrt{|\ln \varepsilon|}\}$, for any given $L > 0$, and hence, as we shall see, by $l_\varepsilon(r, 2t_a; \delta_{\sigma^*})$, with r varying in the same interval. On the other hand, by (4.4), $l_\varepsilon(r, 2t_a; \delta_{\sigma^*})$ can be expressed as one of the magnetization fields studied in the previous section and this will be used in the proof of the following lemma.

LEMMA 4.9. *Given any $\zeta > 0$, there is $d > 0$ (sufficiently small) so that for any r,*

$$\lim_{\varepsilon \to 0} \mathbb{P}^{\varepsilon}_{\mu^\varepsilon}\left(|l_\varepsilon(r, 2t_a; \delta_{\sigma^*})| \le d \frac{1}{|\ln \varepsilon|^{1/4}} \right) \le \zeta. \tag{4.14}$$

By (4.5) and (4.8), given any positive ζ and d', there is $L > 0$ so that for any r and ε,

$$\sup_{|r - r'| \le L\sqrt{|\ln \varepsilon|}} |l_\varepsilon(r, 2t_a; \delta_{\sigma^*}) - l_\varepsilon(r', 2t_a; \delta_{\sigma^*})| \le d'|\ln \varepsilon|^{-1/4}, \tag{4.15}$$

with probability larger than $1 - \zeta$.

From (4.14) and (4.15) it follows that: *given any $\zeta > 0$, $n \ge 1$ and any n-tuple $x_1, \dots, x_n$ of distinct sites, there are L, d and ε_0 so that for all $0 < \varepsilon \le \varepsilon_0$ the probability of the set*

$$\{|l_\varepsilon(r, 2t_a; \delta_{\sigma^*})| > d|\ln \varepsilon|^{-1/4} \text{ for all } r \text{ in all the intervals of length } L|\ln \varepsilon|^{1/2} \text{ centered at the points } \varepsilon x_i\}$$

is larger than $1 - \zeta$.

By (4.11), choosing $3\eta > 1/4$ (the only constraint on η, so far, was $\eta < 1/8$), we deduce that $|m_\varepsilon(r, 2t_a; \delta_{\sigma^*})|$ also satisfies the previous lower bound. Then by Lemma 4.8 it will be easy to prove that $m_\varepsilon(\varepsilon x_i, t_f - t^*; \delta_{\sigma^*})$ becomes close (for ε small) to $\pm m^*$, the sign being the same as that of $l_\varepsilon(\varepsilon x_i, 2t_a; \delta_{\sigma^*})$. By using Theorem 3.1 we shall also determine the limiting distribution of these signs, and will be able, in this way, to conclude the proof of Theorem 2.1. The proof of Theorem 2.1 appears after those of Lemmas 4.1–4.9, which we present next.

In the sequel we will often use the following classical property of our reaction diffusion equation (2.4) [see, e.g., Fife (1979)].

Monotonicity properties of (2.4). Let $m(r,t)$ and $\tilde{m}(r,t)$ be two solutions of (2.4). Suppose that $m(r,0) \geq \tilde{m}(r,0)$ for all r. Then $m(r,t) \geq \tilde{m}(r,t)$ for all r and $t \geq 0$. The same property holds if the equation is defined in an interval with periodic boundary conditions.

As a consequence, $|m(r,t)| \leq 1$ for all r and $t \geq 0$ if $|m(r,0)| \leq 1$ for all r, by the choice (2.6) for the reactive potential.

PROOF OF LEMMA 4.1. By the monotonicity properties of (2.4), $|m_\varepsilon(r,t; \delta_{\sigma^*})| \leq 1$. Hence there is a constant c such that

$$|m_\varepsilon(r, \varepsilon^{1/4}; \delta_{\sigma^*}) - l_\varepsilon(r, \varepsilon^{1/4}; \delta_{\sigma^*})| \leq c\varepsilon^{1/4}. \tag{4.16}$$

By the arbitrariness of $b > 0$ and because $2\alpha a < 1/4$, it will be enough to prove (4.2) with m_ε replaced by l_ε. To reduce the sup in (4.2) to a sup over a countable set, we notice that

$$\frac{\partial}{\partial r} l_\varepsilon(r,t;\delta_{\sigma^*}) = -\frac{e^{\alpha t}}{\sqrt{t}} \int dr'\, K_t(r-r')\sigma^*([\varepsilon^{-1}r']), \tag{4.17a}$$

where

$$K_t(r) = \frac{r}{\sqrt{t}} \frac{e^{-r^2/2t}}{\sqrt{2\pi t}}. \tag{4.17b}$$

Then

$$\left|\frac{\partial}{\partial r} l_\varepsilon(r,t;\delta_{\sigma^*})\right| \leq \frac{c}{\sqrt{t}}, \tag{4.17c}$$

and it follows that for any $\delta > 0$,

$$\begin{aligned} \|l_\varepsilon(\cdot, \varepsilon^{1/4}; \delta_{\sigma^*})\| &\equiv \sup_{0 \leq r \leq |\ln \varepsilon|} |l_\varepsilon(r, \varepsilon^{1/4}; \delta_{\sigma^*})| \\ &\leq \sup_{0 \leq r \leq |\ln \varepsilon| r \in \varepsilon^\delta \mathbb{Z}} |l_\varepsilon(r, \varepsilon^{1/4}; \delta_{\sigma^*})| + \varepsilon^\delta \frac{c}{\sqrt{\varepsilon^{1/4}}}. \end{aligned}$$

Therefore, Lemma 4.1 is reduced to the proof that for any positive b and u

there is c so that

$$\text{(4.18a)}\quad \mathbb{P}^{\varepsilon}_{\mu^\varepsilon}\Bigg(\sigma(\cdot,t^*)\equiv\sigma^*:\ \sup_{\substack{0\le r\le|\ln\varepsilon|\\ r\in\varepsilon^\delta\mathbb{Z}}}\big|l_\varepsilon(r,\varepsilon^{1/4};\delta_{\sigma^*})\big|\le\varepsilon^{2\alpha a-b}\Bigg)\ge 1-c\varepsilon^u,$$

where

$$\text{(4.18b)}\qquad \delta>\tfrac{1}{8}+2\alpha a-b,$$

so that $\varepsilon^{\delta-1/8}\le\varepsilon^{2\alpha a-b}$. On the other hand, the left-hand side of (4.18a) is bounded by

$$\text{(4.18c)}\quad \varepsilon^{-\delta}|\ln\varepsilon|\sup_r\mathbb{P}^{\varepsilon}_{\mu^\varepsilon}\big(\sigma(\cdot,t^*)\equiv\sigma^*:\ \big|l_\varepsilon(r,\varepsilon^{1/4};\delta_{\sigma^*})\big|>\varepsilon^{2\alpha a-b}\big).$$

By using the Chebyshev inequality with power $2n$, we get that the probability in (4.18c) is bounded by

$$\text{(4.19)}\quad \varepsilon^{-(2\alpha a-b)2n}e^{\alpha 2n\varepsilon^{1/4}}\int dr_1\dots dr_{2n}\prod_{i=1}^{2n}G_{\varepsilon^{1/4}}(r-r_i)\mathbb{E}^{\varepsilon}_{\mu^\varepsilon}\Bigg(\prod_{i=1}^{2n}\sigma^*\big([\varepsilon^{-1}r_i]\big)\Bigg).$$

We shall use Proposition 2.3 to estimate the expectation in (4.19). For this we must split the region of integration according to the number of different sites in $\mathbb{Z}_\varepsilon$ among $[\varepsilon^{-1}r_i]$, $i=1,\dots,2n$. Thus we say that r_i is isolated if $[\varepsilon^{-1}r_i]\neq[\varepsilon^{-1}r_j]\ (\mathrm{mod}\ |\ln\varepsilon|)$ for all $j\neq i$, and set

$$A_k=\big\{(r_1,\dots,r_{2n}):\text{there are exactly } k \text{ isolated } r_j \text{ among } r_1,\dots,r_{2n}\big\}$$

for $k=0,\dots,2n$, $\mathbf{1}_{A_k}$ being its characteristic function. Then, by (2.14) and the definition of t^* [cf. (4.1)], for any k there is c so that

$$\text{(4.20)}\quad \mathbf{1}_{A_k}(r_1,\dots,r_{2n})\Bigg|\mathbb{E}^{\varepsilon}_{\mu^\varepsilon}\Bigg(\prod_{i=1}^{2n}\sigma^*\big([\varepsilon^{-1}r_i]\big)\Bigg)\Bigg|\le c\Gamma_{t^*}(\varepsilon)^k\le c\varepsilon^{2\alpha ak}.$$

On the other hand, since $\int G_t(r)\,dr=1$ and $G_t(r)\le 1/\sqrt{2\pi t}$, we easily see that for any k there is a c so that

$$\text{(4.21)}\quad \int dr_1\dots dr_{2n}\,\mathbf{1}_{A_k}(r_1,\dots,r_{2n})\prod_{i=1}^{2n}G_{\varepsilon^{1/4}}(r-r_i)\le c\Bigg(\frac{\varepsilon}{\sqrt{\varepsilon^{1/4}}}\Bigg)^{(2n-k)/2}.$$

Going back to (4.19) and recalling that $(1-1/8)\ge 2(2\alpha a)$, by choosing n sufficiently large we prove that the expression in (4.18c) vanishes when $\varepsilon\to0$ as fast as any given power of ε; from this the lemma follows. □

PROOF OF LEMMA 4.2. We start by proving (4.3a). By the monotonicity properties of (2.4), $|m_\varepsilon(r,t;\delta_{\sigma^*})|$ is bounded for $t\ge\varepsilon^{1/4}$ by $z(t-\varepsilon^{1/4})$ where $z(t)$ solves

$$\text{(4.22)}\qquad \frac{dz}{dt}=\alpha z-\beta z^3,\qquad z(0)=\varepsilon^{2\alpha a-b}.$$

Hence

$$|m_\varepsilon(r,t;\delta_{\sigma^*})| \le z(t) \le e^{\alpha t} z(0) \tag{4.23}$$

and this, for $t = t_a$, proves (4.3a).

We notice that from (2.4) it follows that

$$\begin{aligned} m_\varepsilon(r,t;\delta_{\sigma^*}) &= \int dr'\, e^{\alpha t} G_t(r-r')\sigma^*([\varepsilon^{-1}r']) \\ &\quad + \int_0^t ds \int dr'\, e^{\alpha(t-s)} G_{t-s}(r-r')\big(-\beta m_\varepsilon(r',s;\delta_{\sigma^*})^3\big). \end{aligned} \tag{4.24}$$

Let

$$h_\varepsilon(t) = \| m_\varepsilon(\cdot,t;\delta_{\sigma^*}) - l_\varepsilon(\cdot,t;\delta_{\sigma^*})\|. \tag{4.25}$$

Then, using (4.23) and (4.4), we have

$$h_\varepsilon(t) \le e^{\alpha(t-\varepsilon^{1/4})} h_\varepsilon(\varepsilon^{1/4}) + c\int_{\varepsilon^{1/4}}^t ds\, e^{\alpha(t-s)}\big[e^{\alpha s}\varepsilon^{2\alpha a - b}\big]^3. \tag{4.26}$$

Since, by the choice of a and b,

$$h_\varepsilon(\varepsilon^{1/4}) \le c\varepsilon^{1/4} \le c e^{-\alpha t_a} \varepsilon^{3(\alpha a - b)},$$

we then get (4.3b) and complete the proof of the lemma. □

In Section 5 we shall need the following corollary of the proofs of Lemmas 4.1 and 4.2.

Lemma 4.10. *For any $0 < \tau_0 < \tau < 1/(2\alpha)$ and any b and u positive, there is c so that, denoting by $\tilde\sigma$ the configuration at time $\tau_0|\ln \varepsilon|$,*

$$\mathbb{P}^\varepsilon_{\mu^\varepsilon}\big(\| m_\varepsilon(\cdot,t;\delta_{\tilde\sigma})\| \le e^{\alpha t} S_\varepsilon,\ \varepsilon^{1/4} \le t \le (\tau-\tau_0)|\ln\varepsilon|\big) \ge 1 - c\varepsilon^u,$$

where

$$S_\varepsilon = \varepsilon^{-b}\max\{e^{\alpha\tau_0|\ln\varepsilon|}\varepsilon^{1/2}, \varepsilon^{1/4}\}.$$

Proof. If $1/(4\alpha) < \tau_0 < 1/(2\alpha)$, we may write $\tau_0 = 1/(2\alpha) - 2a$ with $\alpha a < 1/8$ and the estimate follows at once from the proofs of Lemmas 4.1 and 4.2, since $S_\varepsilon = \varepsilon^{-b+2\alpha a}$ in this case. Otherwise, $S_\varepsilon = \varepsilon^{1/4-b}$ and the estimate follows from (4.16), (4.19) and (4.21) for $t = \varepsilon^{1/4}$ and from (4.23) for the other values of t. □

Proof of Lemma 4.4. From (4.17a) we have

$$\left|\frac{\partial}{\partial r} l_\varepsilon(r,t;\delta_{\sigma^*})\right| \le \frac{e^{\alpha t}}{\sqrt{t}} \int dr'\, K_t(|r-r'|)\sigma^*([\varepsilon^{-1}r']), \tag{4.27}$$

where, according to (4.17b), $0 \le K_t(|r|)$ and there is a constant c so that

$$\sup_r K_t(|r|) \le \frac{c}{\sqrt{t}} \int K_t(|r|)\, dr \le c. \tag{4.28}$$

To prove (4.7), we use the Chebyshev inequality with power n. By (4.6) and (4.27) we get

$$\begin{aligned}
&\mathbb{P}^{\varepsilon}_{\mu^{\varepsilon}}\left(N_{\varepsilon}(r,L,t_a) > \left\{\frac{\varepsilon^{\alpha a}}{|\ln\varepsilon|^{\eta}}\right\}^2\right)\\
&\quad\le \left\{\frac{\varepsilon^{\alpha a}}{|\ln\varepsilon|^{\eta}}\right\}^{-2n}\left[2L\sqrt{|\ln\varepsilon|}\right]^n\int_{r-L|\ln\varepsilon|}^{r+L|\ln\varepsilon|}dr_1\dots\int_{r-L|\ln\varepsilon|}^{r+L|\ln\varepsilon|}dr_n\,e^{2\alpha n t_a}\\
&\qquad\times\left[\frac{c}{\sqrt{t_a}}\right]^{2n}\prod_{i=1}^{n}\int dr_i'\,dr_i''\,K_{t_a}(|r_i-r_i'|)K_{t_a}(|r_i-r_i''|)\\
(4.29)\qquad&\qquad\qquad\times\mathbb{E}^{\varepsilon}_{\mu^{\varepsilon}}\left(\prod_{i=1}^{n}\sigma^*\left([\varepsilon^{-1}r_i']\right)\sigma^*\left([\varepsilon^{-1}r_i'']\right)\right)\\
&\quad\le c\varepsilon^{-2n\alpha a}|\ln\varepsilon|^{2n\eta}\left(\sqrt{|\ln\varepsilon|}\right)^{2n}\varepsilon^{-2n\alpha a}\left(\frac{1}{\sqrt{|\ln\varepsilon|}}\right)^{2n}\\
&\qquad\times\sup_{r_1,\dots,r_n}\prod_{i=1}^{n}\int dr_i'\,dr_i''\,K_{t_a}(|r_i-r_i'|)K_{t_a}(|r_i-r_i''|)\\
&\qquad\qquad\times\mathbb{E}^{\varepsilon}_{\mu^{\varepsilon}}\left(\prod_{i=1}^{n}\sigma^*\left([\varepsilon^{-1}r_i']\right)\sigma^*\left([\varepsilon^{-1}r_i'']\right)\right).
\end{aligned}$$

By (4.28), using the same argument as in the proof of Lemma 4.1, we conclude that the right-hand side of (4.29) is bounded by

$$c\varepsilon^{-4n\alpha a}|\ln\varepsilon|^{2n\eta}\sum_{k=0}^{2n}\left(\frac{\varepsilon}{\sqrt{|\ln\varepsilon|}}\right)^{(2n-k)/2}\left(\frac{\varepsilon^{2n\alpha a}}{|\ln\varepsilon|^{\eta_0}}\right)^k.$$

Hence, choosing η_0 in (2.15) larger than η in (4.29), we get (4.7).

Proof of (4.8). The proof of (4.8) is similar: We again use (4.29) with t_a replaced by $2t_a$, $n=1$ and setting $\eta\equiv 1/4$. We denote by $\mathbf{1}(\cdot)$ the characteristic function of $(\cdot)$. Then the left-hand side of (4.8) is bounded by

$$\begin{aligned}
&d^{-1}\left\{\frac{1}{|\ln\varepsilon|^{1/4}}\right\}^{-2}\left[2L\sqrt{|\ln\varepsilon|}\right]\int_{r-|\ln\varepsilon|^{1/2}L}^{r+|\ln\varepsilon|^{1/2}L}dr_1\,e^{4\alpha t_a}\\
&\qquad\times\left[\frac{c}{\sqrt{t_a}}\right]^2\int dr'\,dr''\,K_{2t_a}(|r_1-r'|)K_{2t_a}(|r_1-r''|)\\
(4.30)\qquad&\qquad\qquad\times\left\{\frac{e^{2\alpha t^*}\varepsilon}{|\ln\varepsilon|^{1/2}}\mathbf{1}(|r'-r''|>\varepsilon)+\mathbf{1}(|r'-r''|\le\varepsilon)\right\}\\
&\quad\le cd^{-1}|\ln\varepsilon|^{1/2}L^2|\ln\varepsilon|\left(\frac{1}{\sqrt{|\ln\varepsilon|}}\right)^2\varepsilon^{-4\alpha a}\left[\frac{\varepsilon^{4\alpha a}}{\sqrt{|\ln\varepsilon|}}+\frac{\varepsilon}{\sqrt{|\ln\varepsilon|}}\right],
\end{aligned}$$

where we have used (2.16) and (4.1). Given $d > 0$, by choosing L small enough we can make the expression in (4.30) smaller than any given $\zeta > 0$. Therefore, the lemma is proven. □

PROOF OF LEMMA 4.5. We again use the Chebyshev inequality with power $2n$. The left-hand side of (4.9) is then bounded by

$$\left\{\frac{\varepsilon^{\alpha a}}{|\ln \varepsilon|^{\eta}}\right\}^{-2n} \int dr_1 \dots \int dr_{2n}\, e^{2\alpha n t_a} \prod_{i=1}^{2n} \int dr_i'\, dr_i'' G_{t_a}(|r_i - r_i'|)$$

$$\times \mathbb{E}^{\varepsilon}_{\mu^\varepsilon}\left(\prod_{i=1}^{2n} \sigma^*([\varepsilon^{-1} r_i'])\sigma^*([\varepsilon^{-1} r_i''])\right)$$

$$\le c\varepsilon^{-2n\alpha a} |\ln \varepsilon|^{2n\eta} \varepsilon^{-2n\alpha a} \sum_{k=0}^{2n} \left(\frac{\varepsilon}{\sqrt{|\ln \varepsilon|}}\right)^{(2n-k)/2} \left(\frac{\varepsilon^{2\alpha a}}{|\ln \varepsilon|^{\eta_0}}\right)^{k},$$

where we have used the bound on G_{t_a} which is equal to the one in (4.28), and we have also used the same estimate as in the proof of Lemma 4.1. By choosing n sufficiently large and $\eta_0 > \eta$, we then see that the last expression can be made smaller than $c|\ln \varepsilon|^{-u}$, for any given u. The lemma is therefore proven. □

PROOF OF LEMMA 4.6. We have (i below denotes an integer)

$$\|l_\varepsilon(\cdot, t_a; \delta_{\sigma^*})\| \le \sup_{0 \le i \le \sqrt{|\ln \varepsilon|}} \left| l_\varepsilon\left(i\sqrt{|\ln \varepsilon|}, t_a; \delta_{\sigma^*}\right)\right|$$

$$+ \sup_{0 \le i \le \sqrt{|\ln \varepsilon|}} \left[N_\varepsilon\left(i\sqrt{|\ln \varepsilon|}, \frac{1}{2}, t_a\right)\right]^{1/2},$$

so that

$$\mathbb{P}^{\varepsilon}_{\mu^\varepsilon}\left(\|l_\varepsilon(\cdot, t_a; \delta_{\sigma^*})\| > \frac{\varepsilon^{\alpha a}}{|\ln \varepsilon|^{\eta}}\right) \le \sqrt{|\ln \varepsilon|}\left\{\sup_r \mathbb{P}^{\varepsilon}_{\mu^\varepsilon}\left(|l_\varepsilon(r, t_a; \delta_{\sigma^*})| > \frac{1}{2}\frac{\varepsilon^{\alpha a}}{|\ln \varepsilon|^{\eta}}\right)\right.$$

$$\left. + \sup_r \mathbb{P}^{\varepsilon}_{\mu^\varepsilon}\left(\left[N_\varepsilon\left(r, \frac{1}{2}, t_a\right)\right]^{1/2} > \frac{1}{2}\frac{\varepsilon^{\alpha a}}{|\ln \varepsilon|^{\eta}}\right)\right\}.$$

Hence by (4.7) and (4.9) the lemma follows. □

PROOF OF LEMMA 4.7. The hypothesis and (4.3b) imply that there is a c so that

$$\|m_\varepsilon(\cdot, t_a; \delta_{\sigma^*})\| \le c\frac{\varepsilon^{\alpha a}}{|\ln \varepsilon|^{\eta}}$$

and so, by monotonicity, we then have, as in the proof of Lemma 4.2, for $t > t_a$,

$$\| m_\varepsilon(\cdot, t; \delta_{\sigma^*}) \| \le c e^{\alpha(t-t_a)} \frac{\varepsilon^{\alpha a}}{|\ln \varepsilon|^\eta},$$

which shows the first inequality in (4.11). To prove the second inequality in (4.11), we observe that, with the notation introduced in (4.25),

$$h_\varepsilon(t) \le e^{\alpha(t-t_a)} h_\varepsilon(t_a) + c \int_{t_a}^t ds\, e^{\alpha(t-s)} \left\{ \frac{e^{\alpha s} \varepsilon^{2\alpha a}}{|\ln \varepsilon|^\eta} \right\}^3.$$

Now, using (4.3), the lemma easily follows. □

PROOF OF LEMMA 4.8. We set

$$D(r,t) \equiv |m(r,t;\psi) - m(r,t;\psi')|. \tag{4.31}$$

Hence $D(r,0) = 0$ for $|r| \le L\sqrt{|\ln \varepsilon|}$ since $\psi' =_L \psi$. Since $|m| \le 1$ we have that there is c so that

$$\begin{aligned} D(r,t) &\le 2 \int dr'\, G_t(r-r') 1\big(|r'| > L\sqrt{|\ln \varepsilon|}\big) \\ &\quad + c \int_0^t ds \int dr'\, G_{t-s}(r-r') D(r',s). \end{aligned} \tag{4.32}$$

Iterating (4.32) and using the semigroup property of $G_\cdot(r)$, we get, for $|r| \le L/2\sqrt{|\ln \varepsilon|}$,

$$D(r,t) \le 2e^{ct} \int dr'\, G_t(r-r') 1\big(|r'| > L\sqrt{|\ln \varepsilon|}\big) \le \bar{c} e^{ct} \frac{e^{-(L/2\sqrt{|\ln \varepsilon|})^2/2t}}{\sqrt{2\pi t}}.$$

The right-hand side vanishes for $t = |\ln \varepsilon|^{1/3}$ and when $\varepsilon \to 0$. The proof of the lemma is thus concluded. □

PROOF OF LEMMA 4.9. The law of $\sigma(\cdot, t)$ under $\mathbb{P}^\varepsilon_{\mu^\varepsilon}$ is obviously shift invariant and so it suffices to consider $r = 0$. In this case, we define

$$\phi(r) = \frac{1}{\sqrt{4\pi a}} e^{-r^2/4a}$$

and notice that

$$\begin{aligned} &|\ln \varepsilon|^{1/4} l_\varepsilon(0, 2t_a, \delta_{\sigma^\varepsilon(\cdot, t^*)}) \\ &= |\ln \varepsilon|^{1/4} e^{2\alpha t_a} \int \frac{e^{-r^2/4t_a}}{\sqrt{4\pi t_a}} \sigma^\varepsilon([\varepsilon^{-1} r], t^*)\, dr \\ &= \frac{\varepsilon^{-2\alpha a}}{|\ln \varepsilon|^{1/4}} \int \frac{\exp(-r^2/(4a|\ln \varepsilon|))}{\sqrt{4\pi a}} \sigma^\varepsilon([\varepsilon^{-1} r], t^*)\, dr \\ &= \varepsilon \frac{\varepsilon^{-2\alpha a}}{|\ln \varepsilon|^{1/4}} \sum_x \left(\frac{1}{\varepsilon} \int_{\varepsilon x}^{\varepsilon x + \varepsilon} \frac{\exp(-r^2/(4a|\ln \varepsilon|))}{\sqrt{4\pi a}}\, dr \right) \sigma^\varepsilon(x, t^*), \end{aligned}$$

so that we easily see that

$$(4.33)\qquad \left| |\ln \varepsilon|^{1/4} l_\varepsilon(0, 2t_a, \delta_{\sigma^\varepsilon(\cdot, t^*)}) - Y^\varepsilon_{t^*}(\phi) \right| \le c \frac{\varepsilon^{1-2\alpha a}}{|\ln \varepsilon|^{1/4}}.$$

In particular, (4.14) with $r = 0$ follows from Theorem 3.1. □

PROOF OF (2.9a). For $\tau < 1/(2\alpha)$, (2.9a) follows at once from (2.14). To consider $t = t_c$ and $t = t_f$, we write $\mu^\varepsilon_t(\prod_{i=1}^n \sigma(x_i))$ for $t > t^*$ by conditioning on the process up to time t^* and using the Markov property, that is,

$$(4.34)\qquad \mu^\varepsilon_t\left(\prod_{i=1}^n \sigma(x_i)\right) = \mathbb{E}^\varepsilon_{\mu^\varepsilon}\left(\mathbb{E}^\varepsilon_{\delta_{\sigma(\cdot,t^*)}}\left(\prod_{i=1}^n \sigma(x_i, t - t^*)\right)\right).$$

Setting $\sigma^* = \sigma(\cdot, t^*)$ for $\sigma \in D([0, +\infty), X_\varepsilon)$, we write

$$\begin{aligned}
&\mathbb{E}^\varepsilon_{\delta_{\sigma^*}}\left(\prod_{i=1}^n \sigma(x_i, t - t^*)\right)\\
&\quad = \mathbb{E}^\varepsilon_{\delta_{\sigma^*}}\Bigg(\prod_{i=1}^n \big[\sigma(x_i, t - t^*) - m_\varepsilon(\varepsilon x_i, t - t^*; \delta_{\sigma^*})\\
&\qquad\qquad + m_\varepsilon(\varepsilon x_i, t - t^*; \delta_{\sigma^*})\big]\Bigg)\\
(4.35)\qquad &\quad = \sum_{\substack{J \subseteq \{1, \dots, n\}\\ J \ne \varnothing}} \mathbb{E}^\varepsilon_{\delta_{\sigma^*}}\left(\prod_{i \in J} \big[\sigma(x_i, t - t^*) - m_\varepsilon(\varepsilon x_i, t - t^*; \delta_{\sigma^*})\big]\right)\\
&\qquad \times \prod_{i \notin J} m_\varepsilon(\varepsilon x_i, t - t^*; \delta_{\sigma^*})\\
&\qquad + \prod_{i=1}^n m_\varepsilon(\varepsilon x_i, t - t^*; \delta_{\sigma^*}).
\end{aligned}$$

Now, if $0 < t - t^* \le 3t_a$ and a has been chosen in such a way that $3a < a^*$, then, by Proposition 2.2, all the terms in the sum of the r.h.s. of (4.35) must vanish as $\varepsilon \to 0$. Thus, from (4.35),

$$(4.36)\qquad \lim_{\varepsilon \to 0} \left| \mathbb{E}^\varepsilon_{\delta_{\sigma^*}}\left(\prod_{i=1}^n \sigma(x_i, t - t^*)\right) - \prod_{i=1}^n m_\varepsilon(\varepsilon x_i, t - t^*; \delta_{\sigma^*}) \right| = 0$$

and we need to study the behavior of

$$\mathbb{E}^\varepsilon_{\mu^\varepsilon}\left(\prod_{i=1}^n m_\varepsilon(\varepsilon x_i, t - t^*; \delta_{\sigma^*})\right).$$

Let A_{t^*} be the set $\{\sigma(\cdot, t^*) \equiv \sigma^*: \|m_\varepsilon(\cdot, \varepsilon^{1/4}; \delta_{\sigma^*})\| \le \varepsilon^{2\alpha a - b}\}$ where $b > 0$. From Lemma 4.1 it follows that for any $u > 0$ there is a $c > 0$ so that

$$\mathbb{P}^\varepsilon_{\mu^\varepsilon}(A_{t^*}) \ge 1 - c\varepsilon^u.$$

Now, if $t = t_c$, that is, $t - t^* = 2t_a$, we get from Lemmas 4.6 and 4.7: If $\eta < 1/8$,

$$\lim_{\varepsilon \to 0} \mathbb{P}^\varepsilon_{\mu^\varepsilon}\left(\left\| m_\varepsilon\left(\cdot, \varepsilon^{1/4}; \delta_{\sigma^*}\right)\right\| > c|\ln \varepsilon|^{-\eta}\right) = 0,$$

so that (2.9a) for $t = t_c$ follows from this, (4.34) and (4.36). □

To prove (2.9b), we first study the behavior of $m_\varepsilon(r,t;\delta_{\sigma^*})$ in the time interval $2t_a \le t \le t_f - t^*$. We shall see that as $\varepsilon \to 0$, $m_\varepsilon(r, t_f - t^*; \delta_{\sigma^*})$ converges either to m^* or to $-m^*$ for all r such that $m_\varepsilon(\cdot, 2t_a; \delta_{\sigma^*})$ is suitably bounded away from 0 (and positive, respectively negative) in a sufficiently large region around r. The argument will use the monotonicity properties of (2.4).

After that, we prove that, given any $\mathbf{x} = (x_1, \dots, x_n) \in M_n^\varepsilon$, the probability that $m_\varepsilon(\cdot, 2t_a; \delta_{\sigma^*})$ satisfies the above conditions relative to each point εx_i goes to 1 as $\varepsilon \to 0$. We will then be close to the end of the proof of (2.9b).

Given ε, $d > 0$, $L > 0$ and x, we introduce the following set of configurations σ^* (at time t^*).

$$\mathscr{S}_+(\varepsilon, d, L, x) = \Big\{ \inf_{|r-\varepsilon x| \le L|\ln \varepsilon|^{-1/2}} l_\varepsilon(r, 2t_a; \delta_{\sigma^*}) \ge \frac{d}{|\ln \varepsilon|^{1/4}},$$
$$\sup_{|r-\varepsilon x| \le L|\ln \varepsilon|^{1/2}} \left| l_\varepsilon(r, 2t_a; \delta_{\sigma^*}) - m_\varepsilon(r, 2t_a; \delta_{\sigma^*}) \right| \le c|\ln \varepsilon|^{-3\eta} \Big\}, \tag{4.37}$$

where c is as in Lemma 4.7, equation (4.11).

We define $\mathscr{S}_-(\varepsilon, d, L, x)$ analogously, with the first inequality replaced by $\sup l_\varepsilon(r, 2t_a; \delta_{\sigma^*}) \le -d|\ln \varepsilon|^{-1/4}$.

LEMMA 4.11. *There is $\theta_1(\varepsilon, d, L)$, $d > 0$, $L > 0$, vanishing as $\varepsilon \to 0$ for any given d and L, such that for any $\sigma^* \in \mathscr{S}_\pm(\varepsilon, d, L, x)$,*

$$\sup_{|r-\varepsilon x| \le L/(2|\ln \varepsilon|^{1/2})} \left| m_\varepsilon(r, t_f - t^*; \delta_{\sigma^*}) \mp m^* \right| \le \theta_1(\varepsilon, d, L). \tag{4.38}$$

PROOF. Suppose for notational simplicity that $\sigma^* \in \mathscr{S}_+(\varepsilon, d, L, x)$. Then, for ε small enough and recalling the definition of $\mathscr{S}_+$,

$$m^* \ge m_\varepsilon(r, 2t_a; \delta_{\sigma^*}) \ge_L d/(2|\ln \varepsilon|^{-1/4}) \tag{4.39}$$

[see (4.12b) for notation]. Equation (4.38) is then a consequence of Lemma 4.8, the monotonicity properties of (2.4) and the fact that

$$\lim_{\varepsilon \to 0} z(t_f - 2t_a) = m^*, \tag{4.40}$$

where $z(t)$ is the solution of (4.22) with initial condition $z(0) = d/(2|\ln \varepsilon|^{-1/4})$. □

We define

$$\mathscr{S}(\varepsilon, d, L, x) = \mathscr{S}_+(\varepsilon, d, L, x) \cup \mathscr{S}_-(\varepsilon, d, L, x) \tag{4.41}$$

and for $\mathbf{x} \in M_n^\varepsilon$,

$$\mathscr{S}(\varepsilon, d, L, \mathbf{x}) = \bigcap_{i=1}^{n} \mathscr{S}(\varepsilon, d, L, x_i). \tag{4.42}$$

As a consequence of (4.11), (4.14) and (4.15), there is a function $\theta_2(\varepsilon, d, L, n)$,

$n \geq 1$, such that for all n and all $\mathbf{x} \in M_n^\varepsilon$,

$$\mathscr{P}_{\mu^\varepsilon}^\varepsilon(\mathscr{G}(\varepsilon, d, L, \mathbf{x})) \geq 1 - \theta_2(\varepsilon, d, L, n) \tag{4.43}$$

and furthermore, for any n,

$$\limsup_{d+L\to 0} \limsup_{\varepsilon\to 0} \theta_2(\varepsilon, d, L, n) = 0.$$

Proof of (2.9b). We fix below $n \geq 1$ and $R > 0$. Given $L > 0$ there is a finite set Γ, which is $L/2$-dense in $[-R, R]$. That is, for any $|r| \leq R$, there is $r' \in \Gamma$ such that $|r - r'| \leq L/2$.

Given $\mathbf{x} \in M_n^\varepsilon$ and such that $|x_i| \leq R_\varepsilon \equiv \varepsilon^{-1}\sqrt{|\ln \varepsilon|}$, we denote by $\mathbf{x}' = (x_1', \dots, x_n')$ any n-tuple such that for all $i = 1, \dots, n$, $|x_i' - x_i| \leq \varepsilon^{-1}\sqrt{|\ln \varepsilon|}\,L/2$ and $\varepsilon|\ln \varepsilon|^{-1/2}x_i \in \Gamma$.

We then have, by (4.34),

$$\mu_{t_f}^\varepsilon\left(\prod_{i=1}^n \sigma(x_i)\right) = \mathbb{E}_{\mu^\varepsilon}^\varepsilon\left(\mathbb{E}_{\delta_{\sigma^*}}^\varepsilon\left(\prod_{i=1}^n \sigma(x_i, t_f - t^*)\right)\right)$$

and, by (4.36),

$$\left|\mu_{t_f}^\varepsilon\left(\prod_{i=1}^n \sigma(x_i)\right) - \mathbb{E}_{\mu^\varepsilon}^\varepsilon\left(\prod_{i=1}^n m_\varepsilon(\varepsilon x_i, t_f - t^*; \delta_{\sigma^*})\right)\right| \leq \theta_3(\varepsilon, n),$$

where $\theta_3(\varepsilon, n) \to 0$ as $\varepsilon \to 0$, for each fixed n. Then, by (4.43),

$$\begin{aligned}\left|\mu_{t_f}^\varepsilon\left(\prod_{i=1}^n \sigma(x_i)\right) - \mathbb{E}_{\mu^\varepsilon}^\varepsilon\left(1_{\mathscr{G}(\varepsilon, d, L, \mathbf{x}')}\left(\prod_{i=1}^n m_\varepsilon(\varepsilon x_i, t_f - t^*; \delta_{\sigma^*})\right)\right)\right| \\ \leq \theta_2(\varepsilon, d, L, n) + \theta_3(\varepsilon, n)\end{aligned}$$

and, by (4.38),

$$\begin{aligned}\left|\mu_{t_f}^\varepsilon\left(\prod_{i=1}^n \sigma(x_i)\right) - \mathbb{E}_{\mu^\varepsilon}^\varepsilon\left(1_{\mathscr{G}(\varepsilon, d, L, \mathbf{x}')}\left(\prod_{i=1}^n m^* \operatorname{sign}\{l_\varepsilon(\varepsilon x_i', 2t_a; \delta_{\sigma^*})\}\right)\right)\right| \\ \leq c\theta_1(\varepsilon, d, L) + \theta_2(\varepsilon, d, L, n) + \theta_3(\varepsilon, n),\end{aligned} \tag{4.44}$$

where c is a suitable constant which depends on n. We thus have, again using (4.43),

$$\begin{aligned}\left|\mu_{t_f}^\varepsilon\left(\prod_{i=1}^n \sigma(x_i)\right) - \mathbb{E}_{\mu^\varepsilon}^\varepsilon\left(\prod_{i=1}^n m^* \operatorname{sign}\{l_\varepsilon(\varepsilon x_i', 2t_a; \delta_{\sigma^*})\}\right)\right| \\ \leq c\theta_1(\varepsilon, d, L) + 2\theta_2(\varepsilon, d, L, n) + \theta_3(\varepsilon, n).\end{aligned} \tag{4.45}$$

By (4.33) we have

$$\left||\ln \varepsilon|^{1/4} l_\varepsilon(\varepsilon x_i', 2t_a; \delta_{\sigma^*}) - Y_{t^*}^\varepsilon(\phi_{r_i'})\right| \leq c\varepsilon^{1-2\alpha a}|\ln \varepsilon|^{-1/4}, \tag{4.46}$$

where

$$\phi_{r_i'}(r) = \phi(r - r_i'), \qquad r_i' = \varepsilon|\ln \varepsilon|^{-1/2}x_i' \in \Gamma. \tag{4.47}$$

By Theorem 3.1 we have

$$\lim_{\varepsilon \to 0} \mathbb{E}^{\varepsilon}_{\mu^{\varepsilon}}\left(\prod_{i=1}^{n} m^* \operatorname{sign}\left\{ Y^{\varepsilon}_{t^*}(\phi_{r'_i}) \right\} \right) = \tilde{E}\left(\prod_{i=1}^{n} \rho(r'_i) \right). \tag{4.48}$$

In fact, $\operatorname{sign}\{X^{\varepsilon}_{t^*|\ln \varepsilon|^{-1}}(\phi)\}$ [cf. (3.1) for notation] is almost surely continuous with respect to the limiting law defined by Theorem 3.1, which is the same as the law defined in Theorem 2.1.

By the continuity properties of the limiting process,

$$\left| \tilde{E}\left(\prod_{i=1}^{n} \rho(r'_i) \right) - \tilde{E}\left(\prod_{i=1}^{n} \rho\left(\varepsilon |\ln \varepsilon|^{-1/2} x_i \right) \right) \right| \le \theta_4(L, n), \tag{4.49}$$

where, for any n, $\theta_4(L, n) \to 0$ as $L \to 0$. Collecting the above estimates and letting first $\varepsilon \to 0$ and then $L + d \to 0$, we obtain the proof of (2.9b), thus concluding the proof of Theorem 2.1. □

5. Estimates of the v-functions. In this section we prove Proposition 2.3. Its proof is based on the analysis of an integral equation for the v-functions that we derive in Lemma 5.1 below, but first we need a definition.

DEFINITION [The stirring process and the transition probability $P^{\varepsilon}_t(\mathbf{x} \to \mathbf{y})$]. For any $n \ge 1$ and for any function f on M^{ε}_n, let

$$Lf(\mathbf{x}) = \frac{1}{2} \sum_{i=1}^{n} \sum_{b=\pm 1} \left[f(\mathbf{x}^{i,b}) - f(\mathbf{x}) \right], \tag{5.1}$$

where $\mathbf{x}^{i,b} = (x'_1, \dots, x'_n)$ is defined below. Let $\mathbf{x} = (x_1, \dots, x_n)$. Then if $x_i + b \ne x_j$ for all j, we set $x'_l = x_l$ for $l \ne i$, while $x'_i = x_i + b$. If, on the other hand, there is j such that $x_j = x_i + b$, then if $b = -1$, $x'_l = x_l$ for all l and if $b = 1$, $x'_l = x_l$ for all $l \ne i, j$, while $x'_i = x_i + 1$ and $x'_j = x_j - 1$. We now define $P^{\varepsilon}_t(\mathbf{x} \to \mathbf{y})$ as the transition probability of the Markov process on M^{ε}_n with generator $(\varepsilon^{-2} + 4\gamma)L$, where γ is the parameter appearing in (2.5).

The process defined above is the stirring process; its marginal over the symmetric functions is the symmetric simple exclusion process. For each $n \ge 1$ it has the same law as the process on X_{ε} with generator $(\varepsilon^{-2} + 4\gamma)L_0$ if we identify a configuration σ with the set $\mathbf{x}$ of the sites in Λ_{ε} where the spin has value 1 and if we restrict this process to the set of configurations which have just n spins equal to 1. The jump intensities in (5.1) have been chosen in such a way that the marginals $(x_{i_1}(t), \dots, x_{i_k}(t))$, for any given subset $(i_1, \dots, i_k)$ of $(1, \dots, n)$, have again the law of the stirring process (with k particles).

LEMMA 5.1. *For any $t > 0$ and $\mathbf{x} \in M^{\varepsilon}_{2n}$,*

$$v^{\varepsilon}_{2n}(\mathbf{x}, t) = \int_0^t ds\, e^{2n\alpha(t-s)} \sum_{\mathbf{y} \in M^{\varepsilon}_{2n}} P^{\varepsilon}_{t-s}(\mathbf{x} \to \mathbf{y}) \mathscr{R}^{\varepsilon}(\mathbf{y}, s), \tag{5.2}$$

where

$$(5.3)\quad \begin{aligned}\mathscr{R}^\varepsilon(\mathbf{x},t) = \sum_{x\in\mathbf{x}}\Bigg[&-2\gamma^2 v^\varepsilon_{2n+2}(\mathbf{x}+\delta_{x+1}+\delta_{x-1},t)\mathbf{1}(x\pm 1\notin\mathbf{x})\\ &+2\gamma\sum_{b=\pm1}\mathbf{1}(x+b\in\mathbf{x})\big\{v^\varepsilon_{2n-2}(\mathbf{x}-\delta_x-\delta_{x+b},t)-v^\varepsilon_{2n}(\mathbf{x},t)\\ &\qquad -\gamma\mathbf{1}(x-b\notin\mathbf{x})v^\varepsilon_{2n}(\mathbf{x}-\delta_{x+b}+\delta_{x-b},t)\\ &\qquad -\gamma\mathbf{1}(x-b\in\mathbf{x})v^\varepsilon_{2n-2}(\mathbf{x}-\delta_{x+1}-\delta_{x-1},t)\big\}\Bigg],\end{aligned}$$

where $\mathbf{x}\pm\delta_y$ *is the configuration obtained from* $\mathbf{x}$ *by adding, respectively subtracting,* y *and* $\mathbf{1}(\cdot)$ *is the characteristic function of* $(\cdot)$.

PROOF. Recall that $v^\varepsilon_m(\mathbf{x},t)=\mathbb{E}^\varepsilon_{\mu^\varepsilon}(\prod_{i=1}^m\sigma(x_i,t))$ and so

$$\frac{d}{dt}v^\varepsilon_m(\mathbf{x},t)=\mathbb{E}^\varepsilon_{\mu^\varepsilon}\left[\left(\varepsilon^{-2}L_0+L_G\right)\prod_{i=1}^m\sigma(x_i,t)\right].$$

Using (2.2) and (2.5) to compute $L_G(\prod_{i=1}^m\sigma(x_i))$, after some simple algebra we obtain $(m=2n)$,

$$(5.4)\quad \frac{d}{dt}v^\varepsilon_m(\mathbf{x},t)=\left(\varepsilon^{-2}+4\gamma\right)L_0v^\varepsilon_m(\mathbf{x},t)+\alpha m v^\varepsilon_m(\mathbf{x},t)+\mathscr{R}^\varepsilon(\mathbf{x},t).$$

Since this is a simple algebraic manipulation, we omit the details and refer to Chapter 9 of De Masi and Presutti (1991). Equation (5.2) follows at once from (5.4) □

The proof of Proposition 2.3 is based on (a) proving that the leading contribution to (5.2) comes only from the terms on the right-hand side of (5.3) which contain v^ε_{n-2}, and (b) proving that the contribution of these terms gives the estimates in (2.14) and (2.16). It would not be difficult to see that the term $v^\varepsilon_{n-2}\mathbf{1}(x\pm1\in\mathbf{x})$ can also be neglected. We start with point (b) and give the following definition.

DEFINITION (The w-functions). We define $w^\varepsilon_0(\mathbf{x},t)\equiv1$ and for any $n\geq1$, $\mathbf{x}\in M^\varepsilon_{2n}$ and $t\geq0$ we set recursively

$$(5.5)\quad \begin{aligned}w^\varepsilon_{2n}(\mathbf{x},t)=2\gamma(1+\gamma)\int_0^t ds\, e^{2n\alpha(t-s)}&\sum_{\mathbf{y}\in M^\varepsilon_{2n}}P^\varepsilon_{t-s}(\mathbf{x}\to\mathbf{y})\\ &\times\sum_{i,j}\mathbf{1}\big(|y_i-y_j|\leq2\big)w^\varepsilon_{2n-2}(\mathbf{y}^{i,j},s),\end{aligned}$$

where $\mathbf{y}^{i,j}\in M^\varepsilon_{2n-2}$ is the configuration obtained from $\mathbf{y}$ by dropping y_i and y_j. Notice that the w-functions are nonnegative.

Therefore, w^ε_{2n} satisfies the equation obtained from (5.2) by neglecting the terms with v^ε_{2n} and v^ε_{2n+2} and by suitably bounding the others. More precisely, call $\tilde{w}^{\varepsilon}_{2n}(\mathbf{x}, t)$ the solution of

$$\frac{d}{dt}\tilde{w}^{\varepsilon}_{2n}(\mathbf{x},t) = (\varepsilon^{-2} + 4\gamma)L_0\tilde{w}^{\varepsilon}_{2n}(\mathbf{x},t) + \tilde{R}^{\varepsilon}(\mathbf{x},t), \qquad \tilde{w}^{\varepsilon}_{2n}(\mathbf{x},0) \equiv 0,$$

where

$$\tilde{R}^{\varepsilon}(\mathbf{x},t) = \sum_{x\in\mathbf{x}} 2\gamma \sum_{b=\pm 1} \mathbf{1}_{(x+b\in\mathbf{x})}\big\{\tilde{w}^{\varepsilon}_{2n-2}(\mathbf{x} - \delta_x - \delta_{x+b}, t) + \gamma\mathbf{1}_{(x-b\in\mathbf{x})}\tilde{w}^{\varepsilon}_{2n-2}(\mathbf{x} - \delta_{x+1} - \delta_{x-1}, t)\big\}.$$

Then $\tilde{w}^{\varepsilon}_{2n}(\mathbf{x}, t) \le w^{\varepsilon}_{2n}(\mathbf{x}, t)$ for all n, all $\mathbf{x}$ and all t.

LEMMA 5.2. *For all $\delta_0 > 0$, $n \ge 1$ and $\tau < 1/(2\alpha)$, there exists a constant c such that*

$$\sup_{t\le\tau|\ln\varepsilon|} \big(\Gamma_t(\varepsilon)\big)^{-2n}\|w^{\varepsilon}_{2n}(t)\| \le c, \tag{5.6a}$$

where

$$\|w^{\varepsilon}_{2n}(t)\| = \sup_{\mathbf{x}\in M^{\varepsilon}_{2n}} w^{\varepsilon}_{2n}(\mathbf{x},t) \tag{5.6b}$$

and

$$\Gamma_t(\varepsilon) = \sqrt{\varepsilon}\, e^{\alpha t}(1+t)^{-1/8+\delta_0}.$$

Furthermore,

$$\sup_{t\le\tau|\ln\varepsilon|}\left(\frac{\varepsilon e^{2\alpha t}}{(1+t)^{1/2}}\right)^{-1}\|w^{\varepsilon}_{2}(t)\| \le c. \tag{5.7}$$

PROOF. From (5.5) we get

$$\begin{aligned} w^{\varepsilon}_{2n}(\mathbf{x},t) &\le 2\gamma(1+\gamma)\int_0^t ds\, e^{\alpha 2n(t-s)} \sum_{\mathbf{y}\in M^{\varepsilon}_{2n}} P^{\varepsilon}_{t-s}(\mathbf{x}\to\mathbf{y}) \\ &\quad\times \sum_{i,j}\mathbf{1}\big(|y_i - y_j| \le 2\big)\|w^{\varepsilon}_{2n-2}(s)\|. \end{aligned} \tag{5.8}$$

We have that given any $\tau > 0$ there is c so that for all $t \le \tau|\ln\varepsilon|$,

$$\begin{aligned} &\sum_{\mathbf{y}\in M^{\varepsilon}_{2n}} P^{\varepsilon}_t(\mathbf{x}\to\mathbf{y})\mathbf{1}\big(|y_i - y_j| \le 2\big) \\ &\quad= \sum_{y_i,y_j} P^{\varepsilon}_t\big((x_i,x_j)\to(y_i,y_j)\big)\mathbf{1}\big(|y_i-y_j|\le 2\big) \\ &\quad\le \frac{c}{\sqrt{\varepsilon^{-2}(t+1)}} = \frac{c\varepsilon}{\sqrt{t+1}}, \end{aligned} \tag{5.9}$$

where c is a suitable constant (the value of the constant c will be changing from line to line). Equation (5.9) follows from classical estimates on random walks. In fact, $|x_i(t) - x_j(t)|$ has the law of the distance between two symmetric random walks on the circle with $\varepsilon^{-1}|\ln \varepsilon|$ sites, each jumping by ± 1 with intensity $\varepsilon^{-2} + 4\gamma$ when $|x_i(t) - x_j(t)| > 1$. When they are at distance 1, they leave this state to go to $\{|x_i(t) - x_j(t)| = 2\}$ with intensity $\varepsilon^{-2} + 4\gamma$. For $t \le \tau|\ln \varepsilon|$ the fact of being on the circle is not relevant; in fact, the bound (5.9) is the same as for unbounded volumes. The estimate (5.7) is then a straightforward consequence of (5.8) with $n = 1$ (recall that $w_0^\varepsilon = 1$) and (5.9). Iterating (5.8), we would easily get the estimate (5.6) but without the factor $(1 + t)^{2n(1/8-\delta_0)}$. The whole problem is to recover such a factor.

From (5.5) we obtain

$$
\begin{aligned}
w_{2n}^\varepsilon(\mathbf{x}, t) \le c\int_0^t ds_1\, e^{2n\alpha(t-s_1)}\int_0^{s_1} ds_2\, e^{2(n-1)\alpha(s_1-s_2)} \cdots \int_0^{s_{n-1}} ds_n\, e^{2\alpha(s_{n-1}-s_n)} \\
\times \sum_{\mathbf{y}_1,\dots,\mathbf{y}_n} \sum_{(i_1,j_1),\dots,(i_n,j_n)} P_{t-s_1}^\varepsilon(\mathbf{x} \to \mathbf{y}_1) \cdots P_{s_{n-1}-s_n}^\varepsilon(\mathbf{y}_{n-1} \to \mathbf{y}_n) \\
\times \prod_{l=1} \mathbf{1}\big(|y_{i_l} - y_{j_l}| = 1\big),
\end{aligned} \tag{5.10}
$$

where $(i_1, j_1), \dots, (i_n, j_n)$ vary over all the partitions of $\{1, \dots, 2n\}$ into disjoint sets of two elements.

We call $\mathscr{I}_1$ the contribution to the right-hand side of (5.10) for $s_1 \le T$, where $T < t$ will be specified later on. We then denote by $\mathscr{I}_2$ the contribution coming from $s_1 > T$, so that

$$w_{2n}^\varepsilon(\mathbf{x}, t) \le \mathscr{I}_1 + \mathscr{I}_2.$$

To bound $\mathscr{I}_2$, we fix $(i_1, j_1), \dots, (i_n, j_n)$, $s_1, \dots, s_n$, $s_1 > T$, and $\mathbf{y}_1, \dots, \mathbf{y}_n$. We then use (5.9) to estimate the sum over $\mathbf{y}_n$, obtaining an estimate uniform in $\mathbf{y}_{n-1}$. After iterating this procedure we get

$$
\begin{aligned}
\mathscr{I}_2 &\le c\varepsilon^n e^{2n\alpha t} e^{-2\alpha T}\int_T^t ds_1 \frac{e^{-2\alpha(s_1-T)}}{\sqrt{t-s_1}} \cdots \int_0^{s_{n-1}} ds_n \frac{e^{-2\alpha s_n}}{\sqrt{s_{n-1}-s_n}} \\
&\le c'\Gamma_t(\varepsilon)^{2n} e^{-2\alpha T}(1+t)^{2n(1/8-\delta_0)},
\end{aligned}
$$

where c and c' are suitable constants. We choose $T = \sqrt{t}$ for $t \ge 2$ (and $= t/2$ when $t < 2$), so that $\mathscr{I}_2$ has the desired bound, because

$$\sup_t e^{-2\alpha\sqrt{t}}(1+t)^{2n(1/8-\delta_0)} < \infty.$$

To estimate $\mathscr{I}_1$, we split the transition probability

$$P_{t-s_1}^\varepsilon(\mathbf{x} \to \mathbf{y}_1) = \sum_{\mathbf{z}} P_{t-T}^\varepsilon(\mathbf{x} \to \mathbf{z}) P_{T-s_1}^\varepsilon(\mathbf{z} \to \mathbf{y}_1);$$

recall that in this case $T > s_1$. We then introduce the characteristic functions:

$$\chi(T) = \prod_{l=1}^n \chi_l(T), \qquad \chi_l(T) = \mathbf{1}\big(|z_{i_l} - z_{j_l}| \le \varepsilon^{-1}T^{1/2+\delta}\big),$$

where $\delta > 0$ will be specified later on. For any fixed $(i_1, j_1), \dots, (i_n, j_n)$ and $T > s_1 > \cdots > s_n$, we have that

$$(1 - \chi(T)) \sum_{\mathbf{y}_1, \dots, \mathbf{y}_n} P^{\varepsilon}_{T-s_1}(\mathbf{z} \to \mathbf{y}_1) \dots P^{\varepsilon}_{s_{n-1}-s_n}(\mathbf{y}_{n-1} \to \mathbf{y}_n) \prod_{h=1}^{n} \mathbf{1}\big(|y_{i_h} - y_{j_h}| \le 2\big)$$

$$\le \sum_{l=1}^{n} (1 - \chi_l(T)) \sum_{\mathbf{y}_l} P^{\varepsilon}_{T-s_l}(\mathbf{z} \to \mathbf{y}_l) \mathbf{1}\big(|y_{i_l} - y_{j_l}| \le 2\big)$$

$$\le \sum_{l=1}^{n} c \exp\left\{ - \frac{\varepsilon^{-2} T^{1+2\delta}}{4(\varepsilon^{-2} + 4\gamma)T} \right\},$$

which follows from classical estimates on random walks [see the remark just after (5.9)]. Therefore, the contribution of the above term vanishes faster than any given power of $1/T = 1/\sqrt{t}$ as $t \to \infty$; hence this term is also bounded as desired.

We are left with the contribution to $\mathscr{I}_1$ coming from configurations $\mathbf{z}$ such that $\chi(T) = 1$. Proceeding as when estimating $\mathscr{I}_2$, we get the bound

$$c\varepsilon^n e^{2n\alpha t} \int_0^T ds_1 \frac{e^{-2\alpha s_1}}{\sqrt{T - s_1}} \cdots \int_0^{s_{n-1}} ds_n \frac{e^{-2\alpha s_n}}{\sqrt{s_{n-1} - s_n}} \sum_{\mathbf{z}} P^{\varepsilon}_{t-T}(\mathbf{x} \to \mathbf{z}) \chi(T).$$

To bound the last factor, we recall a suitable coupling between the stirring process $\mathbf{x}(t)$ on M^{ε}_{2n} and independent random walks $\mathbf{x}^0(t) = (x^0_1(t), \dots, x^0_{2n}(t))$ with the same one-dimensional marginals, that is, each $x^0_i(t)$ describes a symmetric random walk on Λ_ε with rate $(\varepsilon^{-2} + 4\gamma)$.

It has been proven in De Masi, Ianiro, Pellegrinotti and Presutti (1984) [cf. also Proposition 6.6.3 in De Masi and Presutti (1991)] that for any $\mathbf{x} \in M^{\varepsilon}_{2n}$ and $\mathbf{x}_0 \in \mathbb{Z}^n$ a coupling $\mathscr{P}^{\varepsilon}_{\mathbf{x}, \mathbf{x}^0}$ can be constructed in such a way that for any $\delta > 0$ and $k > 0$ there is a c depending on δ, k and n so that uniformly in $t > 0$:

$$\mathscr{P}^{\varepsilon}_{\mathbf{x}, \mathbf{x}_0}\Big(\big\|(\mathbf{x}(t) - \mathbf{x}) - (\mathbf{x}^0(t) - \mathbf{x}^0)\big\| \ge (\varepsilon^{-2} t)^{1/4 + \delta}\Big) \le C(\varepsilon^{-2} t)^{-k},$$

where $|\mathbf{x}| = \max_i |x_i|$. Choosing $\delta < 1/4$, we have

$$\sum_{\mathbf{x}} P^{\varepsilon}_{t-T}(\mathbf{x} \to \mathbf{z}) \chi(T) \le \left[\frac{\varepsilon^{-1} T^{1/2+\delta} + \big[\varepsilon^{-2}(t - T)\big]^{1/4+\delta}}{\varepsilon^{-1}(t - T)^{1/2}} \right]^n,$$

with $2\delta_0 - \delta/2 > 0$, and the lemma follows. □

To prove Proposition 2.3, we derive first a bound for $v^{\varepsilon}_{2n}(\mathbf{x}, t)$ of the form $c\varepsilon^n e^{2n\alpha t}$. Once we have such a bound we will then easily reduce the estimate of v^{ε}_{2n} to that of w^{ε}_{2n}. We start by proving the bound for $t \le \varepsilon^{\beta^*}$, with β^* as in Proposition 2.2. After that we extend it first to $t \le a|\ln \varepsilon|$ with $a > 0$ small enough, and then, by an induction argument, to all $t \le \tau|\ln \varepsilon|$, $\tau < 1/(2\alpha)$. This is done in the following lemmas.

LEMMA 5.3. *For any $n \ge 1$ there is a constant c so that for all $\mathbf{x} \in M^\varepsilon_{2n}$ and $t \le \varepsilon^{\beta^*}$,*

$$|v^\varepsilon_{2n}(\mathbf{x}, t)| \le c\varepsilon^n. \tag{5.11}$$

PROOF. Let us consider an arbitrarily fixed value of n. We define

$$d_m = \begin{cases} \sup_{t \le \varepsilon^{\beta^*}} \|v^\varepsilon_{2m}(t)\| \varepsilon^{-m}, & \text{if } m \le n, \\ \sup_{t \le \varepsilon^{\beta^*}} \|v^\varepsilon_{2m}(t)\| \varepsilon^{-n+\zeta(m-n)}, & \text{otherwise} \end{cases} \tag{5.12}$$

[recall that $\|v^\varepsilon_{2m}(t)\|$ denotes the sup norm of $v^\varepsilon_{2m}(\mathbf{x}, t)$]. We choose $0 < \zeta < \beta^*$ and we let N be the first integer such that $\zeta(N - n) \ge n$. With this definition $d_N \le 1$. Finally, define

$$d = \max_{1 \le m < N} d_m. \tag{5.13}$$

From (5.2) and (5.3) we have for a suitable constant c ($d_0 \equiv 1$ below),

$$\begin{aligned} |v^\varepsilon_{2m}(\mathbf{x}, t)| \le ce^{2m\alpha t} \int_0^t ds\, e^{-2m\alpha s} \Big[& 2m \|v^\varepsilon_{2m+2}(s)\| \\ & + \sum_{i,j} \sum_{\mathbf{y}} P^\varepsilon_{t-s}(\mathbf{x} \to \mathbf{y}) \mathbf{1}(|y_i - y_j| \le 2) \\ & \times \{\|v^\varepsilon_{2m-2}(s)\| + \|v^\varepsilon_{2m}(s)\|\} \Big]. \end{aligned} \tag{5.14}$$

From (5.14) with $m < N$, using (5.9), we get, for a suitable constant c, which depends on n through N and ζ,

$$d_m \le \begin{cases} c\varepsilon^{\beta^*/2}[d_{m-1} + \varepsilon d_m] + \varepsilon^{\beta^*+1} d_{m+1}, & \text{if } m < n, \\ c\varepsilon^{\beta^*/2}[d_{n-1} + \varepsilon d_n] + \varepsilon^{\beta^*-\zeta} d_{n+1}, & \text{if } m = n, \\ c\varepsilon^{\beta^*/2}[\varepsilon^{1+\zeta} d_{m-1} + \varepsilon d_m] + \varepsilon^{\beta^*-\zeta} d_{m+1}, & \text{if } m > n. \end{cases} \tag{5.15}$$

Notice that the right-hand side of (5.15) is strictly smaller than d for ε small enough, if $2 \le m \le N - 2$. Therefore, for all ε small enough, $d = \max(d_1, d_{N-1}) \le d_1 + d_{N-1}$, and by (5.15) for $m = 1$ and $m = N - 1$ we have

$$d \le c\{\varepsilon^{\beta^*/2}[1 + \varepsilon d] + \varepsilon^{\beta^*+1} d + \varepsilon^{\beta^*/2}[d + \varepsilon d] + \varepsilon^{\beta^*-\zeta}\}.$$

Therefore, d vanishes when $\varepsilon \to 0$ and the lemma is proven. □

LEMMA 5.4. *There is $a > 0$ such that the following holds. For any $n \ge 1$ there is a constant c so that for all $\mathbf{x} \in M^\varepsilon_{2n}$ and $t \le a|\ln \varepsilon|$,*

$$|v^\varepsilon_{2n}(\mathbf{x}, t)| \le ce^{2n\alpha t}\varepsilon^n. \tag{5.16}$$

PROOF. We proceed as in the proof of Lemma 5.3. We fix n and introduce the new coefficients d_m as follows ($b < 1$ below is a positive number which will be specified later on):

$$(5.17)\quad d_m = \begin{cases} b^m \sup_{\mathbf{x}\in M_{2m}^\varepsilon} \sup_{t\le a|\ln\varepsilon|} e^{-\alpha 2mt}|v_{2m}^\varepsilon(\mathbf{x},t)|\varepsilon^{-m}, & \text{if } m \le n,\\ b^m \sup_{\mathbf{x}\in M_{2m}^\varepsilon} \sup_{t\le a|\ln\varepsilon|} e^{-\alpha 2mt}|v_{2m}^\varepsilon(\mathbf{x},t)|\varepsilon^{-n-\delta^*(m-n)} & \text{otherwise},\end{cases}$$

with $a \le a^*$ and a^* and δ^* being as Proposition 2.2, and we also assume that $\delta^* < 1$. Then by (2.12) (when $\varepsilon^{\beta^*} \le t \le a^*|\ln\varepsilon|$) and by Lemma 5.3 (when $t \le \varepsilon^{\beta^*}$), we deduce the existence of an integer N such that $d_N \le 1$, for ε small enough. We define $d = \max_{1\le m<N} d_m$, and from (5.14) we then get

$$(5.18)\quad d_m \le \begin{cases} cbd_{m-1} + \varepsilon\sqrt{|\ln\varepsilon|}\,d_m + b^{-1}e^{2\alpha a|\ln\varepsilon|}\varepsilon d_{m+1}, & \text{if } m < n,\\ cbd_{n-1} + \varepsilon\sqrt{|\ln\varepsilon|}\,d_n + b^{-1}e^{2\alpha a|\ln\varepsilon|}\varepsilon^{\delta^*} d_{n+1}, & \text{if } m = n,\\ cb\varepsilon^{1-\delta^*}d_{m-1} + \varepsilon\sqrt{|\ln\varepsilon|}\,d_m + b^{-1}e^{2\alpha a|\ln\varepsilon|}\varepsilon^{\delta^*} d_{m+1}, & \text{if } m > n.\end{cases}$$

We choose $a > 0$ so small that $2\alpha a < \delta^*$. By choosing b small enough we can make the right-hand side of (5.18) strictly smaller than d when $2 \le m \le N-2$ for ε small enough. As in the proof of Lemma 5.3, we can then conclude that d is bounded, and this proves the lemma. □

LEMMA 5.5. *Assume that for some $\tau < 1/(2\alpha)$ the following holds: For any $n \ge 1$ there is c so that $\|v_{2n}^\varepsilon(t)\| \le c\varepsilon^n e^{2n\alpha t}$ for all $t \le \tau|\ln\varepsilon|$. Then there is $\zeta > 0$ and for each $n \ge 1$ there is c_n so that for all $\mathbf{x} \in M_{2n}^\varepsilon$ and all $t \le \tau|\ln\varepsilon|$,*

$$(5.19)\qquad |v_{2n}^\varepsilon(\mathbf{x},t)| \le w_{2n}^\varepsilon(\mathbf{x},t) + c_n\varepsilon^\zeta[\varepsilon^n e^{2n\alpha t}].$$

REMARKS. By Lemma 5.2, under the assumptions of Lemma 5.5, it follows that for each $n \ge 1$ there is c so that for all $t \le \tau|\ln\varepsilon|$,

$$(5.20)\qquad \|v_{2n}^\varepsilon(t)\| \le c\Gamma_\varepsilon(t)^{2n}.$$

From Lemmas 5.5 and 5.4 it follows that (5.20) holds for all $t \le a|\ln\varepsilon|$, with a as in Lemma 5.4.

PROOF OF LEMMA 5.5. We prove (5.19) by induction on n. From (5.2) with $n = 1$ we get

$$|v_2^\varepsilon(\mathbf{x},t)| \le w_2^\varepsilon(\mathbf{x},t) + ce^{2t\alpha}\int_0^t ds\, e^{-2s\alpha}\Big\{\sum_{\mathbf{y}} P_{t-s}^\varepsilon(\mathbf{x}\to\mathbf{y})\mathbf{1}(|y_1-y_2|\le 2)\|v_2^\varepsilon(s)\| + \|v_4^\varepsilon(s)\|\Big\},$$

recalling that $\|v_n^\varepsilon(s)\|$ is the sup norm of $v_n^\varepsilon(\mathbf{x},s)$. By using (5.9) and the assumption that $\|v_{2n}^\varepsilon(t)\| \le c\varepsilon^n e^{2n\alpha t}$, it follows that

$$|v_2^\varepsilon(\mathbf{x},t)| \le w_2^\varepsilon(\mathbf{x},t) + c'e^{2t\alpha}\big[\varepsilon^2\sqrt{t} + \varepsilon^2 e^{2t\alpha}\big].$$

We have therefore proven (5.19) for $n = 1$ and any $\zeta < 1$ [having chosen $\tau < 1/(2\alpha)$]. We now assume (5.19) for $n-1$ and want to prove it for n. The argument is completely analogous to the previous one. We use (5.2) and (5.3). The terms with v^{ε}_{2n-2}, by the induction assumption, reconstruct w^{ε}_{2n} plus the term

$$\int_0^t ds\, e^{2n\alpha(t-s)} 2\gamma(1+\gamma) \sum_{i,j} P^{\varepsilon}_{t-s}(\mathbf{x} \to \mathbf{y}) \mathbf{1}(|y_i - y_j| \le 2) c_{n-1} \varepsilon^{\zeta} \varepsilon^{n-1} e^{2(n-1)\alpha s},$$

which by (5.9) is compatible with (5.19). For the terms with v^{ε}_{2n} we use Lemma 5.4 and then again (5.9); for the term with v^{ε}_{2n+2} we use Lemma 5.4. In both cases it is easy to see that we obtain a bound compatible with (5.19) if ζ is suitably small, depending on τ. We omit the details. □

LEMMA 5.6. *For any* $\tau < 1/(2\alpha)$ *and* $n \ge 1$ *there is* c *so that for all* $\mathbf{x} \in M^{\varepsilon}_{2n}$ *and all* $t \le \tau|\ln \varepsilon|$,

$$|v^{\varepsilon}_{2n}(\mathbf{x}, t)| \le c\Gamma_{\varepsilon}(t)^{2n}. \tag{5.21}$$

REMARK. By using (5.21), (5.19) and (5.7) we derive (2.16); therefore, the proof of Proposition 2.3 is completed once we prove Lemma 5.6.

PROOF OF LEMMA 5.6. Given $\tau < 1/(2\alpha)$ we consider a so that $a^{-1}\tau$ is a positive integer and $a \le a^*$, where a^* will be chosen in the sequel. By choosing a^* as required for applying Lemma 5.4, by the remarks after Lemma 5.5, we know that (5.21) holds for $t \le a|\ln \varepsilon|$. We are going to prove that if (5.21) holds for $t \le k|\ln \varepsilon|$, it also holds for $t \le (k+1)|\ln \varepsilon|$, provided that $k + 1 \le k_{\max}$, where $k_{\max} a = \tau$. Given any $n \ge 1$, we set for $m \ge 1$, $\mathbf{x} \in M^{\varepsilon}_{2m}$ and $t_k < t \le t_{k+1}$,

$$u^{\varepsilon}_{2m}(\mathbf{x}, t) = v^{\varepsilon}_{2m}(\mathbf{x}, t) - e^{2m\alpha(t-t_k)} \sum_{\mathbf{y}} P^{\varepsilon}_{t-t_k}(\mathbf{x} \to \mathbf{y}) v^{\varepsilon}_{2m}(\mathbf{y}, t_k), \tag{5.22}$$

$$d_m = \begin{cases} \varepsilon^{-\zeta} \sup\limits_{t_k < t \le t_{k+1}} e^{-2m\alpha t} \| u^{\varepsilon}_{2m}(t) \| \varepsilon^{-m}, & \text{if } m \le n, \\ \varepsilon^{-\zeta} \sup\limits_{t_k < t \le t_{k+1}} e^{-2m\alpha t} \| u^{\varepsilon}_{2m}(t) \| \varepsilon^{-n-\theta(m-n)}, & \text{if } m > n, \end{cases} \tag{5.23}$$

where $\zeta > 0$, $1 > \theta > 0$ and furthermore

$$\begin{gathered} \zeta < 2a\alpha, \qquad \zeta < 1 - 2(k+1)a\alpha, \qquad \theta > 2(k+1)a\alpha, \\ \theta - 2ka\alpha < 2\delta^*, \qquad \theta - 2(k-1)a\alpha < 1/12. \end{gathered} \tag{5.24}$$

Indeed, since $k + 1 \le k_{\max}$, $2(k+1)\alpha a \le 2k_{\max}\alpha a = 2\tau\alpha < 1$. Hence for any given a we can find $\zeta > 0$ so that the first two inequalities in (5.24) are satisfied. On the other hand, the conditions on θ are

$$2ka\alpha + 2a\alpha < \theta < 2ka\alpha + 2\delta^*, \qquad \theta < 2ka\alpha - 2a + 1/12,$$

which can be fulfilled if a is so small that $2a\alpha < 2\delta^*$ and $4a\alpha < 1/12$. Thus for any a sufficiently small ($a \le a^*$), there are solutions ζ and θ to (5.24) for all $k < k_{\max}$. We shall choose any $a < 1/100$ in this set and for which we can apply Lemma 5.4. We also require that $a \le a^*/2$, a^* being as in Proposition 2.2.

We shall prove that $d_n \le 1$ for ε small enough, and hence that there is c' so that $d_n \le c'$ for all $\varepsilon \le 1/2$. By the arbitrariness of n, this proves the induction argument. In fact, by (5.22) we have

$$\|v_{2m}^\varepsilon(t)\| \le \|u_{2m}^\varepsilon(t)\| + e^{2m\alpha(t-t_k)}\|v_{2m}^\varepsilon(t_k)\|. \tag{5.25}$$

Putting $m = n$ and assuming that $d_n \le c'$, we then get by the induction assumption

$$\|v_{2n}^\varepsilon(t)\| \le c'\varepsilon^{\zeta+n}e^{2n\alpha t} + c\Gamma_\varepsilon(t)^{2n}.$$

Therefore, to prove the lemma, it is enough to show that the coefficients d_m in (5.23) are bounded by 1 for ε small enough. We postpone the proof of the existence of an integer $N > n$ such that $d_N \le 1$ for ε small enough. From (5.2), (5.9) and (5.22) we get

$$\begin{aligned} |u_{2m}^\varepsilon(\mathbf{x},t)| \le ce^{2m\alpha t}\int_{t_k}^t ds\, e^{-2m\alpha s}\Big[\|v_{2m+2}^\varepsilon(s)\| &+ \frac{\varepsilon}{\sqrt{t-s}} \\ &\times\{\|v_{2m-2}^\varepsilon(s)\| + \|v_{2m}^\varepsilon(s)\|\}\Big]. \end{aligned} \tag{5.26}$$

From (5.25) and (5.26), using the definition (5.23) of the d_m's, we have that for $m < n$,

$$\begin{aligned} d_m \le c\Big[&e^{-2\alpha t_k}\big[d_{m-1} + \varepsilon^{-\zeta}\big] \\ &+\varepsilon\sqrt{t_{k+1}}\,\big[d_m + \varepsilon^{-\zeta}\big] + \varepsilon e^{2\alpha t_{k+1}}\big[d_{m+1} + \varepsilon^{-\zeta}\big]\Big]. \end{aligned} \tag{5.27}$$

For $m = n$ we get

$$\begin{aligned} d_n \le c\Big[&e^{-2\alpha t_k}\big[d_{n-1} + \varepsilon^{-\zeta}\big] \\ &+\varepsilon\sqrt{t_{k+1}}\,\big[d_n + \varepsilon^{-\zeta}\big] + e^{2\alpha t_{k+1}}\big[\varepsilon^\theta d_{n+1} + \varepsilon^{1-\zeta}\big]\Big]. \end{aligned} \tag{5.28}$$

For $m > n$ we get

$$\begin{aligned} d_m \le c\Big[&e^{-2\alpha t_k}\big[\varepsilon^{1-\theta}d_{m-1} + \varepsilon^{-\zeta+(1-\theta)(m-n)}\big] \\ &+\varepsilon\sqrt{t_{k+1}}\,\big[d_m + \varepsilon^{(1-\theta)(m-n)-\zeta}\big] \\ &+e^{2\alpha t_{k+1}}\big[\varepsilon^\theta d_{m+1} + \varepsilon^{1-\zeta+(1-\theta)(m-n)}\big]\Big]. \end{aligned} \tag{5.29}$$

We define d as the sum of d_m from $m = 1$ to $m = N - 1$, so that d is bounded by the sum of the right-hand sides of (5.27), (5.28) and (5.29). We start by proving that the coefficients which multiply the d_m's vanish as $\varepsilon \to 0$. This is so because (a) $\varepsilon \exp\{2\alpha t_{k+1}\} \to 0$, as $\varepsilon \to 0$ (recall that $k + 1 \le k_{\max}$); (b) $\varepsilon^\theta \exp\{2\alpha t_{k+1}\} \to 0$, by the third inequality in (5.24). The terms which do not contain any d_m also vanish when $\varepsilon \to 0$. In fact, (c) $\varepsilon^{-\zeta}\exp\{-2\alpha t_k\} \to 0$, by the first inequality in (5.24); $\zeta < 1$, by the second inequality in (5.24), (d) $\varepsilon^{1-\zeta}\exp\{2\alpha t_{k+1}\} \to 0$, again by the second inequality in (5.24).

The lemma is therefore proven once we show the existence of N such that $d_N \leq 1$ for ε small enough. By Lemma 5.5 we easily see that, when the right-hand side of (5.22) is substituted for u^ε in (5.23), the contribution of the second term on the right-hand side of (5.22) vanishes as $\varepsilon \to 0$. We therefore need only show that there is N such that

$$\lim_{\varepsilon \to 0} \varepsilon^{-\zeta-(1-\theta)n} \sup_{t_k < t \leq t_{k+1}} \| v_{2N}^\varepsilon(t) \| e^{-2N\alpha t} \varepsilon^{-\theta N} = 0. \tag{5.30}$$

We denote by σ^* the random configurations at time t_{k-1} and call $\mathbb{E}_{\sigma^*}^\varepsilon$ the expectation when the process starts at time 0 from σ^*. Setting $t^* = t - t_{k-1}$, we get

$$\begin{aligned} v_{2N}^\varepsilon(\mathbf{x}, t) = \mathbb{E}_{\mu^\varepsilon}^\varepsilon \Bigg(\mathbb{E}_{\sigma^*}^\varepsilon \Bigg(\prod_{i=1}^{2N} \big[\{ \sigma(x_i, t') - m_\varepsilon(\varepsilon x_i, t'; \delta_{\sigma^*}) \} \\ + m_\varepsilon(\varepsilon x_i, t'; \delta_{\sigma^*}) \big] \Bigg) \Bigg), \end{aligned} \tag{5.31}$$

where m_ε is defined as in Proposition 2.2.

We now expand the product obtaining a sum of terms which are products of curly brackets and m_ε's: the latter are constant with respect to the expectation $\mathbb{E}_{\sigma^*}^\varepsilon$; the others, by (2.12), give a contribution bounded by $c\varepsilon^{k\delta^*}$, if there is a product of k curly bracket terms. By Lemma 4.10 we have that for any u there is c so that

$$\mathbb{P}_{\mu^\varepsilon}^\varepsilon\big(\| m_\varepsilon(t - t_{k-1}; \delta_{\sigma^*}) \| > \varepsilon^{-b} \max\{ \varepsilon^{1/2} e^{\alpha t}, \varepsilon^{1/12} e^{\alpha(t - t_{k-1})} \} \big) < c\varepsilon^u \tag{5.32}$$

for all $t_k \leq t \leq t_{k+1}$.

We thus get from (5.31) a vanishing contribution because of the last two inequalities in (5.24).

6. Concluding remarks. The extension of our results to the case when the system is defined in the whole space for all the values of ε is an interesting but rather technical question, as we do not expect physically relevant changes. For this reason and to make the paper shorter, we have avoided the issue. The really interesting question, in our opinion, concerns the behavior of the system at longer times. Interactions between clusters of different phases and tunnelling effects should then have a relevant role, but the techniques we have presented here do not seem adequate for such an analysis.

In a paper in preparation by De Masi, Orlandi, Presutti and Triolo, the same phenomenology of phase separation is observed in a spin system which evolves by the Glauber dynamics with Kac potentials. For the definition of the model, see Penrose (1991).

The behavior should be quite different, however, when, for the same interaction, the dynamics are conservative Kawasaki dynamics [see Penrose (1991) for the definition of the process]. The analysis is then much harder and no results are known so far.

A short survey on phase separation phenomena has been presented by Pellegrinotti (1991). Giacomin (1991) has generalized our techniques and results to two and three dimensions. The conclusions of Giacomin (1991) are similar to those found here. The phases separate on the time scale $|\ln \varepsilon|$ into clusters which, in units $\varepsilon^{-1}\sqrt{|\ln \varepsilon|}$, have smooth boundaries. Their geometry is described in terms of a Gaussian distribution, as in Theorem 2.1.

As already mentioned, the most interesting open question concerns the motion of the clusters after the phases separate. This problem has been studied by Bonaventura (1992), who considers the spin model presented here in two dimensions. The initial state is again a product measure. The average spin at x equals $m(\varepsilon^{1+b}x)$, for $b > 0$ and sufficiently small. The function $m(r)$ is chosen close to $\pm m^*$ inside, respectively outside, of a smooth region Λ. There are technical conditions on $m(r)$ for which we refer to the original paper.

At times $t\varepsilon^{-2b}$, Bonaventura (1991) proves that $\mu^{\varepsilon}_{t\varepsilon^{-2b}}$ is still close, in the sense of our Theorem 2.1, to a product measure with averages close to $\pm m^*$ inside, respectively outside, of Λ_t. Λ_t is obtained from Λ by letting the points of the boundary move with velocity proportional to the curvature and directed toward the interior of the region. We refer to the original paper for a precise statement.

It is conjectured that the behavior found by Bonaventura (1991) also describes the evolution of the clusters of the different phases after they separate, but no proof has appeared so far.

Acknowledgments. We thank M. Bramson and R. Holley for valuable discussions, G. Giacomin for pointing out some mistakes present in a previous version of the paper, and a referee for a very careful reading and useful criticism.

REFERENCES

BONAVENTURA, L. (1992). Motion by curvature in an interacting spin system. Preprint.

CALDERONI, P., PELLEGRINOTTI, A., PRESUTTI, E. and VARES, M. E. (1989). Transient bimodality in interacting particle systems. *J. Statist. Phys.* **55** 523–578.

DE MASI, A., FERRARI, P. A. and LEBOWITZ, J. L. (1986). Reaction diffusion equations for interacting particle systems. *J. Statist. Phys.* **44** 589–644.

DE MASI, A., IANIRO, N., PELLEGRINOTTI, A. and PRESUTTI, E. (1984). A survey of the hydrodynamical behavior of many particle systems. In *Studies in Statistical Mechanics. Non-equilibrium Phenomena. II. From Stochastics to Hydrodynamics* (J. L. Lebowitz and E. W. Montroll, eds.) **11** 124–294. North-Holland, Amsterdam.

DE MASI, A. and PRESUTTI, E. (1991). *Mathematical Methods for Hydrodynamic Limits. Lecture Notes in Math.* **1501**. Springer, New York.

FIFE, P. C. (1979). Mathematical aspects of reacting and diffusing systems. *Lecture Notes in Biomath.* **28**. Springer, New York.

GIACOMIN, G. (1991). Spatial patterns of phase separation in more than one dimension: Study of a model. Preprint.

HOLLEY, R. and STROOCK, D. W. (1978). Generalized Ornstein Uhlenbeck processes and infinite branching Brownian motions. *Kyoto Univ. RIMSA* **14** 741.

MITOMA, I. (1983). Tightness of probabilities on $C([0,1], \mathscr{S}')$ and $D([0,1], \mathscr{S}')$. *Ann. Probab.* **11** 989–999.

PELLEGRINOTTI, A. (1991). Phase separation in an interacting particle system. CARR 22/91. To appear in *Proc. Internat. Workshop on Probabilistic Methods, Siena*.

PENROSE, O. (1991). A mean field equation of motion for the dynamic Ising model. *J. Statist. Phys.* **63** 975–986.

REBOLLEDO, R. (1980). Sur l'éxistence de solution à certains problems de semimartingales. *C. R. Acad. Sci. Paris* **280** 843.

SPOHN, H. (1991). *Large Scale Dynamics of Interacting Particles*. Springer, New York.

VARES, M. E. (1990). A note on small random perturbations of dynamical systems. *Stochastic Process. Appl.* **35** 225–230.

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