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ON THE UPPER AND LOWER CLASSES FOR A STATIONARY GAUSSIAN STOCHASTIC PROCESS[1]

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We give a complete and rather explicit characterization of the upper and lower classes for a family of stationary Gaussian stochastic processes.

1. Introduction. We shall assume that our probability space $(\Omega, \mathscr{F}, \mathbf{P})$ is complete and that $\{\xi(t)\}_{t\in T}$ is an $\mathbb{R}$-valued separable stochastically continuous standardized Gaussian random field on a pseudometric unbounded space (T, ρ). Let (T, ρ) be equipped with an abelian group-operation $+$ such that the covariance $r(s,t) \equiv \mathbf{E}\{\xi(s)\xi(t)\}$ satisfies $r(s+u, t+u) = r(s,t)$ for $s, t, u \in T$ and whose bounded subsets are totally bounded in the canonical pseudometric $d(s,t) \equiv [\mathbf{E}\{(\xi(t)-\xi(s))^2\}]^{1/2}$. We also define the entropy $N_S(\varepsilon)$ as the minimum number of closed d-balls $\mathscr{O}_\varepsilon$ of radius ε needed to cover $S \subseteq T$ and $M_S(\varepsilon)$ as the largest n for which there exist $t_1, \dots, t_n \in S$ satisfying $d(t_i, t_j) > \varepsilon$ for each $i \neq j$, and we write $\mathbf{P}_0\{S\} \equiv \sup\{\mathbf{P}\{B\}: S \supseteq B \in \mathscr{F}\}$, $\mathbf{P}^0\{S\} \equiv \inf\{\mathbf{P}\{B\}: S \subseteq B \in \mathscr{F}\}$, Φ for the standard Gaussian d.f., $\underline{\Phi} \equiv 1 - \Phi$, $0 \cdot \infty \equiv 0$, $S_\rho(t,\varepsilon) \equiv \{s \in T: \rho(s,t) < \varepsilon\}$, $S(t,\varepsilon) \equiv \{s \in T: d(s,t) \leq \varepsilon\}$ and $\sigma(t,\varepsilon) \equiv \sup\{0 \vee r(s,t): s \in T - S_\rho(t,\varepsilon)\}$.

In view of recent tight tail-estimates for local suprema over d-compacts of general Gaussian processes (cf., e.g., [1], [2], [3], [7], [15], [22], [25] and [28]), one is motivated to study also the global behaviour of suprema. Here the only tractable approach seems to be upper and lower classes.

Let Ψ be the class of functions $\psi: T \to [-\infty, \infty]$. Provided that $\sigma(t, \Delta) \to 0$ not too slowly as $\Delta \to \infty$, we prove a zero–one law for the sets

$$E(\psi) \equiv \{\omega \in \Omega: \text{the set } \{t \in T: \xi(\omega;t) > \psi(t)\} \text{ is } \rho\text{-unbounded}\}, \qquad \psi \in \Psi.$$

We also give an explicit characterization of when the different values for $\mathbf{P}\{E(\psi)\}$ occur, that is, we determine the upper and lower classes for $\xi(t)$.

Consider the Euclidean case $(T, \rho, +) = (\mathbb{R}, |\cdot|, +)$ and assume that

$$(1.1)\qquad \begin{aligned} 0 &< \liminf_{t\to s} |t-s|^{-\alpha}(1 - r(s,t)) \\ &\leq \limsup_{t\to s} |t-s|^{-\alpha}(1-r(s,t)) < \infty \end{aligned}$$

for some $\alpha \in (0, 2]$. Following the discovery of the tail behaviour for the

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suprema of such a process in Pickands [17, 18] and also in [5] and [20], upper and lower classes were studied for increasing ψ's in Ψ by Pathak and Qualls [16], Qualls and Watanabe [19, 20], Watanabe [26] and Weber [27]: Assuming $\lim_{|t-s|\to\infty} r(s,t)\log|t-s| = 0$, they proved that

$$\mathbf{P}\{E(\psi)\} = 0 \quad \Leftrightarrow \quad \int_0^\infty (1 \vee \psi(t))^{2/\alpha} \underline{\Phi}(\psi(t))\, dt < \infty \tag{1.2}$$

for increasing ψ's in Ψ, while $\mathbf{P}\{E(\psi)\} = 1$ when the integral is infinite.

For $(T, \rho, +) = (\mathbb{R}^n, |\cdot|, +)$ Kôno [12] and Qualls and Watanabe [21] showed that, if $\psi = \varphi \circ |\cdot|$ with φ: $[0,\infty) \to (0,\infty)$ increasing, if (1.1) holds and if $r(s,t)(\log|t-s|)^{4+2n/\min\{\alpha, 2-\alpha\}} \to 0$ as $|t-s| \to \infty$, then

$$\mathbf{P}\{E(\psi)\} = 0 \quad \Leftrightarrow \quad \int_{\mathbb{R}^n} \psi(t)^{2n/\alpha} \underline{\Phi}(\psi(t))\, dt < \infty. \tag{1.3}$$

The proofs of (1.2) and (1.3) use crucially that ψ is increasing and for more general ψ's there are no corresponding results in the literature.

The contribution of the present investigation is a characterization of when $\mathbf{P}\{E(\psi)\} = 0$ valid for all $\psi \in \Psi$. Since our methods do not use any order structure we can also prove our results on a general space.

2. The main result. Our main result is the following theorem.

THEOREM 1. *Assume that there is an $R \in (0, \sqrt{2})$ such that*

$$\limsup_{\varepsilon \downarrow 0} N_{\mathscr{O}_R}(x\varepsilon)/N_{\mathscr{O}_R}(\varepsilon) < \infty \quad \text{for some } x \in (0,1), \tag{2.1}$$

and such that for each $C > 0$ and $s \in T$ there is an increasing sequence $\{\varrho_s(n)\}_{n=0}^\infty$, with $\varrho_s(0) = 0$ and $\lim_{n\to\infty} \varrho_s(n) = \infty$ for $s \in T$, satisfying

$$\sup_{s\in T} \sum_{\{n\geq 0:\, \sigma(s,\varrho_s(n))>0\}} N_{S_\rho(s,\varrho_s(n+1))}(R) \exp\{-C/\sigma(s,\varrho_s(n))\} < \infty. \tag{2.2}$$

Then $E(\psi) \in \mathscr{F}$ with $\mathbf{P}\{E(\psi)\}$ zero or one for each $\psi \in \Psi$, and moreover

$$\mathbf{P}\{E(\psi)\} = 0 \quad \Leftrightarrow \quad \sum_{n=1}^\infty N_{\mathscr{O}_{r_n}}\left(\left(1 \vee \inf_{t\in S_n} \psi(t)\right)^{-1}\right) \times \underline{\Phi}\left(\inf_{t\in S_n} \psi(t)\right) < \infty \tag{2.3}$$

for some covering $S_n = S(t_n, r_n)$, $n = 1, 2, \ldots,$ of T with $r_n \leq R$ for all n.

REMARK 1. Note that, by (2.2), given $\varepsilon > 0$ and $t_0 \in T$, we have $r(t, t_0) < \varepsilon$ for $\rho(t, t_0) \geq k$ and k large, which yields $S(t_0, \sqrt{2(1-\varepsilon)}) \subseteq S_\rho(t_0, k)$. Thus $\mathscr{O}_\delta$ is d-totally bounded for $\delta < \sqrt{2}$ so that $N_{\mathscr{O}_R}(\varepsilon) < \infty$ and each covering $\{S(t_n, r_n)\}$ of T with $r_n \leq R$ is infinite. Also observe that (2.1) means O-regularly varying entropy at 0 (cf. e.g., [5a]).

PROOF ($\Leftarrow$). We have, for $\varepsilon \le \delta \le R/3$, (since $N_S(\varepsilon) \le M_S(\varepsilon) \le N_S(\varepsilon/2)$),

$$(2.4)\quad M_{\mathscr{O}_\delta}(\varepsilon) \le N_{\mathscr{O}_\delta}\left(\frac{\varepsilon}{2}\right) \le \frac{N_{\mathscr{O}_{R/3+\delta+\varepsilon}}(\varepsilon/2)}{M_{\mathscr{O}_{R/3}}(2\delta+2\varepsilon)} \le \frac{N_{\mathscr{O}_R}(\varepsilon/2)}{N_{\mathscr{O}_R}(4\delta)/N_{\mathscr{O}_R}(R/3)},$$

and (2.4) trivially extends to $\varepsilon \le \delta \le R$. Letting l be the smallest integer having $x^{-l} \ge 8\delta/\varepsilon$, $K_1 \equiv \sup_{\varepsilon>0} N_{\mathscr{O}_R}(x\varepsilon)/N_{\mathscr{O}_R}(\varepsilon)$ [$<\infty$ by (2.1)], $K_2 \equiv K_1 N_{\mathscr{O}_R}(R/3)$ and $y \equiv -\log K_1/\log x$, we get $K_1^l \le K_1(8\delta/\varepsilon)^y$ and hence

$$(2.5)\quad \begin{aligned} M_{\mathscr{O}_\delta}(\varepsilon) &\le N_{\mathscr{O}_\delta}(\varepsilon/2) \le N_{\mathscr{O}_R}(R/3)\prod_{k=0}^{l-1}\left[N_{\mathscr{O}_R}(4\delta x^{k+1})/N_{\mathscr{O}_R}(4\delta x^k)\right] \\ &\le K_2(8\delta/\varepsilon)^y \quad \text{for } \varepsilon \le \delta \le R. \end{aligned}$$

Now, by (2.5), $\limsup_{\varepsilon\downarrow 0} \log\log N_{\mathscr{O}_R}(\varepsilon)/\log(1/\varepsilon) = 0$ so $\{\xi(t)\}_{t\in\mathscr{O}_R}$ has an a.s. bounded version; compare [6], [8] and [24]. Since $N_{S_\rho(t_0,\delta)}(R) < \infty$ for $t_0 \in T$, $\delta > 0$, ρ-separability yields that $\{\xi(t)\}_{t\in S_\rho(t_0,\delta)}$ is a.s. bounded so

$$\mathbf{E}\left\{\sup_{t\in S_\rho(t_0,\delta)} \xi(t)^2\right\} \le 2\mathbf{E}\left\{\left(\sup_{t\in S_\rho(t_0,\delta)} \xi(t)\right)^2\right\} < \infty;$$

compare [8], [9] and [13]. Since $\xi(t)$ is stochastically continuous, we get

$$d(t,t_0)^2 \le \varepsilon^2 + \int_{G_\varepsilon} (\xi(t)-\xi(t_0))^2\, d\mathbf{P} \le \varepsilon^2 + 4\int_{G_\varepsilon} \sup_{s\in S_\rho(t_0,\delta)} \xi(s)^2\, d\mathbf{P} \to \varepsilon^2$$

as $\rho(t,t_0) \to 0$, where $G_\varepsilon \equiv \{\omega \in \Omega: |\xi(\omega;t) - \xi(\omega;t_0)| > \varepsilon\}$, so $d(t,t_0) \to 0$ as $\rho(t,t_0) \to 0$. Hence d-open sets are ρ-open and so $\{\xi(t)\}_{t\in T}$ is d-separable. In view of $\xi(t)$'s (trivial) d-stochastic continuity, it follows readily that any countable d-dense subset of $\mathscr{O}_\varepsilon$ is a separator for $\{\xi(t)\}_{t\in\mathscr{O}_\varepsilon}$.

Take $a_0 = \min\{(1-x^{1/2})^{1/2}/4, R/2\}$ and $t \in T$, let $C_0 = \{t\}$ and let C_n be a $(a/u)x^n$-net in $S(t,a/u)$ [i.e., for each $s \in S(t,a/u)$ there is a $v \in C_n$ such that $d(s,v) \le (a/u)x^n$] with $d(s_1,s_2) > (a/u)x^n$ for $C_n \ni s_1 \ne s_2 \in C_n$, so $\#C_n \le M_{\mathscr{O}_{a/u}}((a/u)x^n)$. Write $p_n = (1-x^{1/2})x^{(n-1)/2}$ and $C = \bigcup_{n=0}^\infty C_n$ and choose $t_n(s) \in C_n$ with $d(t_n(s),s) \le (a/u)x^n$ for $s \in C$. (Samorodnitsky uses a similarly constructed set C in [22].) Then $\xi(s) = \xi(t) + \sum_{n=1}^N[\xi(t_n(s)) - \xi(t_{n-1}(s))]$ for some N for each $s \in C$. Adapting ([4], the proof of Theorem 6) to the present context, we get

$$\begin{aligned} &\{\xi(s) > u + 1/u, \xi(t) \le u\} \\ &\quad\subseteq \bigcup_{n=1}^N \{\xi(t_n(s)) - \xi(t_{n-1}(s)) > p_n/u, \xi(t_n(s)) > u, \\ &\qquad\qquad \xi(t_{n-1}(s)) \le u + 1/u\}. \end{aligned}$$

Thus, since $d(t_n(s), t_{n-1}(s)) \le d(t_n(s), s) + d(s, t_{n-1}(s)) \le 2(a/u)x^{(n-1)}$,

$$
\begin{aligned}
&\mathbf{P}\Big\{\sup_{s\in S(t,a/u)} \xi(s) > u + 1/u,\, \xi(t) \le u\Big\} \\
&\quad = \mathbf{P}\Big\{\bigcup_{s\in C}\{\xi(s) > u + 1/u\},\, \xi(t) \le u\Big\} \\
(2.6)\qquad &\quad \le \sum_{n=1}^{\infty}\sum_{s_1\in C_{n-1}}\sum_{s_2\in C_n\cap S(s_1, 2(a/u)x^{n-1})} \mathbf{P}\{\xi(s_2) - \xi(s_1) > p_n/u, \\
&\qquad\qquad \xi(s_2) > u,\, \xi(s_1) \le u + 1/u\}.
\end{aligned}
$$

Now take $a \in (0, a_0]$ and $u \ge 1$ so that $r(s_1, s_2) = 1 - d(s_1, s_2)^2/2 \ge 1 - 2(a/u)^2 \ge 3/4$ for $d(s_1, s_2) \le 2(a/u)x^{n-1}$, which yields

$$
\left(\frac{1}{r(s_1,s_2)} - 1\right)\xi(s_1) = \frac{d(s_1,s_2)^2}{2r(s_1,s_2)}\xi(s_1) \le 8\left(\frac{a}{u}\right)^2 x^{2(n-1)}u \le \frac{p_n}{2u}
$$

for $\xi(s_1) \le u + 2/u$. Hence we have, for $d(s_1, s_2) \le 2(a/u)x^{n-1}$,

$$
\begin{aligned}
&\mathbf{P}\left\{\xi(s_2) - \xi(s_1) > \frac{p_n}{u},\, \xi(s_2) \ge u,\, \xi(s_1) \le u + \frac{2}{u}\right\} \\
&\quad \le \mathbf{P}\left\{\xi(s_2) - r(s_1,s_2)^{-1}\xi(s_1) > \frac{p_n}{2u},\, \xi(s_2) \ge u\right\} \\
(2.7)\qquad &\quad = \underline{\Phi}\left(\frac{\sqrt{2}\, r(s_1,s_2)p_n/(2u)}{\sqrt{1 + r(s_1,s_2)}\, d(s_1,s_2)}\right)\underline{\Phi}(u) \\
&\quad \le \underline{\Phi}\left(\frac{3(1 - x^{1/2})}{16ax^{(n-1)/2}}\right)\underline{\Phi}(u).
\end{aligned}
$$

Combining (2.5)–(2.7) we conclude that, uniformly for $u \ge 1$, as $a \downarrow 0$,

$$
\begin{aligned}
&\underline{\Phi}(u)^{-1}\mathbf{P}\left\{\sup_{s\in S(t,a/u)} \xi(s) > u + \frac{1}{u},\, \xi(t) \le u\right\} \\
(2.8)\qquad &\quad \le \sum_{n=1}^{\infty} M_{\mathscr{O}_{a/u}}\left(\left(\frac{a}{u}\right)x^{n-1}\right) M_{\mathscr{O}_{2(a/u)x^{n-1}}}\left(\left(\frac{a}{u}\right)x^{n}\right)\underline{\Phi}\left(\frac{3(1 - x^{1/2})}{16ax^{(n-1)/2}}\right) \\
&\quad \le K_2^2 \sum_{n=1}^{\infty} (128x^{-n})^{y}\,\underline{\Phi}\left(\frac{3(1 - x^{1/2})}{16ax^{(n-1)/2}}\right) = o(a).
\end{aligned}
$$

Arguing as for (2.6) for $\eta_u(s) \equiv 2u + 2/u - \xi(s)$, we deduce for future use that, by (2.5), (2.7) and symmetry, uniformly for $u \geq 1$, as $a \downarrow 0$,

$$
\begin{aligned}
&\underline{\Phi}(u)^{-1}\mathbf{P}\Big\{\inf_{s\in S(t,a/u)} \xi(s) \leq u,\ \xi(t) > u + 2/u\Big\} \\
&\quad \leq \underline{\Phi}(u)^{-1}\mathbf{P}\Big\{\sup_{s\in S(t,a/u)} \eta_u(s) > u + 1/u,\ \eta_u(t) \leq u\Big\} \\
&\quad \leq \underline{\Phi}(u)^{-1}\sum_{n=1}^{\infty}\sum_{s_1\in C_{n-1}}\sum_{s_2\in C_n\cap S(s_1, 2(a/u)x^{n-1})} \\
&\qquad \mathbf{P}\{\eta_u(s_2) - \eta_u(s_1) > p_n/u,\ \eta_u(s_2) > u,\ \eta_u(s_1) \leq u + 1/u\} \\
&\quad = \underline{\Phi}(u)^{-1}\sum_{n=1}^{\infty}\sum_{s_1\in C_{n-1}}\sum_{s_2\in C_n\cap S(s_1, 2(a/u)x^{n-1})} \\
&\qquad \mathbf{P}\{\xi(s_1) - \xi(s_2) > p_n/u,\ \xi(s_1) \geq u + 1/u, \\
&\qquad\qquad \xi(s_2) < u + 2/u\} = o(a).
\end{aligned}
\tag{2.9}
$$

In order to proceed we observe that, by (2.5), for $a \leq 1$ and $\delta \leq R$,

$$
\begin{aligned}
&N_{\mathscr{O}_\delta}(a\varepsilon) \leq N_{\mathscr{O}_\delta}(\varepsilon)N_{\mathscr{O}_\varepsilon}(a\varepsilon) \leq K_2(8/a)^{y}N_{\mathscr{O}_\delta}(\varepsilon) \quad \text{for } \varepsilon \leq R, \\
&N_{\mathscr{O}_\delta}(a\varepsilon) \leq N_{\mathscr{O}_R}(aR) \leq K_2(8/a)^{y}N_{\mathscr{O}_\delta}(\varepsilon) \quad \text{for } \varepsilon > R.
\end{aligned}
\tag{2.10}
$$

Further $u - 2/u \equiv \tilde{u} \geq (1/2)u \geq 1$ for $u \geq 2$, so that $\tilde{u} + 1/\tilde{u} \leq u$, and

$$\underline{\Phi}(\tilde{u}) \leq \frac{1}{\tilde{u}}\phi(\tilde{u}) \leq \frac{2}{u}e^2\phi(u) \leq \tfrac{8}{3}e^2\underline{\Phi}(u),$$

where $\phi(u) = (2\pi)^{-1/2}\exp\{-u^2/2\}$. Now

$$\mathbf{P}\Big\{\sup_{s\in S(t,a/u)} \xi(s) > u + 1/u,\ \xi(t) \leq u\Big\} \leq \underline{\Phi}(u) \quad \text{for } u \geq 1$$

for some sufficiently small $a \in (0, a_0]$ [cf. (2.8)]. Hence we conclude

$$
\begin{aligned}
&\mathbf{P}\Big\{\sup_{s\in\mathscr{O}_\delta} \xi(s) > u\Big\} \\
&\quad \leq N_{\mathscr{O}_\delta}(a/u)\Big[\mathbf{P}\Big\{\sup_{s\in S(t,a/u)} \xi(s) > u,\ \xi(t) \leq \tilde{u}\Big\} + \mathbf{P}\{\xi(t) > \tilde{u}\}\Big] \\
&\quad \leq N_{\mathscr{O}_\delta}(a/u)\Big[\mathbf{P}\Big\{\sup_{s\in S(t,a/\tilde{u})} \xi(s) > \tilde{u} + 1/\tilde{u},\ \xi(t) \leq \tilde{u}\Big\} + \underline{\Phi}(\tilde{u})\Big] \\
&\quad \leq \tfrac{16}{3}e^2K_2(8/a)^{y}N_{\mathscr{O}_\delta}(1/u)\underline{\Phi}(u) \quad \text{for } u \geq 2 \text{ and } \delta \leq R.
\end{aligned}
$$

Obviously the right-hand side is at least 1 for $1 \leq u < 2$, and taking $K_3 \equiv \frac{16}{3}e^2K_2(8/a)^{y}$ it therefore follows that

$$\mathbf{P}\Big\{\sup_{s\in\mathscr{O}_\delta} \xi(s) > u\Big\} \leq K_3N_{\mathscr{O}_\delta}\big((1 \vee u)^{-1}\big)\underline{\Phi}(u) \quad \text{for } \delta \leq R \text{ and all } u. \tag{2.11}$$

Assume that the sum (2.3) is finite for a covering $\{S_n\} = \{S(t_n, r_n)\}$ of T with $r_n \le R$. Taking $m = \sup\{\rho(t_1, t_n)\colon 1 \le n < J\}$ where

$$\sum_{n=J}^{\infty} N_{\mathscr{O}_{r_n}}\left(\left(1 \vee \inf_{t\in S_n} \psi(t)\right)^{-1}\right)\underline{\Phi}\left(\inf_{t\in S_n}\psi(t)\right) < \varepsilon/K_3,$$

completeness yields that $E(\psi) \in \mathscr{F}$ with $\mathbf{P}\{E(\psi)\} = 0$ since, by (2.11),

$$\begin{aligned}\mathbf{P}^0\{E(\psi)\} &\le \mathbf{P}^0\{\xi(t) > \psi(t) \text{ for some } t \in T \text{ with } \rho(t_1, t) > m + R\}\\ &\le \mathbf{P}\left\{\bigcup_{\{n:\,\rho(t_1,t_n)>m\}}\left\{\xi(t) > \inf_{s\in S_n}\psi(s) \text{ for some } t \in S_n\right\}\right\}\\ &\le K_3 \sum_{n=J}^{\infty} N_{\mathscr{O}_{r_n}}\left(\left(1 \vee \inf_{t\in S_n}\psi(t)\right)^{-1}\right)\underline{\Phi}\left(\inf_{t\in S_n}\psi(t)\right) < \varepsilon.\end{aligned}$$

($\Rightarrow$) Write $\Sigma(\{S_n\};\psi)$ for the sum (2.3) and assume that $\Sigma(\{S_n\};\psi) = \infty$ for each covering $S_n = S(t_n, r_n)$, $n = 1, 2, \ldots,$ of T with $r_n \le R$.

Taking $t_0 \in T$ and $2 \le u_1 \le u_2 \le \cdots$ with $\mathbf{P}\{\sup_{t\in S_\rho(t_0,n)} \xi(t) > u_n\} \le n^{-2}$ [recall that $\{\xi(t)\}_{t\in S_\rho(t_0,n)}$ is a.s. bounded], the function $\psi^*(t) \equiv u_1$ for $t \in S_\rho(t_0, 1)$ and $\psi^*(t) \equiv u_n$ for $t \in S_\rho(t_0, n) - S_\rho(t_0, n-1)$, $n \ge 2$, has

$$\mathbf{P}^0\{E(\psi^*)\} \le \lim_{n\to\infty} \mathbf{P}^0\{\xi(t) > \psi^*(t) \text{ for some } t \in T - S_\rho(t_0, n)\} = 0.$$

Clearly $\mathbf{P}_0\{A \cup B\} \le \mathbf{P}^0\{A\} + \mathbf{P}_0\{B\}$ so that $\mathbf{P}_0\{E(\psi \wedge \psi^*)\} = \mathbf{P}_0\{E(\psi) \vee E(\psi^*)\} \le \mathbf{P}_0\{E(\psi)\}$ and so, by completeness, it suffices to prove that

$$\varphi(t) \equiv (\psi(t) \wedge \psi^*(t)) \vee 2 \quad \text{has} \quad \mathbf{P}_0\{E(\varphi)\} = 1. \tag{2.12}$$

Now take $x, y > 0$. Then we have, for $0 \le r(s,t) < 1$,

$$\begin{aligned}&\mathbf{P}\{\xi(s) > x,\, \xi(t) > y\}\\ &\quad\le \mathbf{P}\{\xi(s) > x,\, \xi(t) > y,\, \xi(t) \ge \xi(s)\}\\ &\qquad + \mathbf{P}\{\xi(s) > x,\, \xi(t) > y,\, \xi(t) \le \xi(s)\}\\ &\quad\le \mathbf{P}\{\xi(t) - r(s,t)\xi(s) > (1 - r(s,t))y,\, \xi(s) > x\}\\ &\qquad + \mathbf{P}\{\xi(s) - r(s,t)\xi(t) > (1 - r(s,t))x,\, \xi(t) > y\}\\ &\quad= \underline{\Phi}\left(\sqrt{\frac{1 - r(s,t)}{1 + r(s,t)}}\, y\right)\underline{\Phi}(x) + \underline{\Phi}\left(\sqrt{\frac{1 - r(s,t)}{1 + r(s,t)}}\, x\right)\underline{\Phi}(y).\end{aligned}$$

Further we have, for $-1 < r(s,t) \le 0$,

$$\{\xi(s) > x,\, \xi(t) > y\} \subseteq \begin{cases}\{\xi(t) - r(s,t)\xi(s) > (1 - r(s,t))y,\, \xi(s) > x\}, & y \le x,\\ \{\xi(s) - r(s,t)\xi(t) > (1 - r(s,t))x,\, \xi(t) > y\}, & y \ge x,\end{cases}$$

and repeating the above arguments we therefore readily conclude

$$\text{(2.13)}\qquad \begin{aligned} &\mathbf{P}\{\xi(s) > x, \xi(t) > y\}\\ &\quad \le \underline{\Phi}\big(\tfrac{1}{2}d(s,t)y\big)\underline{\Phi}(x) + \underline{\Phi}\big(\tfrac{1}{2}d(s,t)x\big)\underline{\Phi}(y)\end{aligned}$$

for $x, y > 0$ and $r(s,t) < 1$ [the left-hand side is 0 for $r(s,t) = -1$].

Take a (p/v)-net $\{s_i\}_{i=1}^n$ in $\mathscr{O}_\delta$ with $d(s_i, s_j) > p/v$ for $s_i \ne s_j$. Since

$$\text{(2.14)}\qquad M_{\mathscr{O}_{\delta \wedge (kp/v)}}(p/v) \le K_2\left(8\frac{\delta \wedge (kp/v)}{\delta \wedge (p/v)}\right)^y \le K_2(8k)^y$$

for $\delta \le R$ and $k \ge 1$ [again using (2.5)], we obtain, by (2.13),

$$\begin{aligned} &\sum_{i \ne j} \mathbf{P}\big\{\xi(s_i) > v, \xi(s_j) > v\big\}\\ &\quad \le 2\underline{\Phi}(v) \sum_{i=1}^{n} \sum_{k=1}^{[2\delta v/p]} \sum_{\{1 \le j \le n:\, kp/v < d(s_i,s_j) \le (k+1)p/v\}} \underline{\Phi}\big(\tfrac{1}{2}d(s_i,s_j)v\big)\\ &\quad \le 2n\underline{\Phi}(v) \sum_{k=1}^{\infty} K_2(8(k+1))^y \underline{\Phi}\big(\tfrac{1}{2}kp\big) \le \tfrac{1}{2}n\underline{\Phi}(v)\end{aligned}$$

for $v > 0$, $\delta \le R$ and some $p \ge 1$ (not depending on δ or v). Since, by (2.10), $N_{\mathscr{O}_\delta}(1/v) \le K_2(8p)^y N_{\mathscr{O}_\delta}(p/v) \le K_2(8p)^y n$ for $\delta \le R$, we readily deduce, taking $K_4 \equiv \frac{1}{2}K_2^{-1}\underline{\Phi}(1)(8p)^{-y}$ and $v \equiv u \vee 1$, for $u \in \mathbb{R}$ and $\delta \le R$,

$$\text{(2.15)}\qquad \begin{aligned} \mathbf{P}\Big\{\sup_{t \in \mathscr{O}_\delta} \xi(t) > u\Big\} &\ge \mathbf{P}\Big\{\sup_{1 \le i \le n} \xi(s_i) > v\Big\}\\ &\ge n\underline{\Phi}(v) - \sum_{i \ne j} \mathbf{P}\big\{\xi(s_i) > v, \xi(s_j) > v\big\}\\ &\ge \tfrac{1}{2}n\underline{\Phi}(v)\\ &\ge K_4 N_{\mathscr{O}_\delta}\big((1 \vee u)^{-1}\big)\underline{\Phi}(u).\end{aligned}$$

Now, combining (2.11) and (2.15) we get, for each choice of $\{S_n\}$,

$$\text{(2.16)}\qquad \begin{aligned} K_3 \textstyle\sum(\{S_n\};\varphi) &\ge \displaystyle\sum_{n=1}^{\infty} \mathbf{P}\Big\{\sup_{t \in S_n} \xi(t) > \inf_{t \in S_n} \varphi(t)\Big\}\\ &\ge \sum_{n=1}^{\infty} \mathbf{P}\Big\{\sup_{t \in S_n} \xi(t) > \inf_{t \in S_n} \psi(t) \wedge \psi^*(t)\Big\}\mathbf{P}\Big\{\sup_{t \in S_n} \xi(t) > 2\Big\}\\ &\ge K_4\underline{\Phi}(2) \textstyle\sum(\{S_n\};\psi) = \infty.\end{aligned}$$

Let $r_t \equiv \sup\{r > 0\colon r \inf_{s \in S(t,r)} \varphi(s) < a\}$ for $a \le R \wedge 1$, $t \in T$, so that $a/\psi^*(t) \le r_t \le a/2$. Taking $\delta_k \uparrow r_t$ with $\delta_k \inf_{s \in S(t,\delta_k)} \varphi(s) < a$, we get

$$\text{(2.17)}\qquad \begin{aligned} a\Big/\Big(\inf_{s \in S(t,r_t)} \varphi(s)\Big) &\ge \lim_{k \to \infty} a\Big/\Big(\inf_{s \in S(t,\delta_k)} \varphi(s)\Big) \ge \lim_{k \to \infty} \delta_k = r_t,\\ a\Big/\Big(\inf_{s \in S(t,r_t)} \varphi(s)\Big) &\le \lim_{\varepsilon \downarrow 0} a\Big/\Big(\inf_{s \in S(t,r_t+\varepsilon)} \varphi(s)\Big) \le \lim_{\varepsilon \downarrow 0} r_t + \varepsilon = r_t.\end{aligned}$$

Ordering $\mathscr{S} \equiv \{A \subseteq T\colon A \ni s \ne t \in A \Rightarrow d(s,t) > r_s \wedge r_t\}$ partially by $A \le B \Leftrightarrow A \subseteq B$, a chain $\{A_\alpha\} \subseteq \mathscr{S}$ has upper bound $\bigcup\{A_\alpha\}$ so that, by Zorn's lemma, $\mathscr{S}$ has a maximal element $\mathscr{C}$. Here $\mathscr{C}$'s maximality readily yields $\bigcup_{t\in\mathscr{C}} S_t = T$, where $S_t \equiv S(t, r_t)$. Further, since $\#\mathscr{C} \cap S_\rho(t_0, n) \le M_{S_\rho(t_0,n)}(a/u_n) < \infty$, we have $\#\mathscr{C} \le \aleph_0$ and, by (2.16), $\Sigma(\{S_t\};\varphi) = \infty$. Writing $\varphi_t = \inf_{s\in S_t}\varphi(s)$ we therefore obtain, by (2.17),

$$\sum_{t\in\mathscr{C}} \underline{\Phi}(\varphi_t) = \sum_{t\in\mathscr{C}} N_{S_t}(r_t/a)\underline{\Phi}(\varphi_t) = \sum(\{S_t\};\varphi) = \infty. \tag{2.18}$$

Now let

$$\varphi_t^* \equiv \varphi_t + 2/\varphi_t,$$

$$J_t \equiv \{\omega \in \Omega: \xi(\omega;t) > \varphi_t^*, \inf_{s\in S_t}\xi(\omega;s) > \varphi_t\},$$

$$\mathscr{C}_m^N \equiv \{t \in \mathscr{C}: m \le \rho(t_0,t) < N\}.$$

Letting I_t be the indicator of J_t, we get

$$\begin{aligned}
\mathbf{P}_0\{E(\varphi)\} &= \mathbf{P}_0\left\{\bigcap_{m=1}^{\infty}\bigcup_{N=m}^{\infty}\bigcup_{t\in\mathscr{C}_m^N}\{\xi(s) > \varphi(s) \text{ for some } s \in S_t\}\right\}\\
&\ge \mathbf{P}\left\{\bigcap_{m=1}^{\infty}\bigcup_{N=m}^{\infty}\left\{\sum_{t\in\mathscr{C}_m^N} I_t > 0\right\}\right\}\\
&\ge \limsup_{m\to\infty}\limsup_{N\to\infty}\left(\int_{\{\Sigma_{t\in\mathscr{C}_m^N} I_t > 0\}}\sum_{t\in\mathscr{C}_m^N} I_t\,d\mathbf{P}\right)^2\Bigg/\mathbf{E}\left\{\left(\sum_{t\in\mathscr{C}_m^N} I_t\right)^2\right\}\\
&\ge 1 - \liminf_{m\to\infty}\liminf_{N\to\infty}\operatorname{Var}\left\{\sum_{t\in\mathscr{C}_m^N} I_t\right\}\Bigg/\left(E\left\{\sum_{t\in\mathscr{C}_m^N} I_t\right\}\right)^2,
\end{aligned} \tag{2.19}$$

where the second inequality follows from Hölder's inequality. Write

$$\mu_{s,t} = \mathbf{P}\{\xi(s) > \varphi_s^*, \xi(t) > \varphi_t^*\} - \mathbf{P}\{\xi(s) > \varphi_s^*\}\mathbf{P}\{\xi(t) > \varphi_t^*\} \quad \text{for } s,t \in \mathscr{C}$$

and note that, since $\underline{\Phi}(\varphi_t^*) \ge \frac{1}{2}e^{-5/2}\underline{\Phi}(\varphi_t)$, we have, by (2.9) and (2.17),

$$\mathbf{E}\{I_t\} = \underline{\Phi}(\varphi_t^*) - \mathbf{P}\left\{\xi(t) > \varphi_t^*, \inf_{s\in S_t}\xi(s) \le \varphi_t\right\} \ge \tfrac{1}{4}e^{-5/2}\underline{\Phi}(\varphi_t) \tag{2.20}$$

for $t \in \mathscr{C}$ and $a \le a_1$, for some $a_1 \le R \wedge 1$. Since, again by (2.9) and (2.17),

$$\begin{aligned}
\operatorname{Var}\left\{\sum_{t\in\mathscr{C}_m^N} I_t\right\} &\le \sum_{(s,t)\in\mathscr{C}_m^N\times\mathscr{C}_m^N}[\mathbf{P}\{\xi(s) > \varphi_s^*, \xi(t) > \varphi_t^*\} - \mathbf{P}\{J_s\}\mathbf{P}\{J_t\}]\\
&= \sum_{(s,t)\in\mathscr{C}_m^N\times\mathscr{C}_m^N}\mu_{s,t} - \left(\sum_{t\in\mathscr{C}_m^N}\mathbf{P}\left\{\xi(t) > \varphi_t^*, \inf_{v\in S_t}\xi(v) \le \varphi_t\right\}\right)^2\\
&\quad + 2\sum_{(s,t)\in\mathscr{C}_m^N\times\mathscr{C}_m^N}\underline{\Phi}(\varphi_s^*)\mathbf{P}\left\{\xi(t) > \varphi_t^*, \inf_{v\in S_t}\xi(v) \le \varphi_t\right\}\\
&\le \sum_{(s,t)\in\mathscr{C}_m^N\times\mathscr{C}_m^N}\mu_{s,t} + o(a)\left(\sum_{t\in\mathscr{C}_m^N}\underline{\Phi}(\varphi_t^*)\right)^2,
\end{aligned}$$

(2.19) and (2.20) show that in order to prove (2.12) it suffices to prove

$$(2.21)\qquad \liminf_{m\to\infty}\liminf_{N\to\infty}\Bigg(\sum_{(s,t)\in\mathscr{C}_m^N\times\mathscr{C}_m^N}\mu_{s,t}\Bigg)\Bigg/\Bigg(\sum_{t\in\mathscr{C}_m^N}\underline{\Phi}(\varphi_t)\Bigg)^2\le 0\quad\text{for } a\le a_1.$$

Given an integer $k\ge 1$, partition $\mathscr{C}_m^N\times\mathscr{C}_m^N$ into

$$\mathscr{C}_{m,N}^{k,1}\equiv\Big\{(s,t): d(s,t)>R,\ 0<r(s,t)\le k^{-1}\big[(\varphi_s^*)^2+(\varphi_t^*)^2\big]^{-1}\Big\},$$

$$\mathscr{C}_{m,N}^{k,2}\equiv\Big\{(s,t): d(s,t)>R,\ r(s,t)>k^{-1}\big[(\varphi_s^*)^2+(\varphi_t^*)^2\big]^{-1}\Big\},$$

$$\mathscr{C}_{m,N}^{3}\equiv\{(s,t): 0<d(s,t)\le R,\ r(s,t)>0,\ \tfrac12\varphi_s\le\varphi_t\le 2\varphi_s\},$$

$$\mathscr{C}_{m,N}^{4}\equiv\{(s,t): 0<d(s,t)\le R,\ r(s,t)>0,\ \varphi_t>2\varphi_s \text{ or } \varphi_s>2\varphi_t\},$$

$$\mathscr{C}_{m,N}^{5}\equiv\{(s,t): d(s,t)>0,\ r(s,t)\le 0\}\cup\{(s,t): s=t\}.$$

Now we have, by (an analysis of the proof of) [14, Theorem 4.2.1],

$$(2.22)\qquad |\mu_{s,t}\le\begin{cases}\dfrac{r(s,t)}{2\pi\sqrt{1-r(s,t)^2}}\exp\left\{-\dfrac{(\varphi_s^*)^2+(\varphi_t^*)^2}{2(1+r(s,t))}\right\}, & \text{for } 0\le r(s,t)<1,\\ 0, & \text{for } r(s,t)\le 0,\end{cases}$$

and using that $2\varphi_s^*\varphi_t^*\le(\varphi_s^*)^2+\varphi_t^*)^2$ and $\phi(\varphi_s^*)\le\frac43\varphi_s^*\underline{\Phi}(\varphi_s^*)$ we thus get

$$\mu_{s,t}\le\frac{r(s,t)}{\sqrt2\,\pi d(s,t)\sqrt{1+r(s,t)}}\exp\left\{-\frac{1-r(s,t)}{2}\left((\varphi_s^*)^2+(\varphi_t^*)^2\right)\right\}$$

$$(2.23)\qquad\le\frac{e^{1/(2k)}\phi(\varphi_s^*)\phi(\varphi_t^*)}{\sqrt2\,Rk\varphi_s^*\varphi_t^*}$$

$$\le\frac{16e^{1/(2k)}\underline{\Phi}(\varphi_s)\underline{\Phi}(\varphi_t)}{9\sqrt2\,Rk}\quad\text{for } (s,t)\in\mathscr{C}_{m,N}^{k,1}.$$

Further, again by (2.22), for $\varphi_t^*\ge\varphi_s^*$, $d(s,t)>R$ and $r(s,t)\ge 0$,

$$\mu_{s,t}\le\frac{r(s,t)}{\sqrt2\,\pi R}\exp\left\{-\frac{(\varphi_s^*)^2}{2}-\frac{d(s,t)^2(\varphi_t^*)^2}{4(1+r(s,t))}\right\}$$

$$\le\frac{4\varphi_s^*\underline{\Phi}(\varphi_s^*)}{3\sqrt{\pi}R}\exp\left\{-\frac{R^2(\varphi_t^*)^2}{8}\right\}.$$

Thus, taking $C \equiv R^2/(48k)$ in (2.2) and using $\sqrt{2x}\,e^{-x} \le 1$ and (2.17),

$$\sup_{s\in\mathscr{C}_m^N} \underline{\Phi}(\varphi_s)^{-1} \sum_{\{t\in\mathscr{C}_m^N:\,(s,t)\in\mathscr{C}_{n,N}^{k,2},\ \varphi_t^*\ge\varphi_s^*\}} \mu_{s,t}$$

$$\le \sup_{s\in\mathscr{C}_m^N}\Bigg[\sum_{l=2}^{\infty}\sum_{n=0}^{\infty}\sum_{\{t\in\mathscr{C}_m^N:\,l\le\varphi_t^*<l+1,\ \varrho_s(n)\le\rho(s,t)<\varrho_s(n+1),\ r(s,t)>0\}} \frac{4\varphi_t^*}{3\sqrt{\pi}R} \times\exp\left\{-\frac{R^2(\varphi_t^*)^2}{12}\right\}\exp\left\{-\frac{R^2}{48kr(s,t)}\right\}\Bigg]$$

$$\le \sup_{s\in\mathscr{C}_m^N}\Bigg[\sum_{l=2}^{\infty}\sum_{\{n\ge 0:\,\sigma(s,\varrho_s(n))>0\}}\frac{8}{\sqrt{3\pi}R^2}M_{S_\rho(s,\varrho_s(n+1))}\left(\frac{a}{l+1}\right) \times\exp\left\{-\frac{R^2l^2}{24}\right\}\exp\left\{-\frac{C}{\sigma(s,\varrho_s(n))}\right\}\Bigg]$$

$$\le \sup_{s\in\mathscr{C}_m^N}\Bigg[\sum_{\{n\ge 0:\,\sigma(s,\varrho_s(n))>0\}} N_{S_\rho(s,\varrho_s(n+1))}(R)\exp\left\{-\frac{C}{\sigma(s,\varrho_s(n))}\right\}\Bigg] \times\frac{8K_2}{\sqrt{3\pi}R^2}\sum_{l=2}^{\infty}\left(8R\frac{(l+1)}{a}\right)^{y}\exp\left\{-\frac{R^2l^2}{24}\right\}\equiv K_5<\infty$$

[again using (2.5)]. Since

$$\sum_{t\in\mathscr{C}_0^m}\underline{\Phi}(\varphi_t)\le N_{S_\rho(t_0,m)}(a/u_m)\underline{\Phi}(2)<\infty$$

so that, by (2.18), $\lim_{N\to\infty}\sum_{t\in\mathscr{C}_m^N}\underline{\Phi}(\varphi_t)=\infty$, we deduce, by symmetry,

$$\text{(2.24)}\qquad \liminf_{N\to\infty}\frac{\sum_{(s,t)\in\mathscr{C}_{m,N}^{k,2}}\mu_{s,t}}{\left(\sum_{t\in\mathscr{C}_m^N}\underline{\Phi}(\varphi_t)\right)^2}\le\liminf_{N\to\infty}\frac{2K_5}{\sum_{t\in\mathscr{C}_m^N}\underline{\Phi}(\varphi_t)}=0\quad\text{for } a\le a_1.$$

Clearly we have, by (2.13), (2.14) and (2.17), for $s\in\mathscr{C}_m^N$,

$$\sum_{\{t\in\mathscr{C}_m^N:\,(s,t)\in\mathscr{C}_{m,N}^{3},\ \varphi_t\ge\varphi_s\}}\mu_{s,t}$$

$$\le\sum_{l=1}^{\infty}\sum_{\{t\in\mathscr{C}_m^N:\,la/(2\varphi_s)<d(s,t)\le R\wedge((l+1)a/(2\varphi_s)),\ \varphi_t\le 2\varphi_s\}} \times 2\underline{\Phi}(\varphi_s)\underline{\Phi}(\tfrac{1}{2}d(s,t)\varphi_s)$$

$$\le 2\underline{\Phi}(\varphi_s)\sum_{l=1}^{\infty}M_{\mathscr{O}_{R\wedge((l+1)a/(2\varphi_s))}}(a/(2\varphi_s))\underline{\Phi}(\tfrac{1}{4}la)\le K_6\underline{\Phi}(\varphi_s),$$

where K_6 does not depend on s. Arguing as for (2.24) we thus get

$$\liminf_{N\to\infty}\Bigg(\sum_{(s,t)\in\mathscr{C}^3_{m,N}}\mu_{s,t}\Bigg)\Bigg/\Bigg(\sum_{t\in\mathscr{C}^N_m}\underline{\Phi}(\varphi_t)\Bigg)^2=0\quad\text{for } a\le a_1. \tag{2.25}$$

Further we have, for $s\in\mathscr{C}^N_m$, by (2.5), (2.17), and (2.22) and using the facts that $\varphi_s\ge 2$ and that $x^\beta\exp\{-Kx^2\}\le(\beta/(2K))^{\beta/2}$,

$$\begin{aligned}
&\sum_{\{t\in\mathscr{C}^N_m:\,(s,t)\in\mathscr{C}^4_{m,N},\,\varphi_t>2\varphi_s\}}\mu_{s,t}\\
&\le\sum_{l=2}^{\infty}\sum_{\{t\in\mathscr{C}^N_m:\,l\varphi_s<\varphi_t\le(l+1)\varphi_s,\,r(s,t)>0,\,0<d(s,t)\le R\}}\frac{r(s,t)}{\sqrt{2}\,\pi d(s,t)\sqrt{1+r(s,t)}}\\
&\qquad\qquad\times\exp\left\{-\frac{(\varphi_s^*)^2+(\varphi_t^*)^2}{2(1+r(s,t))}\right\}\\
&\le\sum_{l=2}^{\infty}\frac{(l+1)\varphi_s}{\sqrt{2}\,\pi a}M_{\mathscr{O}_R}\left(\frac{a}{(l+1)\varphi_s}\right)\exp\left\{-\frac{(l^2+1)\varphi_s^2}{4}\right\}\\
&\le\underline{\Phi}(\varphi_s)\sum_{l=2}^{\infty}\frac{4K_2(8R(l+1)\varphi_s/a)^y(l+1)\varphi_s^2}{3\sqrt{\pi}\,a}\exp\left\{-\frac{(l^2-1)\varphi_s^2}{4}\right\}\\
&\le\underline{\Phi}(\varphi_s)\sum_{l=2}^{\infty}\frac{K_2(2+y)^{1+y/2}(8R(l+1)/a)^{1+y}}{6\sqrt{\pi}\,R}\exp\{-(l^2-3)\},
\end{aligned}$$

and invoking a by now familiar argument we thus obtain

$$\liminf_{N\to\infty}\Bigg(\sum_{(s,t)\in\mathscr{C}^4_{m,N}}\mu_{s,t}\Bigg)\Bigg/\Bigg(\sum_{t\in\mathscr{C}^N_m}\underline{\Phi}(\varphi_t)\Bigg)^2=0\quad\text{for } a\le a_1. \tag{2.26}$$

Finally we have, by the lower option in (2.22), for $a\le a_1$,

$$\liminf_{N\to\infty}\frac{\sum_{(s,t)\in\mathscr{C}^5_{m,N}}\mu_{s,t}}{\left(\sum_{t\in\mathscr{C}^N_m}\underline{\Phi}(\varphi_t)\right)^2}\le\liminf_{N\to\infty}\frac{\sum_{t\in\mathscr{C}^N_m}\left(\underline{\Phi}(\varphi_t)-\underline{\Phi}(\varphi_t^*)^2\right)}{\left(\sum_{t\in\mathscr{C}^N_m}\underline{\Phi}(\varphi_t)\right)^2}=0. \tag{2.27}$$

Combining (2.23)–(2.27) we see that (given $a<a_1$) the left-hand side of (2.21) is at most $O(1/k)$, and so (2.21) follows from sending $k\uparrow\infty$. □

COROLLARY 1. *Assume the hypothesis of Theorem* 1 *and that* d *is a complete metric. Then there exists an invariant* ($w.r.t.+$) *Haar measure* μ *on Borel sets of* (T,d) *with* $\mu(\mathscr{O}_\delta)<\infty$ *for* $\delta<\sqrt{2}$. *If* λ *is a version of this Haar measure, then* $\mathbf{P}\{E(\psi)\}=0$ *if and only if there is a covering* $S_n=S(t_n,r_n)$, $n=1,2,\dots,$ *of* T *with* $r_n\le R$ *such that*

$$\sum_{n=1}^{\infty}\left[1+\lambda(\mathscr{O}_{r_n})N_{\mathscr{O}_R}\left(\left(1\vee\inf_{t\in S_n}\psi(t)\right)^{-1}\right)\right]\underline{\Phi}\left(\inf_{t\in S_n}\psi(t)\right)<\infty. \tag{2.28}$$

PROOF. Since $d(t-s,t_0-s_0)\le d(s,s_0)+d(t,t_0)$, the map $(s,t)\to t-s$ is d-continuous. Hence $(T,d,+)$ is a locally compact (Hausdorff) topological group and μ exists and is Radon where, by Remark 1 and local compactness, $\mathscr{O}_\delta$ is compact for $\delta<\sqrt{2}$. Now, by (2.4) and (2.10),

$$N_{\mathscr{O}_\delta}(\varepsilon)\le 1+\frac{K_2N_{\mathscr{O}_R}(\varepsilon/2)}{K_1N_{\mathscr{O}_R}(4\delta)}\le 1+\frac{K_2^3N_{\mathscr{O}_R}(\varepsilon)}{512^{-y}K_1N_{\mathscr{O}_R}(\delta)}\le 1+\frac{K_2^3\lambda(\mathscr{O}_\delta)N_{\mathscr{O}_R}(\varepsilon)}{512^{-y}K_1\lambda(\mathscr{O}_R)}$$

for $\varepsilon>0$ and $\delta\le R$, and so the sum (2.3) is finite when (2.28) holds. Conversely (2.28) holds when the sum (2.3) is finite since, by (2.14),

$$\begin{aligned}\frac{N_{\mathscr{O}_R}(\varepsilon)}{N_{\mathscr{O}_\delta}(\varepsilon)}&\le N_{\mathscr{O}_R}(R/2)M_{\mathscr{O}_{R/2}}((R/2)\wedge(2\delta))M_{\mathscr{O}_{(R/2)\wedge(2\delta)}}(\delta)\\&\le\frac{K_2 16^y N_{\mathscr{O}_R}(R/2)\lambda(\mathscr{O}_R)}{\lambda(\mathscr{O}_{(R/4)\wedge\delta})}\\&\le\frac{K_2 16^y N_{\mathscr{O}_R}(R/2)\lambda(\mathscr{O}_R)^2}{\lambda(\mathscr{O}_{R/4})\lambda(\mathscr{O}_\delta)}\quad\text{for }\delta\le R.\end{aligned}$$

□

REMARK 2. There is no loss of generality in requiring d to be complete (but it is a restriction to require d to be a metric): There is a unique extension of $\xi(t)$ to a separable stochastically continuous Gaussian $\xi^*(t)$ on the d-completion T^* of T, and $N_S^*(\varepsilon)=N_{S\cap T}(\varepsilon)$ for $S\subseteq T^*$. So if $\{\xi(t)\}_{t\in T}$ satisfies the hypothesis of Theorem 1, then $\{\xi^*(t)\}_{t\in T^*}$ satisfies the hypothesis of Theorem 1 with (T^*,d) complete. Given $\psi\in\Psi$ we define $\psi^*(t)=\psi(t)$ for $t\in T$ and $\psi^*(t)=\infty$ for $t\in T^*-T$. Since $\xi^*(t)$ is locally bounded we then have $E(\psi)=E^*(\psi^*)$.

Corollary 2 sharpens [22] and [28] (but they do not require stationarity); the reader easily spots what conditions of Section 1 one can omit.

COROLLARY 2. *Assume that there is an* $R\in(0,\sqrt{2})$ *such that* (2.1) *holds. Then there are constants* $C_1,C_2\in(0,\infty)$ *such that*

$$C_1\le\frac{\mathbf{P}\{\sup_{t\in\mathscr{O}_\delta}\xi(t)>u\}}{N_{\mathscr{O}_\delta}((1\vee u)^{-1})\underline{\Phi}(u)}\le C_2\quad\textit{for } u\in\mathbb{R}\textit{ and }\delta\in[0,R].$$

If in addition d is a complete metric and λ *is a version of the Haar measure, then there are constants* $C_1,C_2\in(0,\infty)$ *such that*

$$C_1\le\frac{\mathbf{P}\{\sup_{t\in\mathscr{O}_\delta}\xi(t)>u\}}{\left[1+\lambda(\mathscr{O}_\delta)N_{\mathscr{O}_R}((1\vee u)^{-1})\right]\underline{\Phi}(u)}\le C_2\quad\textit{for } u\in\mathbb{R}\textit{ and }\delta\in[0,R].$$

In homogeneous space we have the following criterion for (2.2) to hold.

PROPOSITION 1. *If $\rho(s+u,t+u)=\rho(s,t)$ for $s,t,u\in T$ and if there is a function $f\colon \mathbb{R}\to\mathbb{R}$ such that, writing $\mathscr{B}_\varepsilon$ for an open ρ-ball of radius ε,*

$$(2.29)\qquad 1<\liminf_{\Delta\to\infty}\frac{N_{\mathscr{B}_{\Delta+f(\Delta)}}(R)}{N_{\mathscr{B}_\Delta}(R)}\le\limsup_{\Delta\to\infty}\frac{N_{\mathscr{B}_{\Delta+f(\Delta)}}(R)}{N_{\mathscr{B}_\Delta}(R)}<\infty,$$

then (2.2) *holds if* $\sigma(\varepsilon)\equiv\sup\{0\vee r(s,t)\colon \rho(s,t)\ge\varepsilon\}$ *satisfies*

$$(2.30)\qquad \lim_{\Delta\to\infty}\sigma(\Delta)\log N_{\mathscr{B}_\Delta}(R)=0.$$

PROOF. Take $\varepsilon,y,\Delta>0$ with $1+\varepsilon\le N_{\mathscr{B}_{x+f(x)}}(R)/N_{\mathscr{B}_x}(R)\le y$ for $x\ge\Delta$ and let $\varrho(0)=0$, $\varrho(1)=\Delta$ and $\varrho(n+1)=\varrho(n)+f(\varrho(n))$ for $n\ge1$, so that

$$N_{\mathscr{B}_{\varrho(n+1)}}(R)/N_{\mathscr{B}_{\varrho(1)}}(R)=\prod_{k=1}^{n}\left[N_{\mathscr{B}_{\varrho(k+1)}}(R)/N_{\mathscr{B}_{\varrho(k)}}(R)\right]\ge(1+\varepsilon)^n\to\infty$$

as $n\to\infty$, which yields $\lim_{n\to\infty}\varrho(n)=\infty$. Taking n_0 such that $\sigma(\varrho(n))\log N_{\mathscr{B}_{\varrho(n)}}(R)\le C/2$ for $n\ge n_0$, we now readily obtain

$$\begin{aligned}\sup_{s\in T}&\sum_{\{n\ge0\colon \sigma(s,\varrho(n))>0\}}N_{S_\rho(s,\varrho(n+1))}(R)\exp\{-C/\sigma(s,\varrho(n))\}\\&\le\sum_{n=1}^{n_0}N_{\mathscr{B}_{\varrho(n)}}(R)+\sum_{n=n_0}^{\infty}N_{\mathscr{B}_{\varrho(n+1)}}(R)\exp\left\{-2\log N_{\mathscr{B}_{\varrho(n)}}(R)\right\}\\&\le\sum_{n=1}^{n_0}N_{\mathscr{B}_{\varrho(n)}}(R)+\sum_{n=n_0}^{\infty}yN_{\mathscr{B}_{\varrho(1)}}(R)^{-1}(1+\varepsilon)^{-(n-1)}<\infty.\end{aligned}\qquad\square$$

REMARK 3. When (T,ρ) is metrizable there always exists a homogeneous metric generating the topology of T; compare, for example, [11].

3. The Euclidean case. Theorem 2 extends (1.2) and (1.3) to a test for all $\psi\in\Psi$ and (3.1) is also an improvement ((3.2) is essentially due to Kôno [12]). It is easy to derive (1.2) from Theorem 2 for increasing ψ's.

THEOREM 2. *If $\{\xi(t)\}_{t\in\mathbb{R}^n}$ is separable stationary standard Gaussian, if*

$$(3.1)\qquad \lim_{|t-s|\to\infty}(0\vee r(s,t))\log|t-s|=0,$$

and if there are constants $\alpha_1,\dots,\alpha_n,\delta,C_1,C_2\in(0,\infty)$ and functions $f_1,\dots,f_n\ge0$ on $[0,\delta]$ with $\lim_{x\downarrow0}f_i(\lambda x)/f_i(x)=\lambda^{\alpha_i}$ for $\lambda>0$ such that

$$(3.2)\qquad \begin{aligned}C_1\sum_{i=1}^{n}f_i(|t_i-s_i|)&\le1-r(s,t)\\&\le C_2\sum_{i=1}^{n}f_i(|t_i-s_i|)\quad\text{for }0\le|t-s|\le\delta,\end{aligned}$$

then $E(\psi)\in\mathscr{F}$ with $\mathbf{P}\{E(\psi)\}$ equal to 0 or 1 for $\psi\in\Psi$. Moreover, writing λ^n

for the Lebesgue measure on $\mathbb{R}^n$ *and* $f_i^*(x) \equiv \sup\{y \in [0,\delta]: f_i(y) \le x\}$,

$$\mathbf{P}\{E(\psi)\} = 0 \quad \Leftrightarrow \quad \sum_{k=1}^{\infty}\left[1 + \lambda^n(\mathscr{O}_{r_k})\prod_{i=1}^{n} f_i^*\left(\left(1 \vee \inf_{t\in S_k}\psi(t)\right)^{-2}\right)^{-1}\right]$$

$$\times\underline{\Phi}\left(\inf_{t\in S_k}\psi(t)\right) < \infty$$

for some covering $S_k = S(t_k, r_k)$, $k = 1,2,\ldots,$ *of* $\mathbb{R}^n$ *with* $r_k \le 1$.

REMARK 4. Since, by (3.2), $f_i(0) = 0$ and, by (3.1) (cf. below), $|t - s| \to 0$ as $d(s,t) \to 0$, we get $f_i(x) > 0$ for $x > 0$ since otherwise, by (3.2), $d(s,t) = 0$ for some $s \ne t$. Thus $1/f_i$ and $1/f_i^*$ make sense and d is a metric.

PROOF. Here $(T, \rho, +) = (\mathbb{R}^n, \ |\cdot|, +)$ and $R = 1$. Take $\Delta > 0$ with $r(0,t) < \frac{1}{2}$ for $|t| \ge \Delta$ and suppose $|t| \nrightarrow 0$ as $d(0,t) \to 0$. Then $\inf\{d(0,t): |t| \ge \varrho\} = 0$ for some $\varrho \in (0,\Delta]$ and, picking s with $|s| \ge \varrho$ and $d(0,s) < \varrho/(2\Delta)$, we get $d(0,([\Delta/\varrho]+1)s) < 1$ so that $r(0,([\Delta/\varrho]+1)s) > \frac{1}{2}$. This is a contradiction since $|(\Delta/\varrho)s| \ge \Delta$, and so, by homogeneity, $|t-s| \to 0$ as $d(s,t) \to 0$. Further, $\lim_{x\downarrow 0} f_i(x) = 0$, since $\liminf_{x\downarrow 0}(f_i(\lambda x)/f_i(x)) \times \limsup_{x\downarrow 0} f_i(x) \le \sup_{x\in[0,\delta]} f_i(x)$ for all λ. Thus we have stochastic continuity. Taking

$$\underline{R}(t,\varepsilon) \equiv \left\{s \in \mathbb{R}^n: |t_i - s_i| < \tfrac{1}{2}f_i^*\left((2nC_2)^{-1}\varepsilon^2\right), i = 1,\ldots,n\right\},$$

$$\overline{R}(t,\varepsilon) \equiv \left\{s \in \mathbb{R}^n: |t_i - s_i| \le 2f_i^*\left((2C_1)^{-1}\varepsilon^2\right), i = 1,\ldots,n\right\}$$

and $\hat{\varrho}, \varrho > 0$ with $\frac{1}{2}\varepsilon \le f_i^*(f_i(\varepsilon)) \le 2\varepsilon$ for $\varepsilon \le \hat{\varrho}$ (cf. [10, page 11]), $|t-s| \le \hat{\varrho} \wedge \delta$ for $d(s,t) \le \varrho$ and $\frac{1}{2}n^{1/2}f_i^*((2nC_2)^{-1}\varrho^2) \le \hat{\varrho} \wedge \delta$, (3.2) easily yields

$$s \in \underline{R}(t,\varepsilon) \Rightarrow f_i^*\left((2nC_2)^{-1}\varepsilon^2\right) > 2|t_i - s_i| \ge f_i^*\left(f_i(|t_i - s_i|)\right) \Rightarrow s \in S(t,\varepsilon),$$

$$s \in S(t,\varepsilon) \Rightarrow |t_i - s_i| \le 2f_i^*\left(f_i(|t_i - s_i|)\right) \le 2f_i^*\left((2C_1)^{-1}\varepsilon^2\right) \Rightarrow s \in \overline{R}(t,\varepsilon)$$

for $\varepsilon \in (0,\varrho]$. Hence $|\cdot|$-bounded sets are d-totally bounded, (T,d) is locally compact and λ^n is a Haar measure on $(T,d,+)$. Further, since $S(t,1) \subseteq S_{|\cdot|}(t,\Delta)$ and $\lim_{x\downarrow 0} f_i^*(\lambda x)/f_i^*(x) = \lambda^{1/\alpha_i}$ (cf. [10, page 10]), there are $K_1, K_2, x_0 > 0$ such that $K_1\prod_{i=1}^n f_i^*(\varepsilon^2)^{-1} \le N_{\mathscr{O}_1}(\varepsilon) \le K_2\prod_{i=1}^n f_i^*(\varepsilon^2)^{-1}$ for $\varepsilon \in (0,1]$ and $K_1x^n \le N_{\mathscr{B}_x}(1) \le K_2x^n$ for $x \ge x_0$. This proves (2.1), that (2.29) holds for $f(x) = (K_2/K_1)^{1/n}x$ and [using (3.1)] (2.30). □

REMARK 5. Regularly varying r's were first used by Berman [5].

REMARK 6. Theorem 1 also contains the case $T = \mathbb{Z}^n$ for which, if (3.1) holds, $\mathbf{P}\{E(\psi)\} = 0 \Leftrightarrow \sum_{t\in\mathbb{Z}^n}\underline{\Phi}(\psi(t)) < \infty$: Since, by (3.1), $S(t,R) = \{t\}$ for $R > 0$ small, we have $N_{\mathscr{O}_R}(\varepsilon) \equiv 1$ and $N_{\mathscr{B}_x}(R) \sim \text{const.} \times x^n$.

REMARK 7. Theorem 1 also applies if $1 - r(s,t) \sim \exp\{-|\log|t-s||^{\gamma}\}$ as $|t-s| \to 0$, for some $\gamma \in (0,1)$, since then $\lim_{\varepsilon \downarrow 0} N_{\mathscr{O}_R}(x\varepsilon)/N_{\mathscr{O}_R}(\varepsilon) = 1$. See also [27] and [28].

4. The Brownian sheet. Let $\mathbb{R}^n_+ \equiv \{s \in \mathbb{R}^n\colon s_1, \dots, s_n > 0\}$, let Θ be the class of functions $\theta\colon \mathbb{R}^n_+ \to \mathbb{R}$, let $\{W(t)\}_{t \in \mathbb{R}^n_+}$ be separable zero-mean Gaussian with covariance $R(s,t) = \prod_{i=1}^n (s_i \wedge t_i)$, define metrics $p(s,t) \equiv \frac{1}{2}[\sum_{i=1}^n (\log(t_i/s_i))^2]^{1/2}$ and $q(s,t) \equiv \sqrt{2}\,[1 - \prod_{i=1}^n ((s_i \wedge t_i)/(s_i \vee t_i))^{1/2}]^{1/2}$ on $\mathbb{R}^n_+$ and let $F(\theta) \equiv \{\omega \in \Omega\colon \{t \in \mathbb{R}^n_+\colon W(\omega;t) > \theta(t)\}$ is p-unbounded$\}$.

COROLLARY 3. *We have $F(\theta) \in \mathscr{F}$ with $\mathbf{P}\{F(\theta)\}$ equal to 0 or 1 for each $\theta \in \Theta$ and moreover $\mathbf{P}\{F(\theta)\} = 0$ if and only if there is a covering $\{S_k\}_{k=1}^{\infty}$ of $\mathbb{R}^n_+$ with closed q-balls S_k of radius at most 1 such that*

$$\sum_{k=1}^{\infty} \left[1 + \lambda^n \left(\frac{1}{2} \log S_k \right) \left(1 \vee \inf_{t \in S_k} \frac{\theta(t)}{\sqrt{t_1 \times \cdots \times t_n}} \right)^{2n} \right] \times \underline{\Phi}\left(\inf_{t \in S_k} \frac{\theta(t)}{\sqrt{t_1 \times \cdots \times t_n}} \right) < \infty.$$

PROOF. Take $\xi(t) \equiv e^{-(t_1 + \cdots + t_n)} W(e^{2t_1}, \dots, e^{2t_n})$, $t \in \mathbb{R}^n$, to get $r(s,t) = \prod_{i=1}^n e^{-|t_i - s_i|}$, so $\xi(t)$ satisfies the hypothesis of Theorem 2 with $f_i(x) = x$. The corollary now readily follows from applying appropriate changes of variable while keeping track of how these affect the ρ- and d-metrics. □

REMARK 8. Given $s \in \mathbb{R}^n_+$ we have $p(s,t) \to \infty$ if some $t_i \to \infty$ or some $t_i \downarrow 0$. Corollary 3 handles these cases simultaneously: To study only one case, let θ be $+\infty$ on the relevant part of $\mathbb{R}^n_+$ to rule out the other case.

REMARK 9. Sirao [23] studied Lévy's multiparameter Brownian motion $(R(s,t) = |s| + |t| - |t-s|)$ w.r.t. $\Psi \ni \psi = \varphi \circ |\cdot|$ with $\varphi\colon \mathbb{R}^1_+ \to \mathbb{R}^1_+$ increasing.

5. Two non-Euclidean examples.

EXAMPLE 1. Let $g(t) = 1 - 2|t|$ for $|t| < \frac{1}{2}$ and $g(t) = 0$ otherwise. Then $r\colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ given by $r(s,t) = g(t_1 - s_1)g(t_2 - s_2)$ is a covariance function on $\mathbb{R}^2$. Let $\{\xi(t)\}_{t \in \mathbb{R}^2}$ be zero-mean Gaussian with covariance r, put $T = \mathbb{R} \times \mathbb{Z}$ and let ρ be the metric on T generated by that on $\mathbb{R}^2$. Then $(T, \rho, +)$ is an LCA topological group and $\{\xi(t)\}_{t \in T}$ is stochastically continuous standardized stationary Gaussian.

Clearly $S(t,\varepsilon) = \{s_1 \in \mathbb{R}\colon \sqrt{2(1 - g(t_1 - s_1))} \le \varepsilon\} \times \{t_2\}$ for $t = (t_1, t_2) \in T$ and $\varepsilon < \sqrt{2}$. Taking $R = 1$ one therefore easily get $N_{\mathscr{O}_R}(\varepsilon) = \lceil (R/\varepsilon)^2 \rceil$ where $\lceil x \rceil = n$ if $n - 1 < x \le n$. Hence (2.1) holds. It is also evident that ρ-bounded sets are d-totally bounded. Further (2.2) holds trivially since $\sigma(t,\varrho) = \sup\{0 \vee r(s,t)\colon s \in T - S_\rho(t,\varrho)\} = 0$ for $\varrho \geq 2^{-1/2}$.

EXAMPLE 2. Let $\mathscr{C} = \{e^{i\pi x}\colon 0 \leq x < 1\}$ and define $e^{i\pi x} + e^{i\pi y} = e^{i\pi(x+y)}$. Further equip $T \equiv \mathbb{R} \times \mathscr{C}$ with "component-wise" $+$ and with the metric $\rho(s,t) = \max\{|t_1 - s_1|, \operatorname{arc}(s_2,t_2)\}$ where $\operatorname{arc}(s_2,t_2)$ is the (minimal) arclength between $s_2, t_2 \in \mathscr{C}$. Then $(T, \rho, +)$ is an LCA topological group.

Since $r(s,t) = g(t_1 - s_1)g(\operatorname{arc}(s_2,t_2))$ is a covariance function on T there is a zero-mean Gaussian process $\{\xi(t)\}_{t \in T}$ with covariance r, and $\xi(t)$ is stochastically continuous, standardized and stationary. Further $\{s \in T\colon |t_1 - s_1| \leq \frac{1}{8}\varepsilon^2, \operatorname{arc}(s_2,t_2) \leq \frac{1}{8}\varepsilon^2\} \subseteq S(t,\varepsilon) \subseteq \{s \in T\colon |t_1 - s_1| \leq \frac{1}{2}\varepsilon^2, \operatorname{arc}(s_2,t_2) \leq \frac{1}{2}\varepsilon^2\}$ for $\varepsilon < \sqrt{2}$, so that $\lceil \frac{1}{4}\varepsilon^{-1} \rceil^2 \leq N_{\mathscr{O}_1}(\varepsilon) \leq \lceil 4\varepsilon^{-2} \rceil^2$. Hence (2.1) holds (for $R = 1$). It is also evident that ρ-bounded sets are d-totally bounded. Finally (2.2) holds since $\sigma(t,\varrho) = 0$ for $\varrho \geq \frac{1}{2}$.

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