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STRONG DIFFERENTIAL SUBORDINATION AND STOCHASTIC INTEGRATION[1]

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This paper contains sharp norm, maximal, escape and exponential inequalities for stochastic integrals in which the integrator is either a nonnegative submartingale or a nonnegative supermartingale. Analogous inequalities hold for Itô processes and for smooth functions on Euclidean domains.

1. Introduction. How does the size of a stochastic integral vary with the choice of the predictable integrand and the semimartingale integrator? One of our goals here is to throw new light on this question, especially in the case that the integrator is not necessarily a martingale but belongs to some other class of semimartingales.

Consider first the case in which the integrator X is a nonnegative submartingale and Y is the integral of a predictable process H with respect to X:

$$Y_t = H_0 X_0 + \int_{(0,t]} H_s \, dX_s.$$

The underlying probability space $(\Omega, \mathcal{F}, P)$ is complete and is filtered by $(\mathcal{F}_t)_{t \geq 0}$, a nondecreasing right-continuous family of sub-σ-fields of $\mathcal{F}$, where $\mathcal{F}_0$ contains all $A \in \mathcal{F}$ with $P(A) = 0$. Both X and Y are adapted right-continuous processes on $[0, \infty)$ and have limits from the left on $(0, \infty)$.

If $1 < p < \infty$, let p^* be the maximum of p and q, and let p^{**} be the maximum of $2p$ and q, where $q = p/(p-1)$. Set $\|X\|_p = \sup_{t \geq 0} \|X_t\|_p$.

THEOREM 1.1. *Let $1 < p < \infty$. If X is a nonnegative submartingale and Y is the integral of H with respect to X as above, where H is a predictable process with values in $[-1, 1]$, then*

$$\|Y\|_p \leq (p^{**} - 1)\|X\|_p. \tag{1.1}$$

Strict inequality holds if $0 < \|X\|_p < \infty$. The constant $p^{**} - 1$ is the best possible.

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The last sentence is to mean that if $\beta < p^{**} - 1$, then there is a probability space and X, Y as above such that $\|Y\|_p > \beta\|X\|_p$. Note that $p^{**} = 2p$ if $p \geq \frac{3}{2}$, and $p^{**} = q$ if $p \leq \frac{3}{2}$. So $p^{**} \geq 3$ with equality holding only for $p = \frac{3}{2}$.

Inequality (1.1) should be compared with the analogous inequality in which X is a martingale that is not necessarily nonnegative. Then $\|Y\|_p \leq (p^* - 1)\|X\|_p$ and $p^* - 1$ is the best constant [2]. In fact, the constant $p^* - 1$ is already the best possible for the smaller class of nonnegative martingales. In the martingale case, strict inequality holds if $0 < \|X\|_p < \infty$ and $p \neq 2$; see [4]. If $p = 2$, equality can of course hold: take $H = 1$. However, for nonnegative submartingales, equality does not hold in (1.1) for any $p \in (1, \infty)$ unless the right-hand side of (1.1) is zero or the left-hand side is infinite. Another contrast is that duality as used in [4] is not available in the submartingale case.

Here is a simple consequence of Theorem 1.1. If $1 < p \leq \frac{3}{2}$, then $p^{**} = p^*$; so if H is a predictable process as above and X is a nonnegative submartingale with $0 < \|X\|_p < \infty$, then there is a nonnegative martingale with the same norm as X and a predictable process with values in $[-1, 1]$ that yield a strictly larger stochastic integral.

Theorem 1.1 is a special case of Theorem 3.1 to be proved in Section 3. Later sections contain a number of other sharp inequalities for stochastic integrals. Section 5 contains sharp maximal and escape inequalities for integrals with respect to nonnegative submartingales. Section 7 contains, among other things, a sharp exponential inequality. Sections 9 and 11 contain such inequalities for integrals with respect to nonnegative supermartingales.

REMARK 1.1. All of these results carry over to the local case provided the norm $\|\cdot\|_p$ is replaced by the norm $|||\cdot|||_p$ defined by $|||X|||_p = \sup \|X_\tau\|_p$, where the supremum is taken over all bounded stopping times τ. So, for example, if X is a nonnegative local submartingale, then the analogue of (5.1) is

$$\lambda P(Y^* \geq \lambda) \leq \lambda P\left(\sup_{t \geq 0} (X_t + |Y_t|) \geq \lambda\right) \leq 3|||X|||_1, \tag{1.2}$$

where Y^* denotes the maximal function of Y: $Y^*(\omega) = \sup_{t \geq 0} |Y_t(\omega)|$. Of course, if X is a nonnegative local supermartingale, then $|||X|||_1 = \|X\|_1 = EX_0$ and (9.1) is unchanged.

One reason for including the middle part of (1.2) is that the inequality on the right implies that

$$\lambda P(Y^* \geq \lambda) \leq 3\left(\frac{b-a}{2}\right)|||X|||_1, \tag{1.3}$$

where $a \leq 0 \leq b$ and the size condition on H is $a \leq H \leq b$. This follows as in the martingale setting; see the proof of Theorem 3.7 in [8].

We shall also compare the norms of two adapted sequences f and g of integrable functions, where the difference sequence of g is dominated both conditionally and unconditionally by the difference sequence of f. To be precise, let d_n and e_n be $\mathcal{F}_n$-measurable and integrable on a probability space $(\Omega, \mathcal{F}, P)$ filtered by $(\mathcal{F})_{n \geq 0}$. Consider the following two conditions:

$$|e_n| \leq |d_n| \quad \text{if } n \geq 0, \tag{1.DS}$$

$$|E(e_n|\mathcal{F}_{n-1})| \leq |E(d_n|\mathcal{F}_{n-1})| \quad \text{if } n \geq 1. \tag{1.CDS}$$

(If d_n is viewed as the net gain from game n of someone playing a sequence of games that are not necessarily fair, then e_n can be viewed as the net gain of a more cautious player who is playing fairer games.) If the sequences f and g are given by

$$f_n = \sum_{k=0}^{n} d_k \quad \text{and} \quad g_n = \sum_{k=0}^{n} e_k,$$

and (1.DS) is satisfied, then g *is differentially subordinate to* f. If (1.CDS) is satisfied, then g *is conditionally differentially subordinate to* f. If both of the conditions (1.DS) and (1.CDS) are satisfied, then g *is strongly differentially subordinate to* f or, more simply, g *is strongly subordinate to* f. Of course, if f and g are martingales, then both sides of (1.CDS) vanish and (1.CDS) is trivially satisfied.

It will be convenient to allow g to have its values in a space of possibly more than one dimension. So we assume throughout the paper tht g has its values in $\mathbf{R}^\nu$, where ν is a positive integer. The Euclidean norm of $y \in \mathbf{R}^\nu$ is denoted by $|y|$ and the inner product of y and $k \in \mathbf{R}^\nu$ by $y \cdot k$.

Under the further assumption that f is either a nonnegative submartingale or a nonnegative supermartingale, sharp inequalities for g are obtained in Sections 2, 4, 6, 8 and 10. These are used to prove the corresponding inequalities for stochastic integrals but are of interest in their own right. Their proofs are fairly easy once certain special functions are found. These special functions are upper solutions to some nonclassical boundary value problems (see Section 12) and can also be used to compare the sizes of smooth functions on Euclidean domains (see Section 13). They also play a key role in the size comparison of Itô processes (Section 14).

2. A sharp norm inequality for a strong subordinate of a nonnegative submartingale. Set $\|f\|_p = \sup_{n \geq 0} \|f_n\|_p$.

THEOREM 2.1. *Let* $1 < p < \infty$. *If* f *is a nonnegative submartingale and* g *is strongly subordinate to* f, *then*

$$\|g\|_p \leq (p^{**} - 1)\|f\|_p. \tag{2.1}$$

Strict inequality holds if $0 < \|f\|_p < \infty$. *The constant* $p^{**} - 1$ *is the best possible and is already the best possible constant if* $\nu = 1$ *and* g *is a* ± 1*-transform of* f, *that is, if* $e_n = \varepsilon_n d_n$, *where* $\varepsilon_n \in \{1, -1\}$, $n \geq 0$.

If $e_n = H_n d_n$, where H_n is an $\mathcal{F}_{n-1}$-measurable function ($\mathcal{F}_0$-measurable if $n = 0$) with values in the closed unit ball of $\mathbf{R}^\nu$, then g is strongly subordinate to f. In this case, (2.1) follows from Theorem 3.1. However, strong subordination is less restrictive and leads to a wider class of applications (for the martingale case, see [5] and [8]). In fact, Theorem 2.1 will be used to prove Theorem 3.1.

PROOF OF THEOREM 2.1. To prove (2.1), we can assume that $\|f\|_p$ is finite. So every d_k belongs to L^p, and, by (1.DS), so does every e_k, hence every g_n. Define $V: [0, \infty) \times \mathbf{R}^\nu \to \mathbf{R}$ by

$$V(x, y) = |y|^p - (p^{**} - 1)^p x^p. \tag{2.2}$$

Then $EV(f_n, g_n) = \|g_n\|_p^p - (p^{**} - 1)^p \|f_n\|_p^p$ and (2.1) follows from

$$EV(f_n, g_n) \le 0. \tag{2.3}$$

Let $\alpha_p = p(1 - 1/p^{**})^{p-1}$. To prove the inequality (2.3), we shall show that the function $U: [0, \infty) \times \mathbf{R}^\nu \to \mathbf{R}$ defined by

$$U(x, y) = \alpha_p \big[|y| - (p^{**} - 1)\, x\big] \big(x + |y|\big)^{p-1} \tag{2.4}$$

satisfies

$$EV(f_n, g_n) \le EU(f_n, g_n) \le \cdots \le EU(f_0, g_0) \le 0. \tag{2.5}$$

To see that $EU(f_0, g_0) \le 0$, use $|g_0| = |e_0| \le |d_0| = |f_0| = f_0$ to obtain

$$U(f_0, g_0) \le -\alpha_p(p^{**} - 2) f_0^p \le 0. \tag{2.6}$$

The inequality $EV(f_n, g_n) \le EU(f_n, g_n)$ follows from

$$V(x, y) \le U(x, y). \tag{2.7}$$

The latter inequality holds if $x + |y| = 0$, so we can assume that $x + |y| > 0$ and, by homogeneity, that $x + |y| = 1$. Accordingly, set $|y| = 1 - x$ in the expressions on the right-hand sides of (2.2) and (2.4) and consider their difference given by

$$F(x) = \alpha_p(1 - p^{**}x) - (1 - x)^p + (p^{**} - 1)^p x^p. \tag{2.8}$$

We need to show that this is nonnegative for $x \in [0, 1]$. It is easy to check that $F(0) > 0$, an inequality equivalent to

$$\frac{1}{p} \log \frac{1}{p} + \frac{(p-1)}{p} \log \frac{p^{**}}{p^{**} - 1} < 0, \tag{2.9}$$

since, by the strict concavity of the log function and Jensen's inequality, the left-hand side of (2.9) is less than

$$\log\left[\frac{1}{p^2} + \frac{p-1}{p} \frac{p^{**}}{p^{**} - 1}\right] < \log 1 = 0.$$

It is also easy to check that $F(1) > 0$, $F(1/p^{**}) = 0$, $F'(1/p^{**}) = 0$ and $F''(1/p^{**}) > 0$. Furthermore, F'' has at most one zero in $[0,1]$ so there is an interval I containing $1/p^{**}$ and one of the endpoints of $[0,1]$ such that F is convex on I and is concave on $J = [0,1]\setminus I$. Therefore, F is nonnegative on I. For $p \neq 2$, the set J is nonempty and F is positive at its endpoints, hence also positive in the interior of J. This completes the proof of (2.7).

It remains to prove that

$$EU(f_n, g_n) \le EU(f_{n-1}, g_{n-1}), \tag{2.10}$$

for all $n \ge 1$. The first step in the proof is to show that if $x > 0, x + h > 0, y \in \mathbf{R}^\nu$, $k \in \mathbf{R}^\nu$, $|k| \le |h|$ and $|y + kt| > 0$ for all $t \in \mathbf{R}$, then

$$U(x+h, y+k) \le U(x,y) + U_x(x,y)h + U_y(x,y)\cdot k, \tag{2.11}$$

where the partial derivatives $U_x(x,y)$ and $U_y(x,y)$ are given by

$$U_x(x,y) = -\alpha_p\left[(p^{**}-p)|y| + p(p^{**}-1)x\right](x+|y|)^{p-2},$$
$$U_y(x,y) = \alpha_p\left[p|y| - \{(p-1)p^{**}-p\}x\right](x+|y|)^{p-2}y'.$$

Throughout the paper $y' = y/|y|$ provided that $|y|$ is nonzero, as it is here. Note that

$$|U_y| \le -U_x, \tag{2.12}$$

and the function G defined on $\{t \in \mathbf{R}\colon x + ht > 0\}$ by

$$G(t) = U(x+ht, y+kt)$$

is infinitely differentiable. The inequality

$$G(1) \le G(0) + G'(0)$$

is equivalent to (2.11) and follows from the concavity of G, the proof of which can be reduced by translation to showing that $G''(0) \le 0$. However, $G''(0) \le 0$ follows from

$$G''(0) = -\alpha_p(A+B+C)(x+|y|)^{p-3},$$

where

$$A = \{(p-1)p^{**}-p\}|y|^{-1}\left[|k|^2 - (y'\cdot k)^2\right](x+|y|)^2$$

is nonnegative by the Cauchy–Schwarz inequality and the inequality $p^{**} > q$,

$$B = (p-1)p^{**}\left(h^2 - |k|^2\right)(x+|y|)$$

is nonnegative by the condition $|k| \le |h|$, and

$$C = (p-1)\left[(p^{**}-p)|y| + \{(p-1)p^{**}-p\}x\right][h + y'\cdot k]^2$$

is also nonnegative by the definition of p^{**}. This completes the proof of (2.11).

To prove (2.10), we can assume that there is a proper subspace M of $\mathbf{R}^\nu$ such that $g_n(\omega)$ belongs to M for all $\omega \in \Omega$ and $n \geq 0$; otherwise replace $\mathbf{R}^\nu$ by $\mathbf{R}^{\nu+1}$ to make this possible. Choose a in the orthogonal complement of M with $0 < |a| < 1$. Then $y = a + g_{n-1}(\omega)$ and $k = e_n(\omega)$ satisfy $|y + kt| \geq |a| > 0$ for all $t \in \mathbf{R}$. Let $f_n^a = |a| + f_n$ and $g_n^a = a + g_n$. Fix $n \geq 1$. Using (1.DS) and (2.11), we have that

$$U(f_n^a, g_n^a) \leq U(f_{n-1}^a, g_{n-1}^a) + U_x(f_{n-1}^a, g_{n-1}^a)d_n + U_y(f_{n-1}^a, g_{n-1}^a) \cdot e_n.$$

Each of these four terms is integrable and the conditional expectation with respect to $\mathcal{F}_{n-1}$ of the sum of the last two terms is

$$U_x(f_{n-1}^a, g_{n-1}^a)E(d_n \mid \mathcal{F}_{n-1}) + U_y(f_{n-1}^a, g_{n-1}^a) \cdot E(e_n \mid \mathcal{F}_{n-1}).$$

Because f is a submartingale, $E(d_n \mid \mathcal{F}_{n-1}) \geq 0$ and the product on the left is nonpositive. Using the Cauchy–Schwarz inequality, then (2.12) and, finally, condition (1.CDS), we see that the last sum above is nonpositive. This implies that

$$E\Big[U(f_n^a, g_n^a) \mid \mathcal{F}_{n-1}\Big] \leq U(f_{n-1}^a, g_{n-1}^a).$$

Now let $a \to 0$ and use the dominated convergence theorem for conditional expectations to obtain

$$E\big[U(f_n, g_n) \mid \mathcal{F}_{n-1}\big] \leq U(f_{n-1}, g_{n-1}). \tag{2.13}$$

Taking expectations of each side gives (2.10) and completes the proof of (2.1).

Strictness of the inequality. Suppose that $0 < \|f\|_p < \infty$. Let m be the least integer n such that $\|f_n\|_p > 0$. Then, with probability 1, $|g_m| = |e_m| \leq |d_m| = |f_m| = f_m$ so that, if $n \geq m$, then, by (2.10) and the analogue of (2.6) for the pair (f_m, g_m),

$$EV(f_n, g_n) \leq EU(f_n, g_n) \leq EU(f_m, g_m) \leq -\alpha_p(p^{**} - 2)\|f_m\|_p^p < 0.$$

Since $\|g_k\|_p = 0$ if $0 \leq k < m$, this implies that

$$\|g\|_p^p \leq (p^{**} - 1)^p \|f\|_p^p - \alpha_p(p^{**} - 2)\|f_m\|_p^p$$

and gives strict inequality in (2.1).

The constant is the best possible. To prove this, we shall use the following lemma.

LEMMA 2.1. *Let $1 < p < \infty$ and $S = \{(x, y) \in \mathbf{R}^2 : x + y \geq 0\}$. Suppose that $\beta \in [1, \infty)$ satisfies*

$$\|g\|_p \leq \beta \|f\|_p$$

for all pairs (f, g), where f is a nonnegative submartingale on the Lebesgue unit interval and g is a ± 1-transform of f. Then there is a biconcave function

$u\colon S \to \mathbf{R}$ that is nonincreasing in x and y separately, such that $v \le u$ on S, where v is defined on S by

$$v(x,y) = \left|\frac{x-y}{2}\right|^p - \beta^p\left(\frac{x+y}{2}\right)^p.$$

The proof is a slight modification of the proof of the "only if" part of Theorem 2.1 in [8] and is omitted (but see Section 12).

To show that $\beta \ge 2p-1$ and $\beta \ge q-1$, which imply that $\beta \ge p^{**}-1$, we can assume that u satisfies the same homogeneity condition that v satisfies: if $\lambda > 0$, then

$$u(\lambda x, \lambda y) = \lambda^p u(x,y). \tag{2.14}$$

Note that if u does not already satisfy this condition, it can be replaced by the function $(x,y) \mapsto \inf_{\lambda>0} u(\lambda x, \lambda y)/\lambda^p$, which does. Let w be the nonincreasing concave function defined on $[-1,\infty)$ by $w(y) = u(1,y)$, and let $y_0 = (1-\beta)/(1+\beta)$. Then $y_0 \in (-1,0]$ and $w(y_0) \ge v(1,y_0) = 0$ so, by the concavity of w,

$$\begin{aligned} D^+w(-1) = \lim_{y\downarrow -1}\frac{w(y)-w(-1)}{y-(-1)} &\ge \frac{w(y_0)-w(-1)}{y_0-(-1)} \\ &\ge -w(-1)\frac{1}{y_0-(-1)} = -w(-1)\frac{\beta+1}{2}. \end{aligned} \tag{2.15}$$

The function $u(\cdot,-1)$ is nonincreasing on $[1,\infty)$ so, for $x > 1$, $u(1,-1) \ge u(x,-1)$. By (2.14), this is equivalent to $w(-1) \ge x^p w(-1/x)$, which in turn is equivalent to

$$-w(-1)\frac{x^p-1}{x-1} \ge x^{p-1}\frac{w(-1/x)-w(-1)}{(-1/x)-(-1)}.$$

Now let $x \downarrow 1$ to obtain

$$-pw(-1) \ge D^+w(-1). \tag{2.16}$$

By (2.15) and (2.16), $-pw(-1) \ge -w(-1)(\beta+1)/2$; but $w(-1) \ge v(1,-1) = 1$, so $p \le (\beta+1)/2$, which implies that

$$\beta \ge 2p-1. \tag{2.17}$$

To prove $\beta \ge q-1$ we can assume that u, like v, is symmetric: $u(x,y) = u(y,x)$. Symmetry and homogeneity imply that, for $y > 0$,

$$w(y) = y^p w(1/y).$$

This implies that if $0 < y < 1$, then

$$\frac{w(1)-w(y)}{1-y} = w(1)\frac{1-y^p}{1-y} - y^{p-1}\frac{w(1/y)-w(1)}{(1/y)-1}.$$

Now let $y \uparrow 1$ to obtain $D^- w(1) = pw(1) - D^+ w(1)$. Note that $w(0) \le 0$. Otherwise, as $x \to \infty$,

$$u(0,0) \ge u(x,0) = x^p u(1,0) = x^p w(0) \to \infty,$$

which contradicts that $u(0,0) \in R$. Therefore, $w(1) \le -w(-1) + 2w(0) \le -1$. Again, by concavity,

$$\begin{aligned} pw(1) &= D^- w(1) + D^+ w(1) \le 2\frac{w(1) - w(y_0)}{1 - y_0} \\ &\le 2\frac{w(1)}{1 - y_0} = w(1)\frac{\beta + 1}{\beta}, \end{aligned}$$

so $p \ge (\beta + 1)/\beta$, which gives $\beta \ge q - 1$. This, with (2.17), gives $\beta \ge p^{**} - 1$ and completes the proof of Theorem 2.1.

A related inequality. Theorem 2.1 leads to the following variation.

THEOREM 2.2. *Let $1 < p < \infty$. If f and g are sequences given by*

$$f_n = \sum_{k=0}^{n} d_k + \sum_{k=0}^{n} a_k, \tag{2.18}$$

$$g_n = \sum_{k=0}^{n} e_k + \sum_{k=0}^{n} b_k, \tag{2.19}$$

where d and e satisfy (1.DS) *and* (1.CDS), *f is a nonnegative submartingale, $a_0 = 0, b_0 = 0$ and, for all $n \ge 1$, the functions $a_n\colon \Omega \to [0,\infty)$ and $b_n\colon \Omega \to \mathbf{R}^\nu$ are $\mathcal{F}_{n-1}$-measurable and integrable with*

$$|b_n| \le a_n, \tag{2.20}$$

then

$$\|g\|_p \le (p^{**} - 1)\|f\|_p. \tag{2.21}$$

Strict inequality holds if $0 < \|f\|_p < \infty$. The constant $p^{**} - 1$ is the best possible.

Note that $|e_n + b_n| \le |d_n + a_n|$ need not hold. Theorem 2.2 implies Theorem 2.1: take $a = 0$ and $b = 0$. On the other hand, Theorem 2.2 can be proved using Theorem 2.1 by introducing a new filtration and new adapted sequences F and G.

PROOF OF THEOREM 2.2. Let $\mathcal{G}_{2n} = \mathcal{G}_{2n+1} = \mathcal{F}_n$, $D_{2n} = d_n$, $D_{2n+1} = a_{n+1}$, $E_{2n} = e_n$ and $E_{2n+1} = b_{n+1}$ for all $n \ge 0$. It is easy to check that D and E are adapted to $(\mathcal{G}_n)_{n\ge 0}$ and satisfy (1.DS) and (1.CDS) with $E(D_n \mid \mathcal{F}_{n+1}) \ge 0$ for all $n \ge 1$. Let F have D as its difference sequence and G have E. Then $F_{2n} = f_n \ge 0$

and $F_{2n+1} = F_{2n} + a_{n+1} \geq 0$, so F is a nonnegative submartingale and G is strongly subordinate to F. Therefore, by Theorem 2.1,

$$\|G\|_p \leq (p^{**} - 1)\|F\|_p. \tag{2.22}$$

Since F is a nonnegative submartingale,

$$\|f\|_p = \sup_{n\geq 0} \|f_n\|_p = \sup_{n \geq 0} \|F_{2n}\|_p = \|F\|_p. \tag{2.23}$$

Also,

$$\|g\|_p = \sup_{n\geq 0} \|g_n\|_p = \sup_{n\geq 0} \|G_{2n}\|_p \leq \|G\|_p.$$

Inequality (2.21) now follows from (2.22). If $0 < \|f\|_p < \infty$, then strict inequality follows from (2.23) and Theorem 2.1 applied to F. The constant $p^{**} - 1$ is the best possible since it is already the best possible in the special case $a = 0$ and $b = 0$.

REMARK 2.1. With $e = 0$ and b replaced by $(|b_k|)_{k\geq 0}$ in (2.19), inequality (2.21) gives

$$\left\| \sum_{k=0}^{\infty} |b_k| \right\|_p \leq (p^{**} - 1)\|f\|_p. \tag{2.24}$$

If $E(d_k \mid \mathcal{F}_{k-1}) = 0$ for all $k \geq 1$, that is, if the sequence M defined by $M_n = \sum_{k=0}^{n} d_k$ is a martingale, then condition (1.CDS) implies that N defined by $N_n = \sum_{k=0}^{n} e_k$ is also a martingale. So, by Doob's inequality for the maximal function N^* of N, and by (2.21) in the case $b = 0$,

$$\|N^*\|_p \leq q\|N\|_p \leq q(p^{**} - 1)\|f\|_p. \tag{2.25}$$

The special cases (2.24) and (2.25) imply that if g is given by (2.19) and M is a martingale, then

$$\|g^*\|_p \leq (q + 1)(p^{**} - 1)\|f\|_p. \tag{2.26}$$

In particular, if g is the transform of a nonnegative submartingale f by a predictable sequence with values in the closed unit ball $\mathbf{R}^\nu$, then (2.26) holds: use the Doob decomposition of f in (2.18) and note that with the natural choice of the e_k and b_k in (2.19), the pair f and g satisfies the conditions of Theorem 2.2.

REMARK 2.2. As the interested reader can see, Theorems 2.1 and 2.2 carry over to the case in which g has its values in a real or complex Hilbert space $\mathbf{H}$. In the proofs $\mathbf{H}$ can be taken to be the real Lebesgue sequence space ℓ^2. This is also true for the other theorems in this paper.

3. A sharp norm inequality for an integral with respect to a nonnegative submartingale. We shall now prove Theorem 1.1 and a little more. Here Y has its values in $\mathbf{R}^\nu$.

THEOREM 3.1. *Let $1 < p < \infty$. Suppose that X is a nonnegative submartingale and Y is the integral of H with respect to X, where H is a predictable process with values in the closed unit ball of $\mathbf{R}^\nu$. Then*

$$\|Y\|_p \le (p^{**} - 1)\|X\|_p. \tag{3.1}$$

Strict inequality holds if $0 < \|X\|_p < \infty$. The constant $p^{**} - 1$ is the best possible.

PROOF. To prove (3.1), we can assume that $\|X\|_p$ is finite. Consider the family $\mathbf{Y}$ of all Y of the form

$$Y_t = H_0 X_0 + \sum_{k=1}^{n} h_k \big[X(T_k \wedge t) - X(T_{k-1} \wedge t)\big]. \tag{3.2}$$

Here n is a positive integer, the coefficients h_k belong to the closed unit ball of $\mathbf{R}^\nu$, the stopping times T_k take only a finite number of finite values, $0 = T_0 \le \cdots \le T_n$ and $X(t) = X_t$. Consider the nonnegative sequence

$$f = \big(X(T_0), \dots, X(T_n), X(T_n), \dots\big)$$

and let g be its transform by $(H_0, h_1, \dots, h_n, 0, 0, \dots)$. By Doob's optional sampling theorem, f is a submartingale. By Theorem 2.1, if T_n is bounded from above by t, then Y defined by (3.2) satisfies

$$\|Y_t\|_p = \|g_n\|_p \le (p^{**} - 1)\|f_n\|_p \le (p^{**} - 1)\|X_t\|_p. \tag{3.3}$$

For $\nu = 1$, an immediate consequence of Theorem 2.1 is that X and H satisfy the conditions of Proposition 4.1 of Bichteler [1]. So, by (2) of that proposition, if Y is the integral of H with respect to X as in the statement of Theorem 3.1 but with $\nu = 1$, then there is a sequence $Y^j \in \mathbf{Y}$ such that

$$\lim_{j \to \infty} (Y^j - Y)^* = 0 \quad \text{a.s.} \tag{3.4}$$

Using the additivity of the integral, we see that (3.4) holds for $\nu \ge 1$. By (3.4) for this case and Fatou's lemma, the inequality $\|Y_t\|_p \le (p^{**} - 1)\|X_t\|_p$ holds for all Y as in the statement of the theorem. Take the supremum of each side with respect to $t \in [0, \infty)$, to obtain (3.1).

REMARK 3.1. There is an analogue of (2.26) for the maximal function of Y:

$$\|Y^*\|_p \le (q + 1)(p^{**} - 1)\|X\|_p. \tag{3.5}$$

By Remark 2.1, the inequality (3.3) can be replaced by

$$\Big\|\sup_{0 \le k \le n} |Y(T_k)|\Big\|_p = \|g^*\|_p \le (q + 1)(p^{**} - 1)\|X\|_p. \tag{3.6}$$

Without loss of generality, the stopping times T_k^j, $0 \le k \le n(j)$, corresponding to Y^j can be chosen so that, for all $\omega \in \Omega$, the sequence of sets

$$\{T_0^j(\omega), \dots, T_{n(j)}^j(\omega)\}$$

converges upward to a dense set of $[0, \infty)$ as $j \uparrow \infty$. The inequality (3.5) now follows from (3.4), (3.6), Fatou's lemma and the right-continuity of Y.

The constant in 3.1 *is the best possible.* This is an immediate consequence of the last statement in Theorem 2.1.

Strictness of the inequality. Assume that $0 < \|X\|_p < \infty$. Then, by Doob's maximal inequality for nonnegative submartingales and (3.5), both $\|X^*\|_p$ and $\|Y^*\|_p$ are finite. This implies that the process W defined by

$$W_t = U(X_t, Y_t) \tag{3.7}$$

satisfies $EW^* < \infty$. Since U, defined by (2.4), is continuous, W is right-continuous with left limits. We shall now show that W is a supermartingale. Let $0 \le s < t$. Then

$$E(W_t \mid \mathcal{F}_s) \le W_s, \tag{3.8}$$

as can be seen as follows. Choose the stopping times associated with $Y^j \in \mathbf{Y}$ so that $T_{n(j)}^j = t$ and $T_{m(j)}^j = s$ for some $m(j) < n(j)$. This can be done so (3.4) is also satisfied. Then, by the discrete-time analogue of (3.8) that is implied by (2.13),

$$E\Big[U\big(X_t, Y_t^j\big) \mid \mathcal{F}_s\Big] \le U\big(X_s, Y_s^j\big).$$

As $j \to \infty$, the right-hand side converges almost surely to W_s. The integrand on the left is bounded from below by the integrable function $-\alpha_p(p^{**}X_t)^p$: if $U(x, y) \le 0$, then $|y| \le (p^{**} - 1)x$. An application of Fatou's lemma for conditional expectations gives (3.8).

Let $t_0 = \inf\{t > 0 \colon \|X_t\|_p > 0\}$, a number in the interval $[0, \infty)$. If $\|X_{t_0}\|_p > 0$, then the strictness of the inequality (3.1) follows as in Section 2. So we can assume throughout the remainder of the proof that $X_{t_0} = 0$ and, without loss of generality, that $t_0 = 0$. Thus, $X_0 = Y_0 = 0$ but $\|X_t\|_p > 0$ for $t > 0$.

Let v be the function defined on $[0, \infty)$ by $v(t) = EV(X_t, Y_t)$, where V is given in (2.2). By the dominated convergence theorem, v is right-continuous with left limits. Using (2.7) and the supermartingale property of W, we have that $v(t) \le EW_t \le EW_0 = 0$. This implies that $v(t-) \le EW_{t-} \le 0$ for $t > 0$. By Doob's submartingale convergence theorem, the finiteness of $\|X\|_p$ implies that almost surely $\lim_{t\to\infty} X_t$ exists and is finite. Denote this limit by X_∞. Since $p > 1$, it is easy to see that the almost sure limit of Y also exists: use the Doob–Meyer decomposition $X = M + A$ and the inequality $\|N\|_p \le (p^* - 1)\|M\|_p$, where N is the stochastic integral of H with respect to the martingale M. In fact, the existence of both X_∞ and Y_∞ can be seen also as an immediate consequence of Theorem 5.2. Therefore, by the dominated convergence theorem, $v(\infty)$ exists and satisfies $v(\infty) \le EW_\infty \le 0$. If $t > 0$, then, as we shall prove,

$$v(t-) \vee v(t) \vee v(\infty) < 0. \tag{3.9}$$

This implies that strict inequality holds in (3.1). To see this, choose $\delta > 0$ so that, for $t \le \delta$,

$$\|Y_t\|_p \le (p^{**} - 1)\|X\|_p - \delta.$$

Such a number δ exists by the right-continuity of $\|Y_t\|_p$ in t. Let $\gamma = -\sup_{t \ge \delta} v(t)$. By (3.9), since v is right-continuous with left limits, $\gamma > 0$. Therefore, for $t \ge \delta$,

$$\begin{aligned}\|Y_t\|_p^p &= (p^{**} - 1)^p\|X_t\|_p^p + v(t)\\ &\le (p^{**} - 1)^p\|X\|_p^p - \gamma.\end{aligned}$$

Accordingly, strict inequality hods in (3.1).

Turning to (3.9), we note that to prove $v(\infty) < 0$ it is enough to prove that

$$P\big(|Y_\infty| \ne \rho X_\infty\big) > 0, \tag{3.10}$$

where $\rho = p^{**} - 1$: by the proof of (2.7),

$$V(x,y) < U(x,y) \text{ if } |y| \ne \rho x, \tag{3.11}$$

so (3.10) implies that $v(\infty) = EV(X_\infty, Y_\infty) < EU(X_\infty, Y_\infty) = EW_\infty \le 0$.

Suppose that (3.10) is not true. Then $P(W_\infty = 0) = 1$. Since W is a uniformly integrable supermartingale starting at 0 and, in fact, is dominated in absolute value by an integrable function, $P(W = 0) = 1$ or, equivalently,

$$P\big(|Y_t| = \rho X_t \text{ for all } t \ge 0\big) = 1. \tag{3.12}$$

(To see that $P(W = 0) = 1$, note that the nonpositive supermartingale $(W_t \wedge 0)_{t \ge 0}$ satisfies $P(W \ge W \wedge 0 = 0) = 1$, which implies that $\lambda P(W^* \ge \lambda) \le EW_0 = 0$.)

A simple argument shows that X is almost surely continuous, hence also Y. The right-continuity of X implies continuity at $t = 0$, so assume that $t > 0$. Since $\rho > 1$, the conditions $x \ge 0, x + h \ge 0, y \in \mathbf{R}^\nu, y + k \in \mathbf{R}^\nu$, $|k| \le |h|$, $|y| = \rho x$ and $|y + k| = \rho(x + h)$ imply that $h = 0$, where x represents $X_{t-}(\omega)$, h represents $\Delta X_t(\omega) = X_t(\omega) - X_{t-}(\omega)$ and so forth. We assume that ω belongs to the set of full measure where $W(\omega) = 0$ and $|\Delta Y_t(\omega)| \le |\Delta X_t(\omega)|$. If $x = 0$, then $y = 0$, so $h \ne 0$ leads to the contradiction $|h| \ge |k| = \rho|h|$. If $x > 0$, then $y \ne 0$ and with y' denoting $y/|y|$ as before so that here $y = \rho x y'$, the inequality $h \ne 0$ again leads to a contradiction:

$$\begin{aligned}x + h &= -x\left[\frac{\rho^2h^2 + |k|^2 - 2\rho hk \cdot y'}{\rho^2h^2 - |k|^2}\right]\\ &\le -x\left[\frac{\rho|h| - |k|}{\rho|h| + |k|}\right] < 0.\end{aligned}$$

Therefore, we can assume in the following that X and Y are everywhere continuous and, by a stopping time argument, that X has its values in [0, 1]. Using the Doob–Meyer decomposition $X = M + A$, Itô's formula, and (3.12), we have that almost surely

$$\begin{aligned} 0 &= |Y_t|^2 - \rho^2 X_t^2 \\ &= \int_{(0,t]} \left(-2\rho^2 X_s + 2H_s \cdot Y_s\right) dM_s + \int_{(0,t]} \left(-2\rho^2 X_s + 2H_s \cdot Y_s\right) dA_s \\ &\quad + \int_{(0,t]} \left(-\rho^2 + |H_s|^2\right) d[M,M]_s, \end{aligned}$$

where $[M,M]$ is the quadratic variation process of M. Taking expectations of both sides and using $|H_s \cdot Y_s| \le |Y_s|$ together with (3.12), we obtain

$$0 \le -2\rho(\rho - 1)E \int_{(0,t]} X_s \, dA_s - (\rho^2 - 1)E[M,M]_t \le 0.$$

The inequality on the right follows from the nonnegativity of both $E[M,M]_t$ and $E \int_{(0,t]} X_s \, dA_s$. So the inequality on the left implies that both of these expectations must vanish. But if $E[M,M]_t = 0$, then $X_s = A_s$ for all $s \le t$ almost surely, so

$$0 = E \int_{0,t]} X_s \, dA_s = E \int_{(0,t]} A_s \, dA_s = EA_t^2/2 = EX_t^2/2.$$

This contradicts the inequality $EX_t^2 > 0$, which holds for all $t > 0$. So (3.10) must hold and, therefore, $v(\infty) < 0$. The proofs of the two other parts of (3.9) are similar.

4. A sharp maximal inequality for a strong subordinate of a nonnegative submartingale.

The maximal function g is defined by $g^* = \sup_{n \ge 0} |g_n|$. Recall that here g_n has its values in $\mathbf{R}^\nu$.

THEOREM 4.1. *If f is a nonnegative submartingale and g is strongly subordinate to f, then, for all $\lambda > 0$,*

$$\lambda P(g^* \ge \lambda) \le \lambda P\left(\sup_{n \ge 0} (f_n + |g_n|) \ge \lambda\right) \le 3\|f\|_1. \tag{4.1}$$

Strict inequality holds on the right if $\|f\|_1 > 0$. Even for $\lambda P(g^* \ge \lambda) \le 3\|f\|_1$, the constant 3 is the best possible and is already the best if $\nu = 1$ and g is a ± 1-transform of f.

If f is a martingale that is not necessarily nonnegative, one that can even have its values in $\mathbf{R}^\mu$, where μ is a positive integer, then $\lambda P(\sup_{n \ge 0}(|f_n| + |g_n|) \ge \lambda) \le 2\|f\|_1$ and $\lambda P(g^* \ge \lambda) = 2\|f\|_1 > 0$ can hold [6]. As we shall see in Section

8, the best constant in the nonnegative supermartingale setting is also 2. These maximal inequalities imply escape inequalities; see [7] for the martingale case and the proof of Theorem 5.2.

REMARK 4.1. Theorem 4.1 also holds for f and g satisfying the conditions of Theorem 2.2: if F and G are as in proof of Theorem 2.2, then $\|f\|_1 = \|F\|_1$ and

$$\sup_{n\geq 0}\big(f_n + |g_n|\big) = \sup_{n\geq 0}\big(F_{2n} + |G_{2n}|\big) \leq \sup_{n\geq 0}\big(F_n + |G_n|\big).$$

PROOF OF THEOREM 4.1. To prove (4.1), we can assume that $\|f\|_1$ is finite. A stopping-time argument leads to a further reduction: it is enough to prove that if $n \geq 0$, then

$$P\big(f_n + |g_n| \geq 1\big) \leq 3Ef_n. \tag{4.2}$$

Here let U and V be defined on $[0,\infty)\times\mathbf{R}^\nu$ by

$$\begin{aligned} &U(x,y) = \big(|y| - 2x\big)\sqrt{x+|y|}, && V(x,y) = -3x && \text{if } x+|y| < 1;\\ &U(x,y) = 1-3x, && V(x,y) = 1-3x && \text{if } x+|y| \geq 1. \end{aligned}$$

On the set where $x + |y| < 1$, the function U above is simply the function U of (2.4) in the special case $p = \frac{3}{2}$ but without the factor α_p.

Inequality (4.2) is equivalent to $EV(f_n, g_n) \leq 0$. The latter inequality follows from $V \leq U$ and $U(f_0, g_0) \leq 0$, both easy to check, and from

$$EU(f_n, g_n) \leq EU(f_{n-1}, g_{n-1}), \tag{4.3}$$

which holds for all $n \geq 1$. To prove (4.3), let S be the set of all (x, y) with $x > 0$ and $0 \neq y \in \mathbf{R}^\nu$. Define φ and ψ on S by

$$\begin{aligned} &\varphi(x,y) = -\frac{6x+3|y|}{2\sqrt{x+|y|}}, && \psi(x,y) = \frac{3y}{2\sqrt{x+|y|}} && \text{if } x+|y|<1;\\ &\varphi(x,y) = -3, && \psi(x,y) = 0 && \text{if } x+|y| \geq 1. \end{aligned}$$

The condition $(x,y)\in S$ and the further condition that $x + |y| \neq 1$ imply that

$$U_x(x,y) = \varphi(x,y) \quad \text{and} \quad U_y(x,y) = \psi(x,y); \tag{4.4}$$

see Section 2. Also, note that $|\psi| \leq -\varphi$, an inequality analogous to (2.12).

We shall show that if $x > 0$, $x + h > 0$, $y \in \mathbf{R}^\nu$, $k \in \mathbf{R}^\nu$, $|k| \leq |h|$ and $|y + kt| > 0$ for all $t \in \mathbf{R}$, then

$$U(x+h, y+k) \leq U(x,y) + \varphi(x,y)h + \psi(x,y)\cdot k. \tag{4.5}$$

The inequality (4.3) follows from (4.5) in the same way that (2.10) follows from (2.11). To prove (4.5), we can assume that $h \neq 0$ as well as the conditions on (x, y, h, k) given above. We define G on $I = \{t \in \mathbf{R}\colon x + ht > 0\}$ by $G(t) =$

$U(x+ht, y+kt)$. The function G is concave on I. To see this, let G_1 and G_2 be defined on I by

$$G_1(t) = \left[|y+kt| - 2(x+ht)\right]\sqrt{r(t)},$$
$$G_2(t) = 1 - 3(x+ht),$$

where $r(t) = x + ht + |y+kt|$. Both G_1 and G_2 are concave on I. In fact, G_1 is a positive multiple of the function G of Section 2 for the special case $p = \frac{3}{2}$. Also, $G_1(t) \le G_2(t)$ if $r(t) \le 1$. So, using the definition of U, we see that G is concave. Therefore, (4.5) must hold since, by (4.4), the inequality (4.5) is equivalent in the case $x + |y| \ne 1$ to

$$G(1) \le G(0) + G'(0), \tag{4.6}$$

a consequence of the concavity of G, and the case $x+|y| = 1$ follows by replacing x by $x + 2^{-j}$ in (4.5) and taken taking the limit of both sides as $j \to \infty$. This completes the proof of (4.1).

Strict inequality follows as in the proof of Theorem 2.1.

To see that 3 is the best possible constant, consider Example 2 of [9], which shows that 3 is the best possible constant in the analogous inequality for Itô processes X and Y. Let N be a positive integer. Example 2 of [9] implies the existence of a nonnegative submartingale f and a transform g of f by a predictable sequence that takes its values in $\{1, -1\}$ such that $P(g^* \ge 1) = 1$ and $\|f\|_1 = 1/3 + 1/6N$. An application of the ideas on page 60 of [8] gives a nonnegative submartingale F and a ± 1-transform G of F such that $\|F\|_1 = \|f\|_1$ and $g^* \le G^*$. This completes the proof of Theorem 4.1. □

5. Sharp maximal and escape inequalities for integrals with respect to nonnegative submartingales. Recall that $Y^* = \sup_{t\ge 0} |Y_t|$.

THEOREM 5.1. *If X is a nonnegative submartingale and Y is the integral of H with respect to X, where H is a predictable process with values in the closed unit ball of $\mathbf{R}^\nu$, then, for all $\lambda > 0$,*

$$\lambda P(Y^* \ge \lambda) \le \lambda P\left(\sup_{t\ge 0}(X_t + |Y_t|) \ge \lambda\right) \le 3\|X\|_1. \tag{5.1}$$

Even for the inequality $\lambda P(Y^* \ge \lambda) \le 3\|X\|_1$, the constant 3 is the best possible.

The proof of (5.1) follows from (4.1) in virtually the same way that (3.1) follows from (2.1). Here approximation is possible because the finiteness of $\|X\|_1$, which we can assume, implies, by (4.1), that X is an $L^{1,\infty}$ integrator in the sense of Bichteler [1].

Theorem 5.1 implies an inequality for $C_\varepsilon(Y)$, the number of ε-escapes of Y. Here $\varepsilon > 0$ and $C_\varepsilon(Y)$ is the number of nonnegative integers j satisfying $\tau_j < \infty$, where

$$\tau_0 = \inf\{t \ge 0: |Y_t| \ge \varepsilon\} \quad \text{and} \quad \tau_{j+1} = \inf\{t > \tau_j: |Y_t - Y_{\tau_j}| \ge \varepsilon\}.$$

Note that Y converges a.s. if and only if $P(C_\varepsilon(Y) = \infty) = 0$ for all $\varepsilon > 0$.

THEOREM 5.2. *If X is a nonnegative submartingale and Y is the integral of H with respect to X, where H is a predictable process with values in the closed unit ball of $\mathbf{R}^\nu$, then, for all $j \ge 1$,*

$$P\big(C_\varepsilon(Y) \ge j\big) \le 3\|X\|_1/(\varepsilon j^{1/2}). \tag{5.2}$$

Both the constant 3 and the exponent $\frac{1}{2}$ are the best possible.

PROOF. Fix $j \ge 1$. Consider the predictable process K defined by $K_t = (V_t^i H_t)_{0 \le i < j}$, where $V_t^0 = 1$ $(0 \le t \le \tau_0)$ and $V_t^i = 1$ $(\tau_{i-1} < t \le \tau_i)$ if $1 \le i < j$. The process K has its values in the closed unit ball of the $j\nu$-dimensional Euclidean space

$$\mathbf{K} = \big\{x = (x^0, \dots, x^{j-1}) \colon x^i \in \mathbf{R}^\nu \text{ if } 0 \le i < j\big\},$$

where the norm of $x \in \mathbf{K}$ is given by $|x|_{\mathbf{K}} = (\sum_{i=0}^{j-1} |x^i|^2)^{1/2}$. The integral Z of K with respect to X has its values in $\mathbf{K}$ and, on the set where $C_\varepsilon(Y) \ge j$ and $t \ge \tau_{j-1}$,

$$\begin{aligned} |Z_t|_{\mathbf{K}} &= \left(|Y_{\tau_0}|^2 + \sum_{i=1}^{j-1} |Y_{\tau_i} - Y_{\tau_{i-1}}|^2\right)^{1/2} \\ &\ge \varepsilon j^{1/2}. \end{aligned}$$

Consequently, by (5.1),

$$P\big(C_\varepsilon(Y) \ge j\big) \le P(Z^* \ge \varepsilon j^{1/2}) \le 3\|X\|_1/(\varepsilon j^{1/2}).$$

To see that the constant 3 is the best possible, observe that, for $\varepsilon < \lambda$,

$$P(Y^* \ge \lambda) \le P\big(C_\varepsilon(Y) \ge 1\big) \le 3\|X\|_1/\varepsilon$$

and use the fact that 3 is the best constant in $P(Y^* \ge \lambda) \le 3\|X\|_1/\lambda$.

To see that the exponent $\frac{1}{2}$ is the best possible, consider $Y = X$, where X is the nonnegative martingale satisfying $X_t = f_n$ if $t \in [n, n+1)$ and f is the simple random walk started at 1 and stopped at 0. There is a number $\alpha > 0$ such that, for all $j \ge 1$,

$$P\big(C_1(Y) > j\big) = P(f_n > 0 \text{ for } 0 \le n < j) \ge \alpha j^{1/2}.$$

6. A sharp exponential inequality for a strong subordinate of a nonnegative bounded submartingale. Suppose that Φ is a nondecreasing convex function on $[0, \infty)$ with $\Phi(0) = 0$ and $\int_0^\infty \Phi(t) e^{-t}\, dt$ finite and positive.

Suppose also that Φ is twice differentiable on $(0, \infty)$ and Φ' is convex on this open interval with $\Phi'(0+) = 0$.

THEOREM 6.1. *If f is a nonnegative submartingale bounded from above by 1, and g is strongly subordinate to f, then*

$$\sup_{n \geq 0} E\Phi\left(\frac{|g_n|}{2}\right) < \frac{2}{3}\int_0^\infty \Phi(t)e^{-t}\,dt \tag{6.1}$$

and the bound on the right is the best possible and is already the best if $\nu = 1$ and g is a ± 1-transform of f.

For example, if $\Phi(t) = 2^p t^p e^{2\beta t}$, where $0 \leq \beta < \frac{1}{2}$ and $2 \leq p < \infty$, then

$$\sup_{n \geq 0} E|g_n|^p \exp(\beta|g_n|) < \frac{2^{p+1}\Gamma(p+1)}{3}\frac{1}{(1-2\beta)^{p+1}} \tag{6.2}$$

and the bound on the right is the best possible. As is true throughout the paper, the best possible bound does not depend on the dimension ν of the Euclidean space in which g_n has its values.

REMARK 6.1. The above theorem also holds for f and g satisfying the conditions of Theorem 2.2: if F and G are as in the proof of Theorem 2.2, then the assumption here that f is a submartingale with its values in $[0, 1]$ implies the same for F, and, as in the L^p setting of Section 2,

$$\sup_{n \geq 0} E\Phi\left(\frac{|g_n|}{2}\right) \leq \sup_{n \geq 0} E\Phi\left(\frac{|G_n|}{2}\right).$$

PROOF OF THEOREM 6.1. Let $S = \{(x, y)\colon 0 \leq x \leq 1 \text{ and } y \in \mathbf{R}^\nu\}$. Here U and V are defined on S by $V(x, y) = \Phi(|y|/2)$ and

$$\begin{aligned} U(x,y) &= \frac{2\alpha}{3} + \frac{\alpha}{3}(|y| - 2x)\sqrt{x + |y|}, && x + |y| \leq 1, \\ &= (1-x)A\left(\frac{x + |y| + 1}{2}\right) + xB\left(\frac{x + |y| + 1}{2}\right), && x + |y| > 1, \end{aligned}$$

where $\alpha = \int_0^\infty \Phi(t)e^{-t}\,dt$ and, for all $t \geq 1$,

$$A(t) = e^t \int_t^\infty B(s)e^{-s}\,ds \quad \text{and} \quad B(t) = \Phi(t-1).$$

Note that $A(1) = \alpha$ and $B(1) = 0$ so U is continuous. Let $F = A - B$. Then, on $(1, \infty)$, $F = A'$ and $F' = F - B'$. If $t > 1$, then $F(t) > 0$, $F'(t) > 0$, $F''(t) \geq 0$ and $tF'(t) \geq F(t)$.

Define φ and ψ on S by $\varphi(0,0) = \psi(0,0) = 0$,

$$\begin{aligned}
\varphi(x,y) &= -\frac{\alpha}{2}\frac{2x+|y|}{\sqrt{x+|y|}}, && 0 < x+|y| \le 1,\\
&= -\frac{1}{2}\left[F\left(\frac{x+|y|+1}{2}\right) + xF'\left(\frac{x+|y|+1}{2}\right)\right], && x+|y| > 1,\\
\psi(x,y) &= \frac{\alpha}{2}\frac{y}{\sqrt{x+|y|}}, && 0 < x+|y| \le 1,\\
&= \frac{1}{2}\left[F\left(\frac{x+|y|+1}{2}\right) - xF'\left(\frac{x+|y|+1}{2}\right)\right]y', && x+|y| > 1.
\end{aligned}$$

Note that $x+|y| > 1$ implies that $y \neq 0$, so $y' = y/|y|$ is well defined. Also note that φ and ψ are continuous and $|\psi| \le -\varphi$ on S.

We shall show that if (x,y) and $(x+h, y+k)$ are in S, and $|k| \le |h|$, then

$$U(x+h, y+k) \le U(x,y) + \varphi(x,y)h + \psi(x,y)\cdot k. \tag{6.3}$$

This and the strong subordination of g imply that $EU(f_n, g_n) \le EU(f_{n-1}, g_{n-1})$ as before. The integrability follows from the boundedness of f_{n-1} and g_{n-1} although, of course, g need not be uniformly bounded. Note that (6.3) gives $V \le U$:

$$V(x,y) = U(1,y) = U\big(x+(1-x), y\big) \le U(x,y) + \varphi(x,y)(1-x) \le U(x,y).$$

Inequality (6.3) and the definition of U imply that if $|y| \le x$, then $(x/2)+|y/2| \le x \le 1$ and

$$U(x,y) \le U\left(\frac{x}{2},\frac{y}{2}\right) \le \frac{2\alpha}{3} - \frac{\alpha}{3}\left(\frac{x}{2}\right)^{3/2}. \tag{6.4}$$

The inequality (6.1) holds if $\|f\|_1 = 0$, so we can assume that $Ef_0^{3/2} > 0$. Then, by (6.4), $EU(f_0, g_0) < 2\alpha/3$ and (6.1) will follow [see (2.5)] once we have proved (6.3).

By the continuity of U, φ and ψ, it is enough to prove (6.3) for $|k| < |h|$ with x and $x+h$ in $(0,1)$. Let $G(t) = U(x+ht, y+kt)$. Then G has a continuous first derivative on an open interval containing $[0,1]$, and (6.3) is equivalent to

$$G(1) \le G(0) + G'(0). \tag{6.5}$$

However, (6.5) follows from the fact that G' is noincreasing on (0, 1). To see that G' is nonincreasing, let $r(t) = m(t) + N(t)$, where $m(t) = x + ht$ and $N(t) = |y + kt|$, and

$$I = \{t \in (0,1): r(t) < 1\} \quad \text{and} \quad J = \{t \in (0,1): r(t) > 1\}.$$

A number $t \in (0,1)$ satisfies $r(t) = 1$ only if it is a zero of the polynomial

$$\big[(1-x) - ht\big]^2 - |y+kt|^2 = (h^2 - |k|^2)t^2 + \cdots.$$

Here $h^2 - |k|^2 > 0$ so the complement of $I \cup J$ with respect to $(0,1)$ is finite and it will be enough to show that G' is nonincreasing on each component of $I \cup J$. On I, the second derivative G'' exists and satisfies $G'' \le 0$, as can be seen in Section 4. The second derivative exists on J also since $N(t) > 1 - m(t) > 0$ for all $t \in J$. On J,

$$G = A\left(\frac{r+1}{2}\right) - mF\left(\frac{r+1}{2}\right)$$

and $NN'' = |k|^2 - (N')^2$, so that

$$\begin{aligned} 4G'' = &-\left[2(h^2 - |k|^2) + (r')^2\right]F'\left(\frac{r+1}{2}\right) \\ &- 2N''\left[rF'\left(\frac{r+1}{2}\right) - F\left(\frac{r+1}{2}\right)\right] \\ &- m(\cdot)(r')^2F''\left(\frac{r+1}{2}\right). \end{aligned}$$

The first term on the right-hand side is nonpositive: $F' \ge 0$ and $|k| \le |h|$. The third term is nonpositive since $F'' \ge 0$ and $m(t) \in (0,1)$ for all $t \in (0,1)$. The second term is also nonpositive: by the triangle inequality, N is convex so $N'' \ge 0$ on J, and

$$rF'\left(\frac{r+1}{2}\right) \ge \frac{r+1}{2}F'\left(\frac{r+1}{2}\right) \ge F\left(\frac{r+1}{2}\right).$$

Therefore, on each component of $I \cup J$, the derivative G'' is nonpositive and G' is nonincreasing. This completes the proof of (6.1).

We shall now describe an example showing that the bound on the right is the best possible. It is Example 6.3 and will be constructed using the following two examples.

EXAMPLE 6.1. Suppose that m is a positive integer divisible by 3. Then there is a nonnegative submartingale F that is bounded from above by 1, and a transform G of F by a predictable sequence that takes its values in $\{1, -1\}$ such that

$$\begin{aligned} P(F_m = 0,\ G_m = 1) &= \frac{1}{3} - \frac{1}{4m}, \\ P(F_m = 1,\ G_m = 1) &= \frac{1}{3} + \frac{1}{2m}, \\ P(F_m = 0,\ G_m = -1) &= \frac{1}{3} - \frac{1}{4m}. \end{aligned}$$

This follows at once from Example 2 of [9].

EXAMPLE 6.2. If $\gamma > 0$ and $\varepsilon \in \{1, -1\}$, then there is a nonnegative submartingale F starting at 0 and bounded from above by 1, and a ± 1-transform

G of F, such that

$$\sup_{n \geq 0} E\Phi\left(\frac{|\varepsilon + G_n|}{2}\right) > \int_0^\infty \Phi(t)e^{-t}\,dt - \gamma. \tag{6.6}$$

It is enough to show this for $\varepsilon = 1$. Let $0 < \delta < \frac{1}{2}$, and let $(d_n)_{n \geq 0}$ be an independent sequence of random variables such that $d_0 = 0$ and, for all $n \geq 1$, $d_{2n-1} = \delta$,

$$P(d_{2n} = 1 - \delta) = \delta \quad \text{and} \quad P(d_{2n} = -\delta) = 1 - \delta.$$

Let $d_n^* = \sup_{0 \leq k \leq n} |d_k|$, and let F be the submartingale with the difference sequence D defined by $D_0 = d_0$ and $D_n = 1(d_{n-1}^* \leq \delta)d_n$ for $n \geq 1$. Define G by

$$G_n = \sum_{k=0}^{n} (-1)^{k-1} D_k.$$

It is easy to see that $0 \leq F_n \leq 1$ for all $n \geq 0$ with probability 1 and that the distribution of the almost everywhere limit of $1 + G$ is given by

$$P(1 + G_\infty = 2k\delta) = \delta(1 - \delta)^{k-1}, \qquad k \geq 1.$$

Therefore, by Fatou's lemma,

$$\sup_{n \geq 0} E\Phi\left(\frac{|1 + G_n|}{2}\right) \geq E\Phi\left(\frac{|1 + G_\infty|}{2}\right) = \sum_{k=1}^{\infty} \Phi(k\delta)\delta(1 - \delta)^{k-1},$$

but the right-hand side exceeds

$$\sum_{k=1}^{\infty} \int_{(k-1)\delta}^{k\delta} \Phi(t)e^{-\beta t}\,dt = \int_0^\infty \Phi(t)e^{-\beta t}\,dt,$$

where $\beta = -\delta^{-1}\log(1 - \delta)$. The concavity of the logarithm function and $\log 1 = 0$ imply that $\beta \downarrow 1$ as $\delta \downarrow 0$. Therefore, by the monotone convergence theorem, (6.6) holds for all sufficiently small positive δ.

EXAMPLE 6.3. Let f be a nonnegative submartingale bounded above by 1 and g a transform of f by a predictable sequence that takes its values in $\{1, -1\}$ such that the pair $(f_n)_{0 \leq n \leq m}$ and $(g_n)_{0 \leq n \leq m}$ have the same distribution as the pair $(F_n)_{0 \leq n \leq m}$ and $(G_n)_{0 \leq n \leq m}$, where m is a positive integer divisible by 3 and F and G are as in Example 6.1. On the set where $f_m = 1$ and $g_m = 0$, let $f_n = 1$ and $g_n = 0$ for all $n > m$. On the set where $f_m = 0$ and $g_m = 1$, let the pair $(f_{m+k})_{k \geq 0}$ and $(g_{m+k})_{k \geq 0}$ have the same distribution as the pair F and $1 + G$ of Example 6.2. Similarly, on the set where $f_m = 0$ and $g_m = -1$, let the pair $(f_{m+k})_{k \geq 0}$ and $(g_{m+k})_{k \geq 0}$ have the same distribution as the pair F and $-1 + G$ of Example 6.2. Then, for all large m,

$$\sup_{n \geq 0} E\Phi\left(\frac{|g_n|}{2}\right) > \frac{2}{3}\int_0^\infty \Phi(t)e^{-t}\,dt - 2\gamma,$$

and this implies that the bound on the right in (6.1) is the best possible. An application of the ideas on page 60 of [8] completes the proof of Theorem 6.1. □

REMARK 6.2. Suppose that $|e_n| \le |d_n|$, $n \ge 1$, but that $|e_0| \le 1 - d_0$. We have then, instead of (6.1), the sharp inequality

$$\sup_{n\ge 0} E\Phi\left(\frac{|g_n|}{2}\right) \le (1 - Ef_0)\int_0^\infty \Phi(t)e^{-t}\,dt, \tag{6.7}$$

since $U(x,y) \le U(x,(1-x)y') = \alpha(1-x)$ if $0 < |y| \le 1-x$.

7. A sharp exponential inequality for an integral with respect to a nonnegative bounded submartingale. Suppose that Φ satisfies the conditions of Section 6. Theorem 6.1 and an approximation argument similar to that used in Section 3 yield the following theorem.

THEOREM 7.1. *If X is a nonnegative submartingale bounded from above by 1, and Y is the integral of H with respect to X, where H is a predictable process with values in the closed unit ball of $\mathbf{R}^\nu$, then*

$$\sup_{t\ge 0} E\Phi\left(\frac{|Y_t|}{2}\right) \le \frac{2}{3}\int_0^\infty \Phi(t)e^{-t}\,dt \tag{7.1}$$

and the bound on the right is the best possible.

We now turn to the supermartingale setting, where the results and their proofs can differ substantially from those above.

8. A sharp maximal inequality for a strong subordinate of a nonnegative supermartingale.

THEOREM 8.1. *If f is a nonnegative supermartingale and g is strongly subordinate to f, then, for all $\lambda > 0$,*

$$\lambda P(g^* \ge \lambda) \le \lambda P\Big(\sup_{n\ge 0}(f_n + |g_n|) \ge \lambda\Big) \le 2Ef_0 \tag{8.1}$$

and 2 is the best possible constant. In fact, given $\lambda > 0$, there is a ± 1-transform g of a nonnegative martingale f such that $\lambda P(g^* \ge \lambda) = 2Ef_0 > 0$.

PROOF. It is enough to prove that

$$P\big(f_n + |g_n| \ge 1\big) \le 2Ef_0. \tag{8.2}$$

Here let U and V be defined on $[0,\infty)\times\mathbf{R}^\nu$ by the following:

$$U(x,y) = 2x - x^2 + |y|^2, \qquad V(x,y) = 0 \quad \text{if } x + |y| < 1;$$
$$U(x,y) = 1, \qquad V(x,y) = 1 \quad \text{if } x + |y| \ge 1.$$

Note that U is continuous and is bounded from above by 1. Inequality (8.2) is equivalent to $EV(f_n, g_n) \le 2Ef_0$, which follows from $V \le U$,

$$EU(f_n, g_n) \le EU(f_{n-1}, g_{n-1}), \qquad n \ge 1, \tag{8.3}$$

and $U(f_0, g_0) \le 2f_0$. To prove (8.3), define φ and ψ on the set where $x \ge 0$ and $y \in \mathbf{R}^\nu$ by

$$\begin{aligned} &\varphi(x,y) = 2 - 2x, \qquad \psi(x,y) = 2y \quad \text{if } x + |y| < 1;\\ &\varphi(x,y) = 0, \qquad\qquad \psi(x,y) = 0 \quad \text{if } x + |y| \ge 1. \end{aligned}$$

We shall show that if $x \ge 0, x + h \ge 0, y \in \mathbf{R}^\nu, k \in \mathbf{R}^\nu$ and $|k| \le |h|$, then

$$U(x+h, y+k) \le U(x,y) + \varphi(x,y)h + \psi(x,y) \cdot k. \tag{8.4}$$

The inequality (8.3) follows from (1.DS), (1.CDS), (8.4), $|\psi| \le \varphi$ and the supermartingale condition $E(d_n \mid \mathcal{F}_{n-1}) \le 0$, $n \ge 1$. (See the proof of (2.13), but here no translation by a and $|a|$ is needed.) To prove (8.4), consider the following three cases: (i) $x + |y| \ge 1$; (ii) $x + |y| < 1$ and $x + h + |y + k| < 1$; (iii) $x + |y| < 1$ and $x + h + |y + k| \ge 1$. In case (i), the right-hand side is 1 and the left-hand side is not greater than 1. To check case (ii), observe that the difference between the right-hand side and the left-hand side is $h^2 - |k|^2$, which is nonnegative. In case (iii), this difference is given by

$$\left(h^2 - |k|^2\right)t_0^2 + 2(1 - t_0)\left[(1 - x)h + y \cdot k\right], \tag{8.5}$$

in which the number $t_0 \in (0,1)$ satisfies $r(t_0) = 1$ so that $U(x + ht_0, y + kt_0) = 1$. Here, as above, $r(t) = x + ht + |y + kt|$ and, by (iii), $r(0) < 1$ and $r(1) \ge 1$ so such a number t_0 does exist. Also, $h > 0$, for otherwise, by (iii),

$$|y| + |k| \ge |y + k| \ge 1 - (x + h) > |y| - h = |y| + |h|,$$

a contradiction of $|k| \le |h|$. Therefore, $(1 - x)h \ge |y|\,|k| \ge y \cdot k$ and (8.5) is nonnegative. This completes the proof of (8.1).

To prove that equality can hold throughout (8.1), let f and g be defined on the Lebesgue unit interval by $f_n = \lambda 2^n 1_{[0, 2^{-n}]}$ and $g_n = 2f_0 - f_n$. Then $Ef_0 = \lambda$ and $P(g^* \ge 2\lambda) = 1$, so equality holds throughout (8.1). □

THEOREM 8.2. *If f and g are sequences given by*

$$f_n = \sum_{k=0}^{n} d_k + A_n,$$

$$g_n = \sum_{k=0}^{n} e_k + B_n,$$

where d and e satisfy (1.DS) *and* (1.CDS), *f is a nonnegative supermartingale, $A_0 = 0$, $B_0 = 0$, and, for all $n \ge 1$, the functions $A_n \colon \Omega \to (-\infty, 0]$ and $B_n \colon \Omega \to \mathbf{R}^\nu$ are $\mathcal{F}_n$-measurable and integrable with*

$$|B_n| \le -A_n, \tag{8.6}$$

then, for all $\lambda > 0$,

$$\lambda P(g^* \geq \lambda) \leq \lambda P\Big(\sup_{n \geq 0}(f_n + |g_n|) \geq \lambda\Big) \leq 2Ef_0$$

and equality can hold throughout.

The sequence A need not be monotone, and A and B need not be predictable.

PROOF OF THEOREM 8.2. Fix a nonnegative integer n. If $0 \leq k \leq n$, let $D_k = d_k$ and $E_k = e_k$. Let $D_{n+1} = A_n$ and $E_{n+1} = B_n$. If $k \geq n+2$, let $D_k = 0$ and $E_k = 0$. Then D and E are adapted to the original filtration and satisfy condition (1.DS). They also satisfy (1.CDS): If $1 \leq k \leq n$, then

$$|E(E_k \mid \mathcal{F}_{k-1})| \leq -E(D_k \mid \mathcal{F}_{k-1}).$$

By (8.6), this also holds for $k = n+1$ as well as holding trivially for $k \geq n+2$. Let F have the difference sequence D, and G the difference sequence E. If $1 \leq k \leq n$, then, by the nonpositivity of A_k,

$$F_k = \sum_{j=0}^{k} D_j = \sum_{j=0}^{k} d_j \geq f_k \geq 0;$$

if $k > n$, then $F_k = f_n \geq 0$. So F and G satisfy the conditions of Theorem 8.1 and

$$P(f_n + |g_n| \geq 1) \leq P(F_{n+1} + |G_{n+1}| \geq 1) \leq 2EF_0 = 2Ef_0.$$

This implies the desired inequality. The equality can hold as in Theorem 8.1: take $A = B = 0$. □

9. Sharp maximal and escape inequalities for integrals with respect to nonnegative supermartingales.

THEOREM 9.1. *If X is a nonnegative supermartingale and Y is the integral of H with respect to X, where H is a predictable process with values in the closed unit ball of* $\mathbf{R}^\nu$, *then, for all* $\lambda > 0$,

$$\lambda P(Y^* \geq \lambda) \leq \lambda P\Big(\sup_{t \geq 0}(X_t + |Y_t|) \geq \lambda\Big) \leq 2EX_0 \tag{9.1}$$

and $P(Y^* \geq 1) = 2EX_0 > 0$ *for some X and Y, so the constant 2 is the best possible.*

THEOREM 9.2. *If X is a nonnegative supermartingale and Y is the integral of H with respect to X, where H is a predictable process with values in the closed unit ball of* $\mathbf{R}^\nu$, *then, for all* $j \geq 1$,

$$P(C_\varepsilon(Y) \geq j) \leq 2EX_0/(\varepsilon j^{1/2}). \tag{9.2}$$

Both the constant 2 *and the exponent* $\frac{1}{2}$ *are the best possible.*

Inequality (9.2) follows from (9.1), which follows by approximation from (8.1). Their proofs have the same pattern as those of (5.1) and (5.2). Also, see Example 1 of [9].

10. A sharp exponential inequality for a strong subordinate of a nonnegative bounded supermartingale. Suppose Φ is a convex function satisfying the conditions of Section 6.

THEOREM 10.1. *If f is a nonnegative supermartingale bounded from above by* 1, *and g is strongly subordinate to f, then*

$$\sup_{n\geq 0} E\Phi\left(\frac{|g_n|}{2}\right) \leq Ef_0 \int_0^\infty \Phi(t)e^{-t}\,dt. \tag{10.1}$$

The constant on the right is the best possible and is already the best if $\nu = 1$ and g is a ± 1-transform of f.

If $\Phi(t) = 2^p t^p e^{2\beta t}$, where $0 \leq \beta < \frac{1}{2}$ and $2 \leq p < \infty$, then

$$\sup_{n\geq 0} E|g_n|^p \exp\left(\beta|g_n|\right) \leq \frac{2^p\Gamma(p+1)}{(1-2\beta)^{p+1}} Ef_0 \tag{10.2}$$

and the constant on the right is the best possible. One special case of (10.2) is already known: $\nu = 1$, $p = 2$, $\beta = 0$ and g is the transform of f by a predictable sequence uniformly bounded in absolute value by 1. Using a different method, Edwards [11] proved that, in this case, $\|g\|_2^2 \leq 8Ef_0$. The theorem above implies that the constant 8 is the best possible.

PROOF OF THEOREM 10.1. Let F be the submartingale defined by $F_n = 1 - f_n$. Then F and g satisfy the conditions of Remark 6.2 and the result stated there yields (10.1). The last assertion of the theorem follows easily from a slight modification of the examples in Section 6. □

11. A sharp exponential inequality for an integral with respect to a nonnegative bounded supermartingale. Suppose again that Φ satisfies the conditions of Section 6. Theorem 10.1 and approximation yield the following theorem.

THEOREM 11.1. *Suppose that X is a nonnegative supermartingale bounded from above by* 1, *and Y is the integral of H with respect to X, where H is a predictable process with values in the closed unit ball of $\mathbf{R}^\nu$. Then*

$$\sup_{t\geq 0} E\Phi\left(\frac{|Y_t|}{2}\right) \leq EX_0 \int_0^\infty \Phi(t)e^{-t}\,dt \tag{11.1}$$

and the bound on the right is the best possible.

12. Boundary value problems and the search for U. The aim here is to throw some light on the method used to find the function U of Section 2, the function U of Section 4 and the other such functions in this paper. For simplicity, set $\nu = 1$ so that U will be defined on a subset of $\mathbf{R}^2$, and consider the problem only for ± 1-transforms g of submartingales f. Once this case is understood, suitable functions U for the strongly subordinate and higher-dimensional cases can be conjectured. Furthermore, a simple transformation can be made so that instead of finding first a majorant U of V such that the two mappings $t \mapsto U(x + |h|t, y \pm |h|t)$ are concave and nonincreasing (concave and nondecreasing in the supermartingale case), one can find a biconcave function u that is nonincreasing in each of its two arguments. We shall use the word "bidecreasing" to describe the latter property. (For the martingale setting in which monotonicity is not required, see [2] or, in the Banach-space context, [3].)

Let S be a biconvex subset of $\mathbf{R}^2$, that is, each horizontal and vertical section of S is either empty or an interval. Let S_∞ be a nonempty subset of S. Let F be a real-valued function on S_∞, and let $\mathcal{U}_F$ be the family of all biconcave and bidecreasing functions $u: S \to \mathbf{R}$ such that $u \geq F$ on S_∞. One may think of S_∞ as a boundary, F as the boundary data and $\mathcal{U}_F$ as the upper class. If the upper class is nonempty, let U_F denote the upper solution: for $(x, y) \in S$,

$$U_F(x, y) = \inf\{u(x, y): u \in \mathcal{U}_F\}.$$

A *zigzag submartingale* Z is a sequence $(Z_n)_{n \geq 0}$ with $Z_n = (X_n, Y_n)$, where X and Y are submartingales relative to the same filtration satisfying

$$X_n - X_{n-1} \equiv 0 \quad \text{or} \quad Y_n - Y_{n-1} \equiv 0,$$

for all $n \geq 1$. An example is given by $Z_n = (f_n + g_n, f_n - g_n)$ where g is a ± 1-transform of a submartingale f. A zigzag submartingale Z is *simple* if the Z_n are simple functions and, for some n depending on Z,

$$Z_n = Z_{n+1} = \cdots = Z_\infty.$$

Let $\mathcal{Z}(x, y)$ be the set of all simple zigzag submartingales Z on the Lebesgue unit interval with values in S such that $Z_0 = (x, y)$ and Z_∞ has its values in S_∞. The filtration is allowed to vary with the Z.

THEOREM 12.1. *Assume that $\mathcal{U}_F$ is nonempty and that, for all $(x, y) \in S$, the set $\mathcal{Z}(x, y)$ is also nonempty. Then*

$$U_F(x, y) = \sup\{EF(Z_\infty): Z \in \mathcal{Z}(x, y)\}. \tag{12.1}$$

The proof is similar to the proof of its martingale analogue, Theorem 11.1 of [2]. Note that another way of stating (12.1) is that the following inequalities are sharp: for all $Z \in \mathcal{Z}(x, y)$ and all $u \in \mathcal{U}_F$,

$$EF(Z_\infty) \leq U_F(x, y) \leq u(x, y). \tag{12.2}$$

For example, suppose S is the set of points in the plane satisfying $x+y\geq 0$ and that $S_\infty = S$. Let F be defined on S_∞ by

$$F(x,y)=\left|\frac{x-y}{2}\right|^p-\beta^p\left(\frac{x+y}{2}\right)^p,$$

where the positive number β can be chosen later. If $u \in \mathcal{U}_F$ and u is twice continuously differentiable on some open subset of S, then on this subset,

$$u_x \vee u_y \vee u_{xx} \vee u_{yy} \leq 0. \tag{12.3}$$

If, in addition, $u = U_F$, then its extremality suggests that equality will hold in (12.3) on some subdomain of S. This gives the initial clue, one that leads eventually to the least possible value of β and the functions U and V of Section 2. (The discussion of the martingale analogue in Section 2 of [8] may also be helpful here.) Sometimes it is enough, and simpler, to work with a function u that may not be extremal for every $(x,y)\in S$.

13. Comparing the sizes of smooth functions. Let n be a positive integer and D an open connected set of points $x=(x_1,\dots,x_n)\in\mathbf{R}^n$. Fix $\xi\in D$ and suppose that u and v are continuous functions on D with continuous first and second partial derivatives. Assume that

$$|v(\xi)| \leq |u(\xi)|, \tag{13.1}$$

$$|\nabla v| \leq |\nabla u|, \tag{13.2}$$

$$|\Delta v| \leq |\Delta u|, \tag{13.3}$$

where u is real-valued and $v=(v^1,\dots,v^\nu)$ has its values in $\mathbf{R}^\nu$. So, for example, (13.2) is the condition that $|\nabla v(x)|\leq|\nabla u(x)|$ for all $x\in D$, where

$$|\nabla v|^2=\sum_{k=1}^n |v_k|^2=\sum_{k=1}^n\sum_{j=1}^\nu |v_k^j|^2,$$

$v_k=(v_k^1,\dots,v_k^\nu)$ and $v_k^j=\partial v^j/\partial x_k$. Similarly,

$$|\Delta v|^2=\sum_{j=1}^\nu |\Delta v^j|^2,$$

where $\Delta v^j=\sum_{k=1}^n v_{kk}^j$ and $v_{kk}^j=\partial^2 v^j/\partial x_k^2$. If u and v are harmonic, then both sides of (13.3) vanish and (13.3) is trivially satisfied. Conditions (13.1), (13.2) and (13.3) play a role here analogous to that played by (1.DS) and (1.CDS) above.

Let D_0 be a bounded subdomain satisfying $\xi\in D_0\subset D_0\cup\partial D_0\subset D$, and set

$$\|u\|_p=\sup_{D_0}\left[\int_{\partial D_0}|u|^p\,d\mu\right]^{1/p},$$

where the supremum is taken over all such D_0. Here $\mu = \mu_{D_0}^{\xi}$, the harmonic measure on ∂D_0 with respect to ξ.

THEOREM 13.1. *Let* $1 < p < \infty$. *If* u *and* v *are as above and* u *is nonnegative and subharmonic on* D, *then*

$$\|v\|_p \le (p^{**} - 1)\|u\|_p. \tag{13.4}$$

This should be compared with Theorem 2.1. Also, see [6, Theorem 2.1] in which u and v are harmonic and $\|v\|_p \le (p^* - 1)\|u\|_p$, an inequality that contains the classical conjugate-function inequality of Marcel Riesz. In the Riesz inequality, the domain D is the open unit disk of the complex plane, $\xi = 0$, the function u is harmonic on D, and v is the harmonic conjugate of u satisfying $v(0) = 0$. Therefore, (13.1) is satisfied, (13.2) holds with equality and both sides of (13.3) vanish. For further discussion of the Riesz inequality, the equally classical Kolmogorov inequality and some key references to later work, see [6] and [8]. In [6], the domain D is n-dimensional and the conjugacy condition is replaced by the gradient condition (13.2). Here the harmonicity condition is dropped in favor of the less restrictive condition (13.3).

To prove Theorem 13.1, we shall use the function U of Section 2.

PROOF OF THEOREM 13.1. If U is given by (2.4), then $U(u, v)$ is superharmonic on D. This can be seen by direct calculation or by showing, as we shall do below, that this is a simple consequence of what we have already proved in Section 2. Therefore, with V defined as in (2.2),

$$\int_{\partial D_0} |v|^p \, d\mu = (p^{**} - 1)^p \int_{\partial D_0} |u|^p \, d\mu + \int_{\partial D_0} V(u, v) \, d\mu,$$

where the last integral satisfies

$$\int_{\partial D_0} V(u, v) \, d\mu \le \int_{\partial D_0} U(u, v) \, d\mu \le U\big(u(\xi), v(\xi)\big) \le 0.$$

These three inequalities follow from (2.7), from the superharmonicity of $U(u, v)$ and from (13.1). The desired inequality (13.4) is an immediate consequence.

To prove that $U(u, v)$ is superharmonic, we let $W = U(u, v)$ and assume that both u and $|v|$ are strictly positive on D (otherwise, replace u by $|a| + u$ and v by $a + v$, where a is as in Section 2). Then W is twice continuously differentiable on D and

$$\Delta W = \Delta_1 W + \Delta_2 W,$$

where

$$\Delta_1 W = U_0(u, v)\Delta u + \sum_{j=1}^{\nu} U_j(u, v)\Delta v^j$$

is nonpositive by (2.12), (13.3) and the subharmonicity of u, and

$$\Delta_2 W = U_{00}(u,v)|\nabla u|^2 + 2\sum_{j=1}^{\nu} U_{0j}(u,v)\,\nabla u\cdot\nabla v^j + \sum_{i=1}^{\nu}\sum_{j=1}^{\nu} U_{ij}(u,v)\,\nabla v^i\cdot\nabla v^j$$

is also nonpositive, as we shall show. The U_j and U_{ij} denote the first- and second-order partial derivatives of U on the $(\nu+1)$-dimensional open set where they are defined. If (x,y) belongs to this set, then $G(t) = U(x+ht, y+kt)$ satisfies

$$G''(0) = U_{00}(x,y)h^2 + 2\sum_{j=1}^{\nu} U_{0j}(x,y)hk^j + \sum_{i=1}^{\nu}\sum_{j=1}^{\nu} U_{ij}(x,y)k^ik^j,$$

where $k = (k^1,\dots,k^j)$. Let us write $G''(0) = G''(0;x,y,h,k)$ and use a similar notation for the functions A, B and C of Section 2. Then

$$\begin{aligned}\Delta_2 W &= \sum_{k=1}^{n} G''(0;u,v,u_k,v_k)\\ &= -\alpha_p\sum_{k=1}^{n}\left[A(u,v,u_k,v_k)+B(u,v,u_k,v_k)+C(u,v,u_k,v_k)\right]\big(u+|v|\big)^{p-3}\\ &\le -\alpha_p\sum_{k=1}^{n} B(u,v,u_k,v_k)\big(u+|v|\big)^{p-3}\\ &= -\alpha_p(p-1)p^{**}\big(|\nabla u|^2-|\nabla v|^2\big)\big(u+|v|\big)^{p-2}.\end{aligned}$$

Here we have used the nonnegativity of A and C. Using (13.2), we see that $\Delta_2 W \le 0$. Therefore, $\Delta W \le 0$ so W is superharmonic and Theorem 13.1 is proved. □

The following theorem for smooth functions contains two inequalities that are also analogous to earlier ones in the paper.

THEOREM 13.2. *Suppose, as above, that u and v satisfy* (13.1)–(13.3) *and that μ is the harmonic measure on ∂D_0 with respect to ξ. Let $\lambda > 0$.* (i) *If u is nonnegative and superharmonic on D, then*

$$\lambda\mu\big(u+|v|\ge\lambda\big) \le 2u(\xi) = 2\|u\|_1. \tag{13.5}$$

Even for $\lambda\mu(|v|\ge\lambda)\le 2\|u\|_1$, the constant 2 is the best possible. (ii) *If u is nonnegative and subharmonic on D, then*

$$\lambda\mu\big(u+|v|\ge\lambda\big) \le 3\int_{\partial D_0} u\,d\mu \le 3\|u\|_1. \tag{13.6}$$

This should be compared with Theorem 1.1 of [6] in which u and v are harmonic. That theorem contains the classical Kolmogorov inequality and gives a simple approach to it. The following proof can be easily modified to obtain inequalities for the Brownian maximal function of $u + |v|$ that are analogous to (13.5) and (13.6). See the proof of Theorem 1.2 of [6] or, for a different approach, see Section 14 below.

PROOF OF THEOREM 13.2. To prove (13.5) we can assume that $\lambda = 1$ and use the functions U and V of Section 8. It is easy to check that $U(u,v)$ is superharmonic. Therefore,

$$\begin{aligned}\mu\big(u + |v| \geq 1\big) &= \int_{\partial D_0} V(u,v)\,d\mu \leq \int_{\partial D_0} U(u,v)\,d\mu \\ &\leq U\big(u(\xi), v(\xi)\big) \leq 2u(\xi) = 2\|u\|_1.\end{aligned}$$

Note that $u(\xi) \leq \|u\|_1$ follows from the definition of $\|u\|_1$ and the opposite inequality from the superharmonicity and nonnegativity of u.

The proof of (13.6) is similar and rests on the superharmonicity of $U(u,v)$, where U is the function of Section 4. The expression for $G''(0)$ in (4.7) can be used to prove this superharmonicity.

The following example completes the proof of Theorem 13.2.

EXAMPLE 13.1. Let $n = 1$ and $\nu = 1$. The following example can be easily modified to take care of any pair (n, ν) of positive integers. Let D be the interval $(-1, 3)$, and let $\xi = 0$, $u(x) = 1+x$ and $v(x) = 1-x$. The function u is nonnegative and harmonic on D and the conditions (13.1), (13.2) and (13.3) are satisfied. If $-1 < a < 0 < b < 3$ and $D_0 = (a, b)$, then the harmonic measure μ on ∂D_0 with respect to 0 must satisfy $\int_{\{a,b\}} x\,d\mu = 0$. Therefore, $\|u\|_1 = 1$. If $0 < \lambda < 2$, then $|v(x)| < \lambda$ if and only if $1 - \lambda < x < 1 + \lambda$, so

$$\limsup_{\lambda\uparrow 2}\ \sup_{D_0} \lambda\mu\big(|v| \geq \lambda\big) = \lim_{\lambda\uparrow 2} \lambda = 2\|u\|_1. \tag{13.7}$$

Therefore, the constant 2 in the inequality $\lambda\mu(|v| \geq \lambda) \leq 2\|u\|_1$ of part (i) of Theorem 13.2 cannot be replaced by a smaller number. □

REMARK 13.1. By Theorem 1.1 in [6], the inequality $\lambda\mu(|v| \geq \lambda) \leq 2\|u\|_1$ holds for harmonic functions u and v satisfying (13.1) and (13.2). The function u need not be nonnegative. Indeed both u and v can be vector-valued. The above example shows that the constant 2 is also the best possible in this harmonic function context, thus answering a question implicit in Remark 1.3 of [6]. Here is a closely related example for a more frequently encountered domain. Let D be the open unit disk of the complex plane $\mathbf{C}$, and let $\xi = 0$. Let F be a univalent analytic function mapping D onto the horizontal strip $\{w \in \mathbf{C}\colon -1 < \Im w < 3\}$ with $F(0) = 0$. Then $u = 1 + \Im F$ and $v = 1 - \Im F$ are harmonic functions satisfying (13.1), (13.2) and (13.7).

14. Sharp inequalities for Itô processes. This paper contains sharp inequalities for possibly discontinuous X and Y, where Y is the integral of a predictable process with respect to X. Analogous inequalities hold for pairs of Itô processes satisfying the conditions (14.1)–(14.3), as well as for similar pairs of continuous processes in which dB_s is replaced by dM_s, and ds by dA_s, where M is an adapted continuous martingale and A is an adapted continuous nondecreasing process starting at 0. Their continuity can be exploited in the proofs where the functions U again play a role.

For example, let X and Y be Itô processes [12] defined by

$$X_t = X_0 + \int_0^t \varphi_s \, dB_s + \int_0^t \psi_s \, ds,$$
$$Y_t = Y_0 + \int_0^t \widetilde{\varphi}_s \, dB_s + \int_0^t \widetilde{\psi}_s \, ds.$$

As above, the underlying probability space $(\Omega, \mathcal{F}, P)$ is complete and is filtered by an increasing right-continuous family $(\mathcal{F}_t)_{t \ge 0}$ of sub-σ-fields of $\mathcal{F}$, where $\mathcal{F}_0$ contains all $A \in \mathcal{F}$ with $P(A) = 0$. The real Brownian motion B starts at 0 and is adapted to $(\mathcal{F}_t)_{t \ge 0}$, and the process $(B_{t+s} - B_s)_{t \ge 0}$ is independent of $\mathcal{F}_s$ for all $s \ge 0$. The real processes φ and ψ are predictable,

$$P\left(\int_0^t |\varphi_s|^2 \, ds < \infty \text{ and } \int_0^t |\psi_s| \, ds < \infty \text{ for all } t > 0 \right) = 1,$$

and $\widetilde{\varphi}$ and $\widetilde{\psi}$ are $\mathbf{R}^\nu$-valued processes satisfying the same conditions. Both X and Y are adapted and can be taken to be everywhere continuous.

Under the conditions

$$|Y_0| \le |X_0|, \tag{14.1}$$
$$|\widetilde{\varphi}| \le |\varphi|, \tag{14.2}$$
$$|\widetilde{\psi}| \le |\psi|, \tag{14.3}$$

the nonnegative submartingale and nonnegative supermartingale inequalities of the earlier sections carry over. For the special case in which $\nu = 1$ and X_0 is constant, the analogues of the maximal inequalities (1.2) and (9.1) are in [9]. The proof of the analogue of (1.2), for example, uses Itô's formula and the restriction of the function U of Section 4 to the set where $x \ge 0$ and $x + |y| \le 1$. The proof of the analogue of the norm inequality (3.1) rests similarly on Itô's formula applied to $U(X_t^a, Y_t^a)$, where U is given by (2.4) and a has the same role as in the proof of (2.13). One may also use $U^\varepsilon(X_t, Y_t)$, where $\varepsilon > 0$ and $U^\varepsilon(x, y)$ is the result of substituting $x + \varepsilon$ for x and $(|y|^2 + \varepsilon)^{1/2}$ for $|y|$ in the right-hand side of (2.4).

There are similar inequalities for more general Itô processes that yield alternate proofs of the theorems of Section 13. For example, let B, φ and each component of $\widetilde{\varphi}$ have their values in $\mathbf{R}^n$. Replace $\int_0^t \varphi_s \, dB_s$ by $\int_0^t \varphi_s \cdot dB_s$, with a similar replacement for the corresponding integral of each component of $\widetilde{\varphi}$. Then use the appropriate function U and Itô's formula as above.

REFERENCES

[1] BICHTELER, K. (1981). Stochastic integration and L^p-theory of semimartingales. *Ann. Probab.* **9** 49–89.

[2] BURKHOLDER, D. L. (1984). Boundary value problems and sharp inequalities for martingale transforms. *Ann. Probab.* **12** 647–702.

[3] BURKHOLDER, D. L. (1986). Martingales and Fourier analysis in Banach spaces. *Probability and Analysis. Lecture Notes in Math.* **1206** 61–108. Springer, Berlin.

[4] BURKHOLDER, D. L. (1987). A sharp and strict L^p-inequality for stochastic integrals. *Ann. Probab.* **15** 268–273.

[5] BURKHOLDER, D. L. (1988). Sharp inequalities for martingales and stochastic integrals. *Colloque Paul Lévy sur les processus stochastiques. Astérisque* **157–158** 75–94.

[6] BURKHOLDER, D. L. (1989). Differential subordination of harmonic functions and martingales. *Harmonic Analysis and Partial Differential Equations. Lecture Notes in Math.* **1384** 1–23. Springer, Berlin.

[7] BURKHOLDER, D. L. (1989). On the number of escapes of a martingale and its geometrical significance. In *Almost Everywhere Convergence* (G. A. Edgar and L. Sucheston, eds.) 159–178. Academic, New York.

[8] BURKHOLDER, D. L. (1991). Explorations in martingale theory and its applications. *Ecole d'Eté de Probabilités de Saint-Flour XIX. Lecture Notes in Math.* **1464** 1–66. Springer, Berlin.

[9] BURKHOLDER, D. L. (1993). Sharp probability bounds for Itô processes. In *Current Issues in Statistics and Probability: Essays in Honor of Raghu Raj Bahadur* (J. K. Ghosh, S. K. Mitra, K. R. Parthasarathy and B. L. S. Prakasa Rao, eds.) 135–145. Wiley Eastern, New Delhi.

[10] DELLACHERIE, C. and MEYER, P. A. (1980). *Probabilités et Potentiel: Théorie des Martingales*. Hermann, Paris.

[11] EDWARDS, D. A. (1990). A note on stochastic integrators. *Math. Proc. Cambridge Philos. Soc.* **107** 395–400.

[12] IKEDA, N. and WATANABE, S. (1981). *Stochastic Differential Equations and Diffusion Processes.* North-Holland, Amsterdam.

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