The Annals of Probability
1994, Vol. 22, No. 2, 978–994

ISOPERIMETRIC INEQUALITIES FOR DISTRIBUTIONS OF EXPONENTIAL TYPE

BY S. G. BOBKOV
Syktyvkar State University

An isoperimetric property of exponential distributions with respect to the supremum distance in $\mathbb{R}^n$ is proved and applied to stochastic processes linearly generated by i.i.d. positive random values.

1. Introduction. We consider an isoperimetric problem for probability product measures $\mu_n = \mu \times \cdots \times \mu$ on the n-dimensional space $\mathbb{R}^n$. The problem consists of finding or estimating the value of

$$\inf \mu_n(A^h), \tag{1.1}$$

where the infimum is taken over all sets A, with measure $\mu_n(A) = p$, which belong to some family $\mathfrak{U}$ of measurable subsets in $\mathbb{R}^n$, and A^h denotes the h-neighborhood of $A \subset \mathbb{R}^n$.

In the case when the marginal distribution μ is the standard normal on the real line, the problem (1.1) was solved by Sudakov and Cirel'son (1974) and Borell (1975) in the class $\mathfrak{U}$ of all measurable subsets of $\mathbb{R}^n$: extremal sets at which $\mu_n(A^h)$ attains its minimum are just the half-spaces of measure p. This can be written as the inequality

$$\mu_n(A^h) \geq \mu((-\infty, a+h]), \tag{1.2}$$

where real a is chosen so that $\mu_n(A) = \mu((-\infty, a])$. Thus the extremal sets do not depend on h, that is, Gaussian measure possesses the isoperimetric property. The Bernoulli marginal distribution μ was studied by Talagrand (1988): an estimate obtained for (1.1) in the class $\mathfrak{U}$ of all convex sets of $\mathbb{R}^n$ does not depend on the dimension n as in the Gaussian case. It was also pointed out that the extremal sets in $\mathfrak{U}$ may depend on h.

It should be emphasized that the metric is meant to be Euclidian in the above-mentioned results, and therefore the h-neighborhood A^h is the Minkowski sum of A and l_2-ball B_2. Recently Talagrand (1989) proved an isoperimetric inequality for two-sided exponential distribution μ, with density $\exp(-|x|)/2$, investigating a special kind of enlargement. For arbitrary measurable $A \subset \mathbb{R}^n$, he considered in (1.1) the sets $A + W(h)$ (instead of A^h) involving the mixture $W(h) = h^{1/2}B_2 + hB_1$ of l_2- and l_1–balls. The inequality states that, for any $h \geq 0$,

$$\mu_n(A + W(h)) \geq \mu((-\infty, a + h/K)), \tag{1.3}$$

Received June 1993.
AMS 1991 *subject classification.* 60G70.
Key words and phrases. Isoperimetry, large deviations.

where K is a universal constant, and real a is chosen so that $\mu_n(A) = \mu((-\infty, a])$.

We study the one-sided exponential distribution $\mathbf{E}_n = \mathbf{E}_1 \times \cdots \times \mathbf{E}_1$ with marginal distribution function $\mathbf{E}_1(x) = 1 - \exp(-x)$, $x \geq 0$, and we are interested in the values of $\mathbf{E}_n$ on the sets $A \subset \mathbb{R}_+^n = [0, +\infty)^n$ which satisfy the following condition:

$$\text{if } x = (x_1, \ldots, x_n) \in A,\ y = (y_1, \ldots, y_n) \in \mathbb{R}_+^n,\ y_i \leq x_i \text{ for all } i, \text{ then } y \in A.$$

Considering $\mathbb{R}^n$ as a lattice in the sense of the theory of ordered spaces, such sets A will be called *ideals* in $\mathbb{R}_+^n$. In Bobkov (1989), the following statement was made for the class $\mathcal{U}$ of all ideals: for each fixed $p \in (0,1)$, the infimum in (1.1) is attained at the standard cube $A = [0, a]^n$ of measure $\mathbf{E}_n(A) = p$ [hence, $a = -\log(1 - p^{1/n})$] if A^h denotes the h-neighborhood of A with respect to the uniform metric in $\mathbb{R}^n$: $A^h = A + h[-1, 1]^n$. In other words,

$$\mathbf{E}_n(A^h) \geq \left[e^{-h} p^{1/n} + \left(1 - e^{-h}\right)\right]^n. \tag{1.4}$$

Thus, choosing the appropriate metric and the class $\mathcal{U}$, we have that the extremal sets do not depend on h. It is in this sense that we write about the isoperimetric property of the exponential law. The present paper proves this property [Section 2; here we also consider an infinite-dimensional variant of (1.4)]. In addition, (1.4) is applied to a certain family $\mathcal{F}$ of marginal distributions μ of "exponential type" (Section 3) and then to stochastic processes linearly generated by independent variables with a common law from $\mathcal{F}$ (Sections 4 and 5). Inequalities (1.3)–(1.5) are independent and have applications where they are preferable to existing results. Some relations between (1.4) and (1.2)–(1.3) are discussed in Sections 6 and 7.

2. Isoperimetric property of the exponential distribution. Clearly, all the ideals in $\mathbb{R}_+^n$ are Lebesgue measurable and, moreover, their boundaries are sets of measure 0. For each ideal A in $\mathbb{R}_+^n$ we consider its $2^n - 1$ projections in the coordinate subspaces of $\mathbb{R}^n$, namely,

$$A_{i_1 \ldots i_k} = \left\{x \in \mathbb{R}_+^k : \exists y \in A \text{ such that } \forall s = 1, \ldots, k,\ x_s = y_{i_s}\right\}$$

for any integers $1 \leq i_1 < \cdots < i_k \leq n$. For fixed $k = 1, \ldots, n$ set

$$a_k(A) = \sum \mathbf{V}_k\left(A_{i_1 \ldots i_k}\right), \qquad b_k(A) = \sum \mathbf{E}_k\left(A_{i_1 \ldots i_k}\right),$$

where summing is performed over all possible $1 \leq i_1 < \cdots < i_k \leq n$, and $\mathbf{V}_k$ is the k-dimensional Lebesgue measure on $\mathbb{R}^k$. For $k = 0$ we set $a_0(A) = b_0(A) = 1$. Let A be an arbitrary nonempty ideal in $\mathbb{R}_+^n$, and let $\mathbf{D}_n = [0, 1]^n$ be the unit cube in $\mathbb{R}_+^n$.

LEMMA 2.1. *For all* $\varepsilon \geq 0$,

$$\mathbf{V}_n\left(A + \varepsilon \mathbf{D}_n\right) = \sum_{k=0}^{n} a_{n-k}(A) \varepsilon^k. \tag{2.1}$$

LEMMA 2.2. *For all $h \geq 0$,*

$$\mathbf{E}_n\left(A + h\mathbf{D}_n\right) = e^{-nh} \sum_{k=0}^{n} b_{n-k}(A)\varepsilon^k, \tag{2.2}$$

where $\varepsilon = e^h - 1$.

Expansions in powers of ε such as (2.1) are well known in the theory of convex sets, where such identities are treated for Lebesgue measure $\mathbf{V}_n$ and for convex A. In the following it will be essential that (2.1) also holds for nonconvex sets. Proofs of both Lemma 1 and Lemma 2 are quite similar, so we just prove Lemma 2.

PROOF OF LEMMA 2. In the integral

$$\mathbf{E}_n\left(A + h\mathbf{D}_n\right) = \int_{A+h\mathbf{D}_n} \cdots \int \exp\{-(x_1 + \cdots + x_n)\}\, dx_1 \cdots dx_n$$

let us make the change of variables $y_i = x_i - h$. As result, the set $A + h\mathbf{D}_n$ maps onto the set

$$A' = \left\{(a_1 - h_1, \ldots, a_n - h_n) \colon (a_1, \ldots, a_n) \in A,\ 0 \leq h_i \leq h\right\}.$$

For any $\pi \subset \{1, \ldots, n\}$, define A_π^h as follows. If $\pi = \{i_1, \ldots, i_k\}$, $1 \leq i_1 < \cdots < i_k \geq n$, we set

$$A_\pi^h = \left\{x \in \mathbb{R}^n \colon \left(x_{i_1}, \ldots, x_{i_k}\right) \in A_{i_1 \ldots i_k} \text{ and for all } j \neq i_s,\ -h \leq x_j < 0\right\}.$$

In the case $\pi = \emptyset$, $A_\pi^h = [-h, 0)^n$. Then we have the decomposition $A' = \cup_\pi A_\pi^h$. Since $A_{\pi_1}^h \cap A_{\pi_2}^h = \emptyset$ for $\pi_1 \neq \pi_2$,

$$\begin{aligned}\mathbf{E}_n\left(A + h\mathbf{D}_n\right) &= \exp(-nh) \int_{A'} \cdots \int \exp\left(-y_1 - \cdots - y_n\right) dy_1 \cdots dy_n \\ &= \exp(-nh) \sum_\pi \int_{A_\pi^h} \cdots \int \exp\left(-y_1 - \cdots - y_n\right) dy_1 \cdots dy_n.\end{aligned}$$

It remains to note that for $\pi = \{i_1, \ldots, i_k\}$,

$$\begin{aligned}&\int_{A_\pi^h} \cdots \int \exp\left(-y_1 - \cdots - y_n\right) dy \\ &\quad = \left(e^h - 1\right)^{n-k} \int_{A_{i_1 \ldots i_k}} \cdots \int \exp\left(-y_{i_k} - \cdots - y_{i_k}\right) dy_{i_1} \cdots dy_{i_k} \\ &\quad = \varepsilon^{n-k} \mathbf{E}_k\left(A_{i_1 \ldots i_k}\right).\end{aligned}$$

□

Combining the lemmas, we obtain the following theorem,

THEOREM 2.3. *For any nonempty ideal* $A \subset \mathbb{R}_+^n$ *there exists an ideal* $B \subset \mathbf{D}_n$ *such that, for all* $h \geq 0$,

$$\mathbf{E}_n(A + h\mathbf{D}_n) = \exp(-nh)\mathbf{V}_n(B + \varepsilon\mathbf{D}_n), \tag{2.3}$$

where $\varepsilon = e^h - 1$.

PROOF. It is sufficient to take

$$B = \{(1 - \exp(-a_1), \dots, 1 - \exp(-a_n)) : (a_1, \dots, a_n) \in A\}.$$

Then, for each set of integers $1 \leq i_1 < \cdots < i_k \leq n$,

$$\mathbf{E}_k(A_{i_1 \dots i_k}) = \mathbf{V}_k(B_{i_1 \dots i_k});$$

consequently, $b_k(A) = a_k(A)$ for $k = 0, \dots, n$. □

In view of (2.3), now we can apply the well-known Brunn–Minkowski inequality, according to which for all nonempty measurable sets $B, B' \subset \mathbb{R}^n$ (such that $B + B'$ is measurable too),

$$\mathbf{V}_n^{1/n}(B + B') \geq \mathbf{V}_n^{1/n}(B) + \mathbf{V}_n^{1/n}(B'). \tag{2.4}$$

Taking $B' = \varepsilon\mathbf{D}_n$ in (2.4), we have the following theorem from (2.3).

THEOREM 2.4. *For any nonempty ideal* $A \subset \mathbb{R}_+^n$, *for the standard cube* B *with* $\mathbf{E}_n(B) = \mathbf{E}_n(A)$ *and for all* $h \geq 0$, *the following inequality is valid*:

$$\mathbf{E}_n(A + h\mathbf{D}_n) \geq \mathbf{E}_n(B + h\mathbf{D}_n),$$

or in other words,

$$\mathbf{E}_n(A + h\mathbf{D}_n) \geq \left[e^{-h}\mathbf{E}_n^{1/n}(A) + (1 - e^{-h})\right]^n. \tag{2.5}$$

If n increases and $\mathbf{E}_n(A) = p$ is constant, the right-hand side of (2.5) decreases and tends to the double exponential distribution function of h with a shift parameter:

$$\mathbf{E}_n(A + h\mathbf{D}_n) \geq \exp(-e^{-h}\log(1/p)). \tag{2.6}$$

This inequality does not depend on the dimension n, so it permits a formulation in the infinite-dimensional space $\mathbb{R}^\infty$ with the product measure $\mathbf{E}_\infty = \mathbf{E}_1 \times \mathbf{E}_1 \times \cdots$. Again, $\mathbb{R}_+^\infty$ is considered as a lattice with the same notion of ideal. For a nonempty set A and $h \geq 0$, denote

$$A^h = A + h\mathbf{D}, \qquad A^{-h} = \{a \in A : \{a\} + h\mathbf{D} \subset A\},$$

where $\mathbf{D} = [0,1]^\infty = [0,1] \times [0,1] \times \cdots$ is the infinite-dimensional unit cube in $\mathbb{R}^\infty$. Using the inclusion $(A^{-h})^h \subset A, (h \geq 0)$, we have the following theorem from (2.6).

THEOREM 2.5. *Let A be an ideal in $\mathbb{R}_+^\infty$ with $p = \mathbf{E}_\infty(A) > 0$. Then, for all $h \in \mathbb{R}^1$,*

$$\mathbf{E}_\infty(A^h) \geq \exp\{-e^{-h}\log(1/p)\}, \qquad h \geq 0, \tag{2.7}$$

$$\mathbf{E}_\infty(A^h) \leq \exp\{-e^{-h}\log(1/p)\}, \qquad h \leq 0. \tag{2.8}$$

REMARK 2.6. Inequality (2.7) is accurate in the class $\mathfrak{U}$ of all the ideals of $\mathbb{R}_+^\infty$, that is,

$$\inf_{\substack{A\in\mathfrak{U}\\ \mathbf{E}_\infty(A)=p}} \mathbf{E}_\infty(A^h) = p^\alpha, \qquad \alpha = \exp(-h). \tag{2.9}$$

Indeed, take n-dimensional cubes $A_n = [0,a_n]^n \times \mathbb{R}_+^1 \times \mathbb{R}_+^1 \times \cdots$ of $\mathbf{E}_\infty$-measure p, $a_n = -\log(1-p^{1/n})$. Then $\mathbf{E}_\infty(A_n)$ tends to p^α as $n \to \infty$. On the other hand, (2.7) may fail in the class $\mathcal{B}$ of all measurable sets of $\mathbb{R}_+^\infty$ even if $\mathbf{D}$ is replaced by $B_\infty = [-1,1]^\infty$. This can be easily shown for one-dimensional sets, for example, for intervals $A = (a,+\infty)$.

3. Isoperimetric inequalities for a family of product measures. We consider distributions μ on $\mathbb{R}_+^1$ which satisfy two conditions:

(i) The distribution function F with measure μ, $F(x) = \mu[0,x]$, is continuous and strictly increasing on $[0,b_F)$, where $b_F = \sup\{x\colon F(x) < 1\}$.

(ii) $$\lim_{h\to+\infty} \sup_{0\leq x<b_F} \frac{1-F(x+h)}{1-F(x)} = 0.$$

Let $\mathcal{F}$ denote the family of such distributions. It follows from (i) and (ii) that, for all $F \in \mathcal{F}$, the following hold.

PROPERTY A. The equality

$$1 - F^*(y) = \sup_{0\leq x<b_F} \frac{1-F(x+y)}{1-F(x)}$$

determines a continuous distribution function F^* which is strictly increasing on $[0,b_F) = [0,b_{F^*})$.

PROPERTY B. The function $T_F(x) = F^{-1}(1-e^{-x})$ from $[0,+\infty)$ onto $[0,b_F)$, mapping the measure $\mathbf{E}_1$ to μ, generates a modulus of continuity T_F^*, that is, for all $x \geq 0$,

$$T_F^*(x) = \sup_{y\geq 0}\big(T_F(x+y) - T_F(y)\big) < +\infty.$$

(Here F^{-1} is inverse of F restricted to $[0, b_F)$.)

NOTE 3.1. Provided (i) holds, Property B is equivalent to (ii).

NOTE 3.2. The modulus of continuity T_F^* generates a metric on $\mathbb{R}^1$,

$$\mathbf{d}_F(x, y) = T_F^*(|x-y|),$$

which will be used for a description of the law of the maximum of the processes considered.

PROPERTY C. For all $x \geq 0$ and $h \in [0, b_F)$,

$$T_F^*(x) = h \iff F^*(h) = 1 - e^{-x}.$$

PROPERTY D. There exists a constant C such that, for all x and h large enough,

$$T_F^*(x) \leq Cx, \qquad 1 - F^*(h) \leq \exp(-h/C).$$

Consequently, the distributions F and F^* have finite exponential moments,

$$\int \exp(\varepsilon x)\, dF(x) \leq \int \exp(\varepsilon x)\, dF^*(x) < +\infty \quad \text{for } \varepsilon \text{ small enough.}$$

For the product measures $\mu_n = \mu \times \cdots \times \mu$ on $\mathbb{R}^n$ with marginal law $\mu \in \mathcal{F}$, there are inequalities analogous to those for $\mathbf{E}_n$.

THEOREM 3.3. *For any ideal $A \subset \mathbb{R}_+^n$ with $p = \mu_n(A) > 0$ and $h \geq 0$,*

$$\mu_n(A^h) \geq \exp\left\{ -(1 - F^*(h)) \log\left(\frac{1}{p}\right)\right\}, \tag{3.1}$$

$$\mu_n(A^{-h}) \leq \exp\left\{ -\frac{1}{1 - F^*(h)} \log\left(\frac{1}{p}\right)\right\}. \tag{3.2}$$

REMARK 3.4. Inequalities (3.1) and (3.2) remain true likewise for $n = +\infty$ if, as usual, μ_∞ is the infinite product of μ.

REMARK 3.5. Due to Property C, we may formulate (3.1) and (3.2) with $T_F^*(h)$ instead of h, and e^{-h} instead of $1 - F^*(h)$.

PROOF OF THEOREM 3.3. Define a map $i_n: \mathbb{R}_+^n \to \mathbb{R}_+^n$ as follows:

$$i_n(x_1, \ldots, x_n) = (T_F(x_1), \ldots, T_F(x_n)).$$

Then i_n maps $\mathbf{E}_n$ to μ_n, and the following inclusions are valid:

$$i_n^{-1}(A^x) \supset \left(i_n^{-1}(A)\right)^h, \qquad i_n^{-1}(A^{-x}) \subset \left(i_n^{-1}(A)\right)^{-h},$$

where $x = T_F^*(h)$. It remains to note that $\mathbf{E}_n(i_n^{-1}(A)) = p$ and to use (2.7), (2.8) and Remark 3.5. □

It is now possible to apply (3.1) and (3.2) to individual distributions $F \in \mathcal{F}$ calculating exactly or estimating the functions F^* or T_F^*. However, it is useful to fix some subfamilies of "good" distributions for which F^* and T_F^* can be explored in general.

EXAMPLE 1 (The first subfamily of $\mathcal{F}$). Let $\mathcal{F}_0$ denote the set of those distribution functions F that satisfy conditions (i) and

(iii) for all $x, y \geq 0$, $\quad 1 - F(x+y) \leq \left(1 - F(x)\right)\left(1 - F(y)\right)$.

For such functions $F^* = F$, $T_F^* = T_F$. Hence $1 - F^*(h)$ in (3.1) and (3.2) may be replaced by $1 - F(h)$. In particular, if F is representable as

$$F(x) = 1 - \exp\left(-u(x)\right),$$

where u is a convex, continuous, strictly increasing function on $[0, b_F)$ with $u(0) = 0$, $\lim_{x \to b_F} u(x) = +\infty$, then $F \in \mathcal{F}_0$. For example, the distribution of $|\xi|$, where $\xi \in N(0,1)$, and the uniform distribution on $[0, b]$ possess this property and therefore belong to $\mathcal{F}_0$.

EXAMPLE 2 (The second subfamily of $\mathcal{F}$). Given $C > 0$ and $a \geq 0$, let $\mathcal{F}(C, a)$ denote the set of those distribution functions F which satisfy conditions (i) and

(iv) F is differentiable on $(a, +\infty)$, and for its derivative p,

$$1 - F(x) \leq Cp(x) \quad \text{for all } x > a. \tag{3.3}$$

For example, if F has a density p on $(0, +\infty)$ of the form

$$p(x) = u(x) \exp(-x/C),$$

where u is a continuous, nonincreasing function on $(0, +\infty)$, then $F \in \mathcal{F}(C, 0)$. Note also that if $L(\xi) \in \mathcal{F}(C, 0)$, then $L(\xi/C) \in \mathcal{F}(1, 0)$, where $L(\ )$ denotes the law of a random variable.

LEMMA 3.6. *For $F \in \mathcal{F}(C, a)$ and all $h \geq 0$,*

$$T_F^*(h) \leq Ch + a. \tag{3.4}$$

PROOF. If $F(a) = 1$, then $a \geq b_F$, and (3.4) is obvious. Let $F(a) < 1$. From (3.3) we have that, for all t, $F(a) < t < 1$,

$$1 - t \leq Cp\left(F^{-1}(t)\right). \tag{3.5}$$

Next we can assume that the function p is continuous. Then the function $T_F(x) = F^{-1}(1-\exp(-x))$ is differentiable on $(d,+\infty)$, where $d = -\log(1-F(a))$, and its derivative

$$T'_F(x) = \frac{\exp(-x)}{p\big(F^{-1}(1-\exp(-x))\big)} \leq C,$$

for all $x > d$. [Here we have made use of (3.5) with $t = 1-\exp(-x)$.] Consequently, for all $x > d$ and $h \geq 0$, $T_F(x+h) - T_F(x) \leq Ch$. In the case $0 \leq x \leq d$ and $h \geq 0$,

$$\begin{aligned} T_F(x+h) - T_F(x) &= \big(T_F(x+h) - T_F(d)\big) + \big(T_F(d) - T_F(x)\big) \\ &\leq C\big((x+h)-d\big) + T_F(d) \leq Ch + T_F(d) = Ch + a. \qquad \square \end{aligned}$$

Thus, we may apply Lemma 3.6 for $F \in \mathcal{F}(C,a)$ to estimate the left-hand side in (3.1)–(3.2): For each ideal $A \subset \mathbb{R}^n_+$ with $p = \mu_n(A) > 0$ and $h \geq 0$,

$$\begin{aligned} \mu_n\big(A^{Ch+a}\big) &\geq \exp\big\{-e^{-h}\log(1/p)\big\}, \\ \mu_n\big(A^{-Ch-a}\big) &\leq \exp\big\{-e^{h}\log(1/p)\big\}. \end{aligned}$$

4. Applications to the distribution of the maximum. Let ζ_n, $n \geq 1$, be i.i.d. random variables with a common distribution function $F \in \mathcal{F}$. Let L_F^+ denote the family of all random variables x representable in the form of a.s. convergent series

$$x = \sum_{n=1}^{+\infty} a_n \zeta_n, \tag{4.1}$$

with $a_n \geq 0$. Because F has some finite exponential moment, the a.s. convergence of (4.1) is equivalent to the convergence of $\sum a_n$. Consider the a.s. bounded stochastic process $x(t)$, $t \in T$, consisting of variables from L_F^+, and its supremum

$$\xi = \sup_t x(t).$$

Write $a = \inf\{x \in R\colon F_\xi(x) > 0\}$ and $b = \sup\{x \in R\colon F_\xi(x) < 1\}$, where $F_\xi(x) = \Pr\{\xi \leq x\}$ is the distribution function of ξ.

THEOREM 4.1. *Under the above mentioned assumptions, the following hold*:

(a) $\sigma = \sup_t \mathbf{E}x(t) < +\infty$.

(b) *The function* F_ξ *strictly increases on* (a,b); *hence for each* $p, 0 < p < 1$, *there exists a unique quantile* $m_p = m_p(\xi)$ *of order* p.

(c) *For all* $p, 0 < p < 1$, *and* $h \geq 0$,

$$\Pr\{\xi - m_p \leq Ch\} \geq \exp\left\{-\big(1-F^*(h)\big)\log\left(\frac{1}{p}\right)\right\}, \tag{4.2}$$

$$\Pr\{\xi - m_p < -Ch\} \leq \exp\left\{-\frac{1}{1-F^*(h)}\log\left(\frac{1}{p}\right)\right\}, \tag{4.3}$$

where $C = \sigma/\mathbf{E}\zeta_1$.

COROLLARY 4.2. *For all* $h \geq 0$,

$$\Pr\{\xi - m_p > Ch\} \leq \left(\log\left(\frac{1}{p}\right)\right)(1 - F^*(h)), \tag{4.4}$$

$$\Pr\{\xi - m_p < -Ch\} \leq \frac{1}{\log(1/p)}(1 - F^*(h)), \tag{4.5}$$

If $F \in \mathcal{F}_0$, $1 - F^*(h)$ *may be replaced by* $1 - F(h)$.
If $F \in \mathcal{F}(C_F, 0)$, $1 - F^*(h)$ *may be replaced by* $\exp(-h/C_F)$.

PROOF. It suffices to apply the following inequalities: for any $\varepsilon > 0$, $1 - \varepsilon \leq \exp(-\varepsilon) \leq 1/\varepsilon$. □

COROLLARY 4.3. *For arbitrary* $\alpha > 0$, $p \in (0, 1)$,

$$\mathbf{E}|\xi - m_p(\xi)|^\alpha \leq A(p, \alpha, F)\Big(\sup_t \mathbf{E}\, x(t)\Big)^\alpha, \tag{4.6}$$

where $A(p, \alpha, F) = \big(\log(1/p) + 1/\log(1/p)\big) \int x^\alpha\, dF^*(x)/(\mathbf{E}\zeta_1)^\alpha$.

PROOF. From (4.4) and (4.5) we have that

$$\Pr\big\{|\xi - m_p(\xi)| > Ch\big\} \leq \left(\log\left(\frac{1}{p}\right) + \frac{1}{\log(1/p)}\right)(1 - F^*(h)). \tag{4.7}$$

Inequality (4.6) easily follows from (4.7). □

THEOREM 4.4. *Under the assumptions of Theorem* 4.1, *there exists a Lipschitz function* f *on* $(\mathbb{R}^1, \mathbf{d}_F)$, *with Lipschitz constant less than or equal to* C, *that is, for all* $x, y \in \mathbb{R}^1$,

$$|f(x) - f(y)| \leq C\mathbf{d}_F(x, y) \equiv CT_F^*\big(|x - y|\big), \tag{4.8}$$

such that the random variables ξ *and* $f(\eta)$ *are identically distributed, where* η *has the double exponential distribution.*

NOTE 4.5. In the Gaussian case [$\zeta \sim N(0, 1)$, a_n in (4.1) are arbitrary] there exists an analogous proposition with $\eta \sim N(0, 1)$, $\mathbf{d}(x, y) = |x - y|$ and $C = \sup_t \mathbf{D}x(t))^{1/2}$.

LEMMA 4.6. *Let the distribution function* F_ξ *of the random variable* ξ *be strictly increasing on* (a, b), *where* a *and* b *are defined as in Theorem* 4.1, *and let the distribution function* F_η *of the random variable* η *be continuous and strictly increasing on* $(-\infty, +\infty)$; ξ *and* η *are assumed to satisfy the inequality*

$$\Pr\{\xi - m_p(\xi) \leq u(h)\} \geq \Pr\{\eta - m_p(\eta) \leq h\}, \tag{4.9}$$

for all $h \geq 0$ *and* $p \in (0,1)$, *where* u *is a nonnegative function of* $h \geq 0$. *Then there exists a function* f *defined on* $(-\infty, +\infty)$ *such that, for all* $x, y \in \mathbb{R}^1$,

$$(4.10) \qquad |f(x) - f(y)| \leq u\big(|x-y|\big),$$

and the random variables ξ *and* $f(\eta)$ *are identically distributed.*

PROOF. The functions $F_\xi^{-1}(p) = m_p(\xi)$ and $F_\eta^{-1}(p) = m_p(\eta)$ are well defined on $(0,1)$ and easily seen to be nondecreasing. (However, F_ξ^{-1} is not strictly increasing if F_ξ is not continuous.) If $a > -\infty$ and/or $b < +\infty$, we should extend F_ξ^{-1}: $F_\xi^{-1}(0) = a$ and/or $F_\xi^{-1}(1) = b$. In view of (4.9), for all $h \geq 0$ and $p \in (0,1)$,

$$(4.11) \qquad F_\xi\big(F_\xi^{-1}(p) + u(h)\big) \geq F_\eta\big(F_\eta^{-1}(p) + h\big).$$

Set $f(x) = F_\xi^{-1}(F_\eta(x))$ for all $x \in \mathbb{R}^1$. Applying F_ξ^{-1} to both sides of (4.11), we obtain that

$$(4.12) \qquad f\big(F_\eta^{-1}(p) + h\big) \leq F_\xi^{-1}\big(F_\xi\big(F_\xi^{-1}(p) + u(h)\big)\big).$$

Note that, for all $z \in (a,b)$, $F_\xi^{-1}(F_\xi(z)) = z$. It will be valid likewise for $z = a$ and $z = b$ if $a > -\infty$ or $b < +\infty$. If $b < +\infty$ and $z \geq b$, then $F_\xi^{-1}(F_\xi(z)) = b \leq z$. In any case $F_\xi^{-1}(F_\xi(z)) \leq z$ for all $z > a$ and for $z = a$ if $a > -\infty$. Because $z = F_\xi^{-1}(p) + u(h) \geq a$, it follows from (4.12) that, for all $p \in (0,1)$ and $h \geq 0$,

$$(4.13) \qquad f\big(F_\eta^{-1}(p) + h\big) \leq F_\xi^{-1}(p) + u(h).$$

Taking $p = F_\eta(x)$ in (4.13), with arbitrary $x \in \mathbb{R}^1$, we obtain that $f(x + h) \leq f(x) + h$. Thus f satisfies (4.8). It remains to find the law of $f(\eta)$. If $a < c < b$ and $0 < p < 1$, then $F_\xi^{-1}(p) \leq c \iff F_\xi(c) \geq p$; therefore, $\Pr\{f(\eta) \leq c\} = \Pr\{F_\xi^{-1}(F_\eta(\eta)) \leq c\} = \Pr\{F_\eta(\eta) \leq F_\xi(c)\} = F_\xi(c)$ because $F_\eta(\eta)$ is uniformly distributed on $(0,1)$. If $c < a$ or $c > b$, the set $\{x \in \mathbb{R}^1 : f(x) \leq c\} = \emptyset$ or $\mathbb{R}^1$ and has F_η-measure 0 or 1, respectively. Thus the distribution function of $f(\eta)$ coincides with $F_\xi(c)$ at each $c \in \mathbb{R}^1$, $c \neq a, b$, and hence coincides at each $c \in \mathbb{R}^1$. □

PROOF OF THEOREM 4.4. We may reformulate (4.2) as follows:

$$\Pr\big\{\xi - m_p(\xi) \leq CT_F^*(h)\big\} \geq \exp\big\{-e^{-h}\log(1/p)\big\} = \Pr\big\{\eta - m_p(\eta) \leq h\big\},$$

where η has double exponential distribution with quantile $m_p = -\log(\log(1/p))$, and apply Lemma 4.6 with $u(h) = CT_F^*(h)$. □

COROLLARY 4.7. *There exist constants* $A = A(p, F)$ *and* $R = R(\alpha, F)$, *depending on* $p \in (0,1)$, $\alpha > 0$ *and* $F \in \mathfrak{F}$ *only, such that for arbitrary a.s. bounded* *stochastic process* $x(t)$, $t \in T$, *from* L_F^+,

$$|m_p(\xi) - \mathbf{E}\xi| \le A \sup_t \mathbf{E}\,x(t), \tag{4.14}$$

$$\mathbf{E}|\xi - \mathbf{E}\xi|^\alpha \le R\Big(\sup_t \mathbf{E}\,x(t)\Big)^\alpha, \tag{4.15}$$

where $\xi = \sup_t x(t)$.

PROOF. The function $f(x) = F_\xi^{-1}(\exp(-e^{-x}))$, where F_ξ is the distribution function of ξ, possesses the following properties: for all real x and a,

$$f(x) - f(a) \le CT_F^*(|x-a|), \tag{4.16}$$

$$f(a) - f(x) \le CT_F^*(|x-a|), \tag{4.17}$$

where $C = \sup_t \mathbf{E}x(t)/\mathbf{E}\zeta_1$, and the random variables ξ and $f(\eta)$ are identically distributed if the distribution of η coincides with the double exponential law. In (4.16) and (4.17), setting $x = \eta$ and $a = -\log(\log(1/p))$, and noticing that $f(a) = m_p(\xi)$, we have that

$$\mathbf{E}\xi - m_p(\xi) \le C\,\mathbf{E}T_F^*(|\eta - a|),$$
$$m_p(\xi) - \mathbf{E}\eta \le C\,\mathbf{E}T_F^*(|\eta - a|).$$

Therefore, (4.14) holds with

$$A(p,F) = \frac{\mathbf{E}T_F^*\Big(\big|\eta + \log\big(\log(1/p)\big)\big|\Big)}{\mathbf{E}\zeta_1}.$$

The constant R that satisfies (4.15) may be easily found by combining (4.6) and (4.14). □

REMARK 4.8. For $\alpha = 2$ in (4.15), $R = R(2, F)$ also can be found with help of the identity $\mathbf{D}\xi = \frac{1}{2}\mathbf{E}|\xi - \xi'|^2 = \frac{1}{2}\mathbf{E}|f(\eta) - f(\eta')|^2$, where ξ' and η' are independent copies of ξ and η. In view of (4.8), we may set

$$R = \frac{1}{2}\frac{\mathbf{E}\big(T_F^*(|\eta - \eta'|)\big)^2}{(\mathbf{E}\zeta_1)^2}. \tag{4.18}$$

In particular, if $F \in \mathfrak{F}_0$, then $T_F^*(h) = T_F(h) = F^{-1}(1 - e^{-h})$; hence

$$R = \frac{1}{2}\int\!\!\int \frac{\big(F^{-1}(1 - \exp(-|x-y|))\big)^2\, d\big(\exp(-e^{-x})\big)\, d\big(\exp(-e^{-y})\big)}{(\mathbf{E}\zeta_1)^2}$$

$$= \iint\limits_{0<t<s<\infty} \frac{\big(F^{-1}(1 - t/s)\big)^2 \exp(-t-s)\,dt\,ds}{(\mathbf{E}\zeta_1)^2} \tag{4.19}$$

$$= \int_0^{+\infty} \frac{\big[x^2/(2 - F(x))^2\big]\,dF(x)}{(\mathbf{E}\zeta_1)^2}.$$

In the case where ζ_1 has the standard exponential distribution, $T_F(h) = h$ and, by (4.18), $R = \mathbf{D}\eta = \pi^2/6$, which is not improvable because $\mathbf{D}\max(\zeta_1, \dots, \zeta_n) = \sum_{k=1}^n 1/k^2$ tends to R. Thus (4.19) may give the best interpretation of $R = R(2, F)$ in (4.15). In any case, from (4.19) we have for $F \in \mathcal{F}_0$ that

$$\mathbf{D}\xi \le \mathbf{E}\zeta_1^2 \frac{\left(\sup_t \mathbf{E}x(t)\right)^2}{\left(\mathbf{E}\zeta_1\right)^2}.$$

If $F \in \mathcal{F}(C_F, 0)$, then $T_F^*(h) \le C_F h$. Therefore, likewise, by (4.18), $R \le C_F \pi^2/6(\mathbf{E}\zeta_1)^2$.

PROOF OF THEOREM 4.1. It can be assumed that $\mathbf{D}\zeta_1 = \mathbf{E}\zeta_1 = 1$. We need a lower estimate for $m_p(x)$ via $\mathbf{E}x$. Let $x = a_1\zeta_1 + \dots + a_n\zeta_n$, $a_1 \ge 0$, and let $\mathbf{E}x = a_1 + \dots + a_n = 1$, $n \ge 2$. If there exists $i \in \{1, \dots, n\}$ such that $a_i \ge \frac{1}{2}$, then $m_p(x) \ge m_p(\zeta_1)/2$. If all $a_i \le \frac{1}{2}$, then $\mathbf{D}x \le \frac{1}{2}$. The function $\mathbf{D}x = f(a_1, \dots, a_n) = a_1^2 + \dots + a_n^2$ attains its maximum on the set $0 \le a_i \le \frac{1}{2}, a_1 + \dots + a_n = 1$ at those points $a = (a_1, \dots, a_n)$ for which there exist $i \ne j$ with $a_i = a_j = \frac{1}{2}$, and for all other k, $a_k = 0$. Hence $\mathbf{D}x \le \frac{1}{2}$. By the Chebyshev inequality, for $\alpha \in (0, 1)$,

$$\Pr\{x \le 1 - \alpha\} = \Pr\{\mathbf{E}x - x \ge \alpha\mathbf{E}x\} \le \Pr\{|x - \mathbf{E}x| \ge \alpha\} \le \mathbf{D}x/\alpha^2 \le 1/2\alpha^2.$$

Set $\alpha = \frac{3}{4}$, $p = \frac{1}{2}\alpha^2 = \frac{8}{9}$. Then $m_p(x) \ge 1 - \alpha = \frac{1}{4}$. In any case, $m_p(x) \ge q = \min\{\frac{1}{4}, m_p(\zeta_1)/2\}$. Hence, for all $x \in L_F^+$,

$$m_p(x) \ge q\mathbf{E}x, \qquad p = \tfrac{8}{9},$$

and, for all $t \in T$, we have $\mathbf{E}x(t) \le m_p(x(t))/q \le m_p(\xi)/q$. Finally,

$$\sigma \le m_p(\xi)/q < +\infty.$$

In proving (b) and (c), we may suppose that $C = \sigma/\mathbf{E}\zeta_1 = 1$. Define the function φ from $\mathbb{R}_+^\infty$ to $[0, +\infty]$ as follows. Given $x \in \mathbb{R}_+^\infty$,

$$\varphi(x) = \sup_t \sum_{n=1}^\infty a_n(t)\, x_n,$$

where $a_n(t)$ are the coefficients from the expansions for $x(t)$ in (4.1). Then, for all $c \ge 0$, the set $A(c) = \{x \in \mathbb{R}_+^\infty : \varphi(x) \le c\}$ is a nonempty ideal in $\mathbb{R}_+^\infty$, and in addition, $\mu_\infty(A(c)) = \Pr\{\xi \le c\} \equiv F_\xi(c)$. Because for each $t \in T$, $\mathbf{E}x(t) = \sum a_n(t) \le 1$,

$$A(c) + h\mathbf{D} \equiv A(c)^h \subset A(c + h), \qquad A(c - h) \subset A(c)^{-h}, \tag{4.20}$$

for arbitrary $h \ge 0$. Making use of (3.1) and (3.2) with $n = \infty$ (Remark 3.4), $A = A(c)$ and (4.20), we obtain that

$$F_\xi(c + h) \ge \exp\left\{-(1 - F^*(h))\log\left(\frac{1}{F_\xi(c)}\right)\right\}, \tag{4.21}$$

$$F_\xi(c - h) \le \exp\left\{-\frac{1}{1 - F^*(h)}\log\left(\frac{1}{F_\xi(c)}\right)\right\}. \tag{4.22}$$

In view of (4.21), if $a < c < b$, that is, $0 < F_\xi(c) < 1$, then for every $h > 0$, $F_\xi(c+h) > F_\xi(c)$. Consequently, (b) has been proved. Set $c = m_p(\xi) + \varepsilon$, $\varepsilon > 0$. Then $F_\xi(c) \geq p > 0$, hence the right hand-side of (4.21) is not less than that of (4.2). Letting $\varepsilon \to 0$, we obtain (4.2). Analogously, setting $c = m_p(\xi) - \varepsilon$, $\varepsilon > 0$, and letting $\varepsilon \to 0$, we get (4.3) from (4.22). □

5. On sample behavior of unbounded processes. Let $x(t)$, $t \in I$, be a continuous process from L_F^+, $F \in \mathcal{F}(1,0)$ on $I = \mathbb{N}$ or $I = [1, +\infty)$ such that $\mathbf{E}x(t) \leq \mathbf{E}\zeta_1$ for all $t \in I$. Let

$$\xi(t) = \max_{s \leq t} x(s), \qquad A(t) = \mathbf{E}\xi(t).$$

THEOREM 5.1. *If* $\sup x(t) = +\infty$ *a.s.*, *then a.s.*

$$\limsup_{t \to +\infty} \frac{|\xi(t) - A(t)| - \log(A(t))}{\log\log(A(t))} \leq 1. \tag{5.1}$$

In particular, $\limsup x(t)/A(t) = \lim \xi(t)/A(t) = 1$.

REMARK 5.2. If $F \in \mathcal{F}(C,0)$, we may renormalize the basic variables ζ_n by setting $\zeta_n' = \zeta_n/C$ and considering the new process $y(t) = x(t)/C$. Then the law of ζ_n' will belong to $\mathcal{F}(1, 0)$, and $\mathbf{E}y(t) \leq \mathbf{E}\zeta_1'$.

REMARK 5.3. In view of (4.14), the function $A = A(t)$ may be replaced by the quantiles $m_p(\xi(t))$ for any fixed $p \in (0,1)$.

REMARK 5.4. If $T = \mathbb{N}$ and $x(n) = \zeta_n$ are standard exponential random variables, then (5.1) turns into an equality. Indeed, in this case the quantile $m_p = m_p(\xi)$ of order p has the asymptotic representation

$$m_p = \log n - \log\log(1/p) + O(1/n) \quad \text{as } n \to \infty.$$

On the other hand, applying Corollary 4.3.1 and Theorem 4.3.1 from Galambos (1978), we have

$$\limsup_{n \to \infty} \frac{|\xi(n) - \log n| - \log\log n}{\log\log\log n} = 1 \quad \text{a.s.}$$

PROOF. Let $T = [0, +\infty)$. According to Remark 5.3, we may prove (5.1) with $a(t) = m_{1/e}(\xi(t))$ instead of $A(t)$. By (4.4) and (4.5),

$$\Pr\{|\xi(t) - a(t)| > h\} \leq 2\exp(-h) \quad \text{for any } h \geq 0.$$

Given $1 < q < q'$ set $h_n = \log n + q' \log\log n$. Because the function $a = a(t)$ is continuous and unbounded on T, there exists a sequence $t_n \in T$ such that $a(t_n) = n$. Clearly,

$$\sum_n \Pr\Big\{|\xi(t_n) - a(t_n)| > h_n\Big\} < +\infty;$$

therefore, by the Borell–Cantelli lemma, with probability 1 for some random n_0 and all $n \geq n_0$,

$$\begin{aligned}\left|\xi(t_n) - a(t_n)\right| &\leq \log n + q' \log\log n \\ &= \log\big(a(t_n)\big) + q' \log\log\big(a(t_n)\big).\end{aligned}$$

If $n \geq n_0$, $t_n < t < t_{n+1}$, then

$$\begin{aligned}\xi(t) - a(t) &\leq \xi(t_{n+1}) - a(t_n) \\ &= \big(\xi(t_{n+1}) - a(t_{n+1})\big) + \big(a(t_{n+1}) - a(t_n)\big) \\ &\leq 1 + \log\big(a(t_{n+1})\big) + q' \log\log\big(a(t_{n+1})\big) \\ &\leq \log\big(a(t)\big) + q \log\log\big(a(t)\big)\end{aligned}$$

(the last inequality holds for all n large enough). In the same way, $a(t) - \xi(t) \leq \log(a(t)) + q \log\log(a(t))$ for all t large enough. Thus, with probability 1,

$$|\xi(t) - a(t)| \leq \log\big(a(t)\big) + q \log\log\big(a(t)\big) \quad \text{for } t \text{ large enough},$$

where $q > 1$ is arbitrary, and (5.1) has been proved. To prove (5.1) in the case $T = \mathbb{N}$, we can extend $x(t)$ to $[1, +\infty)$ in such a way that the following hold: (1) $\mathbf{E}x(t) \leq \mathbf{E}\zeta_1$ for any $t \in [1, +\infty)$; (2) $\sup_{t \leq n} x(t) = \max\{x(1), \dots, x(n)\}$ for any $n \in \mathbb{N}$; (3) the function $x = x(t)$ is continuous on $[1, +\infty)$ a.s.

For example, we may set $x(t) = ((n+1) - t)x(n) + (t - n)x(n+1)$, for $t \in [n, n+1]$, and apply (5.1) to $x(t)$. □

6. Comparison with the isoperimetric inequality for Gaussian processes. The isoperimetric property of Gaussian measure implies, in particular, that for the maximum $\xi = \sup_t x(t)$ of a bounded Gaussian process $x(t)$ with $\mathbf{D}x(t) \leq 1$,

$$\Pr\big\{\xi - m_p(\xi) > h\big\} \leq \Pr\big\{\lambda - m_p(\lambda) > h\big\} = 1 - \Phi\big(\Phi^{-1}(p) + h\big), \tag{6.1}$$

where $m_p(\xi)$ and $m_p(\lambda)$ are quantiles of order $p \in (0, 1)$ for ξ and λ, λ is a standard normally distributed variable with the distribution function Φ, Φ^{-1} is the inverse of Φ and $h \geq 0$.

Formally, we may not apply the above results to Gaussian processes because Φ does not belong to $\mathcal{F}$. However, we may apply them in the following situation. Let $\zeta_n, n \geq 1$, be independent $N(0, 1)$ random variables. Their linear combinations generate a Gaussian Hilbert space H, and any Gaussian process can be considered as a subset K of H. Suppose that K possesses the following properties:

(a) If $x = \sum a_n \zeta_n \in K$, $y = \sum b_n \zeta_n$ and $|b_n| \leq |a_n|$ for all n, then $y \in K$.
(b) If $x = \sum a_n \zeta_n \in K$, then $\sum |a_n| \leq 1$.
(c) $\zeta_1 \in K$.

In this case $\xi = \sup_{x \in K} x = \sup_{\sum a_n \zeta_n \in K} \sum |a_n|\, |\zeta_n|$, and in addition,

$$\sup_{x \in k} \mathbf{D}x = \sup_{\sum a_n \zeta_n \in K} \sum |a_n| = 1. \tag{6.2}$$

For example, the random variable $\xi = \max\{|\zeta_1|, (|\zeta_1| + |\zeta_2|)/2\}$ can be considered as the maximum of the Gaussian process

$$K = \left\{\zeta_1, -\zeta_1, \frac{\zeta_1 + \zeta_2}{2}, \frac{\zeta_1 - \zeta_2}{2}, \frac{-\zeta_1 + \zeta_2}{2}, \frac{-\zeta_1 - \zeta_2}{2}\right\},$$

and (a)–(c) are clearly fulfilled for K. Now, in addition to (6.1), one may apply (4.4) to ξ as the supremum of some linear combinations of i.i.d. random variables $|\zeta_i|$, $i \geq 1$. It follows from (a)–(c) that inequality (4.4) is valid for ξ with $C = 1$ and

$$F(x) = \Pr\{|\lambda| \leq x\} = 2\Phi(x) - 1.$$

Since $F \in \mathfrak{F}_0$, we have that $F = F^*$ and, for any $p \in (0,1)$ and $h \geq 0$,

$$\Pr\{\xi - m_p > h\} \leq 2\big(\log(1/p)\big)\big(1 - \Phi(h)\big). \tag{6.3}$$

If we take $p = \frac{1}{2}$, than (6.1) is better then (6.3) because $2\log 2 > 1$; but in the case $p < \frac{1}{2}$, (6.3) is more exact than (6.1) asymptotically as $h \to \infty$ because $\Phi^{-1}(p) < 0$. These observations may show that isoperimetric inequalities for laws from $\mathfrak{F}_0$ are almost exact. Note, however, that (a)–(c) define a very special class of Gaussian processes and require, in particular, that two parameters of the process, the maximal l_1-norm σ_1 and maximal l_2-norm σ_2 of the coefficients, coincide [provided (c) holds, this is equivalent to (6.2)]. In general, $\sigma_2 \ll \sigma_1$, and (4.4) becomes useless for large values of σ_1.

7. Comparison with the isoperimetric inequality for the two-sided exponential distribution. Denote by μ the distribution, on the real line, of the density $\exp(-|x|)/2$, $x \in \mathbb{R}^1$. The increasing map

$$T(x) = \begin{cases} -\ln\left(1 - \frac{1}{2}e^x\right), & x \leq 0, \\ x + \ln(2), & x \geq 0, \end{cases}$$

from $\mathbb{R}^1$ to $\mathbb{R}^1_+$ transforms μ into $\mathbf{E}_1$, that is, $\mu T^{-1} = \mathbf{E}_1$. Analogously, the map $T_\infty((x)_{n\geq 1}) = (T(x_n))_{n\geq 1}$ from $\mathbb{R}^\infty$ to $\mathbb{R}^\infty_+$ transforms μ_∞ into $\mathbf{E}_\infty$, and we can rewrite (1.3) for $\mathbf{E}_\infty$: for any measurable $A \subset \mathbb{R}^\infty$, $h \geq 0$,

$$(\mathbf{E}_\infty)_*\Big(T_\infty\big(A + W(h)\big)\Big) \geq \mu\big((-\infty, a + h/K]\big), \tag{7.1}$$

where $(\mathbf{E}_\infty)_*$ denotes the inner measure, $W(h) = h^{1/2}B_2 + hB_1$,

$$B_i = \left\{x \in \mathbb{R}^\infty : \Sigma_{n\geq 1}|x_n|^i \leq 1\right\}, \qquad i = 1, 2,$$

and a is chosen so that $\mu((-\infty, a]) = \mu_\infty(A)$. The function T is Lipschitz, with Lipschitz constant equal to 1, so

$$T_\infty\big(A + W(h)\big) \subset T_\infty(A) + W(h).$$

Note also that if $\mu((-\infty, a]) = p,\ 0 < p < 1$, and $\alpha = \exp(-h),\ h \geq 0$, then

$$R(p,\alpha) \equiv \mu\big((-\infty, a+h]\big) = \begin{cases} p/\alpha, & \text{if } p \leq \alpha/2, \\ 1-\alpha/(4p), & \text{if } \alpha/2 \leq p \leq 1/2, \\ 1-\alpha(1-p), & \text{if } p \geq 1/2, \end{cases}$$

Therefore, Talagrand's (1989) result (7.1) can be applied to $\mathbf{E}_\infty$ as follows: for any measurable $A \subset \mathbb{R}_+^\infty$ with $\mathbf{E}_\infty(A) = p$,

$$(\mathbf{E}_\infty)_*\big(A + W(Kh)\big) \geq R(p,\alpha), \qquad h \geq 0, \tag{7.2}$$

$$(\mathbf{E}_\infty)_*\big(A + hB_2\big) \geq R\big(p, \alpha^{1/2K}\big), \quad h \geq 1. \tag{7.3}$$

Obviously, (7.2) implies (7.3) because $W(h) \subset 2hB_2$ for $h \geq 1$. On the other hand, inequality (2.7) can be written as

$$\mathbf{E}_\infty(A + hD) \geq p^\alpha, \qquad h \geq 0, \tag{7.4}$$

where A is an arbitrary ideal in $\mathbb{R}_+^\infty$ and $\mathbf{D} = [0,1]^\infty$. Thus the measure of the larger set $(A + h\mathbf{D} \supset A + hB_2)$ is estimated by a larger value $[p^\alpha \geq R(p, \alpha^{1/2K})$; this inequality can be investigated in an elementary way, but to see this it is sufficient to know that (7.4) is accurate in the class of ideals of $\mathbb{R}_+^\infty$].

In order to understand the real difference between (7.2) and (7.4), consider a sequence $\zeta_n,\ n \geq 1$, of independent random variables with common law $\mathbf{E}_1$, and the space L of their linear combinations

$$x = \sum_{n \geq 1} a_n \zeta_n, \tag{7.5}$$

having for simplicity only finitely many nonzero terms. We are interested in the distribution of the supremum $\xi = \sup_t x(t)$ of a bounded stochastic process $x(t)$ consisting of random variables from L. Let

$$\sigma_2^2 = \sup_t \mathbf{D}x(t) = \sup_t \sum_{n \geq 1} a_n(t)^2, \qquad \sigma_\infty = \sup_t \max_{n \geq 1} |a_n(t)|.$$

If $a_n(t)$, the coefficients of $x(t)$ from (7.5), are nonnegative for all $n \geq 1$ and t, then

$$\sigma_1 = \sup_t \mathbf{E}x(t) = \sup_t \sum_{n \geq 1} a_n(t).$$

Now (7.2) allows us to estimate probabilities of deviations ξ from its quantiles $m_p(\xi)$ knowing only the values σ_2 and σ_∞. In particular, for $p = \frac{1}{2}$ $[m_{1/2}(\xi) = m$ is the median of $\xi]$, $h \geq 0$,

$$\Pr\{\xi - m > \sigma_2(Kh)^{1/2} + \sigma_\infty Kh\} \leq \tfrac{1}{2}\exp(-h), \tag{7.6}$$

$$\Pr\Big\{\xi - m < -\big(\sigma_2(Kh)^{1/2} + \sigma_\infty Kh\big)\Big\} \leq \tfrac{1}{2}\exp(-h). \tag{7.7}$$

In the second case (and only in this case), when σ_1 serves a basic characteristic of the process (under the assumption $a_n \geq 0$), one can apply (7.4); that gives, for $p = \frac{1}{2}$ and $h \geq 0$,

$$\Pr\{\xi - m > \sigma_1 h\} \leq 1 - 2^{-\exp(-h)} \leq (\ln 2)\exp(-h), \tag{7.8}$$

$$\Pr\{\xi - m < -\sigma_1 h\} \leq 2^{-\exp(h)}, \tag{7.9}$$

or after change of variable h in (7.9),

$$\Pr\left\{\xi - m < -\sigma_1 \ln\left(1 + \frac{h}{\ln 2}\right)\right\} \leq \frac{1}{2}\exp(-h). \tag{7.10}$$

Thus, in order to estimate the probabilities of the right (resp., left) deviations more exactly by (7.6) or by (7.8) [resp., by (7.7) or by (7.10)], we have to compare the values $\sigma_2(Kh)^{1/2} + \sigma_\infty Kh$ and $\sigma_1 h$ [resp., $\sigma_2(Kh)^{1/2} + \sigma_\infty Kh$ and $\sigma_1 \ln(1 + h/\ln 2)$]. The first case seems much more preferable (at least for the right deviations) in the general situation when $\sigma_\infty \ll \sigma_2 \ll \sigma_1$. On the other hand, let the process $x(t)$ possess the following properties: (a) $a_n(t) \geq 0$ for all t and $n \geq 1$; (b) $\mathbf{E}x(t) \leq 1$ for all t; (c) $x(t_0) = \zeta_1$ for some t_0. Then $\sigma_\infty = \sigma_2 = \sigma_1$ and hence, anyway, (7.8) and (7.9) are more accurate for such a special class of the processes. In addition, (7.9) shows an asymmetric character of the distribution of ξ (more exactly, see Theorem 4.4 on the role of the double exponential law).

REFERENCES

BOBKOV, S. G. (1989). Isoperimetric property of the multidimensional exponential distribution and its applications to random processes linearly generated by independent variables. In *Probab. Theory and Math. Statist., Proc. Fifth Vilnius Conf.* **3** 73–74.

BORELL, C. (1975). The Brunn–Minkowski inequality in Gauss space. *Invent. Math.* **30** 207–216.

GALAMBOS, J. (1978). *The Asymptotic Theory of Extreme Order Statistics.* Wiley, New York.

SUDAKOV, V. N. and CIREL'SON, B. S. (1974). Extremal properties of half-spaces for spherically invariant measures. *Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)* **41** 14–24. (In Russian.)

TALAGRAND, M. (1988). An isoperimetric theorem on the cube and the Khinchine–Kahane inequalities. *Proc. Amer. Math. Soc.* **104** 905–909.

TALAGRAND, M. (1989). A new isoperimetric inequality and the concentration of measure phenomenon. *Lecture Notes in Math.* **1469** 94–124. Springer, Berlin.

CENTER FOR STOCHASTIC PROCESSES
DEPARTMENT OF STATISTICS
CB 3260, PHILLIPS HALL
UNIVERSITY OF NORTH CAROLINA
CHAPEL HILL, NORTH CAROLINA 27599-3260