The Annals of Probability
1994, Vol. 22, No. 2, 875–918

LOCALIZATION OF A TWO-DIMENSIONAL RANDOM WALK WITH AN ATTRACTIVE PATH INTERACTION[1]

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We consider an ordinary, symmetric, continuous-time random walk on the two-dimensional lattice $\mathbb{Z}^2$. The distribution of the walk is transformed by a density which discounts exponentially the number of points visited up to time T. This introduces a self-attracting interaction of the paths. We study the asymptotic behavior for $T \to \infty$. It turns out that the displacement is asymptotically of order $T^{1/4}$. The main technique for proving the result is a refined analysis of large deviation probabilities. A partial discussion is given also for higher dimensions.

1. Introduction. We consider the following model: Let X_t, $t \geq 0$, be a continuous time, symmetric random walk on $\mathbb{Z}^d$, $d \geq 2$, starting at 0, with generator

$$Af(x) = \tfrac{1}{2} \sum_{y:\,|y-x|=1} \big(f(y) - f(x)\big), \qquad x, y \in \mathbb{Z}^d.$$

(The holding times have expectation $1/d$.) If $T > 0$, let N_T be the number of points in $\mathbb{Z}^d$ which are visited by X_t up to $t = T$. We define

$$d\widehat{P}_T = \exp(-N_T)dP/z_T,$$

where $z_T = E(\exp(-N_T))$.

The density $\exp(-N_T)/z_T$ favors paths which typically clump together. This introduces a long-range self-attraction of the paths. We are interested in the effect of this on the typical displacement of the path up to time T for large T.

THEOREM 1.1. *Let $d = 2$ and $\varrho = (\lambda/\pi)^{1/4}$, where λ is the principal eigenvalue of $-\Delta/2$ in the ball of radius* 1. *Then, for any $\varepsilon > 0$,*

$$\lim_{T\to\infty} \widehat{P}_T\Big((\varrho - \varepsilon)T^{1/4} \leq \sup_{s\leq T} |X_s| \leq (2\varrho + \varepsilon)T^{1/4}\Big) = 1.$$

Theorem 1.1 is an obvious consequence of the result that, with $\widehat{P}_T$-probability converging to 1, the set of points visited by the random walk is close to a spherical droplet of radius $\varrho T^{1/4}$. More precisely, let $S_T \subset \mathbb{Z}^2$ be the set of points visited

Received December 1989.
[1]This work was partly supported by the Akademie der Wissenschaften zu Berlin.
AMS 1991 *classifications.* Primary 60K35; secondary 60F10, 60J25.
Key words and phrases. Self-attracting random walk, localization, large deviations.

by the random walk up to time T. If $x \in \mathbb{Z}^d$, $r > 0$, let $D_x(r) = \{y \in \mathbb{Z}^d: |x-y| \le r\}$. Our main result is the following theorem.

THEOREM 1.2. *Let $d = 2$. For any $\varepsilon > 0$,*

$$\lim_{T\to\infty} \widehat{P}_T\left(\bigcup_{x \in D_0(\varrho T^{1/4})} \left\{ D_x\big(\varrho(1-\varepsilon)T^{1/4}\big) \subset S_T \subset D_x\big(\varrho(1+\varepsilon)T^{1/4}\big) \right\} \right) = 1.$$

The problem is strongly connected with the determination of the asymptotic behavior of z_T. A result which is due to Donsker and Varadhan [8] states that, for any dimension d,

$$\lim_{T\to\infty} T^{-d/(2+d)} \log z_T = -\inf\left\{ |\mathrm{supp}\,(\nu)| + I(\nu) \right\}, \tag{1.1}$$

where the infimum is taken over all probability measures on $\mathbb{R}^2$, $|\mathrm{supp}(\nu)|$ denotes the Lebesgue measure of the topological support of ν [$\mathrm{supp}(\nu)$ is the intersection of the closed sets whose complements have ν-measure 0] and

$$I(\nu) = \frac{1}{8} \int \frac{|\nabla f|^2}{f}\, dx$$

where $f = d\nu/dx$ and ∇f is the gradient. The parameter $I(\nu)$ is set to ∞ if the right-hand side of the above expression does not exist. Equation (1.1) is proved in [8] for discrete time random walks. For continuous time walks, (1.1) follows easily from the discrete time result because N_T is also the number of points visited by the imbedded discrete time chain observed up to a random time which, however, is independent of the discrete time walk. The variational problem in (1.1) can be solved. The infimum on the right-hand side is attained by probability measures ν with a density whose square root is the eigenfunction of the largest eigenvalue of the Laplacian in a disk of radius ϱ with Dirichlet boundary conditions. Our theorem then follows from the statement that in dimension 2 the empirical measure

$$\int_0^T \delta_{T^{-1/4}X_s}\, ds$$

has a support which with $\widehat{P}_T$-probability close to 1 is close to the support of one of those ν. Furthermore, one can restrict attention to those which have 0 in their support. Although this is quite plausible, the proof needs a refinement of the analysis of Donsker and Varadhan. They proved that after a suitable rescaling the empirical density satisfies a large deviation principle in the L_1-topology, at least if the random walks live on a torus. This implies that (for a walk on the torus) the empirical density is in $\widehat{P}_T$-probability L_1-close to the density of the solutions of the variational problem. This, however, does not immediately imply that the supports are close to each other, in the sense that the distance between the boundaries is near 0.

The paper is organized as follows. In Section 2 we derive an improved lower bound for z_T, and in Section 3 we discuss slightly improved upper bounds (and some related results). In Section 4 we show that there is, with large $\widehat{P}_T$-probability, a disc of radius approximately ϱ which is completely covered by the empirical measure. This still does not imply that there is no mass outside. The main difficulty is the exclusion of excursions from the optimal droplet (it is actually not unique). To see the point, look at excursions of order $T^{1/4}$ from the disc. Such an excursion contributes at least $T^{1/4}$ to N_T, which, however, is negligible compared with N_T itself. It is easy to see that, under the free measure of the random walk, the probability that such an excursion occurs is at least $\exp(-\text{const.}\, T^{1/4})$, which is very large compared with the probability that the free walk stays inside such a droplet. It is therefore clear that excursions cannot be excluded on the basis of an asymptotic result of type (1.1). The results of Section 3 here, however, show that there cannot be too much mass outside the droplet. Unfortunately, the estimates of Section 3 are not sharp enough to exclude the possibility of excursions. Such excursions, however, would essentially be independent of what is going on inside the disc, so one can manage to exclude them by a suitable conditioning argument. This is done in Section 5. Some estimates are postponed to two appendices.

The main open problem is what happens in higher dimensions. I have no doubts that the result is true in any dimension. More precisely, we have the following conjecture.

CONJECTURE 1.3. *There is a constant $\varrho_d > 0$ (specified in Section 2) such that, for any $\varepsilon > 0$,*

$$\lim_{T\to\infty} \widehat{P}_T\left(\bigcup_{x\in D_0(\varrho_d T^{1/(2+d)})} \left\{D_x\left(\varrho(1-\varepsilon)T^{1/(2+d)}\right) \subset S_T \subset D_x\left(\varrho(1+\varepsilon)T^{1/(2+d)}\right)\right\}\right) = 1.$$

We actually give a partial proof of this by giving a full proof that Lemma 3.1 and Proposition 4.1 imply the conjecture. Lemma 3.1 is crucial. Its proof depends on the fact that a set A of volume 1 in $\mathbb{R}^d$ whose surface is close to that of the unit ball is in fact close (in a certain sense) to a ball. This is easy for $d = 2$. The truth for $d > 3$ must depend on the specification of what "close to a ball" means, because the set can have thin spines which do not contribute much to the surface and the volume. Anyway, to my knowledge, such results do not exist in the literature on isoperimetric problems, except for convex sets A which are useless for our purpose. I also have no doubts that Proposition 4.1 is true, but its proof eludes me, too.

REMARK. A similar confinement property for two-dimensional Brownian motion among Poissonian obstacles has been obtained by Sznitman [13] using different methods.

2. A lower bound for z_T. We use $c, c_1, c_2, c', \dots$ as generic constants which are assumed to be greater than 0 and not necessarily the same at different places. They may depend on the dimension d but on nothing else except when stated explicitly. Whenever we write an inequality, we tacitly require this to hold only for large enough T. The quantity T will always be the time parameter in our main result.

The right-hand side of (1.1) equals

$$-\chi_d \equiv \frac{d+2}{2}\left[\frac{2\lambda_d}{d}\right]^{d/(d+2)} \omega_d^{2/(2+d)},$$

where λ_d is the principal eigenvalue of $-\Delta/2$ in the ball of radius 1 (see [5], Chapter 4.3), and ω_d is the volume of the unit ball.

PROPOSITION 2.1.

$$\liminf_{T\to\infty} T^{-(d-1)/(d+2)}\left(\log z_T + \chi_d T^{d/(2+d)}\right) > -\infty.$$

Here and later on it is convenient to rescale the random walk. Let $\mathbb{L}_T = T^{-1/(d+2)}\mathbb{Z}^d$ and let $\eta_t^{(T)} = T^{-1/(d+2)}X_{tT^{2/(d+2)}}$, $t \le T^{d/(2+d)}$; $\eta_t^{(T)}$ is a jump process on $\mathbb{L}_T$ with generator

$$A_T f(x) = \tfrac{1}{2}\sum_{y\sim x} T^{2/(2+d)}\left(f(y) - f(x)\right), \tag{2.1}$$

where we write $y \sim x$ if y is a nearest neighbor of x on $\mathbb{L}_T$. For notational convenience, we usually drop the index T in $\eta_t^{(T)}$, $\mathbb{L}_T$ and A_T, but the dependence on T should always be kept in mind.

Let us first look a little bit closer at the solution of the variational problem in (1.1). For a careful discussion of the analytical problems, see [5]. If G is a connected bounded open subset of $\mathbb{R}^d$, then $\lambda_G = \inf\{I(\nu): \operatorname{supp}(\nu) \subset G\}$ is the principal (i.e., smallest) eigenvalue of $-\Delta/2$ on G with Dirichlet boundary conditions. By the Faber–Krahn theorem $\inf\{|G| + \lambda_G\}$ is attained when G is a ball $D(r) = \{x: |x| < r\}$. Of course, any translation of $D(r)$ is a minimum, too, but otherwise the solution of the variational problem is unique. In order to determine r, one observes that $\lambda_{D(r)} = \lambda_{D(1)}/r^d = \lambda_d/r^d$. Therefore, one has to minimize $\lambda_d/r^2 + \omega_d r^d$, which is χ_d. The minimizing r is

$$\varrho_d = \left(\frac{2\lambda_d}{d\omega_d}\right)^{1/(d+2)}.$$

We write φ for the eigenfunction on $D(\varrho_d)$, which we take to be normalized, that is, $\int \varphi(x)^2 dx = 1$ and positive on $D(\varrho_d)$. It is well known that $\varphi(x) = aJ_0(b|x|)$, where J_0 is the Bessel function of the first kind and $a, b > 0$ are chosen such that $J_0(br)$, $r \geq 0$ has its first zero at $r = \varrho_d$ and $\int_{D(\varrho_d)} \varphi(x)^2 dx = 1$. We extend φ to $\mathbb{R}^d$ by setting it equal to 0 outside $D(\varrho_d)$.

We use results on discretization of the Laplacian by Weinberger and others. Let $D_T = D(\varrho_d) \cap \mathbb{L}_T$, and define the discrete Laplacian A_T^* with Dirchlet boundary conditions on the set of functions $f: D_T \to \mathbb{R}$ by setting $A_T^* f(x) = A_T f(x)$ if $x \in D_T$, where f is extended to $\mathbb{L}_T$ by setting $f \equiv 0$ outside D_T; $-A_T^*$ is positive definite and symmetric, and we denote the principal (i.e., smallest) eigenvalue by λ_T^*. We note that λ_T^* is a simple eigenvalue and the eigenfunction $\psi: D_T \to \mathbb{R}$ can be chosen such that $\psi(x) > 0$ for $x \in D_T \cap \operatorname{int} D(\varrho_d)$ and $\sum_{x \in D_T} \psi(x)^2 T^{-d/(2+d)} = 1$.

LEMMA 2.1.

(a) $|\lambda_T^* - \lambda_d/\varrho_d^2| = O(T^{-1/(2+d)})$.
(b) $\sup_{x \in \mathbb{L}_T} |\psi(x) - \varphi(x)| = O(T^{-1/(2+d)})$.

PROOF. Part (a) is proved in [14].

Part (b) is proved in [4]. Actually, the latter paper discusses a finite difference operator which is adjusted at the boundary in order to get a faster convergence. If the mesh size is h (in our case $h = T^{-1/(2+d)}$), Bramble and Hubbard obtain estimates for the difference of the eigenvalues and the sup-norm of the difference of the eigenfunctions which are of order h^2. Without the adjustment at the boundary (i.e., in the case we are considering here), Weinberger proved that the differences of the eigenvalues are $O(h)$, but he did not discuss the eigenfunctions. It is, however, straightforward that the discussion in Bramble and Hubbard also covers the (easier) nonadjusted case with an error of order $O(h)$, that is, our claim (b). □

With Lemma 2.1, we can now easily prove Proposition 2.1. We transform the law of $\eta_t^{(T)}$ by transforming the generator A_T:

$$\widetilde{A}_T f(x) = \frac{1}{2} \sum_{y \sim x} T^{d/(2+d)} \frac{\psi(y)}{\psi(x)} \big(f(y) - f(x)\big)$$

when $x \in D_T$. For $S > 0$, we denote by P_S the law of $\eta_t^{(T)}$, $t \leq S$, on the space of right-continuous functions with values in $\mathbb{L}_T$ and by $\widetilde{P}_S$ that of the Markov jump process with starting point 0 and generator $\widetilde{A}_T$. We can calculate $dP_S/d\widetilde{P}_S$ on the set of paths which do not leave D_T. In fact, for such a path $\eta_s, s \leq S$, $\eta_0 = 0$, we have

$$\frac{dP_S}{d\widetilde{P}_S}(\eta) = \frac{\psi(0)}{\psi(\eta_S)} \exp\left(\int_0^s \frac{A_T \psi(\eta_u)}{\psi(\eta_u)} du \right) = \frac{\psi(0)}{\psi(\eta_S)} \exp\big(- \lambda_T^* S \big).$$

For a proof of this see, for example, [12], IV (22.8). Since $\psi(0)/\psi(\eta_S) \geq c > 0$ by

Lemma 2.1(a),

$$
\begin{aligned}
&\frac{dP_S}{d\widetilde{P}_S}(\eta) \geq c\exp\big(-\lambda_T^* S\big),\\
&E\big(e^{-N_T}\big) \geq E\big(\exp(-N_T);\, \eta_u \in D_T \text{ for } u \leq T^{d/(2+d)}\big)\\
&\qquad \geq \exp(-\#D_T)P\big(\eta_u \in D_T \text{ for } u \leq T^{d/(2+d)}\big)\\
&\qquad \geq c\exp(-\#D_T)\exp\big(-\lambda_T^* T^{d/(2+d)}\big)\\
&\qquad\quad \times \widetilde{P}_{T^{d/(2+d)}}\big(\eta_u \in D_T \text{ for } u \leq T^{d/(2+d)}\big)\\
&\qquad = c\exp(-\#D_T)\exp\big(-\lambda_T^* T^{d/(2+d)}\big),
\end{aligned}
\tag{2.2}
$$

where $\#D_T$ denotes the number of lattice points in D_T. Obviously

$$
\#D_T = \omega_d T^{d/(2+d)} \varrho^{d/(2+d)} + O\big(T^{(d-1)/(d+2)}\big). \tag{2.3}
$$

Relations (2.2) and (2.3) and Lemma 2.1 prove Proposition 2.1.

REMARK. It is tempting to conjecture that

$$
\lim_{T\to\infty} T^{-(d-1)/(2+d)}\big(\log z_T + \chi_d T^{d/(2+d)}\big)
$$

exists and is not 0. This would seem to be rather delicate.

3. Upper bounds. The main result of this section is proved here only for $d = 2$. The difficulty of extending the analysis to higher dimensions is a purely analytical one: I am able to show Lemma 3.1 only for $d = 2$. However, if the statement of the lemma is correct for any d, then Proposition 3.1 also is correct for this d. In order to present the results as generally as possible, we give this derivation for arbitrary d.

Let $R = 10\varrho_d$ and set $K = [RT^{1/(d+2)}]$. We define the random walk $X_t^{(K)}$, $t \geq 0$, on the finite set $(\mathbb{Z}_K)^d$ by setting

$$
X_{j,t}^{(K)} = X_{j,t} \mod K
$$

for each of the d components $X_{j,t}$, $1 \leq j \leq d$, of the random walk X_t, where $\mathbb{Z}_K = \{0, 1, \dots, K-1\}$. We again rescale this and set

$$
\eta_t^{(T,R)} = T^{-1/(2+d)} X_{tT^{2/(2+d)}}^{(K)}
$$

for $0 \leq t \leq T^{d/(2+d)}$, which lives on the discrete torus $\mathbb{L}_T^{(R)} = T^{-1/(2+d)}(\mathbb{Z}_K)^d$.

We usually drop the indices R and sometimes also T for notational convenience. Let

$$
l_T(x) = \int_0^{T^{d/(2+d)}} 1_x(\eta_s)\,ds, \qquad x \in \mathbb{L},
$$

where 1_x is the indicator function of x. We use the following notation. If $\varphi: \mathbb{L} \to \mathbb{R}$, we write

$$\int \varphi(x)\,dx = T^{-d/(2+d)} \sum_{x \in \mathbb{L}} \varphi(x).$$

We also write $\langle f,g \rangle$ for $\int f(x)g(x)\,dx$.

We call a function $\varphi: \mathbb{L} \to [0,\infty)$ a density if $\int \varphi(x)\,dx = 1$. Note that l_T is a random density.

Our random walk η_t is defined for all $t \geq 0$ and is for fixed T, R just a random walk on the finite state space $\mathbb{L}_T^{(R)}$ with the generator A_T given in (2.1). The Donsker–Varadhan entropy of a density f is

$$I_T(f) \equiv -\int \sqrt{f(x)}\big(A_T\sqrt{f}\big)(x)\,dx = \tfrac{1}{2}T^{-(d-2)/(d+2)} \sum_{y \sim x} \Big(\sqrt{f(y)} - \sqrt{f(x)}\Big)^2.$$

In Section 2, we introduced the eigenfunction ψ of the difference operator A_T^* on D_T with Dirichlet boundary conditions (there on $T^{-1/(2+d)}\mathbb{Z}^d$). Because $R > 2\varrho_d$, ψ is defined on our torus $\mathbb{L}_T^{(R)}$ as well.

We consider now the translates if ψ: of $\theta \in \mathbb{L}$ we set $\psi_\theta(x) = \psi(x-\theta)$ and $\mathcal{F} := \{\psi_\theta^2 : \theta \in \mathbb{L}\}$. Note that $I_T(\psi_\theta^2) = \lambda_T^*$, which by Lemma 2.1 is $\lambda_d/\varrho_d^2 + O(T^{-1/(2+d)})$. If $g: \mathbb{L} \to \mathbb{R}$, we write

$$\|g - \mathcal{F}\|_1 = \inf_\theta \int |g(x) - \psi_\theta^2(x)|\,dx.$$

LEMMA 3.1. *If $d = 2$, then*

$$\inf\left\{\int 1_{\{f(x)>0\}}\,dx + I_T(f) : \|f - \mathcal{F}\|_1 \geq a\right\} \geq \chi_d + ca^{15}, \tag{3.1}$$

for $T^{-1/8} \leq a \leq 1$ and large enough T.

The proof of this analytical result is given in Appendix A. For the rest of this section, we set $\tau = T^{d/(2+d)}$.

Our main result of the section states that for our transformed measure (now on the torus $\mathbb{L}_T^{(R)}$) the empirical density l_T is polynomially close to an element of $\mathcal{F}$.

PROPOSITION 3.1. *Let $d \in \mathbb{N}$ and assume that* (3.1) *is true with an arbitrary exponent for a on the r.h.s. Then there exists $s > 0$ such that*

$$\lim_{T\to\infty} E\big(e^{-N_T};\ \|l_T - \mathcal{F}\|_1 \geq T^{-s}\big)/z_T = 0,$$

where z_T is $E(e^{-N_T})$, N_T being here the number of points visited by $\eta_t^{(T,R)}$ up to time τ (as remarked, we usually drop R in the notation). We also use the notation $E(X; A)$ for $E(X1_A)$.

PROOF. If $h: [0,\infty) \to [0,1]$ satisfies $h(0) = 0$, then

$$N_T = \tau \int \chi_{(0,\infty)}\big(l_T(x)\big)\,dx \geq \tau \int h\big(l_T(x)\big)\,dx.$$

We choose

$$h(x) = h^{(q,T)}(x) = \min(T^q x, 1), \tag{3.2}$$

where $q > 0$ will be specified later on. We usually drop the indices q and T in $h^{(q,T)}$ and write

$$N_T^{(q)} = \tau \int h\big(l_T(x)\big)\,dx.$$

Let $p_t(x,y)$ be the transition densities of our random walk η_t on $\mathbb{L}$, where the density is taken with respect to the *normalized* counting measure so that, with our counting notation,

$$\int p_t(x,y)\,dy = 1.$$

(In the limit $T \to \infty$, p_t approaches the transition densities of a Brownian motion on a d-dimensional flat torus.)

If $\varphi: \mathbb{L} \to \mathbb{R}$ and $0 < \delta/d < 1/(2+d)$, then we set

$$\overline{\varphi}^{\delta}(x) = \big(P_{T^{2\delta/d - 2/(2+d)}}\varphi\big)(x) = \int_{\mathbb{L}} p_{T^{2\delta/d - 2/(2+d)}}(x,y)\varphi(y)\,dy.$$

(Note that as $p_t(x,\cdot)$ is essentially a normal density with standard deviation $\sqrt{t}$ and the lattice spacing is $T^{-1/(2+d)}$, $\overline{\varphi}^{\delta}$ is essentially just an average over a range of T^{δ} many points.) If $a > s$ and T is so large that $T^{-a} < T^{-s}/2$, we get

$$\begin{aligned}
&E\big(\exp(-N_T);\ \|l_T - \mathcal{F}\|_1 > T^{-s}\big)\\
&\quad\leq E\Big(\exp\big(-N_T^{(q)}\big);\ \|l_T - \mathcal{F}\|_1 \geq T^{-s},\ \big\|l_T - \bar{l}_T^{\delta}\big\|_1 < T^{-a}\Big)\\
&\qquad + P\Big(\big\|l_T - \bar{l}_T^{\delta}\big\|_1 \geq T^{-a}\Big)\\
&\quad\leq \exp(T^{q-a}\tau)E\left(\exp\Big(-\tau\int h\big(\bar{l}_T^{\delta}(x)\big)dx\Big);\ \big\|\bar{l}_T^{\delta} - \mathcal{F}\big\|_1 \geq \frac{T^{-s}}{2}\right)\\
&\qquad + P\Big(\big\|l_T - \bar{l}_T^{\delta}\big\|_1 \geq T^{-a}\Big).
\end{aligned} \tag{3.3}$$

We will estimate the two summands appearing on the right-hand side of this inequality.

LEMMA 3.2. *If a and $\delta > 0$ are small enough, then*

$$\lim_{T\to\infty} \frac{1}{\tau} \log P\Big(\big\|l_T - \bar{l}_T^{\delta}\big\|_1 \geq T^{-a}\Big) = -\infty.$$

(*Remember that* $\tau = T^{d/(2+d)}$.)

To estimate the first summand in (3.3), note that h defined in (3.2) contains a parameter $q > 0$ which we still can choose to our liking.

LEMMA 3.3. *For given* $\varepsilon, a, \delta > 0$, *we can choose* $s < a$ *and* $q > 0$ *such that, with* $h = h^{(q,T)}$,

$$E\left(\exp\left(-\tau \int h\left(\bar{l}_T^{\delta}(x)\right)dx\right);\ \left\|\bar{l}_T^{\delta} - \mathcal{F}\right\|_1 \geq \frac{T^{-s}}{2}\right) \leq \exp\left(-\chi_d\tau - \tau T^{-\varepsilon}\right)$$

for large enough T.

Clearly Lemma 3.2, Lemma 3.3 together with (3.3) and Proposition 2.1 prove Proposition 3.1. □

PROOF OF LEMMA 3.2. No attempt is made to derive the best possible δ and a. We use the same type of arguments as in [3]:

$$\left\|l_T - \bar{l}_T^{\delta}\right\|_1 = \sup_{g:\|g\|_\infty \leq 1} \langle l_T - P_\sigma l_T, g\rangle = \sup_{g:\|g\|_\infty \leq 1} \langle l_T, g - P_\sigma g\rangle,$$

where $\sigma = T^{2\delta/d - 2/(2+d)}$. We abbreviate $g - P_\sigma g$ by $\widehat{g}$ and set $\mathcal{G} = \{\widehat{g}: \|g\|_\infty \leq 1\}$. We can choose a finite set $\mathcal{H}$ of bounded functions defined on $\mathbb{L}$ such that $\sup_{f\in\mathcal{G}} \inf_{h\in\mathcal{H}} \|f - h\|_\infty \leq T^{-a}/4$ which has $|\mathcal{H}| \leq \exp(c(a,\delta)\tau \log T)$ elements, where $c(a,\delta) > 0$ does not depend on T. (Note that $\|f\|_\infty \leq 2$ for $f \in \mathcal{G}$.) For each $h \in \mathcal{H}$, we choose a g with $\|g\|_\infty \leq 1$ satisfying $\|h - \widehat{g}\|_\infty \leq T^{-a}/4$ if such a g exists. Collecting all these functions g, we obtain a finite set $\mathcal{G}_0$ satisfying

$$|\mathcal{G}_0| \leq \exp\left(c(a,\delta)\tau \log T\right), \tag{3.4}$$

$$\sup\nolimits_{f:\|f\|_\infty \leq 1} \inf\nolimits_{g\in\mathcal{G}_0} \left\|\widehat{f} - \widehat{g}\right\|_\infty \leq T^{-a}/2. \tag{3.5}$$

We therefore obtain

$$\begin{aligned} P\left(\left\|l_T - \bar{l}_T^{\delta}\right\|_1 \geq T^{-a}\right) &\leq |\mathcal{G}_0| \sup_{g:\|g\|_\infty \leq 1} P\left(\langle l_T, \widehat{g}\rangle \geq T^{-a}/2\right) \\ &\leq \exp\left(-\tfrac{1}{2} z\tau T^{-a}\right) |\mathcal{G}_0| \sup_{\|g\|_\infty \leq 1} E\left(\exp\left(z \int_0^\tau \widehat{g}(\eta_s)\,ds\right)\right), \end{aligned} \tag{3.6}$$

by the Markov inequality, where $z > 0$ will be chosen later on.

We now estimate $E(\exp(\int_0^T h(\eta_s)\,ds))$ for a bounded function h. By Lemma 2.2 of [7], we get, for $t > 0$,

$$E\left(\exp\left(\int_0^t h(\eta_s)\,ds\right)\right) \leq C \exp\left(2\|h\|_\infty\right) \exp\left[t \sup_\mu\left(\langle \mu, h\rangle - I_T(\mu)\right)\right], \tag{3.7}$$

where the supremum is taken over the probability measure μ on $\mathbb{L}$; C is a constant greater than 0 which does not depend on t and T. The situation here is slightly different from that in [7], as $\eta_s = \eta_s^{(T,R)}$ depends on T. However, by the obvious estimate

$$\sup_{x,y,T} \left|\log p_t^{(T,R)}(x,y)\right| \le c(t) < \infty,$$

for all $t > 0$, the proof of Lemma 2.2 in [7] can be taken literally.

We write

$$\sigma_T(h) = \sup_{\mu} \left(\langle \mu, h\rangle - I_T(\mu)\right), \qquad \overline{\sigma}(z) = \sup_{\|g\|_\infty \le 1} \sigma_T(z\widehat{g}).$$

By (3.4), (3.6) and (3.7), we get

$$P\Big(\big\|l_T - \bar{l}_T^\delta\big\|_1 \ge T^{-a}\Big) \le \exp\Big(c\big[\tau \log T + z + \tau\overline{\sigma}(z)\big] - \tfrac{1}{2} z\tau T^{-a}\Big). \tag{3.8}$$

We take now $z = T^k$ with $a < k < d/(d+2)$. (We did not yet fix a.) Then, in the exponent on the right-hand side of (3.8), z is dominated by $\tau \log T$, which itself is negligible when looking at the negative summand $\frac{1}{2} z\tau T^{-a}$. If we show that

$$\overline{\sigma}(z) \le czT^{-2a}, \tag{3.9}$$

then for large T we get

$$P\Big(\big\|l_T - \bar{l}_T^\delta\big\|_1 \ge T^{-a}\Big) \le \exp\left(-\tfrac{1}{4}\tau T^{-a+k}\right)$$

and we have proved the lemma.

We show now that we can choose $\delta > 0$ and $0 < a < d/(d+2)$ such that (3.9) is true:

$$\begin{aligned} \overline{\sigma}(z) &\le \sup\left\{z\langle \mu, \widehat{g}\,\rangle : I_T(\mu) \le z,\ \|g\|_\infty \le 1\right\} \\ &\le z \sup\left\{\|\mu - \overline{\mu}^\delta\|_v : I_T(\mu) \le z\right\}, \end{aligned} \tag{3.10}$$

where $\|\ \|_v$ denotes total variation norm;

$$\begin{aligned} \|\mu - \overline{\mu}^\delta\|_v &= \|\mu - \mu P_\sigma\|_v \quad \left(\text{remember } \sigma = T^{2\delta/d - 2/(2+d)}\right) \\ &\le \sqrt{8 J_{P_\sigma}(\mu)} \le \sqrt{8\sigma z} \end{aligned} \tag{3.11}$$

if $I_T(\mu) \le z$. Here J is the discrete time Donsker–Varadhan entropy: If K is a Markov kernel on $\mathbb{L}$ and γ a probability measure on $\mathbb{L}$, then $J_K(\gamma) = \sup \int (g - \log K(e^g))\, d\gamma$, the supremum being taken over bounded measurable functions g. The first inequality in (3.11) is by Lemma 4.1 of [6] and the second by Lemma 3.1 in the same paper. Using (3.11) and (3.10) it follows that (3.9) is true for small enough $a, \delta > 0$ and appropriately chosen k, $a < k < d/(d+2)$, therefore proving Lemma 3.2. □

PROOF OF LEMMA 3.3. A direct estimation of the expression does not seem to be possible, the main difficulty being that $\{f \in L_1 : \|f - \mathcal{F}\|_1 \geq c\}$ is not convex. We therefore cut the space of probability densities on $\mathbb{L}$ into small pieces. Given $b > 0$, let $f_1, \dots, f_{m_T}$ be a finite collection of probability densities on $\mathbb{L}$ such that, for any probability density f,

$$\min_{1 \leq j \leq m_T} \int \left| f_j - \bar{f}^\delta \right| dx \leq T^{-b}. \tag{3.12}$$

If $s < b$, $h = h^{(q,T)}$, where $q > 0$, then

$$\begin{aligned} &E\left(\exp\left(-\tau \int h(\bar{l}_T^\delta) dx \right);\ \left\|\bar{l}_T^\delta - \mathcal{F}\right\|_1 \geq T^{-s}/2 \right) \\ &\quad \leq m_T \sup_{j \in J} E\left(\exp\left(-\tau \int h(\bar{l}_T^\delta) dx \right);\ \left\|\bar{l}_T^\delta - f_j\right\|_1 \leq T^{-b} \right), \end{aligned} \tag{3.13}$$

where J is the set of indices j with $\|f_j - \mathcal{F}\|_1 \geq T^{-s}/4$. The above expression is then less than or equal to

$$m_T \exp(\tau T^{q-b}) \sup_{j \in J} \left[\exp\left(-\tau \int h(f_j(x)) dx \right) P\left(\left\|\bar{l}_T^\delta - f_j\right\|_1 \leq T^{-b} \right) \right].$$

To proceed, we have to estimate m_T and $P(\|\bar{l}_T^\delta - f\|_1 \leq T^{-b})$ for $\|f - \mathcal{F}\|_1 \geq T^{-s}/4$.

LEMMA 3.4. *If $db < \delta$, we can find a set $\{f_1, \dots, f_{m_T}\}$ with*

$$m_T \leq \exp\left(c(b,\delta) \tau T^{db-\delta} \log T \right).$$

LEMMA 3.5. *There exists $C > 0$ such that, for any $\varepsilon > 0$,*

$$P(\|l_T - f\|_1 \leq \varepsilon) \leq C \exp(-(\tau - 1)\kappa_T(f, 2\varepsilon))$$

where $\kappa_T(f, \varepsilon) = \inf\{I_T(\nu) : \|\nu - f\|_1 \leq \varepsilon\}$.

We postpone the proof of these two lemmas and proceed with the proof of Lemma 3.3. If $b < a$, we get

$$P\left(\left\|\bar{l}_T^\delta - f_j\right\|_1 \leq T^{-b} \right) \leq P(\|l_T - f_j\|_1 \leq 2T^{-b}) + P\left(\left\|l_T - \bar{l}_T^\delta\right\|_1 \geq T^{-a} \right). \tag{3.14}$$

Using Lemma 3.5 we get

$$\begin{aligned} &\exp\left(-\tau \int h(f_j(x))\, dx \right) P(\|l_T - f_j\|_1 \leq 2T^{-b}) \\ &\quad \leq c \exp\left(-\tau \int h(f_j(x))\, dx - \tau \kappa_T(f_j, 4T^{-b}) \right). \end{aligned} \tag{3.15}$$

If $j \in J$, then

$$\begin{aligned}(3.16)\qquad &\int h\big(f_j(x)\big)\,dx + \kappa_T\big(f_j, 4T^{-b}\big)\\ &\geq \inf\left\{\int h\big(f(x)\big)\,dx + I_T(f) \colon \|f-\mathcal{F}\|_1 \geq T^{-s}/8\right\} - cT^{q-b}.\end{aligned}$$

LEMMA 3.6. *For any $q, \varepsilon > 0$ there exists $\varepsilon' > 0$ with*

$$\inf\left(\int h\big(f(x)\big)dx + I_T(f) \colon \|f-\mathcal{F}\|_1 \geq T^{-\varepsilon'}\right) \geq \chi_d + T^{-\varepsilon}.$$

Postponing the proof of this lemma, the proof of Lemma 3.3 can easily be completed. From (3.13), (3.15), (3.16) and Lemma 3.4, we get

$$\begin{aligned}&E\left(\exp\left(-\tau\int h(\bar{l}_T^\delta dx);\ \left\|\bar{l}_T^\delta - \mathcal{F}\right\|_1 \geq T^{-s}/2\right)\right)\\ &\quad\leq \exp\left(c(b,\delta)\tau T^{-\bar{\delta}} + \tau T^{q-b}\right)P\left(\left\|l_T - \bar{l}_T^\delta\right\|_1 \geq T^{-a}\right)\\ &\qquad + c\,\exp\left(c(b,\delta)\tau T^{-\bar{\delta}} + c_1\tau T^{q-b}\right)\\ &\qquad \times \exp\left(-\tau \inf_{\|f-\mathcal{F}\|_1\geq T^{-s}/8}\left(\int h\big(f(x)\big)\,dx + I_T(f)\right)\right),\end{aligned}$$

where $0 < \bar{\delta} < \delta - db$.

We take now $b = \frac{1}{2}\min(q, \delta/d)$, $q = b/2$, $\bar{\delta} = \frac{1}{2}(\delta - db)$. We apply Lemma 3.6. If we take $\varepsilon < \min(\bar{\delta}, b-q)$ and choose $s < \varepsilon'$ and $s < b$ (which was required above), then

$$\begin{aligned}&E\left(\exp\left(-\tau\int h\left(\bar{l}_T^\delta\right)dx\right);\ \left\|\bar{l}_T^\delta - \mathcal{F}\right\|_1 \geq T^{-s/2}\right)\\ &\qquad \leq \exp(-2\chi_d\tau) + \exp\big(-\chi_d\tau - c\tau T^{-\varepsilon}\big)\end{aligned}$$

and in fact, by choosing $s > 0$ small enough, we can have $\varepsilon > 0$ as small as we like, proving Lemma 3.3. □

It remains to prove Lemmas 3.4–3.6.

PROOF OF LEMMA 3.4. We first divide $\mathbb{L}$ into cubes of side length $\beta = \alpha T^{-b}\sqrt{\sigma}$, where $\alpha > 0$ will be chosen later (remember that $\sigma = T^{2\delta/d - 2/(2+d)}$), containing therefore approximately $T^{\delta - db}$ points. We define $\widehat{f}^\delta(x)$ for x in such a cube B by $(1/|B|)\int_B \bar{f}^\delta(y)\,dy$. Then

$$(3.17)\qquad \int \left|\bar{f}^\delta(x) - \widehat{f}^\delta(x)\right|dx \leq \int \sup_{|z|\leq c\beta} |p_\sigma(y+z) - p_\sigma(y)|\,dy.$$

We have

$$\int \sup_{|z|\le\varepsilon} \left|p_\sigma(y+z)-p_\sigma(y)\right| dy \le \frac{c\varepsilon}{\sqrt{\sigma}}, \tag{3.18}$$

for $\varepsilon > 0$. By a straightforward computation, this is true for the transition probabilities of Brownian motion, and a standard application of the local central limit theorem (see [10], Theorem 1.2.1), (3.18) follows.

Using this, we get from (3.17)

$$\|\overline{f}^\delta - \widehat{f}^\delta\|_1 \le c\alpha T^{-b} \le \frac{T^{-b}}{4} \tag{3.19}$$

if α is small enough. By choosing functions which are constant on our cubes, we find density functions $f_1,\dots,f_{m_T}$ with

$$m_T \le \left(cT^{d/(2+d)+b}\right)^{c_1\tau T^{db-\delta}} \le \exp\left(2c_1\left(\frac{d}{2+d}+b\right)\tau T^{db-\delta}\log T\right)$$

and $\sup_{\|f\|_1=1}\min_{1\le j\le m_T}\|\widehat{f}^\delta - f_j\|_1 \le T^{-b}/4$ (note that $\|\widehat{f}^\delta\|_\infty \le cT^{d/(2+d)}$). Together with (3.19), this proves Lemma 3.4. □

Proof of Lemma 3.5. We keep T fixed for the moment; η_s, $s\ge 0$, is a process on the finite state space $\mathbb{L}_T$. If $x\in\mathbb{L}$, we write P^x for the law of the random walk η_s starting in x. If $x\in\mathbb{L}$, $t>0$, let

$$\widetilde{l}_t(x) = \frac{\tau}{t}\int_0^t \chi_x(\eta_s)\,ds.$$

Remember that $\tau = T^{d/(2+d)}$. We have by definition $l_T = \widetilde{l}_\tau$.

Let

$$\lambda(t,\varepsilon) = \min_{x\in\mathbb{L}} P^x\left(\|\widetilde{l}_t - f\|_1 \le \varepsilon\right).$$

As the set of probability distributions $\{\nu: \|\nu-f\|_1\le\varepsilon\}$ is convex, one sees that $\lambda(t+s,\varepsilon)\ge\lambda(t,\varepsilon)\lambda(s,\varepsilon)$, and from this we get $\lambda(t,\varepsilon)\le\exp(-t\kappa_T(f,\varepsilon))$, where

$$\kappa_T(f,\varepsilon) = -\lim_{t\to\infty}\frac{1}{t}\log P\left(\|\widetilde{l}_t - f\|_1\le\varepsilon\right) = \inf\{I_T(\nu): \|\nu-f\|_1\le\varepsilon\}.$$

The second equality follows from the fact that, for fixed T, η_s, $s\ge 0$ satisfies a uniform large deviation principle with rate function I_T (the state space being finite). Since

$$P^x(\|l_T-f\|_1\le\varepsilon) \le \sum_y p_1(x,y)P^y\left(\|\widetilde{l}_{\tau-1}-f\|_1\le\varepsilon+\frac{2}{\tau}\right),$$

$$P^x(\|l_T-f\|_1\le\varepsilon) \ge \sum_y p_1(x,y)P^y\left(\|\widetilde{l}_{\tau-1}-f\|_1\le\varepsilon-\frac{2}{\tau}\right),$$

where $p_t(x,y)$ is the transition probability of the random walk on $\mathbb{L}_T$ we have

$$P(\|l_T - f\|_1 \le \varepsilon) \le C \exp\left(-(\tau - 1)\kappa_T\left(f, \varepsilon + \frac{4}{\tau}\right)\right),$$

where by a simple computation

$$C = \sup_T \frac{\max_y p_1(x,y)}{\min_y p_1(x,y)} < \infty.$$

[$p_t(x,y)$, of course, depends on T.] □

PROOF OF LEMMA 3.6. We reduce the lemma to the statement of Lemma 3.1 (which itself is proved in Appendix A for $d = 2$).

If f is a density, we put $\bar{f} = ((\sqrt{f} - T^{-q/2}) \vee 0)^2$ ($q > 0$ was used in the definition of h). An elementary calculation gives

$$\|f - \bar{f}\|_1 \le cT^{-q/2}.$$

We put $\tilde{f} = \bar{f}/\|f\|_1$, which then satisfies $\|f - \tilde{f}\|_1 \le cT^{-q/2}$ as well. If $\varepsilon' < q/2$, then, for large enough T, $\|f - \mathcal{F}\|_1 \ge T^{-\varepsilon'}$ implies $\|\tilde{f} - \mathcal{F}\|_1 \ge T^{-\varepsilon'}/2$. For any such f, we also have

$$\int h(f(x))\,dx \ge \int 1_{\{\tilde{f}(x) > 0\}} dx \quad \text{and} \quad I_T(f) \ge (1 - cT^{-q/2}) I_T(\tilde{f}).$$

Using this, the statement of Lemma 3.6 immediately follows from that of Lemma 3.1. □

We will need several refinements of Proposition 3.1 which in the end lead to a proof of our main theorem. Our steps are always of the following type: We consider events which depend on $x \in \mathbb{L}$, $T > 0$ (and maybe some other parameters), say, $A_{x,T}$, and we already know that

$$\widehat{P}_T\left(\bigcup_x A_{x,T}\right) \to 1$$

for $T \to \infty$; we want to show that, for some other family of events $B_{x,T}$, we have

$$\widehat{P}_T\left(\bigcup_x (A_{x,T} \cap B_{x,T})\right) \to 1.$$

For this conclusion, it suffices to show that

$$\begin{aligned} &\lim\nolimits_{T \to \infty} E\left(e^{-N_T};\ \bigcup\nolimits_x \left(A_{x,T} \cap B^c_{x,T}\right)\right) \Big/ z_T \\ &\quad \le c \lim_{T \to \infty} \tau \sup_x E\left(e^{-N_T};\ A_{x,T} \cap B^c_{x,T}\right) \Big/ z_T = 0. \end{aligned}$$

If the last equality is true, we write

$$\underline{A} \rightsquigarrow \underline{B},$$

where $\underline{A}$ is a shorthand for the collection $\{A_{x,T}: x \in \mathbb{L},\ T > 0\}$. We also use the notation

$$\widehat{\Rightarrow} \underline{A} \quad \text{for} \quad \lim_{T\to\infty} \widehat{P}_T\Big(\bigcup_x A_{x,T}\Big) = 1.$$

Note that for proving

$$\underline{A} \rightsquigarrow \underline{B} \cap \underline{C},$$

where $\underline{B} \cap \underline{C} = \{B_{x,T} \cap C_{x,T}: x \in \mathbb{L},\ T > 0\}$, it suffices to have $\underline{A} \rightsquigarrow \underline{B}$ and $\underline{A} \rightsquigarrow \underline{C}$. Obviously $\underline{A} \rightsquigarrow \underline{B}$ and $\underline{B} \rightsquigarrow \underline{C}$ imply $\underline{A} \rightsquigarrow \underline{C}$.

If D is a subset of $\mathbb{L}$, we write $N_T(D)$ for the number of points in D visited by the random walk up to time τ ($= T^{d/(2+d)}$). For $x, y \in \mathbb{L}$, we use $d(x, y)$ as the Euclidean distance on the flat torus.

PROPOSITION 3.2. *If $s > 0$, then there exists $\varepsilon > 0$ such that*

$$\{\|l_T - \psi_\cdot^2\|_1 \le T^{-s}\} \rightsquigarrow \Big\{N_T\Big(D_\cdot^c\big(\varrho_d(1 - T^{-\varepsilon})\big)\Big) \le \tau T^{-\varepsilon}\Big\}.$$

We slightly abuse the above notation. The dots in $\psi_\cdot^2$ and $D_\cdot$ indicate families of events which depend on $x \in \mathbb{L}$ when replacing the dots with x; for example, $D_x(r) \equiv \{y \in \mathbb{L}: d(x,y) \le r\}$.

PROOF OF PROPOSITION 3.2. If $\alpha, \beta, s > 0$, $\beta < 1/(d+1)$, $\alpha < \min(\beta, s - 2\beta)$, then we claim that $N_T \ge \tau\omega_d\varrho_d^d + \tau T^{-\alpha}/2$ on the event

$$\Big\{\big\|l_T - \psi_x^2\big\|_1 \le T^{-s},\ N_T\Big(D_x^c\big(\varrho_d(1 - T^{-\beta})\big)\Big) > \tau T^{-\alpha}\Big\}.$$

To prove this, we compare ψ, which is the eigenfunction of the discrete Laplacian, with that of the continuous one, φ, introduced in Section 2. (We can, of course, take $x = 0$.) We have

$$\varphi \ge cT^{-\beta} \quad \text{on} \quad F \equiv D\Big(\varrho_d\big(1 - T^{-\beta}\big)\Big)$$

and using $\beta < 1/(1+d)$ and Lemma 2.1, $\psi \ge cT^{-\beta}$, too. From this, the above claim follows by a straightforward counting argument. Now

$$\begin{aligned} &E\Big(\exp(-N_T);\ \big\|l_T - \psi_x^2\big\|_1 \le T^{-s},\ N_T(F^c) > \tau T^{-\alpha}\Big) \\ &\quad \le \exp\big(-\omega_d\varrho_d^d\tau - \tfrac{1}{2}\tau T^{-\alpha}\big)P\Big(\big\|l_T - \psi_x^2\big\|_1 \le T^{-s}\Big) \\ &\quad \le c\,\exp\Big(-\omega_d\varrho_d^d\tau - \tfrac{1}{2}\tau T^{-\alpha} - \tau\kappa_T\big(\psi_x^2, 2T^{-s}\big)\Big), \end{aligned}$$

by Lemma 3.5. Proposition 3.2 then immediately follows from the following lemma. □

LEMMA 3.7. *If $s > 0$, there exists $a > 0$ such that*

$$\inf\left\{I_T(f): \|f - \psi_x^2\|_1 \le T^{-s}\right\} \ge \lambda_T - T^{-a}.$$

(λ_T is λ_T^* from Section 2. We drop the asterisk.)

PROOF. The centering x plays no role, so we set $x = 0$ and write ψ for ψ_0. Let $D = D(\varrho_d)$ on the lattice $\mathbb{L}$ and

$$\partial\{x \in \mathbb{L} \setminus D: \exists\, y \in D \text{ with } |x - y| = T^{-1/(2+d)}\}.$$

The function ψ satisfies $A_T\psi = -\lambda_T\psi$ except possibly on ∂D. We write

$$A_T\psi = -\lambda_T\psi + \delta. \tag{3.20}$$

Using Lemma 2.1, we have

$$\sup_x |\delta(x)| \le cT^{1/(2+d)}. \tag{3.21}$$

We consider the "dual torus"

$$\mathbb{L}^* = \left\{0, \frac{2\pi}{R_T}, 2\frac{2\pi}{R_T}, \dots, (K_T - 1)\frac{2\pi}{R_T}\right\}^d,$$

where $R_T = T^{-1/(2+d)}K_T$, $K_T = [RT^{1/(2+d)}]$.

For the sake of notational convenience, we assume that $RT^{1/(2+d)}$ is an integer and so $R_T = R$. The Fourier transform of a function $f: \mathbb{L} \to \mathbb{C}$ is defined by

$$\widehat{f}(\alpha) = \int_{\mathbb{L}} e^{i\langle\alpha, x\rangle} f(x)\, dx,$$

where $\alpha \in \mathbb{L}^*$.

If $g: \mathbb{L} \to \mathbb{R}$ satisfies $\int_{\mathbb{L}} g^2\, dx = 1$, then

$$I_T(g^2) = \frac{R^{-d}}{2} \sum_{\alpha \in \mathbb{L}^*} q_T(\alpha)|\widehat{g}(\alpha)|^2, \tag{3.22}$$

where $q_T(\alpha) = 2\sum_{j=1}^d T^{2/(2+d)}(1 - \cos(T^{-1/(2+d)}\alpha_j))$, $\alpha = (\alpha_1, \dots, \alpha_d)$. We have the elementary estimate

$$c_1|\alpha|^2 \le q_T(\alpha) \le c_2|\alpha|^2, \tag{3.23}$$

where $|\alpha|$ is the Euclidean distance from 0 to α on the flat torus. [We consider $\mathbb{L}^*$ as a subset of the flat torus $[0, RT^{1/(2+d)})^d$.]

Using (3.22), we obtain for any $g\colon \mathbb{L} \to \mathbb{R}$ with $\|g\|_2 = 1$, and for any $l > 0$,

$$
\begin{aligned}
I_T(\psi^2) - I_T(g^2) &= \frac{R^{-d}}{2} \sum_{\alpha} q_T(\alpha)(|\widehat{\psi}(\alpha)|^2 - |\widehat{g}(\alpha)|^2) \\
&\le \frac{R^{-d}}{2} \sum_{\alpha:|\alpha|\ge l} q_T(\alpha)\big|\widehat{\psi}(\alpha)\big|^2 + c \sup_{\alpha:|\alpha|<l} q_T(\alpha)\|\psi - g\|_2 \qquad (3.24)\\
&\le \frac{R^{-d}}{2} \sum_{\alpha:|\alpha|\ge l} q_T(\alpha)\big|\widehat{\psi}(\alpha)\big|^2 + cl^2\|\psi - g\|_2,
\end{aligned}
$$

the last inequality by (3.23).

In order to estimate $\widehat{\psi}$, we use (3.20), which gives

$$
-q_T(\alpha)\widehat{\psi}(\alpha) = -\lambda_T\widehat{\psi}(\alpha) + \widehat{\delta}(\alpha),
$$

and therefore

$$
\big|\widehat{\psi}(\alpha)\big|^2 = \frac{|\widehat{\delta}(\alpha)|^2}{|\lambda_T - q_T(\alpha)|^2}, \tag{3.25}
$$

if $q_T(\alpha) \ne \lambda_T$. We claim that, for any $l > 0$,

$$
\sum_{\alpha:|\alpha|\le l} \big|\widehat{\delta}(\alpha)\big|^2 \le cl. \tag{3.26}
$$

Postponing the proof of (3.26) for the moment, we immediately get from (3.24)–(3.26)

$$
I_T(\psi^2) - I_T(g^2) \le c(R)\left[\frac{1}{l} + l^2\|\psi - g\|_2\right]
$$

for large enough l. Choosing $l = \|\psi - g\|_2^{-1/3}$, we get

$$
I_T(\psi^2) - I_T(g^2) \le c(R)\|\psi - g\|_2^{1/3} \le 4c(R)\|\psi^2 - g^2\|_1^{1/3},
$$

which proves the claim.

It remains to prove (3.26). We can write δ as a fixed finite sum of functions δ' which have the property that there is one component of $x = (x_1, \dots, x_d) \in \mathbb{L}$, say x_i, such that for every value of the other components, the number of x_i's with $\delta'(x) \ne 0$ is bounded above. We assume $i = d$ and write $x' = (x_1, \dots, x_{d-1})$, $x'' = x_d$. For $\alpha \in \mathbb{L}^*$, we do the same splitting $\alpha = (\alpha', \alpha'')$. Using (3.21), we see that the partial transform

$$
\xi(x', \alpha'') = T^{-1/(2+d)} \sum_{x''} e^{i\langle \alpha'', x''\rangle} \delta'(x', x'')
$$

is bounded above, uniformly in x', α''. Using the Parseval equality, we see that

$$
\sum_{\alpha'} \big|\widehat{\delta'}(\alpha', \alpha'')\big|^2
$$

is bounded in α'', and therefore

$$\sum_{\alpha:|\alpha|\leq l} \left|\widehat{\delta}'(\alpha)\right|^2 \leq \sum_{\alpha:|\alpha''|\leq l} \left|\widehat{\delta}'(\alpha)\right|^2 \leq cl.$$

This proves (3.26). □

4. A disc is completely filled. The space $\mathbb{L}$ is still our lattice torus $\mathbb{L}_T^{(R)}$. If $g: \mathbb{L} \to [0, \infty)$, we define

$$s(g) = \{y \in \mathbb{L}: g(y) > 0\}.$$

We also use $\varepsilon > 0$ as a generic small constant, not necessarily the same at different occurrences. All inequalities are required to hold only for large enough T.

We have shown so far that there exists with large $\widehat{P}_T$-probability some translate $f_x = \psi_x^2 \in \mathcal{F}$ such that l_T is L_1-close to f_x. We show in this section that with $\widehat{P}_T$-probability close to 1 every point in $s(f_x)$ is visited, except possibly those near the boundary.

The analysis here is only for $d = 2$ and is actually based on some rather crude arguments which could not work for $d \geq 3$. So, we consider only to $d = 2$ where the distance on $\mathbb{L}$ between points is $T^{-1/4}$ and time for our rescaled random walk η_t runs up to $\tau = \sqrt{T}$. We set $\varrho = \varrho_2$.

PROPOSITION 4.1. *For any $s > 0$, there exists $\iota > 0$, such that we have*

$$\{\|l_T - f_\cdot\|_1 \leq T^{-s}\} \rightsquigarrow \left\{s(l_T) \supset D_\cdot\left(\varrho(1 - T^{-\iota})\right)\right\}.$$

PROOF. We write $A_x(\iota)$ for the event $\{s(l_T) \supset D_x(\varrho(1 - T^{-\iota}))\}$.

We have already remarked in Section 3 that on $\{\|l_T - f_x\|_1 \leq T^{-s}\}$ we have $N_T \geq \pi\varrho^2 T^{1/2} - T^{1/2-\varepsilon}$ for large T. So

$$\begin{aligned} &E\left(\exp(-N_T);\ \|l_T - f_x\|_1 \leq T^{-s}, A_x(\iota)^c\right) \\ &\quad \leq \exp\left(-\pi\varrho^2 T^{1/2} + T^{1/2-\varepsilon}\right) P\left(\|l_T - f_x\|_1 \leq T^{-s},\ A_x(\iota)^c\right). \end{aligned} \tag{4.1}$$

We can assume that $x = 0$ by changing the starting point of the random walk to $x' = -x$. Our analysis will not depend on this starting point.

For a strictly positive $\widetilde{\psi}: \mathbb{L} \to (0, \infty)$ we define the law $\widetilde{P}$ of a random walk on $\mathbb{L}$ by the generator

$$\widetilde{A}g(x) = \frac{1}{2}\sqrt{T}\sum_{y\sim x} \frac{\widetilde{\psi}(y)}{\widetilde{\psi}(x)}\left(g(y) - g(x)\right).$$

Then

$$\begin{aligned} &P^{x'}\left(\|l_T - f\|_1 \leq T^{-s};\ A_0(\iota)^c\right) \\ &\quad = \widetilde{E}^{x'}\left(\frac{\widetilde{\psi}(x')}{\widetilde{\psi}(\eta_{\sqrt{T}})} \exp\left(\int_0^{\sqrt{T}} \frac{A_T\widetilde{\psi}(\eta_s)}{\widetilde{\psi}(\eta_s)}\,ds\right);\ \|l_T - f\|_1 \leq T^{-s},\ A_0(\iota)^c\right). \end{aligned} \tag{4.2}$$

Essentially, we want to take $\widetilde{\psi} \sim \psi$, the latter being, however, not strictly positive and not smooth enough.

We start with the eigenfunction φ of the principal eigenvalue of the continuous Laplacian $\frac{1}{2}\Delta$ in $D(\varrho) = \{x \in \mathbb{R}^2: |x| \leq \varrho\}$ with Dirichlet boundery conditions and set

$$\widetilde{\psi} = \varphi(x) + T^{-\kappa},$$

for some (small) number $\kappa > 0$ and $\widetilde{\psi}(x) = T^{-\kappa}/2$ if $|x| \geq \varrho + 1$. For $\varrho \leq x \leq \varrho + 1$, we put $\widetilde{\psi}(x) = h(|x|)$, where $h: [\varrho, \varrho + 1] \to [\frac{1}{2}T^{-\kappa}, T^{-\kappa}]$ is a smooth monotonically decreasing convex function whose derivative at $1 + \varrho$ is 0 and at ϱ equals that of the radial derivative φ at the boundary of D_ϱ. We can achieve this with

$$\sup_{x \in [\varrho, 1+\varrho]} \frac{d^2}{dx^2} h(x) \leq cT^{\kappa}.$$

We then restrict $\widetilde{\psi}$ to $T^{-1/4}\mathbb{Z}^2$ and then to the lattice torus $\mathbb{L}$, which is no problem, because we assume $R = 10\varrho$. If $x \in \mathbb{L} \cap D(\varrho)$ has all its neighbors inside $D(\varrho)$, we have by Lemma 2.1

$$A_T\widetilde{\psi}(x) = A_T\varphi(x) = -\lambda_2\varphi(x) + O\big(T^{-1/4}\big),$$

where φ is also restricted to $\mathbb{L}$. Therefore

$$A_T\widetilde{\psi}(x) = -\lambda_2\widetilde{\psi}(x) + O\big(T^{-\kappa}\big) \quad \text{if } \kappa < \tfrac{1}{4}$$

and we get

$$\left|\frac{A_T\widetilde{\psi}(x)}{\widetilde{\psi}(x)} + \lambda_2\right| \leq cT^{-\kappa+\kappa'}, \tag{4.3}$$

for $x \in D(\varrho(1 - T^{-\kappa'}))$, and of course

$$\left|\frac{A_T\widetilde{\psi}(x)}{\widetilde{\psi}(x)}\right| \leq cT^{\kappa} \tag{4.4}$$

everywhere.

On $\|l_T - f\|_1 \leq T^{-s}$ we have

$$\begin{aligned}\int_0^{\sqrt{T}} \frac{A_T\widetilde{\psi}(\eta_s)}{\widetilde{\psi}(\eta_s)} ds &= \sqrt{T}\left\langle \frac{A_T\widetilde{\psi}}{\widetilde{\psi}}, f\right\rangle + O\big(T^{1/2+\kappa-s}\big) \\ &= \sqrt{T}\lambda_2 + O\big(T^{1/2-\varepsilon}\big),\end{aligned}$$

by using (4.2) and (4.3), choosing $0 < \kappa' < \kappa < s$. Taking this $\widetilde{\psi}$ in our transformation, we get

$$P^{x'}\big(\|l_T - f\|_1 \leq T^{-s};\, A_0(\iota)^c\big) \leq \exp\big(-\lambda\sqrt{T}\big)\exp\big(T^{1/2-\varepsilon}\big)\widetilde{P}^{x'}\big(A_0(\iota)^c\big).$$

We claim now that for a given $\delta > 0$ we may choose ι small enough such that

$$\widetilde{P}^{x'}(A_0(\iota)^c) \leq \exp\big(- T^{1/2-\delta}\big). \tag{4.5}$$

Having (4.5) and using (4.1), we see that

$$E\big(\exp(-N_T);\ \|l_T - f_x\|_1 \leq T^{-s}, A_x(\iota)^c\big) \leq \exp\big(- 2\sqrt{\pi\lambda}\sqrt{T}\big) \exp\big(- T^{1/2-\varepsilon}\big),$$

which with Proposition 2.1 proves Proposition 4.1.

If $y \in \mathbb{L}$, we define $\tau_y = \inf\{s > 0: \eta_s = y\}$. We prove (4.5) by showing that if $\iota > 0$ is small, then there exists $l > 0$ such that, for large T,

$$\inf_{y \in D(\varrho(1-T^{-\iota}))} \inf_{z \in \mathbb{L}} \widetilde{P}^z(\tau_y \leq l) \geq cT^{-\iota}. \tag{4.6}$$

This implies (4.5). In fact, in the standard way, one proves

$$\widetilde{P}\big(\tau_y \geq \sqrt{T}\big) \leq (1 - cT^{-\iota})^{(\sqrt{T}/l)} \leq \exp\big(- c'T^{1/2-\iota}\big)$$

and

$$\widetilde{P}\Big(\tau_y \geq \sqrt{T} \text{ for some } y \in D_x\big(\varrho(1 - T^{-\iota})\big)\Big) \leq c\sqrt{T} \exp\big(- cT^{1/2-\iota}\big).$$

The first entrance time into $D(\varrho/2)$ has a bounded expectation uniformly in $z \in \mathbb{L}, T$ and, therefore, it suffices to prove (4.6) with the inf over z restricted to $z \in D(\varrho/2)$.

Observe now that for paths of fixed length which stay inside $\overline{D} = D(\varrho(1 - \frac{1}{2}T^{-\iota}))$, the density of $\widetilde{P}$ with respect to P remains within $c[T^{-\iota}, T^{\iota}]$ if ι is small enough.

If $A \subset \mathbb{L}$, let $\sigma_A = \inf\{t > 0: \eta_t \in A\}$ and $\overline{\sigma} = \sigma_{\overline{D}}$. It suffices to prove that

$$P^z(\tau_y \leq l,\ \overline{\sigma} > \tau_y) \geq c(l)T^{-\iota}, \tag{4.7}$$

for $z \in D(\varrho/2), y \in D(\varrho(1 - T^{-\iota}))$, where $c(l)$ does not depend on z, y, T. We first prove that, for $0 < r \leq \varrho$, $a > 0$,

$$P^z\big(\tau_0 < a\sqrt{r},\ \sigma_{D(r)} > \tau_0\big) \geq c(a)(\log rT)^{-1}, \tag{4.8}$$

for $z \in D(2r/3)$, where $c(l) > 0$. We write the l.h.s. in terms of the random walk $\widetilde{\eta}_s$ which is η_s, but killed when leaving $D(r)$. The corresponding hitting time of 0 is denoted by $\widetilde{\tau}_0$. Then

$$E^z\bigg(\int_0^{l\sqrt{r}} 1_0(\widetilde{\eta}_s)\, ds\bigg) \leq P^z\big(\widetilde{\tau}_0 < l\sqrt{r}\big)E^0\bigg(\int_0^{l\sqrt{r}} 1_0(\widetilde{\eta}_s)\, ds\bigg)$$

$$\begin{aligned} P^z\big(\widetilde{\tau}_0 < l\sqrt{r}\big) &\geq \int_0^{l\sqrt{r}} P^z(\widetilde{\eta}_s = 0)\, ds \bigg/ \int_0^{l\sqrt{r}} P^0(\widetilde{\eta}_s = 0)\, ds \\ &\geq \int_0^{l\sqrt{r}} P^z(\widetilde{\eta}_s = 0)\, ds \bigg/ \int_0^{l\sqrt{r}} P^0(\eta_s = 0)\, ds; \end{aligned}$$

$$P^z(\widetilde{\eta}_s = 0) = P^z(\eta_s = 0)P^z\big(\sigma_{D(r)} > s \mid \eta_s = 0\big).$$

However, $P^z(\sigma_{D(r)} > s \mid \eta_s = 0) \geq c(l) > 0$ for $z \in D(2r/3)$, $s \leq l\sqrt{r}$, and therefore

$$P^z(\widetilde{\tau}_0 < l\sqrt{r}) \geq c(l) \int_0^{l\sqrt{r}} P^z(\eta_s = 0)\, ds \Big/ \int_0^{l\sqrt{r}} P^0(\eta_s = 0)\, ds \\ \geq c_1(l)/\log(rT) \vee 1).$$

This proves (4.8). Relation (4.7) is an easy consequence of (4.8). Applying (4.8) to $r = 3\varrho/4$ and $a\sqrt{r} = l/2$, we see that it suffices to prove (4.7) for $z = 0$. Starting at 0, we see that

$$P^0\big(\sigma_{D(|y|)} \leq l\big) \geq c(l) > 0,$$

uniformly in T and $y \in D(\varrho)$. However, with probability $\geq cT^{-\iota}$, the first exit place from $D(|y|)$ is within distance $\frac{1}{4}T^{-\iota}$ from y. If $\tau' \equiv \tau_{D_y}(T^{-\iota}/4)$, we therefore have

$$P^0(\tau' \leq l,\ \tau' < \overline{\sigma}) \geq c(l)T^{-\iota}. \tag{4.9}$$

Using (4.8) again, we see that with probability $\geq c(l)/\log T$ the random walk, starting from a boundary point of $D_y(\frac{1}{4}T^{-\iota})$, hits y in time less than or equal to l before exiting $D_y(\frac{1}{2}T^{-\iota})$. Combining this with (4.9) proves (4.7). □

5. Exclusion of excursions and proof of Theorem 1.2. We have proved so far that in dimension $d = 2$ with $\widehat{P}_T$-probability close to 1, there exists a disc D of radius slightly less than ϱ such that all points in $D \cap \mathbb{L}_T^{(R)}$ are visited by η at least once. More precisely, for $\iota > 0$, let

$$A^0_{x,T,\iota} = \Big\{ s(l_T) \supset \widetilde{D}_x(\iota),\ \#\big(s(l_T) \cap \widetilde{D}_x(\iota)^c\big) \leq T^{d/(2+d)-\iota},\ l_T\big(\widetilde{D}_x(\iota)^c\big) \leq T^{d/(2+d)-\iota} \Big\},$$

where $\widetilde{D}_x(\iota) = D_x(\varrho(1 - T^{-\iota}))$. We usually drop ι and just write $\widetilde{D}_x$.

Propositions 3.2 and 4.1 imply that for some $\iota > 0$ we have, for $d = 2$,

$$\widehat{\Rightarrow} \underline{A}^0. \tag{5.1}$$

Let $\partial\widetilde{D}_x$ be the set of points in $\widetilde{D}_x$ which have at least one neighbor which belongs to the complement $\widetilde{D}_x^c$.

We prove in this section that there are actually, with $\widehat{P}_T$-probability close to 1, no excursions from the disc. In fact, we will prove that, for any $r > \varrho$,

$$\underline{A}^0 \rightsquigarrow \underline{B}, \tag{5.2}$$

where $B_{x,T} = \{D_x(r) \supset s(l_T)\}$. This easily proves Theorem 1.2 (see the lines following the statement of Proposition 5.1). Although (5.1) is proved only for dimension $d = 2$, we prove (5.2) here in any dimension. A full proof of the localization in any dimension therefore depends only on Lemma 3.1 and Proposition 4.1.

At some places it is convenient to switch to a discrete time random walk.

Let $\xi_0 = 0, \xi_1, \xi_2, \dots$ be an ordinary symmetric nearest neighborhood random walk on $\mathbb{L}_T^{(R)}$, and let $\zeta_1, \zeta_2, \dots$ be i.i.d. exponentially distributed random variables with mean $(1/d)T^{-2/(2+d)}$ which are independent of the ξ variables. If $t > 0$, let

$$\sigma(t) = \sup\left\{m: \sum_{j=1}^{m} \zeta_j \le t\right\}.$$

We may assume that $\eta_t = \xi_{\sigma(t)}$.

If $n \in \mathbb{N}$, $x \in \mathbb{L}$, let

$$\lambda_n(x) = \sum_{k=0}^{n} 1_x(\xi_k).$$

By standard estimates for sums of i.i.d. random variables, we know that, for $0 < \varepsilon < \frac{1}{2}$,

$$P\big(|\sigma - dT| \ge T^{1/2+\varepsilon}\big) \le \exp(-cT^{2\varepsilon}),$$

for some $c > 0$, where $\sigma = \sigma(T^{d/(2+d)})$. So we see that if

$$A^1_{x,T,\iota} = A^0_{x,T,\iota} \cap \big\{|\sigma - dT| \le T^{1-\iota}\big\},$$

then, for small enough $\iota > 0$,

$$\underline{A}^0 \rightsquigarrow \underline{A}^1. \tag{5.3}$$

If $r > \varrho$, we denote by $d_x(r)$ the total number of discrete time points spent outside $D_x(r)$,

$$d_x(r) = \lambda_\sigma\big(D_x(r)^c\big).$$

One easily gets

$$\underline{A}^1 \rightsquigarrow \Big\{d_\cdot\big(\varrho(1 - T^{-\iota})\big) \le T^{1-\iota/2}\Big\}. \tag{5.4}$$

To see this, let $\widetilde{\varrho} = (1 - T^{-\iota})\varrho$. If one conditions on $\sigma = k, dT(1 - T^{-\iota}) \le k \le 2T(1 + T^{-\iota})$, and the ξ path, then $\zeta_1, \zeta_2, \dots, \zeta_k, \tau - \sum_{j=1}^k \zeta_j$, where $\tau = T^{d/(2+d)}$, are just the spacings of the order statistics of k uniformly chosen random variables on $[0, \tau]$. Fixing σ and the ξ path fixes N_T, $s(l_T)$ and $d(\widetilde{\varrho})$. If $d(\widetilde{\varrho}) \ge T^{1-\iota/2}$, the conditional probability (given $\sigma = k$ and ξ) of the event $\{l_T(D_x(\widetilde{\varrho})^c) \le \tau T^{-\iota}\}$ is just the probability that $d(\widetilde{\varrho})$ fixed spacings sum up to a length less than or equal to $\tau T^{-\iota}$. That is easily seen to be $O(T^{-m})$ for any m. This implies (5.4).

Let $A_{x,T} = A^1_{x,T} \cap \{d_x(\widetilde{\varrho}) \le T^{1-\iota/2}\}$ and $B_{x,T} = \{D_x(r) \supset s(l_T)\}$.

Summarizing these simple considerations, we have obtained

$$\underline{A}^0 \rightsquigarrow \underline{A},$$

and together with (5.1) we have, for $d = 2$ and small enough ι,

$$\widehat{\Rrightarrow}\underline{A}.$$

Here $(A_{x,T,\iota})$ is the collection of events with which we start now with the most delicate part of the argument, namely, proving that no excursions from the droplet can occur (in the limit $T \to \infty$, of course). Let us repeat what kind of an event $A_{x,T,\iota}$ is: A path (on our discrete rescaled torus) is in $A_{x,T,\iota}$ if a ball with center x and radius slightly less than ϱ_d is completely covered and outside this ball, both the total time spent and the number of points visited are smaller than $T^{d/(2+d)}$ by a factor of at least $T^{-\iota}$, where $T^{d/(2+d)}$ is the value for the free random walk. Furthermore, the same type of inequalities is true for the imbedded discrete time walk.

The hard task is to prove the following proposition.

PROPOSITION 5.1.

(a) *If* $r > \varrho$, *then for sufficiently small* $\iota > 0$, $\underline{A} \rightsquigarrow \underline{B}$.
(b) *For* $d = 2$ *and sufficiently small* ι, *we have* $\widehat{\Rrightarrow}\underline{A} \cap \underline{B}$.

Part (b) follows from (5.1), (5.3), (5.4) and part (a). The main task of this section is to prove (a).

Everything up to now was done on a torus $\mathbb{L}_T^{(R)}$ for a fixed $R > 0$ $(R = 10\varrho)$. However, having proved Proposition 5.1, it is fairly obvious that one has proved the main theorem.

PROOF OF THEOREM 1.2. One has to check that if Proposition 5.1 holds for one R, then it holds true for $R = \infty$, that is, for the random walk on $\mathbb{L}_T^{(\infty)} = T^{-1/(2+d)}\mathbb{Z}^d$. To see this, note that

$$E_T^{(R)}\left(e^{-N_T};\ \bigcup_{x \in L_T^{(R)}} A_{x,T} \cap \{D_x(r) \supset s(l_T)\}\right)$$
$$= E_T^{(\infty)}\left(e^{-N_T};\ \bigcup_{x \in L_T^{(\infty)}} A_{x,T} \cap \{D_x(r) \supset s(l_T)\}\right),$$

where $E_T^{(R)}$ denotes expectation for the random walk on $\mathbb{L}_T^{(R)}$. Using this together with the obvious inequality

$$E_T^{(R)}(e^{-N_T}) \geq E_T^{(\infty)}(e^{-N_T}),$$

the theorem follows. □

We now switch back to the situation on the torus (with $R = 10\varrho$) and start with the proof of Proposition 5.1(a), which will follow from a number of intermediate results (Propositions 5.2–5.5).

Let us introduce some notation:

We use $\widetilde{D}_x(\iota)$ for $D_x(\varrho(1-T^{-\iota}))$ and often just write $\widetilde{D}_x$. An *excursion from* $\widetilde{D}_x$ is a portion $\xi_u, \xi_{u+1}, \ldots, \xi_{v-1}, \xi_v$ of our discrete time random walk $\xi_0, \xi_1, \ldots, \xi_\sigma$ with $\xi_{u+1}, \ldots, \xi_{v-1} \notin \widetilde{D}_x$, ($\xi_u \in \partial\widetilde{D}_x$ or $u = 0$) and ($\xi_v \in \partial\widetilde{D}_x$ or $v = \sigma$). We say that the excursion is on the interval $[u, v]$; $[u+1, v-1]$ is called the interior of the interval. The excursion might have an endpoint not in $\partial\widetilde{D}_x$ but in $\widetilde{D}_x^c$, but only if the excursion starts at 0 or ends at σ. We also speak just of an excursion if it is clear which x is considered. We call $v - u$ the *length* of the excursion. We say that the excursion *reaches* $r > \varrho$ if for some $j \in [u, v] = \{u, u+1, \ldots, v-1, v\}$ one has $\xi_j \notin D_x(r)$.

Our main method for proving the proposition is to compare the probability for certain types of excursion with the probability that the path would stay completely inside $\widetilde{D}_x$.

If $y_1, y_2 \in \widetilde{D}_x, m \in \mathbb{N}$, let $q_m(y_1, y_2)$ be the conditional probability that a path $\xi_0, \ldots, \xi_m$ stays inside $\widetilde{D}_x$ given that $\xi_0 = y_1$ and $\xi_m = y_2$. We need this only if the probability of the latter is not 0. Similarly $q_m(y)$ is defined just as $P(\xi_1, \ldots, \xi_m \in \widetilde{D}_x | \xi_0 = y)$. The probability $q_m(y_1, y_2)$ may be small either because m is large or because $T^{1/(2+d)}|y_2 - y_1|$ is small compared with $\sqrt{m}$. It is, of course, also small because the endpoints y_1 and y_2 may be on the boundary, but this effect will be negligible (only of polynomial order in m). Let $p_m(y_1, y_2) = P(\xi_m = y_2 \mid \xi_0 = y_1)$. In order to avoid repetitions of trivial assumptions, we tacitly always make the following assumptions: We will consider quantities like $p_m(y_1, y_2)$ and $q_m((y_1, y_2)$ only when y_1 and y_2 are in or close to $D_x(\varrho)$, say, inside $D_x(3\varrho/2)$. We will consider them only if there is a path of length m joining y_1 and y_2 which stays inside $D_x(3\varrho/2)$. For such y_1, y_2 we calculate $y_1 - y_2$ by identifying $\mathbb{L}_T^{(R)}$ with a subset of $T^{-1/(2+d)}\mathbb{Z}^d \subset \mathbb{R}^d$ putting x at 0 in this lattice.

Let $h: \mathbb{R}^d \to [0, \infty]$ be the entropy function of our nearest neighbor random walk:

$$h(x) = \sup_{\lambda \in R^d} \left[\langle \lambda, x \rangle - \log \left(\frac{1}{d} \sum_{j=1}^{d} \cosh(\lambda_j) \right) \right];$$

$h(x)$ is finite if and only if $\Sigma|x_i| \leq 1$ and is C^∞ in the interior. We use standard saddle point approximations for $p_m(y_1, y_2)$ and $q_m(y_1, y_2)$. The following results are sufficient for our purpose.

LEMMA 5.1. *There exists $c > 0$ such that, for some $\alpha > 0$,*

$$cm^{-\alpha} \exp\left(-mh\left(\frac{T^{1/(2+d)}(y_1 - y_2)}{m} \right) \right) \leq p_m(y_1, y_2) \leq \exp\left(-mh\left(\frac{T^{1/(2+d)}(y_1 - y_2)}{m} \right) \right)$$

[*of course, only if $p_m(y_1, y_2) \neq 0$, which we tacitly assume*].

LEMMA 5.2. *There exist $k \in \mathbb{N}$ and constants $c_1, c_2 > 0$ such that, for $m \in \mathbb{N}$, we have*

$$q_m(y_1, y_2) \geq c_1 m^{-k} \exp\left(-c_2 m T^{-2/(2+d)}\right),$$

for all $y_1, y_2 \in D_x(r)$, $\varrho/2 \leq r \leq 3\varrho/2$.

We will prove these two lemmas in Appendix B.

Let $n_x(r)$ be the total number of excursions which reach r, and if $a \in \mathbb{R}$, $n_x^{\leq}(r, a)$ $[n_x^{\geq}(r, a)]$ is the number of excursions which reach r and have length less than or equal to T^a [length greater than or equal to T^a]. We first show that no short excursions occur.

PROPOSITION 5.2. *If $r > \varrho$ and $a_0 < 2/(2+d)$, then*

$$\underline{A} \rightsquigarrow \left\{n_\cdot^{\leq}(r, a_0) = 0\right\}.$$

PROOF. We split the event $A_x \cap \{n_x^{\leq}(r, a_0) \neq 0\}$ into disjoint pieces. Let $\mathbb{U}$ be a nonvoid collection of intervals $[u, v]$ in $\mathbb{N}_0$ with $1 \leq v - u \leq T^{a_0}$ and disjoint interiors.

Let $\underline{m}(\mathbb{U}) = \min\{u: [u, v] \in \mathbb{U}\}, \overline{m}(\mathbb{U}) = \max\{v: [u, v] \in \mathbb{U}\}$. We denote by $C_x(\mathbb{U})$ the event that $\sigma \geq \overline{m}(\mathbb{U})$ and that the excursions to r of length less than equal to T^{a_0} occur exactly on these intervals.

Let $N_T(\mathbb{U})$ be the number of points visited by the ξ-path for time points outside the interiors of the intervals of $\mathbb{U}$ (up to time σ). Obviously

$$N_T(\mathbb{U}) \leq N_T. \tag{5.1}$$

We denote by $\overline{S}_T(\mathbb{U})$ the set of points visited by the walk ξ_k for $k \notin \operatorname{int}(\mathbb{U})$, where $\operatorname{int}(\mathbb{U})$ is the union of the interiors of intervals in $\mathbb{U}$.

Let

$$\overline{A}_x = \left\{\overline{S}_T(\mathbb{U}) \supset \widetilde{D}_x\right\} \cap \left\{\sigma \geq \overline{m}(\mathbb{U})\right\}.$$

Clearly

$$A_x \cap C_x(\mathbb{U}) \subset \overline{A}_x \cap C_x^0(\mathbb{U}), \tag{5.2}$$

where $C_x^0(\mathbb{U})$ is the event that on all the intervals in $\mathbb{U}$ there is an excursion which reaches r and is of length less than or equal to T^{a_0} (but there may be others).

Let $\mathcal{F}(\mathbb{U})$ be the σ-field generated by all the ζ variables, by $\sigma(\xi_k) \cap \{\sigma \geq \overline{m}(\mathbb{U})\}$ for $k \notin \operatorname{int}(\mathbb{U}) \cup \{\overline{m}(\mathbb{U})\}$ and by $\sigma(\xi_{\overline{m}(\mathbb{U})}) \cap \{\sigma \geq \overline{m}(\mathbb{U})\}$.

By (5.5) and (5.6) we have

$$E\left(e^{-N_T}; A_x \cap C_x(\mathbb{U})\right) \leq E\left(e^{-N_T(\mathbb{U})} P\left(C_x^0(\mathbb{U}) \mid \mathcal{F}(\mathbb{U})\right); \overline{A}_x\right). \tag{5.3}$$

We assume for the moment that $\underline{m}(\mathbb{U}) \neq 0$ or $\underline{m}(\mathbb{U}) = 0$ and $0 \in \partial\widetilde{D}_x$.

We estimate the conditional probability. On $C_x^0(\mathbb{U})$ we have $\xi_u, \xi_v \in \partial\widetilde{D}_x$ for $[u, v] \in \mathbb{U}$ except possibly on $\overline{m}(\mathbb{U}) = \sigma$, where ξ_v for the last interval may belong to $\widetilde{D}_x^c$.

Let $[\overline{u}, \overline{v}]$ be the last interval in $\mathbb{U}$, and let $\mathbb{U}_1 = \mathbb{U} \setminus \{[\overline{u}, \overline{v}]\}$. Set

$$L_1(\mathbb{U}) = \{\sigma > \overline{m}(\mathbb{U})\} \cap \left[\bigcap_{[u,v]\in\mathbb{U}} \{\xi_u, \xi_v \in \partial\widetilde{D}_x\} \right],$$

$$L_2(\mathbb{U}) = \{\sigma = \overline{m}(\mathbb{U})\} \cap \left[\bigcap_{[u,v]\in\mathbb{U}_1} \{\xi_u, \xi_v \in \partial\widetilde{D}_x\} \cap \{\xi_{\overline{u}} \in \partial\widetilde{D}_x\} \right].$$

Note that $L_1(\mathbb{U})$ and $L_2(\mathbb{U})$ belong to $\mathcal{F}(\mathbb{U})$;

$$\begin{aligned} &P\big(C_x^0(\mathbb{U}) \mid \mathcal{F}(\mathbb{U})\big) \\ &\quad \le 1_{L_1(\mathbb{U})} \prod_{[u,v]\in\mathbb{U}} \overline{q}_{v-u}(\xi_u, \xi_v) + 1_{L_2(\mathbb{U})} \prod_{[u,v]\in\mathbb{U}_1} \overline{q}_{v-u}(\xi_u, \xi_v)\overline{q}_{\overline{v}-\overline{u}}(\xi_{\overline{u}}), \end{aligned} \tag{5.4}$$

where, for $y_1, y_2 \in \partial\widetilde{D}_x \cup \widetilde{D}_x^c$,

$$\overline{q}_m(y_1, y_2) = P\left(\xi_1, \ldots, \xi_{m-1} \notin \widetilde{D}_x, \bigcup_{j=1}^{m-1} \{\xi_j \notin D_x(r)\} \,\middle|\, \xi_0 = y_1, \xi_m = y_2 \right),$$

$$\overline{q}(y_1) = P\left(\xi_1, \ldots, \xi_{m-1} \notin \widetilde{D}_x, \bigcup_{j=1}^{m-1} \{\xi_j \notin D_x(r)\} \,\middle|\, \xi_0 = y_1 \right).$$

LEMMA 5.3. *If $a_0 < 2/(2+d)$, then there exists $\delta > 0$ such that for $r > \varrho$ there are constants $c_1, c_2 > 0$ such that, for $y_1, y_2 \in \partial\widetilde{D}_x, m \le T^{a_0}$,*

$$\text{(a)} \qquad \overline{q}_m(y_1, y_2) \le c_1 \exp(-c_2 T^\delta),$$

and, for all $y \in \partial\widetilde{D}_x$,

$$\text{(b)} \qquad \overline{q}_m(y) \le c_1 \exp(-c_2 T^\delta).$$

Lemma 5.3 is a straightforward consequence of Lemma 5.1 by using the trivial estimate

$$\overline{q}_m(y_1, y_2) \le \sum_{z \notin D_x(r)} \sum_{m'=1}^{m} p_{m'}(y_1, z) p_{m-m'}(z, y_2)/p_m(y_1, y_2),$$

and similarly for $\overline{q}_m(y)$.

We denote by $C_x^{\text{in}}(\mathbb{U})$ the event that $\xi_j \in \widetilde{D}_x$ for all $j \in \operatorname{int}\mathbb{U}$. Using (5.8), Lemma 5.2 and Lemma 5.3 for comparing $\overline{q}_m$ with q_m, we obtain

$$P\big(C_x^0(\mathbb{U}) \mid \mathcal{F}(\mathbb{U})\big) \le c_1^{|\mathbb{U}|} \exp\big(-c_2|\mathbb{U}|T^\delta\big) P\big(C_x^{\text{in}}(\mathbb{U}) \mid \mathcal{F}(\mathbb{U})\big),$$

for some $c_1, c_2, \delta > 0$. Using the fact that $N_T(\mathbb{U}) = N_T$ on $\overline{A}_x \cap C_x^{\text{in}}(\mathbb{U})$, we obtain

$$(5.9)\qquad E\big(\exp(-N_T);\, A_x \cap C_x(\mathbb{U})\big) \le c_1^{|\mathbb{U}|} \exp\big(-c_2|\mathbb{U}|T^\delta\big) z_T.$$

Up to now we have assumed that $0 \ne \underline{m}(\mathbb{U})$ or $0 \in \partial\widetilde{D}_x$. The argument needs a slight modification for $\underline{m}(\mathbb{U}) = 0$ and $0 \notin \widetilde{D}_x$.

Let $\widetilde{v}$ be the endpoint of the interval of $\mathbb{U}$ with starting point 0. The estimate (5.8) remains correct with two small modifications. In the definition of L_1 and L_2 we only intersect with $\{\xi_{\widetilde{v}} \in \partial\widetilde{D}_x\}$ for the first interval of $\mathbb{U}$. Also, the factors for the first interval in $\mathbb{U}$ on the right-hand side of (5.8) have to be replaced by $\overline{q}_{\widetilde{v}}(0, \xi_{\widetilde{v}})$.

We consider

$$E\big(e^{-N_T};\, A_x \cap C_x(\mathbb{U}) \cap \{\xi_{\widetilde{v}} = y\}\big)$$

and then sum over all $y \in \partial\widetilde{D}_x$. This summation will give an additional factor of order $T^{(d-1)/(d+2)}$. Of course, we cannot compare $\overline{q}_m(0, y)$ with $q_m(0, y)$ because the latter is 0 when is not a nearest neighbor of $\widetilde{D}_x$. However, we can compare $\overline{q}_m(0, y)$ with $q_m(y, y)$, the latter being of polynomial order in m. Using the translation invariance of the problem, we get (5.9), too, with an additional factor $T^{(d-1)/(d+2)}$, which is harmless if we decrease $\delta > 0$ slightly. Now, summing (5.9) over all possible choices of $\mathbb{U}$, we get the result. □

Proposition 5.2 is proved by a rather crude argument. We have excluded some types of excursions without taking into account their contribution to N_T. This was possible because these excursions themselves (under the measure P) were sufficiently improbable. It is clear that part of the way of proving Proposition 5.1 must depend on finer arguments. The next two propositions, however, can still be proved in the same crude way as Proposition 5.2. We will not give the proofs of these two propositions, which can be taken nearly word for word from that of Proposition 5.2 by using Lemmas 5.4 and 5.5 stated below. To state the first proposition, we consider two numbers $s > s' > \varrho$. We say that an excursion $\xi_u, \xi_{u+1}, \ldots, \xi_v$ which reaches s has an *a-quick return* to $s', a \in \mathbb{R}$, if for some i, j with $u < i < j \le v$, $j - i \le T^a$, we have $\xi_i \notin D_x(s')$, $\xi_j \in D_x(s)$.

PROPOSITION 5.3. *If $a < 2/(2+d)$, then there exists $b > 2/(2+d)$ such that for all $s > s' > \varrho$ one has*

$\underline{A} \rightsquigarrow$ *there are no excursions of length less than or equal to T^b which reach s and have an a-quick return to s'.*

To formulate the lemma which is needed instead of Lemma 5.3, we consider for $y_1, y_2 \in \partial\widetilde{D}_x$, $m \in \mathbb{N}$, the probability $q'_m(y_1, y_2)$ that a path $\xi_0, \ldots, \xi_m$ stays strictly outside $\widetilde{D}_x$, reaches s and has an a-quick return to s' conditioned on the event that $\xi_0 = y_1$, $\xi_m = y_2$. For $q'_m(y_1)$, one fixes only the starting point y_1.

LEMMA 5.4. *If $a < 2/(2+d)$, there exists $\delta > 0$ such that, for all $s' > s > \varrho$ there exist constants $c_1, c_2 > 0$ with*

(a) $q'_m(y_1, y_2) \le c_1 \exp(-c_2 T^\delta)$ *for all* $y_1, y_2 \in \partial\widetilde{D}_x$, $m \le 3T$,
(b) $q'_m(y) \le c_1 \exp(-c_2 T^\delta)$ *for all* $y \in \partial\widetilde{D}_x$, $m \le 3T$.

For the proof of (a), we use the obvious estimate

$$q'_m(y_1, y_2) \le \frac{\sum_{\substack{1\le m_1<m_2<m \\ m_2-m_1\le T^a}} \sum_{z_1\notin D_x(s')} \sum_{z_2\in D_x(s)} \times p_{m_1}(y_1,z_1)p_{m_2-m_1}(z_1,y_2)p_{m-m_2}(z_2,y_2)}{p_m(y_1,y_2)}$$

and apply Lemma 5.1, and similarly for (b).

As noted above, the proof of Proposition 5.3 is the same as that for Proposition 5.2 with the obvious modification in defining $C_x(\mathbb{U})$ and by using Lemmas 5.2 and 5.4.

PROPOSITION 5.4. *Given $b_0 > 2/(2+d)$ there exists $r_0 > \varrho$ such that*

$$\underline{A} \rightsquigarrow \text{an excursion of length greater than or equal to } T^{b_0} \text{ spends at least half of the time in } D_x(r_0)^c.$$

Let $q''_m(y_1, y_2)$ be the conditional probability given $\xi_0 = y_1$ and $\xi_m = y_2$ of the event that a random walk $\xi_0, \dots, \xi_m$ stays outside $\widetilde{D}_x$ and stays inside $D_x(r_0)$ for at most $m/2$ time points j, $1 \le j \le m-1$; $q''_m(y)$ is defined analogously.

LEMMA 5.5. *For any $K > 0$ and any $b_0 > 2/(2+d)$ there exists $r_0 > \varrho$ such that the following hold*:

(a) *For any* $y_1, y_2 \in \partial\widetilde{D}_x$, $T^{b_0} \le m$,

$$q''_m(y_1, y_2) \le \exp(-KmT^{-2/(2+d)}) \quad \text{for large } T.$$

(b) *For any* $y \in \partial\widetilde{D}_x$, $T^{b_0} \le m$,

$$q''_m(y) \le c\exp(-KmT^{-2/(2+d)})m^k \quad \text{for large } T.$$

The lemma is fairly obvious. We give a sketch of the proof. As $p_m(y_1, y_2)$ is of order $m^{-d/2}$, one needs to prove only (b). If we drop the condition that the walk stays outside $\widetilde{D}_x$ all the time, this gives an upper bound. So we have to estimate the probability that a random walk (on our torus) of length m spends at least $m/2$ time points in $D_x(r_0) \setminus \widetilde{D}_x$. By the usual rough large deviation estimates, this is (logarithmically) equivalent to

$$\exp\Big(-\frac{m}{T^{2/(2+d)}}\alpha(r_0)\Big),$$

where $\alpha(r_0) = \inf\{I(\mu): \mu(D_x(r_0)\setminus D_x(\varrho_d)) \geq \frac{1}{2}\}$. However, obviously $\alpha(r_0) \to \infty$ as $r_0 \downarrow \varrho_d$.

Using this, the proof of Proposition 5.4 again is an easy modification of that of Proposition 5.2 by using the above lemma together with Lemma 5.2(a) and (b).

The point is, of course, that we may take K as large as we like by choosing $r_0 > \varrho$ small. So the q''_m are much smaller than the q_m even for large m.

To proceed further we need some information on how many points in $\mathbb{L}_T^{(R)}$ are visited by independent random walks.

Let $\xi^1, \dots, \xi^n$ be n independent discrete time random walks on $\mathbb{L}_T^{(R)}$ which start at $x_1, \dots, x_n$ and have lengths $l_1, \dots, l_n$. Let $N(\xi^1, \dots, \xi^n)$ be the number of points visited by all these walks.

LEMMA 5.6. *Let $f < 1$. Then if $a < 2/(2+d)$ is sufficiently close to $2/(2+d)$, one has for any $\varepsilon > 0$*

$$E\Big(\exp\big(-N(\xi^1, \dots, \xi^n)\big)\Big) \leq \exp\left(-T^{-\varepsilon}\Big(\sum_i l_i\Big)^{d/(2+d)}\right)$$

if $l_i \geq T^a$, $\sum_i l_i \leq T^f$, for all x_i and for large enough T.

The proof of this will be given in Appendix B.

We define

(5.10) $$\widetilde{A}_{x,T}(\iota, r, a_0, a, b, s, s', b_0, r_0)$$

to be $A_{x,T,\iota}$ intersected with the right-hand sides in the pseudoimplications in Propositions 5.2–5.4. We specify now the bunch of parameters depending on $\iota > 0$.

We take $f = 1 - \iota/4$ in Lemma 5.6 and choose $a < 2/(2+d)$ at least as close to $2/(2+d)$ to be sufficiently close in the sense of this lemma. We choose $\varepsilon, \delta, \kappa > 0$ (close to 0), $a, a_0, a_1 < 2/(2+d)$ and $b = b_0 > 2/(2+d)$ [close to $2/(2+d)$] with the following requirements: for $T^{a_1} \leq m \leq 2T$ we have

(5.11) $$\inf_{y_1, y_2 \in \partial\tilde{D}_x} p_m(y_1, y_2) \geq \exp\big(-cT^{\delta}\big) \quad \text{(use Lemma 5.1);}$$

(5.12) $$\delta + a < \frac{2}{(2+d)};$$

(5.13) $$1 - \frac{\iota}{2} + \frac{(2+d)(\varepsilon + \delta)}{2} < \frac{(2+d)a}{2};$$

for any $\alpha \in [0, d/(2+d) - \iota/4]$,

$$b - \frac{2}{2+d} + \alpha < -\varepsilon + \frac{d}{2+d}(\alpha - \kappa + a), \tag{5.14}$$

$$\kappa + 2a < \frac{4}{2+d}, \tag{5.15}$$

$$a_0 > \max\left(a, a_1, \frac{2}{2+d} - \frac{\iota}{16}\right). \tag{5.16}$$

We select $r_0 > \varrho$ according to Proposition 5.4. We are still free to choose r, s' and s in (1), which are restricted by requiring

$$\varrho < r = s' < s \leq r_0. \tag{5.17}$$

The outcome of these specifications is a set $\widetilde{A}$ which depends on x, T, s', s and ι and is denoted by $\widetilde{A}_x(s', s)$ (as usual, we drop the dependence on T and ι). From Propositions 5.2–5.4 we get

$$\underline{A} \rightsquigarrow \underline{\widetilde{A}}(s', s). \tag{5.18}$$

We denote by $\nu_x(r)$ the number of excursions to r of our discrete time random walk. Note that on $\widetilde{A}_x(s', s)$ each excursion to s' has length greater than or equal to T^{a_0} and as the time spent outside $\widetilde{D}_x$ is at most $T^{1-\iota/2}$ [see (5.4) and the definition of $\underline{A}$ just above], we get the following on $\widetilde{A}_x(s', s)$ [by (5.16)]:

$$\nu_x(s') \leq T^{d/(d+2)-\iota/4}. \tag{5.19}$$

PROPOSITION 5.5. *If* $0 < \alpha \leq d/(2+d) - \iota/4$, *then with the parameters from above*

$$\underline{\widetilde{A}}_\cdot(s', s) \cap \{\nu_\cdot(s') \leq T^\alpha\} \rightsquigarrow \{\nu_\cdot(s) \leq T^{\alpha-\kappa}\}.$$

Proposition 5.5 clearly implies Proposition 5.1(a). In fact, we just select $\alpha_0 = 2/(2+d) - \iota/4 > \alpha_1 > \alpha_2 > \cdots > \alpha_{k-1} > 0 > \alpha_k$ with $\alpha_j - \alpha_{j-1} \leq \kappa, 1 \leq j \leq k$, and for any $r > \varrho$ choose $\varrho < s_0 < s_1 < \cdots < s_k = r \wedge r_0$ (r_0 as specified above). Then from Proposition 5.5 and (5.18) we recursively get

$$\underline{A} \rightsquigarrow \underline{\tilde{A}}(s_{j-1}, s_j) \cap \{\nu_\cdot(s_j) \leq T^{\alpha_j}\},$$

which particularly for $j = k$ gives

$$\underline{A} \rightsquigarrow \underline{\widetilde{A}}(s_{k-1}, s_k) \cap \{\nu_\cdot(r) = 0\},$$

which proves Proposition 5.1.

We now start the proof of Proposition 5.5 by introducing some notation. If $\xi_u, \xi_{u+1}, \dots, \xi_v$ is an excursion to s, let

$$\sigma_1 = \min\{n > u: \xi_n \notin D_x(s)\},$$
$$\tau_1 = \min\{n > \sigma_1: \xi_n \in D_x(s')\},$$
$$\sigma_2 = \min\{n > \tau_1: \xi_n \notin D_x(s)\} \quad \text{and so on.}$$

We call the pieces $\xi_{\sigma_k}, \xi_{\sigma_k+1}, \dots, \xi_{\tau_k}$ the *returns* of the excursion.

We introduce events analogously to $C(\mathbb{U})$ in Proposition 5.2 which essentially specify the places of excursions to s', to s and the returns of the latters.

Let $\mathbb{U}'$, $\mathbb{U}$ and $\mathbb{V}$ be nonvoid sets of intervals with disjoint interiors. We assume the following:

1. $\mathbb{U}' \supset \mathbb{U}$; each $U \in \mathbb{U}'$ of length greater than or equal to T^b is contained in $\mathbb{U}$.
2. For each $V \in \mathbb{V}$ there exists $U \in \mathbb{U}$ with $V \subset U$, and for each $U \in \mathbb{U}$ there exists $V \in \mathbb{V}$ with $V \subset U$.
3. The length of the intervals in $\mathbb{U}'$ is greater than or equal to T^{a_0}.
4. If $U \in \mathbb{U}$ has length greater than or equal to T^b, then the total length of the intervals in $\mathbb{V}$ contained in U is greater than or equal to $|U|/2$.
5. If the length of $U \in \mathbb{U}$ is $< T^b$, then all intervals in $\mathbb{V}$ which are contained in U have length greater than or equal to T^a.

Given such a triple, we consider the event $C_x(\mathbb{U}', \mathbb{U}, \mathbb{V})$ specified by the following conditions:

(a) $\sigma \geq \overline{m}(\mathbb{U}')$.

(b) The excursions to s' occur exactly in the intervals of $\mathbb{U}'$, and those to s in $\mathbb{U}$.

(c) The returns of the excursions to s occur exactly in the intervals in $\mathbb{V}$.

Let $\widetilde{A}_{x,T}(s', s)$ be the disjoint union of its intersection with the $C_x(\mathbb{U}', \mathbb{U}, \mathbb{V})$.

We also consider $C_x^0(\mathbb{U}', \mathbb{U}, \mathbb{V})$, where we change (b) and (c) by requiring that, in the intervals in $\mathbb{U}', \mathbb{U}$ and $\mathbb{V}$, excursions and returns of the prescribed type occur; that is, we drop the word "exactly." (The C_x^0 are no longer disjoint.)

Before proceeding further, let me explain why this somewhat complicated procedure appears to be necessary. The main aim is, of course, to exclude the possibility that excursions occur. Furthermore, it is clear that we must use the fact that such excursions contribute to N_T. One might start to look at what happens when the excursions, say, to $r > \varrho$, are switched inside $\widetilde{D}_x$ but it is difficult to control how much N_T is decreased in this way, because there might be many excursions which just fail to reach s. To handle this problem one looks at two radii $s' < s$ and at what happens when all excursions to s' are switched inside. Then N_T decreases at least by the contribution of the excursions to s inside the shell $D_x(s) \setminus D_x(s')$ or at least by the returns. The contribution of such returns can be estimated if they are not too short. We have already excluded the possibility of short returns on relatively short excursions (length less than or equal to T^b). Long excursions have already been shown to stay outside s quite long (Proposition 5.4); so if there are no long returns, then there must be very many quick ones, which again is highly improbable. By taking all these effects into account one gets an estimate of the number of excursions to s in terms of the ones to s', which is exactly the content of Proposition 5.5.

We define $N_T(\mathbb{U}')$ as in the proof of Proposition 5.2 and N_T' as the number of points visited by the returns of length greater than or equal to T^α. Clearly

$$N_T \geq N_T(\mathbb{U}') + N_T'. \tag{5.20}$$

We now split the expression

$$E\Big(e^{-N_T}; \widetilde{A}_x(s', s) \cap \{\nu_x(s') \leq T^\alpha\} \cap \{\nu_x(s) > T^{\alpha-\kappa}\}\Big)$$

into its restriction to $C(\mathbb{U}', \mathbb{U}, \mathbb{V})$, where the triples satisfy the conditions above and $|\mathbb{U}'| \leq T^\alpha$, $|\mathbb{U}| > \max(T^{\alpha-\kappa}, 0)$. Let Λ_α be the set of these triples. Then we get, by (5.20),

$$\begin{aligned} &E\Big(e^{-N_T}; \widetilde{A}_x(s', s) \cap \{\nu_x(s') \leq T^\alpha\} \cap \{\nu_x(s) > T^{\alpha-\kappa}\}\Big) \\ &\quad \leq \sum_{(\mathbb{U}', \mathbb{U}, \mathbb{V}) \in \Lambda_\alpha} E\Big(1_{\overline{A}_x(\mathbb{U}')} e^{-N_T(\mathbb{U}')} E\big(e^{-N_T'}; C_x^0(\mathbb{U}', \mathbb{U}, \mathbb{V}) \mid \mathcal{F}(\mathbb{U}')\big)\Big), \end{aligned} \tag{5.21}$$

where $\overline{A}_x(\mathbb{U}')$ is the event that $\overline{m}(\mathbb{U}') \leq \sigma$ and the path outside the interiors of the intervals of $\mathbb{U}'$ completely fills $\widetilde{D}_x$.

We divide $\mathbb{V}$ into two subsets $\mathbb{V}_1$ and $\mathbb{V}_2$, where $\mathbb{V}_1$ contains the intervals of length greater than or equal to T^a and $\mathbb{V}_2$ those of length less than T^α. Let $n_1 = |\mathbb{V}_1|$, $n_2 = |\mathbb{V}_2|$, let the length of the intervals in $\mathbb{V}_1$ be $l_1, l_2, \ldots, l_{n_1}$ and let those in $\mathbb{V}_2$, $\bar{l}_1, \ldots, \bar{l}_{n_2}$. Then we get, by summing over all possible choices of the positions of the path at the left endpoints of the intervals in $\mathbb{V}$,

$$\begin{aligned} &E\big(e^{-N_T'}; C_x^0(\mathbb{U}', \mathbb{U}, \mathbb{V}) \mid \mathcal{F}(\mathbb{U}')\big) \\ &\quad \leq \exp\big(c(n_1+n_2)\log T\big) \left[\prod_{U \in \mathbb{U}} \inf_{y_1, y_2 \in \partial \tilde{D}_x} p_{|U|}(y_1, y_2)\right]^{-1} \\ &\qquad \times \max_{x_1, \ldots, x_{n_1}} E\Big(\exp\Big[-N\big(\xi^{1,x_1}, \ldots, \xi^{n_1, x_{n_1}}\big)\Big]\Big) \times \prod_{V \in \mathbb{V}_2} \pi(|V|). \end{aligned} \tag{5.22}$$

Here

$$\pi(m) = \inf_{y \notin D_x(s)} P^y\big(\xi_m \in D_x(s')\big).$$

Furthermore, $\xi^{1,x_1}, \ldots, \xi^{n_1,x_{n_1}}$ are independent random walks starting at x_i and having length l_i. Applying Lemma 5.6 we get

$$\max_x E\Big(\exp\Big[-N(\xi^1, \ldots, \xi^{n_1})\Big]\Big) \leq \exp\left(-cT^{-\varepsilon}\Big(\sum_i l_i\Big)^{d/(2+d)}\right). \tag{5.23}$$

The $\pi(m)$ can easily been estimated by Lemma 5.1, and we get the following for some constant $c > 0$ (depending on s, s' and also on a, which is fixed): for large enough T and $m < T^{a}$,

$$\pi(m) \le \exp\bigl(-cT^{2/(2+d)}m^{-1}\bigr). \tag{5.24}$$

By (5.12), we have

$$\left[\prod_{U\in\mathbb{U}} \inf_{y_1,y_2\in\partial\tilde{D}_x} p_{|U|}(y_1,y_2)\right]^{-1} \le \exp\bigl(c'(n_1+n_2)T^{\delta}\bigr). \tag{5.25}$$

By (5.12) and (5.13) and $n_1T^{a} \le T^{1-\iota/2}$ one gets, for $n_1+n_2 \ge 1$,

$$(n_1+n_2)T^{\delta} = o\left(\sum_{i=1}^{n_2} T^{2/(2+d)}\overline{l}_i^{-1} + T^{-\varepsilon}\left(\sum_{i=1}^{n_1} l_i\right)^{d/(d+2)}\right) \tag{5.26}$$

as $T \to \infty$.

Combining (5.22)–(5.26) gives, for some $c > 0$,

$$\begin{aligned} &E\Bigl(\exp(-N_T'); \; C_x^0(\mathbb{U}',\mathbb{U},\mathbb{V}) \mid \mathcal{F}(\mathbb{U}')\Bigr) \\ &\quad \le \exp\left(-c\left[\sum_{i=1}^{n_2} T^{2/(2+d)}\overline{l}_i^{-1} + T^{-\varepsilon}\left(\sum_{i=1}^{n_1} l_i\right)^{d/(2+d)}\right]\right). \end{aligned} \tag{5.27}$$

Assuming $\underline{m}(\mathbb{U}') \neq 0$ or $0 \in \tilde{D}_x$, we get from Lemma 5.2, for some c',

$$P\bigl(C_x^{\mathrm{in}}(\mathbb{U}') \mid \mathcal{F}(\mathbb{U}')\bigr) \ge \exp\bigl(-c'(l(\mathbb{U}')T^{-2/(2+d)} + |\mathbb{U}'|\log T)\bigr), \tag{5.28}$$

where $l(\mathbb{U}')$ is the total length of the intervals in $\mathbb{U}'$. Combining (5.21), (5.27), and (5.28) with the fact that $N_T(\mathbb{U}') = N_T$ on $\overline{A}_x(\mathbb{U}') \cap C_x^{\mathrm{in}}(\mathbb{U}')$, we get

$$\begin{aligned} &E\Bigl(\exp(-N_T); \tilde{A}_X(s',s) \cap \{\nu_x(s') \le T^{\alpha}\} \cap \{\nu_x(s) > T^{\alpha-\kappa}\}\Bigr) \\ &\quad \le z_T \sum_{(\mathbb{U}',\mathbb{U},\mathbb{V})\in\Lambda_\alpha} \exp\Biggl\{c'\bigl(T^{-2/(2+d)}l(\mathbb{U}') + |\mathbb{U}'|\log T\bigr) \\ &\qquad\qquad - c\left[\sum_{i=1}^{n_2} T^{2/(2+d)}\overline{l}_i^{-1} + T^{-\varepsilon}\left(\sum_{i=1}^{n_1} l_i\right)^{d/(2+d)}\right]\Biggr\}. \end{aligned} \tag{5.29}$$

By the same slight modification as in the proof of Proposition 5.2, we get this estimate for $\underline{m}(\mathbb{U}') = 0$ and $0 \notin \tilde{D}_x$, too.

We will now show that for a suitable constant $\varepsilon_0 > 0$ the right-hand side of (5.29) is less than or equal to $z_T \exp(-T^{\varepsilon_0})$ for large T.

Let $\lambda^{\geq}$ be the total length of the intervals in $\mathbb{V}$ of length greater than or equal to T^a which are inside intervals $U \in \mathbb{U}$ of length greater than or equal to T^b, and let $\lambda^{<}$ be the total length of those intervals in $\mathbb{V}$ of length less than T^a. Since

$$\sum_{i=1}^{n_2} \bar{l}_i = \lambda^{<},$$

$$T^{d/(2+d)} \sum_{i=1}^{n_2} \bar{l}_i^{\,-1} \geq T^{d/(2+d)} \sum_{i=1}^{n_2} \left(\frac{\lambda^{<}}{n_2}\right)^{-1} \geq T^{d/(2+d)} \lambda^{<} T^{-2a}. \tag{5.30}$$

Furthermore,

$$\sum_{i=1}^{n_1} l_i \geq n^{\leq}(\mathbb{U}) T^a + \lambda^{\geq}, \tag{5.31}$$

where $n^{\leq}(\mathbb{U})$ is the number of intervals in $\mathbb{U}$ of length less than or equal to T^b.

Combining (5.26), (5.30) and (5.31), the right-hand side of (5.29) is less than or equal to

$$z_T \sum_{(\mathbb{U}', \mathbb{U}, \mathbb{V})} \exp\Big(c'\big(T^{-d/(2+d)} l(\mathbb{U}') + |\mathbb{U}'| \log T\big) \\ - c\Big[T^{2/(2+d)} \lambda^{<} T^{-2a} + T^{-\varepsilon}\big(\lambda^{\geq} + n^{\leq}(\mathbb{U}) T^a\big)^{d/(2+d)}\Big]\Big).$$

The total number of summands is dominated by

$$|\Lambda_\alpha| \leq \exp\Big(c\big(|\mathbb{V}| \log T + |\mathbb{U}'| \log T\big)\Big);$$

$|\mathbb{V}| \log T$ is dominated by $T^\delta(n_1 + n_2)$. Therefore our last task is to show that

$$T^{-2/(2+d)} l(\mathbb{U}') + |\mathbb{U}'| \log T \\ = o\Big(T^{2/(2+d)} \lambda^{<} T^{-2a} + T^{-\varepsilon}\big(\lambda^{\geq} + n^{\leq}(\mathbb{U}) T^a\big)^{d/(2+d)}\Big) \tag{5.32}$$

as $T \to \infty$, uniformly on Λ_α. We abbreviate the expression $o(\cdot)$ on the right-hand side by X.

We consider two cases:

$$|\mathbb{U}'| \geq \tfrac{1}{2} T^{-b} l(\mathbb{U}'). \tag{I}$$

Then $T^{-2/(2+d)} l(\mathbb{U}') \leq 2T^{b-2/(2+d)} |\mathbb{U}'|$ and we will dominate the right-hand side of this, which dominates $|\mathbb{U}'| \log T$, so we need not take the latter into account.

As $|\mathbb{U}| \geq T^{\alpha-\kappa}$ on Λ_α we have

$$n^{\leq}(\mathbb{U}) \geq \tfrac{1}{2} T^{\alpha-\kappa} \tag{Ia}$$

or

$$\text{(Ib)} \qquad n^{>}(\mathbb{U}) \geq \tfrac{1}{2}T^{\alpha-\kappa}$$

where $n^{>}(\mathbb{U})$ is the number of intervals in $\mathbb{U}$ of length greater than T^b. If (Ia) holds, then $X \geq T^{\alpha/2-\kappa/2+a/2-\varepsilon}$, so (5.32) follows from (5.14).

In case (Ib) we have

$$\tfrac{1}{2}T^{\alpha-\kappa} \leq n^{>}(\mathbb{U}) \leq l^{>}(\mathbb{U})T^{-b} \leq \tfrac{1}{2}\left(\lambda^{<} + \lambda^{\geq}\right)T^{-b},$$

where $l^{>}(\mathbb{U})$ is the total length of the intervals in $\mathbb{U}$ of length greater than T^b. Therefore

$$\max\left(\lambda^{<}, \lambda^{\geq}\right) \geq \tfrac{1}{2}T^{\alpha+b-\kappa}$$

and

$$\begin{aligned} X &\geq T^{-\varepsilon}\left(\lambda^{\geq}\right)^{d/(2+d)} + \lambda^{<}T^{2/(2+d)-2a} \\ &\geq \min\left(\frac{1}{\sqrt{2}}T^{d(\alpha-\kappa+b)/(d+2)-\varepsilon}, \frac{1}{2}T^{\alpha-\kappa+b+2/(2+d)-2a}\right), \end{aligned}$$

and (5.32) follows by (5.14) and (5.15).

It remains to discuss the case

$$\text{(II)} \qquad |\mathbb{U}'| \leq \tfrac{1}{2}T^{-b}l(\mathbb{U}').$$

Here, we have

$$l^{\leq}(\mathbb{U}')T^{-b} \leq n^{\leq}(\mathbb{U}') \leq \tfrac{1}{2}l(\mathbb{U}')T^{-b},$$

where $l^{\leq}(\mathbb{U}')$ is the length of the intervals in $\mathbb{U}'$ of length less than or equal to T^b. This implies

$$l^{>}(\mathbb{U}') \geq \tfrac{1}{2}l(\mathbb{U}'),$$

but $l^{>}(\mathbb{U}') = l^{>}(\mathbb{U}) \leq l(\mathbb{U})$. So $l(\mathbb{U})$, $l(\mathbb{U}')$, $l^{>}(\mathbb{U})$ and $l^{>}(\mathbb{U}')$ essentially coincide. Having this, the left-hand side of (5.32) is dominated by

$$\begin{aligned} T^{-2(2+d)}l^{>}(\mathbb{U}) &\leq \tfrac{1}{2}T^{-2/(2+d)}\left(\lambda^{<} + \lambda^{\geq}\right) \\ &= o\left(T^{d/(2+d)}\lambda^{<}T^{-2a} + \left(\lambda^{\geq}\right)^{d/(2+d)}\right) = o(X), \end{aligned}$$

so we proved (5.32) in this case, too, and therefore Proposition 5.5.

APPENDIX A

PROOF OF LEMMA 3.1. It is convenient to reduce the problem to one in a continuous situation. Let Π be a two-dimensional flat torus with circumference $R = 10\varrho_2$. We identify Π with $[0,R)^2$. As in Section 2, φ is eigenfunction of the principal eigenvalue λ of $-\frac{1}{2}\Delta$ in $D_0(\varrho_2)$ with Dirichlet boundary condition, which is normalized by $\int \varphi^2\,dx = 1$, $\varphi \geq 0$, and is extended to $\mathbb{R}^2$ by setting it equal to 0 outside $D_0(\varrho_2)$. We can interpret φ as a mapping $\Pi \to [0,\infty)$. If $x \in \Pi$, we write φ_x for the translate of φ: $\varphi_x(y) = \varphi(y-x)$.

LEMMA A.1. *If $g \in C^\infty(\Pi)$ satisfies $g \geq 0$, $\int g^2\,dx = 1$, and if*

$$\varepsilon = \inf_{x\in\Pi} \|g - \varphi_x\|_2 > 0$$

is small enough, then

$$\big|\{g > 0\}\big| + \tfrac{1}{2}\int |\nabla g|^2\,dx \geq 2\sqrt{\pi\lambda} + \varepsilon^{15}.$$

PROOF. Of course, we may assume

$$\text{(A.1)} \qquad \big|\{g > 0\}\big| \leq 3\sqrt{\pi\lambda}, \qquad \tfrac{1}{2}\int |\nabla g|^2 dx \leq 3\sqrt{\pi\lambda}.$$

By the Sobolev inequality (see [1], Lemma 5.10), we have, for $2 \leq r < \infty$,

$$\text{(A.2)} \qquad \begin{aligned} \|g\|_r^2 &\leq c(r)\int \big(g^2 + |\nabla g|^2\big)dx \\ &\leq c'(r). \end{aligned}$$

Let $\bar{g} = \max_x g(x)$, $S_g = \{x \in \Pi : \nabla g(x) = 0\}$. According to Sard's theorem $g(S_g)$ is of Lebesgue measure 0. We define $J = J_g = (0,\bar{g}\,] \setminus g(S_g)$. If $y \geq 0$, let $A(y) = \{x \in \Pi : g(x) > y\}$. If $y \in J$, then the boundary $\partial A(y)$ is a C^∞ curve. We set $l(y)$ equal to the length of $\partial A(y)$ and $\Delta(y) = l(y) - 2\sqrt{\pi|A(y)|}$. The symmetrization $g^*: \mathbb{R}^2 \to [0,\infty)$ of g is defined by

$$g^*(x) = \inf\big\{y \geq 0 : |A(y)|/\pi \leq |x|^2\big\}.$$

By (A.1), we may assume that g^* is defined on Π.

If $l^*(y)$ is the length of $\partial A^*(y) = \partial\{x : g^*(x) > y\}$, that is, $l^*(y) = 2\sqrt{\pi|A(y)|}$, and $\delta^*(y)$ is the constant value of $|\nabla g^*|$ on $\partial A^*(y)$, then by Jensen's inequality, if $y \in J$,

$$\int_{\partial A(y)} |\nabla g(z)|\,dz \geq l^2(y)\delta^*(y)/l^*(y),$$

the integral being taken with respect to Lebesgue measure on $\partial A(y)$. Therefore,

$$\begin{aligned}\int |\nabla g|^2\,dx - \int |\nabla g^*|^2\,dx &= \int_J dy\Big(\int_{\partial A(y)} dz|\nabla g(z)| - \delta^*(y)l^*(y)\Big)\\ &\geq \int_J dy\Big(\frac{l^2(y)}{l^*(y)} - l^*(y)\Big)\delta^*(y)\\ &\geq \int_J dy\,\Delta(y)\delta^*(y).\end{aligned}\tag{A.3}$$

Assume for the moment that

$$\Delta(y) \geq \varepsilon^9 \quad \text{for } y \in \Big((0,\varepsilon^3] \cap J\Big),\tag{A.4}$$

then

$$\begin{aligned}&\int |\nabla g|^2\,dx - \int \left|\nabla g^*\right|^2 dx\\ &\geq \varepsilon^9 \int_{J\cap(0,\varepsilon^3]} \delta^*(y)\,dy \geq \varepsilon^{15}\Big\{\int_{J\cap(0,\varepsilon^3]} \frac{dy}{\delta^*(y)}\Big\}^{-1}\\ &= \varepsilon^{15}\big\{|A(0)| - \left|A(\varepsilon^3)\right|\big\}^{-1} \geq \frac{\varepsilon^{15}}{3\sqrt{\pi\lambda}} \geq c\varepsilon^{15}.\end{aligned}$$

Therefore,

$$\big|\{g > 0\}\big| + \tfrac{1}{2}\int |\nabla g|^2\,dx \geq \big|\{g^* > 0\}\big| + \tfrac{1}{2}\int \left|\nabla g^*\right|^2 dx \geq 2\sqrt{\pi\lambda} + c\varepsilon^{15}.\tag{A.5}$$

It remains to investigate the case, where (A.4) is not true. By (A.1), we have

$$|A(y)| \geq c > 0 \quad \text{for } y \in (0,\varepsilon^3],\tag{A.6}$$

for ε small enough. If (A.4) is false, then there exists $y_0 \in (0,\varepsilon^3]$ with $\Delta(y_0) \leq \varepsilon^9$. We claim that there exists $r = r(g) > 0$, $\xi = \xi(g) \in \Pi$ satisfying

$$0 < c \leq r(g)\tag{A.7}$$

$$|A(y_0) \triangle D_\xi(r)| \leq c\varepsilon^9,\tag{A.8}$$

where $\triangle$, here, denotes the symmetric difference.

By (A.6), (A.7) follows if (A.8) is satisfied. Inequality (A.8) is implied by the Bonnesen inequality (see, e.g., Theorem 1 of [11]) as follows: Set $A = A(y_0)$, and if $x \in [0,R)$, $A_x = \{y \in [0,R): (x,y) \in A\}$. Then

$$\begin{aligned}\big|\{x: A_x \neq \emptyset\}\big| &\leq \big|\{x: |A_x| = R\}\big| + \big|\{x: A_x \neq \emptyset,\, A_x \neq [0,R)\}\big|\\ &\leq \frac{|A|}{R} + l(y_0) < R,\end{aligned}$$

by (A.1) and the fact that $\Delta(y_0) \le \varepsilon^9$. Therefore, there exists x with $A_x = \emptyset$.

Similarly, one can cut the torus without touching A in the other coordinate, and therefore we can assume that A is a subset of $\mathbb{R}^2$. By the isoperimetric inequality, there exists a connected component B of A such that $|A \triangle \widetilde{B}| = O(\varepsilon^{18})$, $|\, l(y_0) - |\partial\widetilde{B}|\,| = O(\varepsilon^9)$, where $\widetilde{B}$ is the complement of the unbounded component of B^c. Therefore, we may assume that A is simply connected in $\mathbb{R}^2$, and (A.8) is an immediate consequence of the Bonnesen inequality.

Let h be a C^∞ function: $\Pi \to [0,1]$, which is 0 outside $D_\xi(r+\varepsilon^3)$, 1 on $D_\xi(r)$ and

$$|\nabla h(x)| \le c\varepsilon^{-3}, \qquad x \in \Pi. \tag{A.9}$$

We set

$$\widetilde{g}(x) = \frac{h(g-y_0)^+}{\|h(g-y_0)^+\|_2}.$$

$$\begin{aligned}\|g - h(g-y_0)^+\|_2 &\le \|g-(g-y_0)^+\|_2 + \|(1-h)(g-y_0)^+\|_2\\ &\le c\varepsilon^3, \quad \text{by (A.8) and (A.2).}\end{aligned}$$

Therefore, $\|\widetilde{g}-g\|_2 \le c\varepsilon^3 \le \varepsilon/2$, for small enough ε, and so

$$\inf_{\xi\in\Pi} \|\widetilde{g} - \varphi_\xi\|_2 \ge \varepsilon/2. \tag{A.10}$$

Furthermore, for some $\delta > 0$,

$$\begin{aligned}\int |\nabla\widetilde{g}|^2 dx &\le (1+O(\varepsilon^3))\left\{\int h^2 |\nabla((g-y_0)^+)|^2 dx + \int ((g-y_0)^+)^2 |\nabla h|^2\, dx\right\}\\ &\le \int |\nabla g|^2\, dx + c\varepsilon^{2+\delta},\end{aligned}$$

by (A.8), (A.9), (A.2) and the Hölder inequality.

Let $\widehat{\varphi}_\xi$ be the normalized eigenfunction in $D_\xi(r+\varepsilon^3)$, that is,

$$\widehat{\varphi}_\xi(x) = \varphi_\xi\left(\xi + \frac{(x-\xi)\varrho}{r+\varepsilon^3}\right)\frac{\varrho}{r+\varepsilon^3},$$

for $|x-\xi| \le r+\varepsilon^3$ and 0 elsewhere. Then

$$\begin{aligned}|\{g>0\}| + \tfrac12 \int |\nabla g|^2 dx &\ge |A(y_0)| + \tfrac12 \int |\nabla\widetilde{g}|^2\, dx - c\varepsilon^{2+\delta}\\ &\ge \left|D_\xi\left(r+\varepsilon^3\right)\right| + \tfrac12 \int |\nabla\widetilde{g}|^2\, dx - c_1\varepsilon^{2+\delta}\\ &\ge \left|D_\xi\left(r+\varepsilon^3\right)\right| + \tfrac12 \int |\nabla\widehat{\varphi}_\xi|^2\, dx\\ &\quad + \tfrac12\left[\int |\nabla\widetilde{g}|^2\, dx - \int |\nabla\widehat{\varphi}_\xi|^2\, dx\right] - c_1\varepsilon^{2+\delta};\end{aligned} \tag{A.11}$$

$$|D_\xi(r+\varepsilon^3)| + \tfrac{1}{2}\int |\nabla\widehat{\varphi}_\xi|^2 dx \geq 2\sqrt{\lambda\pi} + c_2|r+\varepsilon^3 - \varrho|^2. \tag{A.12}$$

The second summand on the r.h.s. of (A.11) is estimated in the following way. Let $\delta(r)$ be the difference of the first and the second eigenvalues of $\frac{1}{2}\Delta$ in $D_\xi(r+\varepsilon^3)$ with Dirichlet boundary condition.

From (A.7), one gets $\delta(r(y)) \geq c > 0$, and therefore

$$\begin{aligned}\frac{1}{2}\int |\nabla\widetilde{g}(y)|^2 dy - \frac{1}{2}\int |\nabla\widehat{\varphi}_\xi|^2 dx &\geq \delta(r)\|\widetilde{g} - \widehat{\varphi}_\xi\|_2^2 \\ &\geq c\left[\left(\frac{\varepsilon}{2} - \|\varphi_\xi - \widehat{\varphi}_\xi\|_2\right)\vee 0\right]^2 \\ &\geq c\left[\left(\frac{\varepsilon}{2} - c'|r+\varepsilon^3-\varrho|\right)\vee 0\right]^2.\end{aligned} \tag{A.13}$$

The sum of the expressions on the l.h.s. of (A.12) and (A.13) is therefore greater than or equal to $2\sqrt{\pi\lambda} + c\varepsilon^2$, and therefore, if (A.4) does not hold, we have from (A.11)

$$|\{g>0\}| + \tfrac{1}{2}\int |\nabla g|^2 \geq 2\sqrt{\pi\lambda} + c\varepsilon^2,$$

for small $\varepsilon > 0$. This prove Lemma A.1. □

Let f be a probability density on the discrete lattice $\mathbb{L}$ satisfying $\|f - \mathcal{F}\|_1 \geq a$, $a \geq T^{-1/8}$. If $\mathcal{F}_{\text{cont}}$ is $\{\varphi_x^2 \colon x \in \Pi\}$, φ_x and Π are as introduced before Lemma A.1, we get from Lemma 2.1

$$\|\mathcal{F} - \mathcal{F}_{\text{cont}}\|_1 \leq cT^{-1/4}.$$

Let us agree that when writing $\|g - h\|_1$, where g is defined on the continuous torus and h on the discrete, we understand this by extending h, making it constant on the squares.

We define a function $\widetilde{f}$ by interpolating f first linearly in the "x-direction", that is, on $\{(x, kT^{-1/4})\colon 0 \leq x \leq R\}$ and then also linearly in the other. It is easily checked that

$$\tfrac{1}{2}\int |\nabla\widetilde{f}|^2 dx \leq I_T(f), \tag{A.14}$$

the l.h.s. being an ordinary integral, and

$$\|f - \widetilde{f}\|_1 \leq cT^{-1/4}\sqrt{I_T(f)}.$$

Of course, we may assume that $I_T(f) \le 100$, and therefore we see that if $\widehat{f} = \widetilde{f}/\|\widetilde{f}\|_1$,

$$\|\widehat{f} - \mathcal{F}_{\mathrm{cont}}\|_1 \ge \frac{a}{2}.$$

If we write $\widehat{f} = g^2$, we have

$$\inf_x \| g - \varphi_x\|_2 \ge a/4.$$

Lemma 3.1 now obviously follows from Lemma A.1 and (A.14). □

APPENDIX B

Proofs of the technical lemmas in Section 5. Let $(X_n)_{n\in\mathbb{N}_0}$ be an ordinary symmetric discrete time random walk on $\mathbb{Z}^d$, satisfying $X_0 = 0$. For notational reasons, we formulate everything in terms of (X_n).

Let $F = \{x = (x_1, \dots, x_d) \in \mathbb{R}^d \colon \sum_i |x_i| \le 1\}$ and F_e be the set of extreme points of this set: $F_e = \{x \in \mathbb{Z}^d \colon \sum_i |x_i| = 1\}$.

The following facts are well known from elementary large deviation theory (see, e.g., [9]). If $a \in F$, then there exists a unique probability distribution q_a and F_e satisfying

(B.1) $$h(a) = \sum_{x\in F_l} q_a(x) \log q_a(x) - \log 2d,$$

(B.2) $$\sum_{x\in F_e} x q_a(x) = a;$$

q_a is actually minimizing the r.h.s. of (B.1) among probability measures satisfying (B.2). Furthermore, if $x \in \mathbb{Z}^d$, $a = x/n \in F$, one has

(B.3) $$P(X_n = x) = e^{-nh(a)} \widehat{P}_a(X_n = x),$$

where $\widehat{P}$ is the law of a random walk having one jump distribution q_a. Using a standard local central limit theorem (Bhattacharya and Rao [2], Corollary 22.3), one has

(B.4) $$\widehat{P}_a(X_n = x) \ge cn^{-d/2}$$

uniformly in x for x having the same parity as n (which we tacitly always assume in such statements) and $x/n \in F$. [For x/n on the boundary of F, the covariance matrix of q_a becomes degenerate and $\widehat{P}_a(X_n = x)$ is actually of order $n^{-d'/2}$ for some $d' < d$].

PROOF OF LEMMA 5.1. We tacitly assumed that $T^{1/(2+d)}(y_1 - y_2)$ has the same parity as m, and $|y_1 - y_2| \le 3\varrho_d/2$. Then

$$\begin{aligned} &P(X_m = T^{1/(2+d)}(y_1 - y_2)) \\ &\quad \le p_m(y_1, y_2) \le P(X_m = T^{1/(2+d)}(y_1 - y_2)) + P(|X_m| \ge 3\varrho_d T^{1/(2+d)}). \end{aligned}$$

Using (B.3), (B.4) and the estimate

$$P\big(|X_m| \geq 3\varrho_d T^{1/(2+d)}\big) \leq c \exp\left(-mh\left(\frac{3\varrho_d T^{1/(2+d)}}{m}\right)\right)$$

from Cramér's theorem, the statement follows. □

PROOF OF LEMMA 5.2. We write P_x for the law of the walk starting at x. If $n \leq m$, let $X_{[n,m]} = \{X_j : n \leq j \leq m\}$.

Let K be the intersection of $\mathbb{Z}^d$ with a centered ball in $\mathbb{R}^d$ of radius $rT^{1/(2+d)}$, where $\varrho_d/2 \leq r \leq 2\varrho_d$. Lemma 5.2 claims that

$$\text{(B.5)} \qquad P_x\big(X_{[0,m]} \subset K \mid X_m = y\big) \geq c_1 m^{-k} \exp\left(-c_2 \frac{m}{T^{2/(2+d)}}\right)$$

for some $c_1, c_2 > 0$, $k > 0$, uniformly in $x, y \in K$ and r. Of course, as remarked before, we tacitly assume that m has the correct parity and is large enough for $P_x(X_m = y) > 0$.

We first treat the case where $m \leq T^{2/(2+d)}$ and where we can incorporate the exponential factor on the r.h.s. into c_1. The claim is probably true with $k = 1$, but as there is no need for this here, we leave k unspecified. Let $a = x/m$. By the bound we obtained in Lemma 5.1, it suffices to prove

$$\text{(B.6)} \qquad \widehat{P}^a_x\big(X_{[0,m]} \subset K, X_m = y\big) \geq cm^{-k}.$$

If $0 \leq r \leq m$, $z = x + (r/m)(y - x)$, we have

$$\text{(B.7)} \qquad \begin{aligned} &\widehat{P}^a_x\big(X_{[0,m]} \subset K,\ X_m = y\big) \\ &\quad \geq \widehat{P}^a_x\big(X_{[0,r]} \subset K,\ X_r = z\big)\widehat{P}^a_z\big(X_{[0,m-r]} \subset K,\ X_{m-r} = y\big) \\ &\quad = \widehat{P}^a_x\big(X_{[0,r]} \subset K,\ X_r = z\big)\widehat{P}^{-a}_y\big(X_{[0,m-r]} \subset K,\ X_{m-r} = z\big). \end{aligned}$$

Of course, z has to be adjusted slightly in order to be in $\mathbb{Z}^d$ with the correct parity. We omit mentioning such trivial corrections. By (B.7) it suffices to consider the case where

$$\text{(B.8)} \qquad \langle x - y, y\rangle \geq 0.$$

It should be noted here that the proof of the bound is nontrivial only for x near the boundary of the ball. The lemma is actually only needed in this case.

Let Σ_a be the covariance matrix of q_a, $B_a = \sqrt{\Sigma_a} D_0(1)$ [$D_0(1)$ is the unit Euclidean disc] and $|x|_a = \inf\{t \geq 0 : x \in tB_a\}$

Let $\xi \in \mathbb{R}^d$ satisfying $|\xi|_a = 1$ and

$$\langle -x, \xi\rangle = \max\big\{\langle -x, \eta\rangle : |\eta|_a = 1\big\}.$$

An elementary geometric argument using (B.8) reveals that for some $\delta > 0$, which depends on the dimension d only,

$$F_k(\delta) \cap \mathbb{Z}^d \subset K \quad \text{for } 1 \leq k \leq m,$$

where

$$F_k(t) = x + ka + \sqrt{\frac{k(m-k)}{m}}\xi + t\sqrt{\frac{k(m-k)}{m}}B_a.$$

If $1 \le u < v \le m$, let

$$G_{u,v}(\delta) = \bigcup_{k=u}^{v} F_k(\delta).$$

For notational convenience, we pretend that m is divisible by 3 and write $m = 3n$. Then

$$\begin{aligned}
&\widehat{P}_x^a(X_{[0,m]} \subset K,\ X_m = y)\\
&\quad \ge \widehat{P}_x^a\left(X_{[0,n]} \subset G_{0,n}(\delta),\ X_n \in F_n\left(\frac{\delta}{2}\right),\right.\\
&\qquad\qquad \left. X_{[n,2n]} \subset G_{n,2n}(\delta),\ X_{2n} \in F_{2n}\left(\frac{\delta}{2}\right), X_{[2n,m]} \subset G_{2n,m}(\delta),\ X_m = y\right)\\
&\quad \ge \widehat{P}_x^a\left(X_{[0,n]} \subset G_{0,n}(\delta),\ X_n \in F_n\left(\frac{\delta}{2}\right)\right)\\
&\qquad \times \widehat{P}_y^{-a}\left(X_{[0,n]} \subset G_{2n,m}(\delta),\ X_n \in F_{2n}\left(\frac{\delta}{2}\right)\right)\\
&\qquad \times \inf\left\{\widehat{P}_u^a(X_{[0,n]} \subset G_{n,2n}(\delta),\ X_n = v): u \in F_n\left(\frac{\delta}{2}\right),\ v \in F_{2n}\left(\frac{\delta}{2}\right)\right\}.
\end{aligned} \tag{B.9}$$

By Donsker's invariance principle, we have that there exists $\alpha > 0$ (depending only on the dimension d and δ, the latter depending also only on d) such that, for all $l \in \mathbb{N}$,

$$\widehat{P}_x^a(X_{2l} \in F_{2l}(\delta),\ X_{[l,2l]} \subset G_{l,2l}(\delta) \mid X_l = z) \ge \alpha$$

for $z \in F_l(\delta)$. If $2^{a-1} \le n \le 2^a$, this implies by iteration

$$\widehat{P}_x^a(X_n \in F_n(\delta),\ X_{[0,n]} \subset G_{0,n}(\delta)) \ge \alpha^a \ge cm^{-k}.$$

The same estimate applies to the second factor on the r.h.s. of (B.9). The last factor is easily estimated by the local central limit theorem and Donsker's theorem. By the central limit theorem we have

$$\widehat{P}_u^a(X_n = v) \ge cn^{-d/2}$$

(see [2], Corollary 22.3) and Donsker's for a tied-down random walk gives

$$\widehat{P}_u^a(X_{[0,n]} \subset G_{n,2n}(\delta) \mid X_n = v) \ge c > 0.$$

Implementing these estimates in (B.9) proves (B.6).

Treating (B.5) for $m \geq T^{2/(2+d)}$ is now easy: If $x \in K/2$

$$P_x\left(X_{[0,m]} \subset K,\ X_m \in \frac{K}{2}\right) \geq c_1 \exp\left(\frac{-c_2 m}{T^{2/(2+d)}}\right)$$

is just a rough lower bound of standard large deviation estimates (see Section 2 for a much finer analysis), and this can then be combined with the estimates for $m \leq T^{2/(2+d)}$ in a straightforward way to give the desired estimate also for x, y near or at the boundary of K. □

PROOF OF LEMMA 5.6. The statement is probably true without the factor $T^{-\varepsilon}$. With it, a rather crude argument suffices. We chose $a < 2/(2+d)$ such that $a > 2f/(2+d)$. Let

$$m = \left[T^{-\varepsilon/2d}\left(\sum_i l_i\right)^{1/(2+d)}\right].$$

By a trivial adjustment, we may assume that m divides $K = [RT^{1/(2+d)}]$, and therefore there is a natural projection

$$\mathbb{L}_T^{(R)} = T^{-1/(2+d)}(\mathbb{Z}_K)^d \to T^{-1/(2+d)}(\mathbb{Z}_m)^d = \widetilde{\mathbb{L}}.$$

We denote the images of the walks ξ^i by $\widetilde{\xi}^i$. Obviously, the $\widetilde{\xi}^i$ are simple random walks on $\widetilde{\mathbb{L}}$ and

$$N(\widetilde{\xi}^1, \ldots, \widetilde{\xi}^n) \leq N(\xi^1, \ldots, \xi^n).$$

If $A \subset \widetilde{\mathbb{L}}$, we denote by $\Lambda_i(A)$ the event that $\widetilde{\xi}^i$ does not visit A. Then

$$\begin{aligned} E\Big(\exp\big(-N(\xi^1, \ldots, \xi^n)\big)\Big) &\leq E\Big(\exp\big(-N(\widetilde{\xi}^1, \ldots, \widetilde{\xi}^n)\big)\Big) \\ &\leq \exp\left(-\tfrac{1}{2}m^d\right) + \sum_A \prod_{j=1}^n P(\Lambda_j(A)), \end{aligned} \tag{B.10}$$

where the summation over A extends over those sets containing at least $m^d/2$ points. The first summand on the r.h.s. of (B.10) is of the desired order, so we only have to cope with the second. There are at most 2^{m^d} A's, so it suffices to show that, for large m,

$$\prod_{j=1}^n P(\Lambda_j(A)) \leq \exp(-\beta m^d), \tag{B.11}$$

with arbitrary large β, uniformly in A with $|A| \geq m^d/2$. The local central limit theorem implies that for some constant $\alpha > 0$ we have

$$P\left(\widetilde{\xi}_{m^2-1} \notin A,\ \widetilde{\xi}_{m^2} \notin A\right) \leq 1 - \alpha$$

uniformly in the starting point of the walk $\tilde{\xi}$ and in A if $|A| \geq m^d/2$. Note now that our condition $a > 2f/(2+d)$ ensures that all our l_j are greater than or equal to m^2. Therefore, we get

$$\prod_{j=1}^{n} P(\Lambda_j(A)) \leq \exp\left(-c\sum_{j=1}^{n}\frac{l_j}{m^2}\right),$$

which is more than enough for (B.11). □

Acknowledgment. I would like to thank the referee for a number of helpful comments.

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