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ASYMPTOTIC BEHAVIOR OF THE TWO LEVEL MEASURE BRANCHING PROCESS[1]

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In this paper we consider a multilevel branching diffusion particle system and its diffusion approximation, which can be characterized as an $M(M(R^d))$-valued process. The long term behavior of the limiting process is studied. The main results are that if $d \leq 4$, then the two level $M(M(R^d))$-valued process suffers local extinction, and if $d = 4$, then the process has a self-similarity property.

1. Introduction. The study of one level measure-valued branching processes which arise as small particle limits of one level branching diffusion particle systems in R^d, $d \geq 1$, has developed over the last 20 years. During the last few years considerable interest has also developed in dynamic multilevel models [see, e.g., Dawson, Hochberg and Wu (1990), Wu (1993a, b)]. A dynamic multilevel system consists of objects at different levels. At any given level, it is assumed that each object (particle) of that level can be deleted or copied. Collections of objects (particles) at one level comprise objects at the next higher level. Once an object is copied, it is assumed that subsequent alterations may cause it to differ from the object from which it was copied.

In this paper, we consider a multilevel critical branching diffusion particle system and its continuous limit which was formulated by Dawson and Hochberg (1991). The question of the asymptotic behavior of critical measure-valued processes is of particular interest. If a one level measure-valued critical branching process has finite initial measure, then it follows from elementary properties of critical branching processes that the total mass converges to zero in probability as $t \to \infty$. On the other hand, Dawson (1977) showed that if the spatial mechanism of the one level measure-valued process is Brownian motion or a symmetric stable process on R^d, and the initial state of the process is Lebesgue measure on R^d, then existence or nonexistence of a nontrivial limiting distribution is equivalent to transience or recurrence of the spatial motion. Etheridge (1990) obtained a similar result for a more general class of spatial mechanisms. In this paper, we study the asymptotic behavior of the two level $M(M(R^d))$-valued critical branching process $Y(t)$. Again, if the total initial mass $\int_{M(R^d)} \mu(R^d)Y(0, d\mu)$ is finite, then the total mass $\int_{M(R^d)} \mu(R^d)Y(t, d\mu)$ converges

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to zero in probability as $t \to \infty$. In this paper we consider a class of initial conditions for the two level process which are locally finite but have infinite total mass. The first of these intuitively corresponds to infinitely many level-2 particles (or "colonies") each of which contains one level-1 particle $\Sigma_{x \in Z^d} \delta_{\delta_x}$. For technical reasons we prefer to work with a translation invariant analogue of this measure, which leads us to the following choice of initial measure:

$$Y(0, A) = \nu_0(A) \equiv \int_{R^d} \int_{M(R^d)} 1_A(\mu)\delta_{\delta_x}(d\mu)\,dx,$$

where A is a measurable subset of $M(R^d)$.

DEFINITION 1.1. The two level $M(M(R^d))$-valued process $Y(t)$ is said to suffer local extinction if, for every compact set B in R^d and $\varepsilon > 0$,

$$\lim_{t \to \infty} P\left\{ \int_{M(R^d)} \mu(B) Y(t, d\mu) > \varepsilon \right\} = 0, \tag{1.1}$$

that is, the total mass contained in a compact subset of R^d converges to zero in probability.

We now present the main results of this article. The first result proves Dawson's conjecture about extinction of the two level measure critical branching process.

THEOREM 1.1. *Let $Y(0) = \nu_0$ as defined above. If $d \leq 4$, then the two level process $Y(t)$ suffers local extinction.*

We should underline an important difference between the cases of $d \leq 2$ and $d \geq 3$. In the former case the one level process itself suffers local extinction and the proof is somewhat simpler. In the case $d \geq 3$ Dawson and Perkins proved that the one level process has an infinitely divisible stationary distribution with canonical measure $R_\infty \in M(M(R^d))$. In fact the proof in Section 3 also shows that the system experiences local extinction if the initial measure is given by either $Y(0) = \nu_0$ or $Y(0) = R_\infty$. Moreover, properties of R_∞ also play an important role in the proof of Theorem 1.1.

In the next theorem we establish a self-similarity property of the process $Y(t)$. To do so we let

$$Y_k(t, A) = Y\left(k^2 t, k^{-d/2}\left\{\mu: \exists \tilde{\mu} \in A, \text{ s.t. } \mu\left(x: \frac{x}{k} \in \cdot\right) = \tilde{\mu}(\cdot)\right\}\right) k^{-d/2} \tag{1.2}$$

for each $k \geq 1$, where $A \in \mathfrak{B}(M_p(R^d))$ and $\mathfrak{B}(M_p(R^d))$ denotes all Borel subsets of $M_p(R^d)$. Under this rescaling transformation, we obtain the following theorem:

THEOREM 1.2 (Self-similarity property). *If $d = 4$ and the initial value of process $Y(t)$ is the canonical equilibrium measure R_∞ of the one level measure* *branching process, then the rescaled process $Y_k(t)$ has the same distribution as $Y(t)$.*

Since R_∞ is an invariant measure of the one level measure branching process, it is natural to choose R_∞ as the initial value of the process $Y(t)$ in Theorem 1.2.

2. Description of the model. In order to describe the model in this paper, we introduce some additional notation:

1. $N(R^d)$ denotes all integer-valued measures on R^d;
2. $M(R^d)$ denotes all positive measures on R^d;
3. $M(M(R^d))$ is the collection of all positive measures on $M(R^d)$;
4. $C_b(R)$ is the Banach space of bounded continuous functions on R with $\|f\| = \sup_{x\in R} |f(x)|$;
5. $C_b^2(R) \subset C_b(R)$ is the subspace of bounded twice-differentiable functions on R;
6. $C_c(R^d)$ denotes the collection of continuous functions on R^d with compact support;
7. $C_c^2(R^d)$ denotes the collection of twice-differentiable functions on R^d with compact support;
8. $C_c^+(R^d)$ is the collection of positive continuous functions on R^d with compact support.

We define $\dot{R}^d = R^d \cup \{\tau\}$, τ being an isolated adjoined point. Let $M_\rho(\dot{R}^d)$ be the space of ρ-tempered measures introduced by Iscoe (1986) with the ρ-vague topology, that is, the smallest topology making the maps $\mu \to \langle\phi, \mu\rangle$ continuous for $\phi \in C_c(R^d) \cup \{\phi_\rho\}$, where $d < \rho \le d + 2$ and, for $x \in R^d$,

$$\phi_\rho(x) = \frac{1}{1 + |x|^\rho} \tag{2.1}$$

and

$$\phi_\rho(\tau) = 1.$$

Iscoe (1986) verified that M_ρ is locally compact with the ρ-vague topology.

Let

$$M_\rho^2(\dot{R}^d) \equiv \left\{\nu \in M\Big(M_p(\dot{R}^d)\Big): \int\!\!\int \phi_\rho(x)\mu(dx)\nu(d\mu) < \infty\right\}. \tag{2.2}$$

We endow $M_\rho^2(\dot{R}^d)$ with the smallest topology which makes continuous the maps

$$\nu \to \int\!\!\int \phi(x)\mu(dx)\nu(d\mu)$$

for all $\phi \in C_c(R^d) \cup \{\phi_\rho\}$.

The basic process in this paper is an $M_\rho^2(\dot{R}^d)$-valued process $Y(t)$. Since the process suffers extinction in a finite time when started from a finite measure, interesting asymptotic behavior of the process can only occur when the initial measure of process $Y(t)$ is infinite. It will turn out that the state space $M_\rho^2(\dot{R}^d)$ contains a rich enough class of infinite measures for our needs.

We will not discuss in detail the construction of the process in this paper but explain briefly how it can be characterized as the continuous limit of a two level branching diffusion particle system.

Now we consider a multilevel branching random field which is a natural extension of the multilevel branching model introduced by Dawson and Hochberg (1991) and can be described as follows.

We consider a two level branching random field. By "two level" we mean that the system consists of particles at two different levels. Each level-2 particle (i.e., superparticle) is a collection of level-1 particles (i.e., particle) in R^d. We say that a superparticle is of size i if it consists of exactly i particles. We suppose that after an exponentially distributed time, each particle undergoes a level-1 binary branching process, that is, it either dies or produces a copy, each with probability $\frac{1}{2}$, with the branching rate γ_1, in which case we note that the total number of superparticles is unchanged. We also assume that each superparticle performs a level-2 binary branching process with the branching rate γ_2 after an exponentially distributed holding time with the parameter γ_2. During the holding times of both level 1 and level 2, each particle moves in R^d according to the d-dimensional Brownian motion. The entire system can be represented as a random atomic measure on $N(R^d)$:

$$Y_a(t) \equiv \sum_{i=1}^{\infty} \sum_{k=1}^{n_i} \delta_{\sum_{r=1}^{i} \delta_{x_{i,k,r}(t)}} + n_0(t)\delta_{\delta_\phi} \tag{2.3}$$

where $x_{i,k,r}(t)$ denotes the location in R^d of the rth particle in the kth superparticle $X_{i,k}(t)$ of size i at time t, n_i is the number of superparticles of size i at time t and $n_0(t)$ denotes the number of null superparticles at time t, where the term "null superparticle" means that it does not contain any particles.

To study the continuous limit of the two level branching diffusion particle system, we rescale the system as follows: If $\mathfrak{B}(M_\rho(\dot{R}^d))$ denotes the *Borel σ-algebra* of $M_\rho(\dot{R}^d)$ [Iscoe (1986), page 90], then for $A \in \mathfrak{B}(M_\rho(\dot{R}^d))$ and $t > 0$, we define

$$Y_n(t, A) = \tfrac{1}{n} Y_a(nt, A_n) \text{ where } A_n = \left\{\mu : \tfrac{1}{n}\mu \in A\right\}.$$

We can show that when $n \to \infty$, $Y_n(t)$ converges weakly in $D_{[0,\infty)}(M_\rho^2(R^d))$ to an M_ρ^2-valued process $Y(t)$ which can be characterized as the unique solution of the martingale problem for the limiting generator $G_c^{(2)}$ which is given by

$$G_c^{(2)} F(\nu) = \left\langle\left\langle \mathcal{L}F'(\nu, \cdot), \nu \right\rangle\right\rangle + \gamma_2 \left\langle\left\langle F''(\nu), \delta_{\mu_1}(d\mu_2)\nu(d\mu_1) \right\rangle\right\rangle, \tag{2.4}$$

where $\langle\langle g(\mu), \nu\rangle\rangle \equiv \int g(\mu)\nu(d\mu)$ and we take the class of test functions $F(\nu)$ on M_ρ^2 in the following form:

$$F(\nu) = f\left(\left\langle\left\langle h_1(\langle h_2, \cdot\rangle), \nu \right\rangle\right\rangle\right), \tag{2.5}$$

where $\nu \in M_\rho^2$, h_1, $f \in C_b^2(R)$, $h_2 \in C_c^2(R^d)$, and

$$\begin{aligned} F'(\nu,\mu) &= \frac{\partial F(\nu)}{\partial \nu(\mu)} = \frac{d}{d\varepsilon}\left[F(\nu+\varepsilon\delta_\mu)\right]_{\varepsilon=0} \\ &= f'\Big(\Big\langle \langle h_1(\langle h_2,\cdot\rangle),\nu\rangle\Big\rangle\Big) h_1\big(\langle h_2,\mu\rangle\big). \end{aligned} \tag{2.6}$$

In (2.4), $\mathcal{L}$ denotes the generator of the $M_\rho(R^d)$-valued branching process, that is,

$$\begin{aligned} \mathcal{L}F'(\nu,\mu) &= \mathcal{L}_1 F'(\nu,\mu) + \mathcal{L}_2 F'(\nu,\mu) \\ &= f'\Big(\Big\langle \langle h_1(\langle h_2,\cdot\rangle),\nu\rangle\Big\rangle\Big) h_1'(\langle h_2,\mu\rangle)\langle \Delta h_2,\mu\rangle \\ &\quad + \gamma_1 f'\Big(\Big\langle \langle h_1(\langle h_2,\cdot\rangle),\nu\rangle\Big\rangle\Big) h_1''(\langle h_2,\mu\rangle)\langle h_2^2,\mu\rangle, \end{aligned} \tag{2.7}$$

where Δ is the d-dimensional Laplacian and

$$\mathcal{L}_1 F'(\nu,\mu) = \int \Delta \frac{\delta F'(\nu,\mu)}{\delta\mu(x)}\mu(dx), \tag{2.8}$$

$$\mathcal{L}_2 F'(\nu,\mu) = \gamma_2 \iint \frac{\delta^2 F'(\nu,\mu)}{\delta\mu(x)\delta\mu(y)}\delta_x(dy)\mu(dx). \tag{2.9}$$

The Laplace transition functional of the M_ρ^2-valued process $Y(t)$ is given by

$$\begin{aligned} L_{t,\,\nu(H)} &= E\left[\exp\Big(-\int_{M_\rho(R^d)} H(\mu)Y(t,d\mu)\Big)\,\Big|\, Y(0)=\nu\right] \\ &= \exp\Big\{-\int u(t,\mu)\nu(d\mu)\Big\}, \end{aligned} \tag{2.10}$$

where $u(t,\mu)$ is a solution of the weak form of the following differential equation:

$$\begin{aligned} &\frac{\partial u(t,\mu)}{\partial t} = \mathcal{L}u(t,\mu) - \gamma_2 u^2(t,\mu), \\ &u(0,\mu) = H(\mu) \end{aligned} \tag{2.11}$$

and $H(\mu) = h(\langle \phi,\mu\rangle)$, $\phi \in C_c(R^d)$ and $h \in C_b(R)$.

REMARK 2.1. If $\int_{M_\rho(R^d)} \mu(R^d)\nu(d\mu) < \infty$, then (2.10) and (2.11) can be extended to $\phi \in C_b(R^d)$ and in this case if $H(\mu) = \langle 1,\mu\rangle$, then we obtain

$$\begin{aligned} \frac{\partial u(t,\mu)}{\partial t} &= -\gamma_2 u^2(t,\mu), \\ u(0,\mu) &= \langle 1,\mu\rangle = \mu(R^d). \end{aligned}$$

Solving the above differential equation, we have

$$\begin{aligned} u(t,u) &= \frac{u(0,\mu)}{1+t\gamma_2 u(0,\mu)} \\ &= \frac{\mu(R^d)}{1+t\gamma_2\mu(R^d)}. \end{aligned} \tag{2.12}$$

Therefore,

$$
\begin{aligned}
&\lim_{t\to\infty} E\left[\exp\Big(-\int_{M_\rho(R^d)} \mu(R^d)Y(t,\,d\mu)\Big)\Big| Y(0)=\nu\right]\\
&\quad= \lim_{t\to\infty}\exp\Big(-\int_{M_\rho(R^d)} u(t,\,\mu)\nu(d\mu)\Big)\\
&\quad= \lim_{t\to\infty}\exp\Big(-\int_{M_\rho(R^d)} \frac{\mu(R^d)}{1+t\gamma_2\mu(R^d)}\nu(d\mu)\Big)\\
&\quad= 1.
\end{aligned}
\tag{2.13}
$$

This verifies the fact that the two level process $Y(t)$ goes to extinction if the initial measure is finite.

3. The extinction of the two level measure-valued branching process. In this section we will prove Theorem 1.1. The method of the proof for local extinction we use here was suggested by Dawson and is different from the arguments used previously in the one level case. Before presenting the proof, we consider some preliminary facts.

Let $u(t,\,\mu)$ denote the solution of the equation (2.11) with initial values $u(0,\,\mu)=\langle\phi,\,\mu\rangle$ for $\phi\in C_c(R^d)$, $\nu_0=\delta_{\delta_x}\,dx$ and define T_t^* by

$$
\begin{aligned}
&\int_{M_\rho(R^d)}\big[1-\exp\{-\langle\phi,\,\cdot\rangle\}\big](\mu)T_t^*\nu_0(d\mu)\\
&\quad\equiv \int_{M_\rho(R^d)} T_t\big[1-\exp\{-\langle\phi,\,\cdot\rangle\}\big](\mu)\nu_0(d\mu)\\
&\quad= \int_{R^d} T_t\big[1-\exp\{-\langle\phi,\,\cdot\rangle\}\big](\delta_x)dx\\
&\quad= \int_{R^d}\big[1-\exp\{-\langle v(t),\,\delta_x\rangle\}\big]dx\\
&\quad= \int_{R^d}\big[1-\exp\{-v(t,\,x)\}\big]dx.
\end{aligned}
\tag{3.1}
$$

Here T_t denotes the semigroup of operators associated with the one level $M_\rho(R^d)$-valued branching process with generator $\mathcal{L}$, and $v(t,\,x)$ is the unique solution of the following nonlinear differential equation:

$$
\begin{aligned}
\frac{\partial v(t,x)}{\partial t} &= \Delta v(t,x) - v^2(t,x),\\
v(0,x) &= \phi(x).
\end{aligned}
\tag{3.2}
$$

Note that

$$
\begin{aligned}
\lim_{t\to\infty}\sup_x v(t,\,x) &= 0,\\
1-\exp\{-v(t,\,x)\} &\approx v(t,\,x)\quad \text{as } t\to\infty,
\end{aligned}
\tag{3.3}
$$

and hence

$$
\begin{aligned}
&\int [1-\exp\{-\langle \phi, \cdot \rangle\}](\mu) T_t^* \nu_0(d\mu) \\
(3.4) \qquad &\approx \int v(t, x) dx \\
&\to \int [1-\exp\{-\langle \phi, \mu \rangle\}] R_\infty(d\mu).
\end{aligned}
$$

Therefore we conclude that

$$
T_t^* \nu_0 \Rightarrow R_\infty, \tag{3.5}
$$

where R_∞ denotes the canonical equilibrium measure of the one level $M_\rho(R^d)$-valued branching process [cf. Dawson and Perkins (1991), Chapter 6, for the definition and properties of R_∞].

As mentioned above we need to consider the two level process with an infinite initial measure. In particular, we will now verify that the state space $M_\rho^2(\dot{R}^d)$ of the process $Y(t)$ contains infinite measures $\nu_0(d\mu) \equiv \int_{R^d} \delta_{\delta_x}(d\mu)\, dx$ and R_∞.

If A is a measurable subset of $M(R^d)$, then it is easy to see that $\nu_0 \in M(M(R^d))$ and

$$
\begin{aligned}
\nu_0(A) &= \int_{R^d} \int_{M_\rho(R^d)} 1_A(\mu) \delta_{\delta_x}(d\mu)\, dx \\
&= \int_{R^d} 1_A(\delta_x)\, dx.
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
&\int_{M_\rho(R^d)} \int_{R^d} \phi_\rho(y) \mu(dy) \nu_0(d\mu) \\
(3.6) \qquad &= \int_{R^d} \int_{R^d} \phi_\rho(y) \delta_x(dy)\, dx = \int_{R^d} \phi_\rho(x)\, dx \\
&< \infty
\end{aligned}
$$

and therefore, $\nu_0 \in M_\rho^2(\dot{R}^d)$.

As stated in the introduction, the one level measure branching process with spatial Brownian motion in R^d and with initial value given by d-dimensional Lebesgue measure converges to an equilibrium invariant probability measure in dimension $d \geq 3$ and suffers local extinction in dimension $d \leq 2$. Theorem 1.1 states that if $d \leq 4$, then the two level $M(M(R^d))$-valued process goes to extinction. Our proof in dimensions $d = 3, 4$ in fact uses the invariant probability measure R_∞ of the one level process. Let S_t denote the Brownian semigroup.

Then

$$
\begin{aligned}
&\int_{M_\rho(R^d)}\int_{R^d}\phi_\rho(x)\mu(dx)R_\infty(d\mu)\\
&\quad=\lim_{t\to\infty}\int_{M_\rho(R^d)}\int_{R^d}\phi_\rho(x)\mu(dx)T_t^*\nu_0(d\mu)\\
&\quad=\lim_{t\to\infty}\int_{R^d}T_t\langle\phi_\rho,\,\cdot\rangle(\delta_x)dx\\
&\quad=\lim_{t\to\infty}\int_{R^d}\langle S_t\phi_\rho,\,\delta_x\rangle dx\\
&\quad=\lim_{t\to\infty}\int_{R^d}S_t\phi_\rho(x)dx\\
&\quad=\int_{R^d}\phi_\rho(x)dx<\infty,
\end{aligned}
\tag{3.7}
$$

and therefore $R_\infty \in M_\rho^2(\dot{R}^d)$.

Our proof of Theorem 1.1 requires the following lemmas.

LEMMA 3.1. *Let $\mathfrak{R}_d$ denote the Borel σ-algebra in R^d. Then for each $B \in \mathfrak{R}_d$ and $d \geq 3$, we have*

$$
R_\infty\left(\left\{\mu\colon \frac{1}{t}\mu\big(\sqrt{t}B\big) > a\right\}\right) = t^{(d-2)/2}R_\infty\big(\{\mu\colon \mu(B) > a\}\big),
\tag{3.8}
$$

where R_∞ denotes the canonical equilibrium measure of the one level measure branching process.

PROOF. See Dawson and Perkins (1991). □

LEMMA 3.2. *For $d \geq 3$, $s \geq 1$, we have*

$$
\int_{\{\mu\colon\langle S_s\phi,\,\mu\rangle\leq\varepsilon/s^{(d-2)/2}\}}\langle S_s\phi,\,\mu\rangle R_\infty(d\mu)\to 0
\tag{3.9}
$$

as $\varepsilon \to 0$, where $\phi \in C_c^+(R^d)$ and S_s is the Brownian semigroup.

PROOF. Observe that for $\phi \in C_c^+(R^d)$, $\phi \neq 0$,

$$
\begin{aligned}
S_s\phi(x) &= \int(2\pi s)^{-d/2}\exp\left\{-\frac{|y-x|^2}{2s}\right\}\phi(y)dy\\
&= \int(2\pi s)^{-d/2}\exp\left\{-\frac{|x|^2}{2s}\right\}\exp\left\{-\frac{y^2-2xy}{2s}\right\}\phi(y)\,dy.
\end{aligned}
\tag{3.10}
$$

Therefore there exist two positive constants $0 < C_1 \le C_2 < \infty$, such that

$$C_1 s^{-d/2}\exp\left\{-\frac{|x|^2}{2s}\right\} \le S_s\phi(x) \le C_2 s^{-d/2}\exp\left\{-\frac{|x|^2}{2s}\right\}$$

uniformly in x for fixed nontrivial ϕ.

Now we define a new measure $\mu_s \in M_\rho(R^d)$ in terms of $\mu \in M_\rho(R^d)$ by

$$\mu_s(B) = \frac{1}{s}\mu\left(\left\{x: \frac{x}{\sqrt{s}} \in B\right\}\right), \tag{3.11}$$

where $B \in \mathfrak{R}_d$.

Since (3.10) implies that

$$C_1 s^{-d/2}\left\langle \exp\left\{-\frac{|x|^2}{2s}\right\}, \mu\right\rangle \le \langle S_s\phi, \mu\rangle \le C_2 s^{-d/2}\left\langle \exp\left\{-\frac{|x|^2}{2s}\right\}, \mu\right\rangle$$

and

$$C_1 s^{-(d-2)/2}\left\langle \exp\left\{-\frac{|x|^2}{2}\right\}, \mu_s\right\rangle \le \langle S_s\phi, \mu\rangle \le C_2 s^{-(d-2)/2}\left\langle \exp\left\{-\frac{|x|^2}{2}\right\}, \mu_s\right\rangle,$$

thus, together with Lemma 3.1, we have

$$\begin{aligned} &R_\infty\left(\{\mu: \langle S_s\phi, \mu\rangle > a\}\right) \\ &\quad\le R_\infty\left(\left\{\mu: C_2 s^{-(d-2)/2}\left\langle \exp\left\{-\frac{|x|^2}{2}\right\}, \mu_s\right\rangle > a\right\}\right) \\ &\quad= s^{(d-2)/2}R_\infty\left(\left\{\mu: C_2 s^{-(d-2)/2}\left\langle \exp\left\{-\frac{|x|^2}{2}\right\}, \mu\right\rangle > a\right\}\right) \\ &\quad= s^{(d-2)/2}R_\infty\left(\left\{\mu: \left\langle \exp\left\{-\frac{|x|^2}{2}\right\}, \mu\right\rangle > \frac{1}{C_2}s^{(d-2)/2}a\right\}\right). \end{aligned} \tag{3.12}$$

For $a \ge 0$, we define

$$\nu((a, \infty)) = R_\infty\left(\left\{\mu: C_2 s^{-(d-2)/2}\left\langle \exp\left\{-\frac{|x|^2}{2}\right\}, \mu_s\right\rangle > a\right\}\right)$$

and

$$\begin{aligned} &\tilde{\nu}\left(\left(\frac{1}{C_2}s^{(d-2)/2}a, \infty\right)\right) \\ &\quad= R_\infty\left(\left\{\mu: \left\langle \exp\left\{-\frac{|x|^2}{2}\right\}, \mu\right\rangle > \frac{1}{C_2}s^{(d-2)/2}a\right\}\right). \end{aligned} \tag{3.13}$$

Then it is easy to see that

$$\nu\big((a,\,\infty)\big) = s^{(d-2)/2}\tilde{\nu}\left(\left(\frac{1}{C_2}s^{(d-2)/2}a,\,\infty\right)\right) \tag{3.14}$$

and

$$dF(a) = s^{(d-2)/2}d\tilde{F}\left(\frac{1}{C_2}s^{(d-2)/2}a\right) \tag{3.15}$$

if we define $F(a) = \nu([0,\,a])$ and $\tilde{F}(a) = \tilde{\nu}([0,\,a])$.

Using this notation, we have

$$\begin{aligned}
&\int_{\{\mu:\langle S_s\phi,\,\mu\rangle\le \varepsilon/s^{(d-2)/2}\}}\langle S_s\phi,\,\mu\rangle R_\infty(d\mu)\\
&\quad\le \int_{\{\mu:C_1 s^{-(d-2)/2}\langle \exp\{-|x|^2/2\},\,\mu_s\rangle\le \varepsilon/s^{(d-2)/2}\}} C_2 s^{-(d-2)/2}\\
&\qquad\qquad\times\left\langle \exp\left\{-\frac{|x|^2}{2}\right\},\,\mu_s\right\rangle R_\infty(d\mu)\\
&\quad= \int_{\{\mu:C_2 s^{-(d-2)/2}\langle \exp\{-|x|^2/2\},\,\mu_s\rangle\le C_2\varepsilon/(C_1 s^{(d-2)/2})\}} C_2 s^{-(d-2)/2}\\
&\qquad\qquad\times\left\langle \exp\left\{-\frac{|x|^2}{2}\right\},\,\mu_s\right\rangle R_\infty(d\mu)\\
&\quad= \int_0^{C_2\varepsilon/(C_1 s^{(d-2)/2})} a\nu(da)\\
&\quad= \int_0^{C_2\varepsilon/(C_1 s^{(d-2)/2})} a s^{(d-2)/2}d\tilde{F}\left(\frac{1}{C_2}s^{(d-2)/2}a\right).
\end{aligned} \tag{3.16}$$

If we let $b = (1/C_2)s^{(d-2)/2}a$, then

$$\int_{\{\mu:\langle S_s\phi,\,\mu\rangle\le\varepsilon/s^{(d-2)/2}\}}\langle S_s\phi,\,\mu\rangle R_\infty(d\mu)\le C_2\int_0^{\varepsilon/C_1} b\,d\tilde{F}(b). \tag{3.17}$$

Since

$$\begin{aligned}
\int\langle S_s\phi,\,\mu\rangle R_\infty(d\mu) &= \int T_s\langle\phi,\,\cdot\rangle(\mu)R_\infty(d\mu)\\
&= \int\langle\phi,\,\mu\rangle R_\infty(d\mu)\equiv C_0<\infty,
\end{aligned} \tag{3.18}$$

and for $\varepsilon/C_1 < 1$ we have

$$
\begin{aligned}
\int_0^{\varepsilon/C_1} b\, d\widetilde{F}(b) &\le \int_0^1 b\, d\widetilde{F}(b) \\
&= \int_{\{\mu:\langle \exp\{-|x|^2/2\},\, \mu\rangle \le 1\}} \left\langle \exp\left\{-\frac{|x|^2}{2}\right\},\, \mu\right\rangle R_\infty(d\mu) \\
&= \frac{1}{C_1}\int_{\{\mu: C_2\langle \exp\{-|x|^2/2\},\, \mu\rangle \le C_2\}} C_1\left\langle \exp\left\{-\frac{|x|^2}{2}\right\},\, \mu\right\rangle R_\infty(d\mu) \\
&\le \frac{1}{C_1}\int_{\{\mu:\langle S_1\phi,\, \mu\rangle \le C_2\}} \langle S_1\phi,\, \mu\rangle R_\infty(d\mu) \\
&\le \frac{1}{C_1}\int \langle S_1\phi,\, \mu\rangle R_\infty(d\mu) \\
&< \infty,
\end{aligned}
\tag{3.19}
$$

hence

$$
\begin{aligned}
&\int_{\{\mu:\langle S_s\phi,\, \mu\rangle \le \varepsilon/s^{(d-2)/2}\}} \langle S_s\phi,\, \mu\rangle R_\infty(d\mu) \\
&\quad \le C_2 \int_0^{\varepsilon/C_1} b\, d\widetilde{F}(b) \\
&\quad \to 0 \quad \text{as } \varepsilon \to 0.
\end{aligned}
\tag{3.20}
$$

□

PROOF OF THEOREM 1.1. We consider the two cases (i) $d = 1, 2$ and (ii) $d = 3, 4$ separately.

CASE (i) ($d = 1$ and $d = 2$). Recall that the one level $M_\rho(R^d)$-valued process with the generator $\mathcal{L}$ suffers local extinction for $d \le 2$ [see Dawson (1977)]. If $d \le 2$, then from (2.10) we have

$$
\begin{aligned}
&E\left[\exp\left\{-\int_{M_\rho(R^d)} (1-\exp(-\langle\phi,\, \mu\rangle))\, Y(t, d\mu)\right\}\middle| Y(0) = \nu_0\right] \\
&\quad = \exp\left\{-\int_{M_\rho(R^d)} u(t,\, \mu)\nu_0(d\mu)\right\},
\end{aligned}
\tag{3.21}
$$

where $\phi \in C_c^+(R^d)$ and $u(t,\, \mu)$ satisfies equation (2.11) with

$$u(0,\, \mu) = 1 - \exp\{-\langle\phi,\, \mu\rangle\}.$$

Note that

$$\begin{aligned}
&\exp\left\{-\int_{M_\rho(R^d)} u(t,\mu)\nu_0(d\mu)\right\} \\
(3.22)\qquad &= \exp\left\{-\int_{R^d}\int_{M_\rho(R^d)} u(t,\mu)\delta_{\delta_x}(d\mu)\,dx\right\} \\
&= \exp\left\{-\int_{R^d} u(t,\delta_x)\,dx\right\}.
\end{aligned}$$

Since for each $t > 0$, and $\mu \in M_\rho(R^d)$,

$$u(t,\mu) = T_t u(0,\mu) - \gamma_2 \int_0^t T_{t-s}\big[u(s)\big]^2(\mu)ds \geq 0,$$

it is clear that

$$u(t,\mu) \leq T_t u(0,\cdot)(\mu).$$

Therefore together with (3.1) we have

$$\begin{aligned}
&\exp\left\{-\int_{R^d} u(t,\delta_x)\,dx\right\} \\
&\geq \exp\left\{-\int_{R^d} T_t u(0,\cdot)(\delta_x)\,dx\right\} \\
(3.23)\qquad &= \exp\left\{-\int_{R^d}\Big[1-\exp\Big(-\big\langle v(t),\delta_x\big\rangle\Big)\Big]\,dx\right\} \\
&= \exp\left\{-\int_{R^d}\Big[1-\exp\big(-v(t,x)\big)\Big]\,dx\right\},
\end{aligned}$$

where $v(t, x)$ is a solution of (3.2). From (3.3), we see that

$$\begin{aligned}
&\exp\left\{-\int_{R^d}\Big[1-\exp\big(-v(t,x)\big)\Big]\,dx\right\} \\
(3.24)\qquad &\geq \exp\left\{-\int_{R^d} v(t,x)\,dx\right\} \\
&\to 1 \quad \text{as } t\to\infty.
\end{aligned}$$

Therefore from (3.22)–(3.24) we conclude that

$$\exp\left\{-\int_{M_\rho(R^d)} u(t,\mu)\nu_0(d\mu)\right\} \to 1 \quad \text{as } t\to\infty.$$

This implies that the theorem is true for the case of $d \leq 2$.

CASE (ii) ($d = 3$ and $d = 4$). The method we will use in this case is based on properties of the canonical measure R_∞ of the one level measure branching process. Again from (2.10), for $\phi \in C_c^+(R^d)$,

$$E\left[\exp\left\{-\int_{M_\rho(R^d)}\int_{R^d}\phi(x)\mu(dx)Y(t,d\mu)\right\}\middle|Y(0)=\nu\right] = \exp\left\{-\int_{M_\rho(R^d)}u(t,\mu)\nu(d\mu)\right\}, \tag{3.25}$$

where $u(t, \mu)$ satisfies (2.11) with initial value $u(0, \mu) = \langle\phi, \mu\rangle$.

In the remainder of this section, we always assume that $d \geq 3$. It is easy to see from (3.23) that for each $t > 0$, and $\mu \in M(R^d)$,

$$\gamma_2\int_0^t T_{t-s}\big[u(s)\big]^2(\mu)ds \leq T_t u(0)(\mu) \tag{3.26}$$

and so

$$\begin{aligned}\gamma_2\int\int_0^t & T_{t-s}\big[u(s)\big]^2(\mu)ds\, R_\infty(d\mu)\\ &\leq \int T_t u(0)(\mu)R_\infty(d\mu) = \int u(0,\mu)R_\infty(d\mu)\\ &= \int\langle\phi,\mu\rangle R_\infty(d\mu) < \infty;\end{aligned} \tag{3.27}$$

therefore we obtain that

$$\begin{aligned}\lim_{t\to\infty}\int\int_0^t & u^2(s,\mu)ds\, R_\infty(d\mu)\\ &= \lim_{t\to\infty}\int\int_0^t T_{t-s}\big[u(s)\big]^2(\mu)ds\, R_\infty(d\mu)\\ &< \infty.\end{aligned} \tag{3.28}$$

Next we show that

$$\lim_{t\to\infty}\int u(t,\mu)R_\infty(d\mu) = 0 \tag{3.29}$$

by contradiction. Suppose that

$$\lim_{t\to\infty}\int u(t,\mu)R_\infty(d\mu) = l > 0. \tag{3.30}$$

By Lemma 3.2, for any fixed $s \geq 1$, there exists $\varepsilon_0 > 0$ such that

$$\int_{\{\mu:\langle S_s\phi,\mu\rangle\leq\varepsilon_0/s^{(d-2)/2}\}}\langle S_s\phi,\mu\rangle R_\infty(d\mu) < \frac{l}{2}. \tag{3.31}$$

From (3.23) we know that, for each $s > 0$,

$$u(s,\mu) \leq \langle S_s\phi,\mu\rangle, \tag{3.32}$$

so we have

$$\int_{\{\mu:\langle S_s\phi,\,\mu\rangle\le\varepsilon_0/s^{(d-2)/2}\}} u(s,\,\mu)R_\infty(d\mu) < \frac{l}{2}. \tag{3.33}$$

Since

$$\begin{aligned}\int \frac{\partial u(s,\,\mu)}{\partial s}R_\infty(d\mu) &= \int \mathcal{L}u(s,\,\mu)R_\infty(d\mu) - \gamma_2\int u^2(s,\,\mu)R_\infty(d\mu)\\ &= -\gamma_2\int u^2(s,\,\mu)R_\infty(d\mu) < 0,\end{aligned} \tag{3.34}$$

we obtain that $\int u(s,\,\mu)R_\infty(d\mu)$ is monotone decreasing. Thus from the assumption in (3.30), for any $s>0$ we have

$$\int u(s,\,\mu)R_\infty(d\mu) \ge \lim_{t\to\infty}\int u(t,\,\mu)R_\infty(d\mu) = l. \tag{3.35}$$

Therefore

$$\int_{\{\mu:\langle S_s\phi,\,\mu\rangle>\varepsilon_0/s^{(d-2)/2}\}} u(s,\,\mu)R_\infty(d\mu) > \frac{l}{2} \tag{3.36}$$

and

$$\begin{aligned}&\int_{\{\mu:u(s,\,\mu)\le\varepsilon/s^{(d-2)/2}\}\cap\{\mu:\langle S_s\phi,\,\mu\rangle>\varepsilon_0/s^{(d-2)/2}\}} u(s,\,\mu)R_\infty(d\mu)\\ &\quad\le \int_{\{\mu:u(s,\,\mu)\le\varepsilon/s^{(d-2)/2}\}\cap\{\mu:\langle S_s\phi,\,\mu\rangle>\varepsilon_0/s^{(d-2)/2}\}} \frac{\varepsilon}{s^{(d-2)/2}}R_\infty(d\mu)\\ &\quad\le \frac{\varepsilon}{s^{(d-2)/2}}\int_{\{\mu:\langle S_s\phi,\,\mu\rangle>\varepsilon_0/s^{(d-2)/2}\}} R_\infty(d\mu)\\ &\quad\le \frac{\varepsilon}{s^{(d-2)/2}}\int_{\{\mu:\langle S_s\phi,\,\mu\rangle>\varepsilon_0/s^{(d-2)/2}\}} \frac{\langle S_s\phi,\,\mu\rangle}{\varepsilon_0/s^{(d-2)/2}}R_\infty(d\mu)\\ &\quad= \frac{\varepsilon}{\varepsilon_0}\int_{\{\mu:\langle S_s\phi,\,\mu\rangle>\varepsilon_0/s^{(d-2)/2}\}} \langle S_s\phi,\,\mu\rangle R_\infty(d\mu)\\ &\quad\le \frac{\varepsilon}{\varepsilon_0}C_0 < \infty.\end{aligned} \tag{3.37}$$

We choose ε small enough that

$$\frac{\varepsilon}{\varepsilon_0}C_0 < \frac{l}{4}; \tag{3.38}$$

then

$$\int_{\{\mu:u(s,\,\mu)\le\varepsilon/s^{(d-2)/2}\}\cap\{\mu:\langle S_s\phi,\,\mu\rangle>\varepsilon_0/s^{(d-2)/2}\}} u(s,\,\mu)R_\infty(d\mu) < \frac{l}{4} \tag{3.39}$$

and so

$$
\begin{aligned}
&\int_{\{\mu:\, u(s,\,\mu)>\varepsilon/s^{(d-2)/2}\}} u(s,\,\mu)R_\infty(d\mu)\\
&\quad\geq \int_{\{\mu:\, u(s,\,\mu)>\varepsilon/s^{(d-2)/2}\}\cap\{\mu:\langle S_s\phi,\,\mu\rangle>\varepsilon_0/s^{(d-2)/2}\}} u(s,\,\mu)R_\infty(d\mu)\\
&\quad= \int_{\{\mu:\langle S_s\phi,\,\mu\rangle>\varepsilon_0/s^{(d-2)/2}\}} u(s,\,\mu)R_\infty(d\mu)\\
&\qquad - \int_{\{\mu:\, u(s,\,\mu)\leq\varepsilon/s^{(d-2)/2}\}\cap\{\mu:\langle S_s\phi,\,\mu\rangle>\varepsilon_0/s^{(d-2)/2}\}} u(s,\,\mu)R_\infty(d\mu)\\
&\quad\geq \frac{l}{4}.
\end{aligned}
\tag{3.40}
$$

Hence

$$
\begin{aligned}
&\int u^2(s,\,\mu)R_\infty(d\mu)\\
&\quad\geq \int_{\{\mu:\, u(s,\,\mu)>\varepsilon/s^{(d-2)/2}\}} u^2(s,\,\mu)R_\infty(d\mu)\\
&\quad\geq \frac{\varepsilon}{s^{(d-2)/2}}\int_{\{\mu:\, u(s,\,\mu)>\varepsilon/s^{(d-2)/2}\}} u(s,\,\mu)R_\infty(d\mu)\\
&\quad\geq \frac{\varepsilon}{s^{(d-2)/2}}\frac{l}{4}.
\end{aligned}
\tag{3.41}
$$

Therefore

$$
\begin{aligned}
&\lim_{t\to\infty}\int_0^t\int u^2(s,\,\mu)R_\infty(d\mu)(ds)\\
&\quad\geq \lim_{t\to\infty}\int_1^t \frac{\varepsilon}{s^{(d-2)/2}}\frac{l}{4}ds\\
&\quad= \begin{cases} \dfrac{\varepsilon l}{4}\lim\limits_{t\to\infty}\ln s\Big|_1^t, & \text{if } d = 4\\[2ex] \dfrac{\varepsilon l}{4}\lim\limits_{t\to\infty}\dfrac{1}{-(d-2)/2+1}s^{-(d-2)/2+1}\Big|_1^t, & \text{if } d\neq 4,\end{cases}\\
&\quad= \begin{cases} \infty, & \text{if } d = 4,\\[1ex] \dfrac{\varepsilon l}{4}\dfrac{2}{4-d}\lim\limits_{t\to\infty}\left[t^{(4-d)/2}-1\right], & \text{if } d\neq 4,\end{cases}\\
&\quad= \begin{cases} \infty, & \text{if } d = 4,\\ \infty, & \text{if } d = 3,\\ \text{finite}\,, & \text{if } d > 4.\end{cases}
\end{aligned}
\tag{3.42}
$$

This is a contradiction of (3.30) if $d = 3$ and $d = 4$. Hence we conclude that, for $d = 3$ and $d = 4$,

$$\lim_{t\to\infty}\int u(t,\mu)R_\infty(d\mu) = 0. \tag{3.43}$$

On the other hand, for each $t > 0$,

$$\begin{aligned}
&\int u(t_1+t,\mu)\nu_0(d\mu)\\
&\quad= \int T_{t_1+t}u(0,\mu)\nu_0(d\mu) - \gamma_2\int\int_0^{t+t_1} T_{t_1+t-s}\big[u(s)\big]^2(\mu)ds\,\nu_0(d\mu)\\
&\quad= \int T_{t_1}T_t u(0,\mu)\nu_0(d\mu) - \gamma_2\int\int_0^{t} T_{t_1}T_{t-s}\big[u(s)\big]^2(\mu)ds\,\nu_0(d\mu)\\
&\qquad - \gamma_2\int\int_t^{t+t_1} T_{t_1+t-s}\big[u(s)\big]^2(\mu)ds\,\nu_0(d\mu)\\
&\quad\le \int T_t u(0,\mu)T^*_{t_1}\nu_0(d\mu) - \gamma_2\int\int_0^{t} T_{t-s}\big[u(s)\big]^2(\mu)ds\,T^*_{t_1}\nu_0(d\mu).
\end{aligned} \tag{3.44}$$

This implies that, for each $t > 0$,

$$\begin{aligned}
&\limsup_{t_1\to\infty}\int u(t_1,\mu)\nu_0(d\mu)\\
&\quad= \limsup_{t_1\to\infty}\int u(t_1+t,\mu)\nu_0(d\mu)\\
&\quad\le \lim_{t_1\to\infty}\left[\int T_t u(0,\mu)T^*_{t_1}\nu_0(d\mu) - \gamma_2\int\int_0^{t} T_{t-s}\big[u(s)\big]^2(\mu)ds\,T^*_{t_1}\nu_0(d\mu)\right]\\
&\quad= \int T_t u(0,\mu)R_\infty(d\mu) - \gamma_2\int\int_0^{t} T_{t-s}\big[u(s)\big]^2(\mu)ds\,R_\infty(d\mu)\\
&\quad= \int u(t,\mu)R_\infty(d\mu).
\end{aligned} \tag{3.45}$$

Thus, together with (3.43), we obtain

$$\limsup_{t_1\to\infty}\int u(t_1,\mu)\nu_0(d\mu) = 0. \tag{3.46}$$

Hence Theorem 1.1 is proved. □

4. The rescaled process and the self-similarity property. Renormalization theory has been studied for many years in the case of one level measure branching processes [see, e.g., Dawson (1977), (1978)]. In this section, we consider the same kind of question in the two level case. Specifically, we prove Theorem 1.2 by using the Laplace functionals of the process $Y(t)$ and the rescaled process $Y_k(t)$ defined in (1.2).

PROOF OF THEOREM 1.2. At first, we are going to calculate the Laplace functional of the rescaled process $Y_k(t)$. Let

$$\widetilde{H}(\mu) = h_1\left(\langle h_2, \mu\rangle\right)$$
$$H(\mu) = h_1\Big(k^{-d/2}\Big\langle h_2\Big(\frac{\cdot}{k}\Big), \mu\Big\rangle\Big)k^{-d/2},$$

for $h_1 \in C_b^2(R)$, $h_2 \in C_c^2(R^d)$. Then it is easy to see that the Laplace functional of $Y_k(t)$ has the following form:

$$\begin{aligned} L_{Y_k(t)}(\widetilde{H}) &= E\left[\exp\left\{-\int \widetilde{H}(\mu)Y_k(t, d\mu)\right\}\Big|Y_k(0)\right] \\ &= \exp\left\{-\int u_k(t, \mu)Y_k(0, d\mu)\right\}, \end{aligned} \tag{4.1}$$

where $u_k(t, \mu)$ satisfies a nonlinear initial value problem.

We already know that the Laplace functional of the process $Y(t)$ is given by

$$\begin{aligned} L_{Y(t)}(H) &= E\left[\exp\left\{-\int H(\mu)Y(t, d\mu)\right\}\Big|Y(0)\right] \\ &= \exp\left\{-\int u(t, \mu)Y(0, d\mu)\right\}, \end{aligned} \tag{4.2}$$

where $u(t, \mu)$ is the solution of the following nonlinear initial value problem:

$$\begin{aligned} \frac{\partial u(t, \mu)}{\partial t} &= \mathcal{L}u(t, \mu) - \gamma_2 u^2(t, \mu), \\ u(0, \mu) &= H(\mu). \end{aligned} \tag{4.3}$$

So we have

$$\begin{aligned} L_{Y_k(t)}(\widetilde{H}) &= E\left[\exp\left\{-\int h_1\left(\langle h_2, \mu\rangle\right) Y_k(t, d\mu)\right\}\Big|Y_k(0)\right] \\ &= E\left[\exp\left\{-\int k^{-d/2}h_1\left(k^{-d/2}\int h_2\Big(\frac{x}{k}\Big)\mu(dx)\right)Y(k^2t, d\mu)\right\}\Big|Y_k(0)\right] \\ &= E\left[\exp\left\{-\int H(\mu)Y(k^2t, d\mu)\right\}\Big|Y(0)\right] \\ &= \exp\left\{-\int u(k^2t, \mu)Y(0, d\mu)\right\}, \end{aligned} \tag{4.4}$$

where we use the fact that $Y(0) = Y_k(0)$ for any $k \geq 1$ if $Y(0) = R_\infty$, which will be shown later. On the other hand, the rescaling transformation (1.2) implies that

$$\begin{aligned} &\exp\left\{-\int u_k(t, \mu)Y_k(0, d\mu)\right\} \\ &\quad = \exp\left\{-\int k^{-d/2}u_k\big(t, k^{-d/2}\mu(k\,\cdot)\big)Y(0, d\mu)\right\}. \end{aligned} \tag{4.5}$$

Equating (4.4) to (4.5) we obtain

$$u\big(k^2t,\ \mu(\cdot)\big) = k^{-d/2}u_k\big(t,\ k^{-d/2}\mu(k\,\cdot)\big). \tag{4.6}$$

Therefore

$$u_k\big(0,\ k^{-d/2}\mu(k\,\cdot)\big) = k^{d/2}u\big(0,\ \mu(\cdot)\big) = h_1\left(k^{-d/2}\left\langle h_2\left(\frac{\cdot}{k}\right),\ \mu\right\rangle\right), \tag{4.7}$$

and by taking the derivative with respect to t in (4.6) we have, for $t > 0$,

$$\begin{aligned}
&\frac{\partial u_k\left(t,\ k^{-d/2}\mu(k\,\cdot)\right)}{\partial t}\\
&= k^{d/2}\frac{\partial u(k^2t,\ \mu)}{\partial t}\\
&= k^{d/2}\frac{\partial u(k^2t,\ \mu)}{\partial k^2t}\frac{\partial k^2t}{\partial t}\\
&= k^{d/2+2}\big[\mathcal{L}u(k^2t,\ \mu) - \gamma_2u^2(k^2t,\ \mu)\big]\\
&= k^{d/2+2}\mathcal{L}_1u(k^2t,\ \mu) + k^{d/2+2}\mathcal{L}_2u(k^2t,\ \mu) - k^{d/2+2}\gamma_2u^2(k^2t,\ \mu).
\end{aligned} \tag{4.8}$$

Now let us discuss the three terms on the right-hand side of (4.8) respectively. First, note that

$$\begin{aligned}
\frac{\delta u(k^2t,\ \mu)}{\delta\mu(x)} &= \lim_{\varepsilon\to 0}\frac{u(k^2t,\ \mu+\varepsilon\delta_x) - u(k^2t,\ \mu)}{\varepsilon}\\
&= \lim_{\varepsilon\to 0}\frac{k^{-d/2}\left[u_k\Big(t,\ k^{-d/2}\big(\mu(k\,\cdot)+\varepsilon\delta_x(k\,\cdot)\big)\Big) - u_k\big(t,\ k^{-d/2}\mu(k\cdot)\big)\right]}{\varepsilon}\\
&= k^{-d/2}\lim_{\varepsilon\to 0}\frac{u_k\big(t,\ k^{-d/2}\mu(k\,\cdot)+k^{-d/2}\varepsilon\delta_{x/k}(\cdot)\big) - u_k\big(t,\ k^{-d/2}\mu(k\,\cdot)\big)}{\varepsilon}\\
&= k^{-d}\lim_{\varepsilon\to 0}\frac{u_k(t,\ \widetilde{\mu}+k^{-d/2}\varepsilon\delta_{x/k}) - u_k(t,\ \widetilde{\mu})}{k^{-d/2}\varepsilon}\\
&= k^{-d}\frac{\delta u_k(t,\ \widetilde{\mu})}{\delta\widetilde{\mu}(x/k)},
\end{aligned} \tag{4.9}$$

where $\widetilde{\mu}(\cdot) = k^{-d/2}\mu(k\,\cdot)$, and

$$\begin{aligned}
\Delta\frac{\delta u(k^2t,\ \mu)}{\delta\mu(x)} &= \frac{\partial^2}{\partial x^2}\left[k^{-d}\frac{\delta u_k(t,\ \widetilde{\mu})}{\delta\widetilde{\mu}(x/k)}\right]\\
&= k^{-d}\frac{\partial^2}{\partial(x/k)^2}\frac{\delta u_k(t,\ \widetilde{\mu})}{\delta\widetilde{\mu}(x/k)}\left[\frac{d(x/k)}{dx}\right]^2\\
&= k^{-d-2}\frac{\partial^2}{\partial(x/k)^2}\frac{\delta u_k(t,\ \widetilde{\mu})}{\delta\widetilde{\mu}(x/k)}
\end{aligned}$$

imply that

$$
\begin{aligned}
\mathcal{L}_1 u(k^2t,\, \mu) &= \int \Delta \frac{\delta u(k^2t,\, \mu)}{\delta\mu(x)}\mu(dx) \\
&= k^{-d-2}\int \frac{\partial^2}{\partial(x/k)^2}\frac{\delta u_k(t,\, \widetilde{\mu})}{\delta\widetilde{\mu}(x/k)}\mu(dx) \\
&= k^{-d/2-2}\int \frac{\partial^2}{\partial x^2}\frac{\delta u_k(t,\, \widetilde{\mu})}{\delta\widetilde{\mu}(x)}\widetilde{\mu}(dx) \qquad (4.10)\\
&= k^{-d/2-2}\int \Delta\frac{\delta u_k(t,\, \widetilde{\mu})}{\delta\widetilde{\mu}(x)}\widetilde{\mu}(dx) \\
&= k^{-d/2-2}\mathcal{L}_1 u_k\big(t,\, k^{-d/2}\mu(k\,\cdot)\big).
\end{aligned}
$$

Next, by the calculation

$$
\begin{aligned}
&\frac{\delta^2 u(k^2t,\, \mu)}{\delta\mu(x)\delta\mu(y)} \\
&\quad= \lim_{\varepsilon_1,\,\varepsilon_2\to 0}\left\{\frac{u(k^2t,\, \mu+\varepsilon_1\delta_x+\varepsilon_2\delta_y)-u(k^2t,\, \mu+\varepsilon_2\delta_y)}{\varepsilon_1\varepsilon_2} - \frac{u(k^2t,\, \mu+\varepsilon_1\delta_x)-u(k^2t,\, \mu)}{\varepsilon_1\varepsilon_2}\right\} \\
&\quad= \lim_{\varepsilon_1,\,\varepsilon_2\to 0} k^{-d/2}\left\{\frac{u_k\Big(t,\, k^{-d/2}\big(\mu(k\,\cdot)+\varepsilon_1\delta_x(k\,\cdot)+\varepsilon_2\delta_y(k\,\cdot)\big)\Big)}{\varepsilon_1\varepsilon_2} - \frac{u_k\Big(t,\, k^{-d/2}\big(u(k\,\cdot)-\varepsilon_2\delta_y(k\,\cdot)\big)\Big)}{\varepsilon_1\varepsilon_2} - \frac{u_k\Big(t,\, k^{-d/2}\big(\mu(k\,\cdot)+\varepsilon_1\delta_x(k\,\cdot)\big)\Big)-u_k\big(t,\, k^{-d/2}\mu(k\,\cdot)\big)}{\varepsilon_1\varepsilon_2}\right\} \\
&(4.11)\\
&\quad= k^{-d/2}\lim_{\varepsilon_1,\,\varepsilon_2\to 0}\left\{\frac{u_k\big(t,\, k^{-d/2}\mu(k\,\cdot)+k^{-d/2}\varepsilon_1\delta_{x/k}(\cdot)+k^{-d/2}\varepsilon_2\delta_{y/k}(\cdot)\big)}{\varepsilon_1\varepsilon_2} - \frac{u_k\big(t,\, k^{-d/2}\mu(k\,\cdot)-k^{-d/2}\varepsilon_2\delta_{y/k}(\cdot)\big)}{\varepsilon_1\varepsilon_2} - \frac{u_k\big(t,\, k^{-d/2}\mu(k\,\cdot)+k^{-d/2}\varepsilon_1\delta_{x/k}(\cdot)\big)-u_k\big(t,\, k^{-d/2}\mu(k\,\cdot)\big)}{\varepsilon_1\varepsilon_2}\right\} \\
&\quad= k^{-d/2-d}\lim_{\varepsilon_1,\,\varepsilon_2\to 0}\left\{\frac{u_k(t,\, \tilde{\mu}+k^{-d/2}\varepsilon_1\delta_{x/k}+k^{-d/2}\varepsilon_2\delta_{y/k})-u_k(t,\, \tilde{\mu}+k^{-d/2}\varepsilon_2\delta_{y/k})}{k^{-d/2}\varepsilon_1 k^{-d/2}\varepsilon_2} - \frac{u_k(t,\, \tilde{\mu}+k^{-d/2}\varepsilon_1\delta_{x/k})-u_k(t,\, \tilde{\mu})}{k^{-d/2}\varepsilon_1 k^{-d/2}\varepsilon_2}\right\} \\
&\quad= k^{-d/2-d}\frac{\delta^2 u_k(t,\, \tilde{\mu})}{\delta\tilde{\mu}(x/k)\delta\tilde{\mu}(y/k)},
\end{aligned}
$$

we obtain the relation

$$
\begin{aligned}
\mathcal{L}_2 u(k^2 t, \mu) &= \gamma_1 \int\int \frac{\delta^2 u(k^2 t, \mu)}{\delta\mu(x)\delta\mu(y)} \delta_x(dy)\mu(dx) \\
&= k^{-d/2-d}\gamma_1 \int\int \frac{\delta^2 u_k(t, \widetilde{\mu})}{\delta\widetilde{\mu}(x/k)\delta\widetilde{\mu}(y/k)} \delta_x(dy)\mu(dx) \\
&= k^{-d}\gamma_1 \int\int \frac{\delta^2 u_k(t, \widetilde{\mu})}{\delta\widetilde{\mu}(x)\delta\widetilde{\mu}(y)} \delta_{kx}(dky)\mu(dkx) \qquad (4.12)\\
&= k^{-d}\gamma_1 \int\int \frac{\delta^2 u_k(t, \widetilde{\mu})}{\delta\widetilde{\mu}(x)\delta\widetilde{\mu}(y)} \delta_x(dy)\widetilde{\mu}(dx) \\
&= k^{-d}\mathcal{L}_2 u_k(t, \widetilde{\mu}) \\
&= k^{-d}\mathcal{L}_2 u_k\big(t, k^{-d/2}\mu(k\,\cdot)\big).
\end{aligned}
$$

Thus using (4.8) and relations (4.10) and (4.12), we can continue to calculate:

$$
\begin{aligned}
&\frac{\partial u_k\big(t, k^{-d/2}\mu(k\,\cdot)\big)}{\partial t} \\
&\quad = k^{d/2+2}k^{-d/2-2}\mathcal{L}_1 u_k\big(t, k^{-d/2}\mu(k\,\cdot)\big) \\
&\qquad + k^{d/2+2}k^{-d}\mathcal{L}_2 u_k\big(t, k^{-d/2}\mu(k\,\cdot)\big) \qquad (4.13)\\
&\qquad - k^{d/2+2}\gamma_2 k^{-d} u_k^2\big(t, k^{-d/2}\mu(k\,\cdot)\big) \\
&\quad = \mathcal{L}_1 u_k\big(t, k^{-d/2}\mu(k\,\cdot)\big) + k^{2-d/2}\mathcal{L}_2 u_k\big(t, k^{-d/2}\mu(k\,\cdot)\big) \\
&\qquad - k^{-(d-4)/2}\gamma_2 u_k^2\big(t, k^{-d/2}\mu(k\,\cdot)\big).
\end{aligned}
$$

That is,

$$
\frac{\partial u_k(t, \mu)}{\partial t} = \mathcal{L}_1 u_k(t, \mu) + k^{2-d/2}\mathcal{L}_2 u_k(t, \mu) - \gamma_2 k^{-(d-4)/2} u_k^2(t, \mu). \qquad (4.14)
$$

In particular, if we take $d = 4$, then (4.14) becomes

$$
\begin{aligned}
\frac{\partial u_k(t, \mu)}{\partial t} &= \mathcal{L}_1 u_k(t, \mu) + \mathcal{L}_2 u_k(t, \mu) - \gamma_2 u_k^2(t, \mu) \\
&= \mathcal{L} u_k(t, \mu) - \gamma_2 u_k^2(t, \mu).
\end{aligned}
$$

Moreover, if we let $Y(0) = R_\infty$, where R_∞ is defined as in the last section, and apply Lemma 3.1 to the case of $d = 4$ and $t = k^2$,

$$
R_\infty\big(\{\mu: k^{-2}\mu(kB) > a\}\big) = k^2 R_\infty\big(\{\mu: \mu(B) > a\}\big),
$$

which implies that

$$
R_\infty(A) = k^{-2}R_\infty\left(k^{-2}\left\{\mu: \exists\widetilde{\mu} \in A, \text{ s.t. } \mu\left(\frac{\cdot}{k}\right) = \widetilde{\mu}(\cdot)\right\}\right);
$$

therefore we have $Y_k(0) = Y(0) = R_\infty$. Thus if the initial value of process $Y(t)$ is R_∞, then, for each $k \geq 1$, $Y_k(t)$ and $Y(t)$ have the same Laplace functional. Hence the proof of the Theorem 1.2 is complete. □

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