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LAWS OF THE ITERATED LOGARITHM FOR THE LOCAL TIMES OF SYMMETRIC LEVY PROCESSES AND RECURRENT RANDOM WALKS[1]

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Both standard and functional laws of the iterated logarithm are obtained for the local time of a symmetric Lévy process, at a fixed point in its state space, as time goes to infinity. Similar results are also obtained for the difference of the local times at two points in the state space. These results are sharp if the exponent of the characteristic function that defines the Lévy process is regularly varying at zero with index $1 < \beta \leq 2$. The results are given in terms of the α-potential density at zero, considered as a function of α. Without additional effort our methods give essentially the same results for the number of visits of a symmetric random walk to a point in its state space and for the difference of the number of visits to two points in the state space. A limit theorem for the sequence of times that a random walk returns to its initial point is obtained as an application of the functional laws.

1. Introduction. Let $X = \{X(t),\ t \in R^+\}$ be a symmetric Lévy process and set

$$E^0 \exp\big(i\lambda X(t)\big) = \exp\big(-t\psi(\lambda)\big), \tag{1.1}$$

where

$$\psi(\lambda) = 2\int_0^\infty (1 - \cos\lambda u)\nu(du), \tag{1.2}$$

for ν a Lévy measure, that is, $\int_0^\infty (1 \wedge x^2)\nu(dx) < \infty$. We also include the case $\psi(\lambda) = \lambda^2/2$, which gives us standard Brownian motion.

The process X has a local time if and only if $(\gamma + \psi(\lambda))^{-1} \in L^1(R^+)$, for some $\gamma > 0$ and, consequently, for all $\gamma > 0$ (see [16]). For symmetric Lévy processes the transition probability density $p_t(x,y)$ is a function of $|x-y|$, and we denote $p_t(0,v)$ by $p_t(v)$. The α-potential density $u^\alpha(x,y)$ of X is similarly a function of the difference of its arguments. We denote $u^\alpha(0,v)$ by $u^\alpha(v)$. For symmetric Lévy processes we have

$$u^\alpha(x) = \int_0^\infty e^{-\alpha t} p_t(x)\,dt = \frac{1}{\pi}\int_0^\infty \frac{\cos\lambda x}{\alpha + \psi(\lambda)}\,d\lambda \quad \forall\,\alpha > 0. \tag{1.3}$$

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In general $u^0(0)$ does not exist. Nevertheless we can define

$$\sigma^2(x) = \lim_{\alpha\to 0}\left(u^\alpha(0) - u^\alpha(x)\right) = \frac{1}{\pi}\int_0^\infty \frac{1-\cos\lambda x}{\psi(\lambda)}\,d\lambda. \tag{1.4}$$

We also define

$$\kappa(\alpha) = u^\alpha(0)\frac{1}{\pi}\int_0^\infty \frac{d\lambda}{\alpha+\psi(\lambda)} \quad \forall\, \alpha > 0. \tag{1.5}$$

In all that follows we will assume that the Lévy processes considered here satisfy the following two conditions:

$$\int_0^1 \frac{d\lambda}{\psi(\lambda)} = \infty; \tag{1.6}$$

$$\int_0^\infty \frac{\log(1+\lambda)}{1+\psi(\lambda)}\,d\lambda < \infty. \tag{1.7}$$

Condition (1.6) is equivalent to the Lévy processes being recurrent (see, e.g., [2], 13.23). Clearly this condition is required in order to say anything interesting about the growth of the local time at a fixed level, as t goes to infinity. It follows from (1.6) that

$$\lim_{\alpha\to 0}\kappa(\alpha) = \infty. \tag{1.8}$$

Also, to understand the function κ better, let us note, as we indicate in Remark 2.7, that if ψ is regularly varying at zero with index $1 < \beta \le 2$, then κ is regularly varying with index $-1/\overline{\beta}$, where $1/\beta + 1/\overline{\beta} = 1$. In fact,

$$\lambda\kappa(\lambda) \sim \frac{\Gamma(1/\overline{\beta})\Gamma(1+1/\beta)}{\pi}\psi^{-1}(\lambda) \quad \text{as } \lambda\to 0, \tag{1.9}$$

where we use the notation $f(\lambda)\sim g(\lambda)$ as $\lambda\to 0$ to mean $\lim_{\lambda\to 0} f(\lambda)/g(\lambda) = 1$ and similarly for the limit as $\lambda\to\infty$. [When ψ is regularly varying one can make sense of ψ^{-1} in (1.9) even when ψ is not monotone; see Remark 2.7.]

Since (1.7) implies that $(1+\psi(\lambda))^{-1} \in L^1(R^+)$, X has local times which we denote by $\{L_t^y, (t,y)\in R^+\times R\}$ and normalize by setting

$$E^x\left(\int_0^\infty e^{-t}\,dL_t^y\right) = u^1(x,y). \tag{1.10}$$

THEOREM 1.1. *Let X be a symmetric Lévy process as defined in* (1.1) *for which* (1.6) *and* (1.7) *are satisfied, and let $\{L_t^y,\ (t,y)\in R^+\times R\}$ denote the local times of X. Let ψ be regularly varying at zero with index $1<\beta\le 2$. Then*

$$\limsup_{t\to\infty}\frac{L_t^0}{(\log\log t)\kappa\left((\log\log t)/t\right)} = \beta^{1/\beta}\,\overline{\beta}^{\,1/\overline{\beta}} \quad a.s. \tag{1.11}$$

and

$$\limsup_{t\to\infty} \frac{L_t^0 - L_t^x}{(\log\log t)\kappa^{1/2}((\log\log t)/t)} = \sqrt{2}\sigma(x)\alpha^{1/\alpha}\overline{\alpha}^{\,1/\overline{\alpha}} \quad a.s., \tag{1.12}$$

where $\overline{\alpha} = 2\overline{\beta}$ *and* $1/\alpha + 1/\overline{\alpha} = 1$. *Furthermore,* (1.12) *also holds with the numerator on the left-hand side replaced by* $\sup_{0\le u\le t} L_u^0 - L_u^x$.

We refer to (1.11) as a first-order law of the iterated logarithm because the local time appears alone, whereas we call (1.12) a second-order law of the iterated logarithm because it involves differences of local times.

Condition (1.7), which is only slightly stronger than the necessary condition for the existence of the local time, is a technical condition. We do not understand it because it depends on $\psi(\lambda)$ for λ at infinity, whereas, as one can see from (1.8) and (1.9), the rate of growth of the local times, as a function of t, depends on $\psi(\lambda)$ [and hence on $\kappa(\lambda)$] for λ near zero. Generally speaking, the behavior of $\psi(\lambda)$ for λ near infinity determines the moduli of continuity of the local times for fixed t, whereas the behavior of $\psi(\lambda)$ for λ near zero gives a measure of how recurrent the Lévy process is.

The work done to prove Theorem 1.1, with very little additional effort, gives us analogous results for the local times of symmetric random walks. Let $X = \{X_n,\ n \ge 0\}$ be a symmetric random walk on the integer lattice Z, that is,

$$X_n = \sum_{i=1}^{n} Y_i, \tag{1.13}$$

where the random variables $\{Y_i,\ i \ge 1\}$ are symmetric, independent and identically distributed with values in Z. We assume for convenience that the law of Y_1 is not supported on a proper subgroup of Z. The process X has symmetric transition probabilities $p_n(x-y) \equiv p_n(x,y) = p_n(0,x-y)$. In this case we define the α-potential

$$u^{\alpha}(x) = \sum_{n=0}^{\infty} e^{-\alpha n} p_n(x). \tag{1.14}$$

(We will use the same notation as we did when considering Lévy processes since it will always be clear to which processes we are referring.) The local time $L = \{L_n^y, (n,y) \in N \times Z\}$ of X is simply the family of random variables $L_n^y = \{\text{number of times } j\colon X_j = y,\ 0 \le j \le n\}$. Note that, analogous to (1.10),

$$E^x \sum_{n=0}^{\infty} e^{-\alpha n}\left(L_n^y - L_{n-1}^y\right) = u^{\alpha}(x-y). \tag{1.15}$$

Just as we did for Lévy processes we set

$$\kappa(\alpha) = u^{\alpha}(0). \tag{1.16}$$

Let

$$\phi(\lambda) = Ee^{i\lambda Y_1}, \qquad \lambda \in [-\pi, \pi]. \tag{1.17}$$

It follows that

$$p_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos\lambda x\, \phi^n(\lambda)\, d\lambda$$

and

$$u^\alpha(x) = \frac{1}{\pi}\int_0^{\pi} \frac{\cos\lambda x\, d\lambda}{1-\exp(-\alpha)\phi(\lambda)}. \tag{1.18}$$

Thus

$$\kappa(\alpha) = \frac{1}{\pi}\int_0^{\pi} \frac{d\lambda}{1-\exp(-\alpha)\phi(\lambda)}, \tag{1.19}$$

and we define

$$\sigma^2(x) = \lim_{\alpha\to 0}\left(u^\alpha(0) - u^\alpha(x)\right) = \frac{1}{\pi}\int_0^{\pi} \frac{1-\cos\lambda x}{1-\phi(\lambda)}\, d\lambda. \tag{1.20}$$

We now set

$$\psi(\lambda) = 1 - \phi(\lambda).$$

It is easy to see that (1.9) also holds for κ and ψ as we defined them for symmetric random walks. Note that when ψ is regularly varying at zero the random walk is in the domain of attraction of a stable process. Theorem 1.1 carries over almost verbatim for symmetric random walks.

THEOREM 1.2. *Let X be a symmetric random walk as defined above and let κ, ψ and σ be as defined for X. Assume that* (1.6) *is satisfied and let $\{L_n^y, (n,y) \in N \times Z\}$ denote the local times of X. Then* (1.11), (1.12) *and the comment immediately following it of Theorem* 1.1 *hold as stated except that the limit superior and the supremum is taken over the integers.*

The techniques of Theorem 1.1, along with a clever idea of Finkelstein [11] used to prove functional laws of the iterated logarithm for empirical distributions, enable us to obtain analogous laws for the local times of symmetric Lévy processes and recurrent random walks.

Let B^β denote the set of functions $f \in C(0,1)$ such that $f(0) = 0$, f is absolutely continuous with respect to Lebesgue measure and $\int_0^1 |f'(x)|^\beta\, dx \le 1$. Let B_M^β denote the monotonically increasing functions in B^β. It follows, as in the classical case of $\beta = 2$, that B_M^β is compact in $C(0,1)$. Set

$$L_t(x) = \frac{L_{xt}^0}{\gamma(\overline{\beta})(\log\log t)\kappa((\log\log t)/t)}, \qquad 0 \le x \le 1, \tag{1.21}$$

where

$$\gamma(\overline{\beta}) \equiv \beta^{1/\beta}\overline{\beta}^{1/\overline{\beta}}. \tag{1.22}$$

THEOREM 1.3. *Let X be a symmetric Lévy process as defined in* (1.1) *for which* (1.6) *and* (1.7) *are satisfied and for which ψ is regularly varying at zero with index $1 < \beta \le 2$. Then the set of functions $\{L_t(\cdot)\colon\ 1 \le t < \infty\} \subset C(0,1)$ is relatively compact in $C(0,1)$ and the set of its limit points is B_M^β almost surely.*

Essentially the same result is valid for symmetric random walks in the domain of attraction of a stable process of index $\beta > 1$.

THEOREM 1.4. *Let X be a symmetric random walk and let κ and ψ be as defined for X. Assume that* (1.6) *is satisfied and that ψ is regularly varying at zero with index $1 < \beta \le 2$. Set*

$$L_n(x) = \frac{L_k^0 + (nx-k)\big(L_{k+1}^0 - L_k^0\big)}{\gamma\big(\overline{\beta}\big)(\log\log n)\kappa\big((\log\log n)/n\big)}, \qquad \frac{k}{n} \le x \le \frac{k+1}{n}. \tag{1.23}$$

Then the set of functions $\{L_n(\cdot)\colon\ 1 \le n < \infty\} \subset C(0,1)$ is relatively compact in $C(0,1)$ and the set of its limit points is B_M^β almost surely.

As a corollary of Theorem 1.4, we obtain a limit theorem for the regeneration times of symmetric random walks in the domain of attraction of a stable process of index $\beta > 1$.

THEOREM 1.5. *Let X be a symmetric random walk and let κ and ψ be as defined for X. Assume that* (1.6) *is satisfied and that ψ is regularly varying at zero with index $1 < \beta \le 2$. Let $\rho_1 < \rho_2 < \cdots$ denote the sequence of times k for which $X_k = 0$. Then, for any $q > 0$, we have*

$$\begin{aligned} &\limsup_{n\to\infty} \frac{1}{n^q(\log\log n)\kappa\big((\log\log n)/n\big)} \sum_{\{k:\rho_k \le n\}} \rho_k^q \\ &\qquad = \gamma(\overline{\beta})(1+q\overline{\beta})^{-1/\overline{\beta}} \quad a.s. \end{aligned} \tag{1.24}$$

Another corollary of Theorem 1.4 gives us a refinement of one of the results in Theorem 1.2. Let $\{L_n^0,\ n \in N\}$ be the local time of a symmetric random walk at zero. Consider the set of times j such that, for a fixed $0 \le c \le 1$,

$$L_j^0 > c\gamma\big(\overline{\beta}\big)(\log\log j)\kappa\left(\frac{\log\log j}{j}\right). \tag{1.25}$$

Set

$$a_j = \begin{cases} 1, & \text{if } L_j^0 > c\gamma\big(\overline{\beta}\big)(\log\log j)\kappa\left(\dfrac{\log\log j}{j}\right), \\ 0, & \text{otherwise.} \end{cases} \tag{1.26}$$

The next theorem follows from the methods of Strassen ([22], example (v) in Section 3).

THEOREM 1.6. *Let X be symmetric random walk that satisfies the hypotheses of Theorem* 1.4, *and let L_j^0 and a_j be as defined in* (1.25) *and* (1.26). *Then*

$$\limsup_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}a_j = 1-\exp\left(-\overline{\beta}^{\beta}\left(\frac{1}{c^{\beta}}-1\right)\right)\quad a.s.$$

We were attracted to questions about the rate of growth of local times of Lévy processes by the second-order law of the iterated logarithm of Csáki and Földes [5], where this result is obtained for the local times of Brownian motion. A similar result for the simple random walk was obtained earlier by Csörgő and Révész [7].

Our approach is essentially the same for first- and second-order laws. The first-order results we give in Theorem 1.1 are contained in a 1971 paper by Fristedt and Pruitt [12] although they do not obtain the value of the constant in the limit. Their result is a consequence of the fact that the local times of Lévy processes are the inverses of subordinators and they use the fact that subordinators are independent increment processes. This approach does not extend to second-order laws. The earliest first-order theorem with the constant evaluated was proved by Kesten [16] for Brownian local time. Donsker and Varadhan [9] obtain first-order laws for the local times of stable processes also with the constant evaluated. Our analogue of (1.11) for symmetric random walks, which we state in Theorem 1.2, was obtained earlier by different methods by Jain and Pruitt [13]. Our work on functional laws of the iterated logarithm was stimulated by the results of Csáki and Révész [6], who consider the local times of Brownian motion and the simple random walk.

2. Estimates of moment generating functions. Let

$$f(s,t) = E^0\exp(sL_t^0)\quad\text{and}\quad g(s,t) = E^0\exp\left[\varepsilon s(L_t^0 - L_t^x)\right], \tag{2.1}$$

where ε is a Rademacher random variable independent of X, that is,

$$P(\varepsilon = 1) = P(\varepsilon = -1) = \tfrac{1}{2}. \tag{2.2}$$

Continuing the notation of Theorem 1.1, we define

$$\kappa(\alpha,x) = u^{\alpha}(x)\quad\text{and}\quad h_x(\alpha) = \kappa^2(\alpha) - \kappa^2(\alpha,x). \tag{2.3}$$

Note that for x fixed, $h_x(\alpha)$ is strictly increasing as α decreases to zero. Also, we have by (1.5) that

$$\kappa(\alpha) = \kappa(\alpha,0) = u^{\alpha}(0)\quad\text{and}\quad \sigma^2(x) = \kappa(0) - \kappa(0,x). \tag{2.4}$$

The Laplace transforms of $f(s,t)$ and $g(s,t)$, considered as functions of t, with s fixed are

$$\mathcal{L}\big(f(s,t)\big) = \int_0^\infty e^{-\alpha t} f(s,t)\,dt = \frac{1}{\alpha\big(1-\kappa(\alpha)s\big)}, \qquad \kappa(\alpha)s < 1 \tag{2.5}$$

and

$$\mathcal{L}\big(g(s,t)\big) = \int_0^\infty e^{-\alpha t} g(s,t)\,dt = \frac{1}{\alpha\big(1-h_x(\alpha)s^2\big)}, \qquad h_x(\alpha)s^2 < 1. \tag{2.6}$$

These relations can be obtained by standard Markov process techniques. We obtained them in [17] by a simple application of the Dynkin isomorphism theorem. We proceed to estimate the inverse Laplace transforms of the right-hand sides of (2.5) and (2.6). The next lemma is an application of the monotone density theorem for regularly varying functions (see [3], Theorem 1.7.2).

LEMMA 2.1. *Assume that κ is regularly varying at zero with index $-1/\overline{\beta}$. Then*

$$-\lambda\kappa'(\lambda) \sim \frac{1}{\overline{\beta}}\kappa(\lambda) \quad \text{as } \lambda \to 0. \tag{2.7}$$

Let us also note that it follows from (1.7) that, for all $T > 0$,

$$\int_T^\infty \frac{\kappa(y)}{y}\,dy < \infty. \tag{2.8}$$

THEOREM 2.2. *Let $X = \{X(t), t \in [0,\infty)\}$ be a symmetric Lévy process with local times L_t^0 and L_t^x at 0 at x, respectively. Assume that $\kappa(\alpha)$ is regularly varying at zero with index $-1/\overline{\beta}$ and that (2.8) is satisfied. Then there exists an $s_0 > 0$ such that, for all $s \in [0, s_0]$,*

$$\left| E^0 \exp(sL_t^0) - w(s)\exp\left[\kappa^{-1}\left(\frac{1}{s}\right)t\right]\right| \le C\exp\left[\frac{1}{2}\kappa^{-1}\left(\frac{1}{s}\right)t\right] \quad \forall\, t > 0, \tag{2.9}$$

and

$$\begin{aligned} &\left| E^0 \exp[\varepsilon s(L_t^0 - L_t^x)] - v(s)\exp\left[h_x^{-1}\left(\frac{1}{s^2}\right)t\right]\right| \\ &\qquad \le C'\exp\left[\frac{1}{2}h_x^{-1}\left(\frac{1}{s^2}\right)t\right] \quad \forall\, t > 0, \end{aligned} \tag{2.10}$$

where κ^{-1} is the inverse of κ and h_x^{-1} is the inverse of h_x, with x fixed. Also, w and v are real-valued functions satisfying

$$\lim_{s\to 0} w(s) = \lim_{s\to 0} v(s) = \overline{\beta}, \tag{2.11}$$

and C and C' are constants depending only on s_0 and $\overline{\beta}$.

For $z = x + iy$, $x > 0$, set

$$\kappa(z) = \frac{1}{\pi}\int_0^\infty \frac{d\lambda}{\psi(\lambda)+z}. \tag{2.12}$$

Let $N = N(\overline{\beta})$ be a fixed positive number for which

$$\sqrt{2}\left(\frac{2}{N+1}\right)^{1/\overline{\beta}} < \frac{1}{2}. \tag{2.13}$$

We will be concerned with $\kappa(z)$ for $z = x_0 + iy$, for some $x_0 > 0$ fixed. Under the assumption that κ is regularly varying at zero with index $-1/\overline{\beta}$, given $\epsilon > 0$, let x_0' be small enough so that for all $x_0 \in (0, x_0']$

$$1-\epsilon \le \frac{v^{1/\overline{\beta}}\kappa(vx_0)}{\kappa(x_0)} \le 1+\epsilon \quad \forall\, 1 \le v \le 2N. \tag{2.14}$$

In what follows we will use the symbol C to indicate a nonzero constant not necessarily the same in each occurrence.

LEMMA 2.3. *Let κ be regularly varying at zero with index $-1/\overline{\beta}$ and assume that* (2.8) *holds. Let $z = x_0 + iy$, $x_0 > 0$, where x_0 satisfies* (2.14) *and let N satisfy* (2.13). *Then*

$$|\kappa(z)| \le \sqrt{2}\kappa(x_0+y) \qquad \forall\, y \ge 0, \tag{2.15}$$

$$\int_{Nx_0}^\infty \frac{\kappa(y)}{y}\,dy \le C\kappa(Nx_0), \tag{2.16}$$

$$|\kappa(x_0+iy)| \ge \kappa\left(\frac{5x_0}{4}\right), \qquad 0 \le y \le \frac{x_0}{2}, \tag{2.17}$$

$$|\mathrm{Im}\,\kappa(x_0+iy)| \ge C\kappa\big((N+1)x_0\big), \qquad \frac{x_0}{2} \le y \le Nx_0. \tag{2.18}$$

PROOF. Inequality (2.15) is immediate since

$$|\kappa(z)| \le \int_0^\infty \frac{d\lambda}{\left(\left(\psi(\lambda)+x_0\right)^2+y^2\right)^{1/2}} \le \sqrt{2}\int_0^\infty \frac{d\lambda}{\psi(\lambda)+x_0+y}.$$

The inequality in (2.16) follows by (2.8) and the regular variation of κ at zero. To get (2.17) we note that, since $y \le x_0/2$,

$$\begin{aligned} |\kappa(x_0+iy)| &> \mathrm{Re}\,\kappa(x_0+iy) \\ &\ge \int_0^\infty \frac{d\lambda}{\psi(\lambda)+x_0+x_0^2/4(\psi(\lambda)+x_0)} > \kappa\left(\frac{5x_0}{4}\right); \end{aligned} \tag{2.19}$$

finally, to get (2.18) we note that, by (2.7),

$$
\begin{aligned}
\left|\operatorname{Im} \kappa(z)\right| &= \int_0^\infty \frac{y}{\left(\psi(\lambda)+x_0\right)^2+y^2}\,d\lambda \ge \frac{x_0}{2}\int_0^\infty \frac{d\lambda}{\left(\psi(\lambda)+x_0\right)^2+(Nx_0)^2} \\
&\ge \frac{x_0}{2}\int_0^\infty \frac{d\lambda}{\left(\psi(\lambda)+(N+1)x_0\right)^2} = \frac{x_0}{2}\Big(-\kappa'\big((N+1)x_0\big)\Big) \\
&\ge C\kappa\big((N+1)x_0\big).
\end{aligned} \tag{2.20}
$$

□

PROOF OF (2.9) IN THEOREM 2.2. We will estimate $f(s,t)$ by estimating the inverse Laplace transform of $(\alpha(1-\kappa(\alpha)s))^{-1}$. Since

$$
\frac{1}{\alpha\big(1-\kappa(\alpha)s\big)} = \frac{1}{\alpha} + \frac{\kappa(\alpha)s}{\alpha\big(1-\kappa(\alpha)s\big)} \tag{2.21}
$$

and since $\mathcal{L}(1) = (1/\alpha)$, we have

$$
\mathcal{L}\big(f(s,t)-1\big) = \frac{\kappa(\alpha)s}{\alpha\big(1-\kappa(\alpha)s\big)}, \tag{2.22}
$$

and so

$$
f(s,t)-1 = \frac{1}{2\pi i}\int_\Gamma \frac{e^{zt}\kappa(z)s}{z\big(1-\kappa(z)s\big)}\,dy, \tag{2.23}
$$

where $\Gamma = x' + iy$, for some fixed x' for which $\kappa(x')s < 1$. Note that $\lim_{x\downarrow 0}\kappa(x) = \infty$ and $\kappa(x)$ is strictly decreasing as x increases. For each s choose x_0 such that $\kappa(2x_0)s = 1$. However, only consider those $s \in (0, s_0]$, where s_0 is small enough so that (2.14) holds for x_0. Take $x' > 2x_0$. [The expression on the right-hand side of the equality sign in (2.23) is equal to

$$
\bar{f}(s,t)-1 = e^{x't}\mathcal{F}^{-1}\mathcal{F}\Big(e^{-x't}\big(f(s,t)-1\big)\Big),
$$

where $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the Fourier and inverse Fourier transforms. As we shall see, the integral is in L^1 and so $\bar{f}(s,t) = f(s,t)$, for all t.] To evaluate this integral we consider a rectangular contour consisting of the straight line segments $\{x'+iy,\ -M \le y \le M\}$, $\{x_0+iy,\ -M\le y\le M\}$ and the horizontal lines that join them at y equal M and $-M$. Passing to the limit as M goes to infinity and taking into account the fact that value of the integrals along the sides of the rectangle parallel to the x axis go to zero, we get

$$
\begin{aligned}
f(s,t)-1 &= \operatorname{Res}\left\{\frac{\kappa(z)s}{z\big(1-\kappa(z)s\big)}e^{zt} : z = \kappa^{-1}\left(\frac{1}{s}\right)\right\} \\
&\quad + e^{x_0 t}\operatorname{Re}\frac{1}{\pi i}\int_0^\infty \frac{e^{iyt}}{(x_0+iy)}\left(\frac{\kappa(x_0+iy)s}{1-\kappa(x_0+iy)s}\right)dy,
\end{aligned} \tag{2.24}
$$

where

$$\operatorname{Res}\left\{\frac{\kappa(z)s}{z\big(1-\kappa(z)s\big)}e^{zt}: z=\kappa^{-1}\left(\frac{1}{s}\right)\right\}=e^{2x_0t}\lim_{z\to\kappa^{-1}(1/s)}\frac{z-\kappa^{-1}(1/s)}{z\big(1-\kappa(z)s\big)}.$$

By l'Hôpital's rule this last term equals $e^{2x_0t}(-(\kappa^{-1}(1/s)s\kappa'(\kappa^{-1}(1/s)))^{-1})$. Let $w(s) = -(\kappa^{-1}(1/s)s\kappa'(\kappa^{-1}(1/s)))^{-1}$. Then setting $s = 1/\kappa(\alpha)$ we see by (2.7) that

$$\lim_{s\to 0} w(s)=\lim_{\alpha\to 0}-\frac{\kappa(\alpha)}{\alpha\kappa'(\alpha)}=\overline{\beta}.$$

Thus we get the term in the absolute value in (2.9). This is the principal term in our estimate. We complete the proof of (2.9) by showing that

$$\int_0^\infty \frac{1}{(x_0^2+y^2)^{1/2}}\left|\frac{\kappa(x_0+iy)s}{1-\kappa(x_0+iy)s}\right|dy<C. \tag{2.25}$$

[The 1 can be absorbed into the right-hand side of (2.9).] We show that

$$\left|\frac{\kappa(x_0+iy)s}{1-\kappa(x_0+iy)s}\right|\le C,\quad 0\le y\le Nx_0, \tag{2.26}$$

which will give us

$$\int_0^{Nx_0} \frac{1}{(x_0^2+y^2)^{1/2}}\left|\frac{\kappa(x_0+iy)s}{1-\kappa(x_0+iy)s}\right|dy\le NC. \tag{2.27}$$

We first obtain (2.26) for $0\le y\le x_0/2$. We have

$$\operatorname{Re}\kappa(z)-\frac{1}{s}\ge C\operatorname{Re}\kappa(z),\qquad 0\le y\le\frac{x_0}{2}, \tag{2.28}$$

since, by (2.14) and (2.19),

$$1-\frac{\kappa(2x_0)}{\operatorname{Re}\kappa(z)}\ge 1-\frac{\kappa(2x_0)}{\kappa(5x_0/4)}\ge(1-\varepsilon)\left(1-\left(\frac{5}{8}\right)^{1/\overline{\beta}}\right).$$

Using (2.28) we see that, for $0\le y\le x_0/2$,

$$\begin{aligned}\left|\frac{s\kappa(z)}{1-s\kappa(z)}\right|^2&=\frac{\big(\operatorname{Re}\kappa(z)\big)^2+\big(\operatorname{Im}\kappa(z)\big)^2}{\big(\operatorname{Re}\kappa(z)-1/s\big)^2+\big(\operatorname{Im}\kappa(z)\big)^2}\\&\le\frac{\big(\operatorname{Re}\kappa(z)\big)^2+\big(\operatorname{Im}\kappa(z)\big)^2}{C^2\big(\operatorname{Re}\kappa(z)\big)^2+\big(\operatorname{Im}\kappa(z)\big)^2}\le\frac{1}{C^2}.\end{aligned}$$

When $x_0/2\le y\le Nx_0$, by (2.15) and (2.18),

$$\left|\frac{s\kappa(z)}{1-s\kappa(z)}\right|^2\le 1+\frac{\big(\operatorname{Re}\kappa(z)\big)^2}{\big(\operatorname{Im}\kappa(z)\big)^2}\le 1+\frac{2\kappa\big((N+1)x_0\big)}{C\kappa\big((N+1)x_0\big)}\le C.$$

Thus we get (2.26) and hence (2.27).

Now note that, for $y \geq Nx_0$, by (2.14), (2.15) and the fact that κ is decreasing on $(0, \infty)$,

$$|s\kappa(z)| \leq \frac{\sqrt{2}\kappa((N+1)x_0)}{\kappa(2x_0)} \leq \sqrt{2}\left(\frac{2}{N+1}\right)^{1/\beta}.$$

By (2.13) this is less than $\frac{1}{2}$. Thus $|s\kappa(z)/(1-s\kappa(z))| \leq Cs|\kappa(z)|$. Therefore,

$$\int_{Nx_0}^{\infty} \frac{1}{(x_0^2+y^2)^{1/2}} \left| \frac{\kappa(z)s}{1-\kappa(z)s} \right| dy \leq Cs \int_{Nx_0}^{\infty} \frac{\kappa(x_0+y)}{y}\, dy$$

$$\leq Cs \int_{Nx_0}^{\infty} \frac{\kappa(y)}{y}\, du.$$

Using (2.16) we see that this is less than or equal to

$$Cs\kappa(Nx_0) \leq C.$$

Thus we obtain (2.25) and hence (2.9). □

The proof of (2.10) mimics the proof of (2.9). For fixed w set

$$p(z) = \int_0^{\infty} \frac{1+\cos \lambda w}{\psi(\lambda)+z}\, d\lambda = \kappa(z) + \kappa(z,w), \tag{2.29}$$

$$q(z) = \int_0^{\infty} \frac{1-\cos \lambda w}{\psi(\lambda)+z}\, d\lambda = \kappa(z) - \kappa(z,w), \tag{2.30}$$

$$h(z) = p(z)q(z). \tag{2.31}$$

We will take $h(z)$ as the analytic extension of $(\kappa(\alpha)+\kappa(\alpha,w))(\kappa(\alpha)-\kappa(\alpha,w))$. Analogously to Lemma 2.3, we have the following lemma.

LEMMA 2.4. *Let N and x_0 be as given in* (2.13) *and* (2.14). *For w fixed and $z = x_0 + iy$, we have*

$$h(x) \sim 2\kappa(x)q(0), \qquad as\ x \to 0, \tag{2.32}$$

$$|h(z)| \leq 4\kappa(x_0+y)q(0), \qquad \forall\, y > 0, \tag{2.33}$$

$$|h(x_0+iy)| \geq (1-2\epsilon)h\left(\frac{5x_0}{4}\right), \qquad 0 \leq y \leq \frac{x_0}{2}, \tag{2.34}$$

$$|\mathrm{Im}\, h(x_0+iy)| \geq Ch\big((N+1)x_0\big), \qquad \frac{x_0}{2} \leq y \leq Nx_0, \tag{2.35}$$

where $\epsilon = \epsilon(x_0) > 0$ can be taken arbitrarily small for $x_0 > 0$ sufficiently small.

PROOF. Clearly $q(x) \uparrow q(0) < \infty$ as $x \downarrow 0$. Let us take x_0 smaller, if necessary, so that we also have

$$(1-\varepsilon)q(0) \leq q\Big(\big(N^2+1\big)x_0\Big) < q(0). \tag{2.36}$$

On the other hand $p(x) > \kappa(x)$ goes to infinity as $x \downarrow 0$. Thus, since

$$h(x) = \big(2\kappa(x) - q(x)\big)q(x), \tag{2.37}$$

we get (2.32). To obtain (2.33) we note that, as in (2.15),

$$|p(z)| \le \sqrt{2}\, p(x_0 + y) \le 2\sqrt{2}\, \kappa(x_0 + y),$$
$$|q(z)| \le \sqrt{2}\, q(x_0 + y) \le \sqrt{2}\, q(0).$$

Note that proceeding as in the proof of (2.17) and using (2.36) we see that, for $0 \le y \le x_0/2$,

$$\big|q(x_0 + iy)\big| \ge q\left(\frac{5x_0}{4}\right) \ge (1 - \varepsilon)q(0). \tag{2.38}$$

Also by (2.36) and (2.37) we have, for $0 \le y \le x_0/2$,

$$h\left(\frac{5x_0}{4}\right) \le 2(1 + \varepsilon)\kappa\left(\frac{5x_0}{4}\right)q(0) \tag{2.39}$$

and again by (2.37), with x replaced by $x_0 + iy$, and also by (2.17) and (2.38) we have, for $0 \le y \le x_0/2$,

$$h(x_0 + iy) \ge 2(1 - \varepsilon)\kappa(x_0 + iy)q(0) \ge 2(1 - \varepsilon)\kappa\left(\frac{5x_0}{4}\right)q(0). \tag{2.40}$$

Combining (2.39) and (2.40) we obtain (2.34). To obtain (2.35) we note that, for $x_0/2 \le y \le Nx_0$, as in (2.20),

$$\begin{aligned}|\mathrm{Im}\, p(z)| &\ge \frac{x_0}{2}\int_0^\infty \frac{1 + \cos\lambda w}{\big(\psi(\lambda) + (N+1)x_0\big)^2}\, d\lambda \\ &= \frac{x_0}{2}\Big(- \kappa'\big((N+1)x_0\big) - \kappa'\big((N+1)x_0, w\big)\Big) \\ &\ge \frac{x_0}{2}\Big(- \kappa'\big((N+1)x_0\big)\Big) \ge C\kappa\big((N+1)x_0\big),\end{aligned}$$

and it follows, as in (2.19), that

$$\mathrm{Re}\, q(x_0 + iy) \ge \int_0^\infty \frac{1 - \cos\lambda w}{\psi(\lambda) + x_0 + N^2x_0^2/[\psi(\lambda) + x_0]}\, d\lambda > q\Big(\big(N^2 + 1\big)x_0\Big).$$

Therefore by the above, (2.36) and (2.37) we have

$$\begin{aligned}|\mathrm{Im}\, h(z)| &= \mathrm{Re}\, p(z)|\mathrm{Im}\, q(z)| + |\mathrm{Im}\, p(z)|\mathrm{Re}\, q(z) \\ &\ge |\mathrm{Im}\, p(z)|\mathrm{Re}\, q(z) \\ &\ge C\kappa\big((N+1)x_0\big)q(0) \ge Ch\big((N+1)x_0\big).\end{aligned}$$

□

PROOF OF (2.10) IN THEOREM 2.2. The proof is the same as the proof of (2.9). For s fixed, we define x_0 by $h(2x_0)s^2 = 1$. Then

$$g(s,t) - 1 = \frac{1}{2\pi i} \int_\Gamma \frac{e^{zt} h(z) s^2}{z\left(1 - h(z)s^2\right)}\, dy$$

where, as in the proof of (2.9), $\Gamma = x' + iy$ for some fixed $x' > 2x_0$. We evaluate this integral by using the same contour that gives rise to (2.24) (although the values of x_0 and x' are different). Analogously to (2.24), we get a pole at

$$z = h^{-1}\left(\frac{1}{s^2}\right)$$

and see that

$$v(s) = -\left(h^{-1}\left(\frac{1}{s^2}\right) s^2 h'\left(h^{-1}\left(\frac{1}{s^2}\right)\right)\right)^{-1}.$$

Setting $s^2 = 1/h(\alpha)$, we have

$$\lim_{\alpha\to 0} v(s) = \lim_{\alpha\to 0} \frac{h(\alpha)}{\alpha h'(\alpha)} = \lim_{\alpha\to 0} \frac{\kappa(\alpha)}{\alpha\kappa'(\alpha)} = \overline{\beta}$$

since

$$\lim_{\alpha\to 0} \frac{h(\alpha)}{2\kappa(\alpha)q(0)} = 1 \quad \text{and} \quad \lim_{\alpha\to 0} \frac{h'(\alpha)}{2\kappa'(\alpha)q(0)} = 1.$$

The integral corresponding to (2.27) is bounded because (2.33)–(2.35) are essentially the analogues of (2.15), (2.17) and (2.18), which is what we used to bound (2.27). Finally, we note that, by (2.33),

$$\int_{Nx_0}^\infty \frac{h(y)}{y}\, dy \le C \int_{Nx_0}^\infty \frac{\kappa(y)}{y}\, dy\, q(0) \le C\kappa(Nx_0)q(0) \le Ch(Nx_0),$$

and since $s^2 h(Nx_0) < C$ we can follow all the steps used to get (2.9) to obtain (2.10) as well. □

COROLLARY 2.5. *In the notation of Theorem* 2.2 *we have that, for all* $\epsilon > 0$, *there exists an* $s_0 = s_0(\epsilon) > 0$ *such that, for all* $s \in [0, s_0]$ *and all* t *greater than* *or equal to zero,*

$$E^0 \exp\left(\varepsilon \frac{s^{1/2}}{\sqrt{2}\sigma(x)}(L_t^0 - L_t^x)\right) \leq v(\gamma(s)) \exp\left((1+\epsilon)\kappa^{-1}\left(\frac{1}{s}\right)t\right) + o\left(\exp\left(\kappa^{-1}\left(\frac{1}{s}\right)t\right)\right), \tag{2.41}$$

$$E^0 \exp\left(\varepsilon \frac{s^{1/2}}{\sqrt{2}\sigma(x)}(L_t^0 - L_t^x)\right) \geq v(\gamma(s)) \exp\left((1-\epsilon)\kappa^{-1}\left(\frac{1}{s}\right)t\right) + o\left(\exp\left(\kappa^{-1}\left(\frac{1}{s}\right)t\right)\right), \tag{2.42}$$

where

$$\gamma(s) = \frac{s^{1/2}}{\sqrt{2}\sigma(x)}$$

and $\sigma^2(x)$ *is as given in* (1.4). *The "little o" limit is taken as* $\kappa^{-1}(1/s)t$ *goes to infinity.*

PROOF. This follows from (2.4), (2.10) and (2.32). □

LEMMA 2.6. *For any symmetric Lévy process, for which the local times exist, the functions* $f(s,t)$ *and* $g(s,t)$ *defined in* (2.1) *are increasing functions of* t. *Furthermore, for all* $s, t \geq 0$,

$$E^y \exp\left(sL_t^0\right) \leq E^0 \exp\left(sL_t^0\right), \tag{2.43}$$

$$E^y \exp\left(sL_t^0\right) \geq 1 + s\int_0^{\epsilon t} p_v(y)\,dv E^0 \exp\left(sL_{t(1-\epsilon)}^0\right), \quad 0 < \epsilon < 1, \tag{2.44}$$

$$E^y \exp\left[\varepsilon s\left(L_t^0 - L_t^x\right)\right] \leq 2E^0 \exp\left[\varepsilon s\left(L_t^0 - L_t^x\right)\right], \tag{2.45}$$

and for each $x \in R$ *and* $\epsilon > 0$, *sufficiently small, there exists a* $t_0 = t_0(\epsilon)$ *such that*

$$E^y \exp\left[\varepsilon s\left(L_t^0 - L_t^x\right)\right] \geq 1 + \frac{s^2\sigma^2(x)}{2}\int_0^{\epsilon t/2} \left(p_v(y) + p_v(y-x)\right) dv E^0 \exp\left[\varepsilon s\left(L_{t(1-\epsilon)}^0 - L_{t(1-\epsilon)}^x\right)\right], \tag{2.46}$$

for all $s \geq 0$ *and* $t \geq t_0$.

PROOF. The statement about the monotonicity of $f(s,t)$ is trivial. Monotonicity is easily verified for $g(s,t)$ by considering the even moments of $L_t^0 - L_t^x$, which can be expressed as integrals from 0 to t of a nonnegative integrand (see [21] or Remark 2.4 in [17]). Inequalities (2.43)–(2.46) follow from the equations

$$E^y\Big(\exp\big(sL_t^0\big)\Big) = 1 + s\int_0^t p_{t-v}(y)f(s,v)\,dv \tag{2.47}$$

and

$$E^y \exp\Big[\varepsilon s\big(L_t^0 - L_t^x\big)\Big] = 1 + s^2\int_0^t\int_0^v h_{t-v}(y,x)k_{v-u}g(s,u)\,du\,dv, \tag{2.48}$$

where

$$h_{t-v}(y,x) = p_{t-v}(y) + p_{t-v}(y-x) \quad \text{and} \quad k_{t-v} = p_{t-v}(0) - p_{t-v}(x).$$

These are obtained by inverting the Laplace transforms given in [17, (1.5), (2.11) and (2.13)]. [In obtaining the inequalities we use the monotonicity of $f(s,t)$ and $g(s,t)$ in t and the fact that $p_t(0) \geq p_t(y)$ for all y).] □

REMARK 2.7. If ψ is regularly varying at zero with index $1 < \beta \leq 2$, then it is asymptotic at zero to a monotonic function (see, e.g., [3], Theorem 1.5.3). Hence, for the purposes of obtaining (1.9), we can assume that ψ^{-1} exists. Therefore we can write

$$p_t(0) = \frac{1}{\pi}\int_0^\infty e^{-t\lambda}\,d\psi^{-1}(\lambda)$$

and by [3], Theorem 1.7.1, we see that

$$p_t(0) \sim \frac{\Gamma(1+1/\beta)}{\pi}\psi^{-1}(1/t) \quad \text{as } t \to \infty. \tag{2.49}$$

Since $\kappa(\lambda)$ is the Laplace transform of $p_t(0)$, we get (1.9) from (2.49) and Theorem 1.7.6 of [3].

The next lemma will be used in the proof of (1.11) and (1.12).

LEMMA 2.8. *Assume that* ψ *is regularly varying at zero with index* $1 < \beta \leq 2$, *and let* p_t *be the corresponding transition probability density function. Then there exists a* t_0 *such that, for all* $t > t_0$,

$$\sup_{y\in R}\left|\int_0^t \big(p_s(y-x) - p_s(y)\big)\,ds\right| \leq F(x)\log t, \tag{2.50}$$

where $F(x) < \infty$.

PROOF. We have

$$\begin{aligned}\left|\int_0^t \big(p_s(y-x)-p_s(y)\big)\,ds\right| &\le \frac{1}{\pi}\int_0^t\int_0^\infty \left|1-e^{i\lambda x}\right|e^{-s\psi(\lambda)}\,d\lambda\,ds \\ &= \frac{1}{\pi}\int_0^\infty \frac{\left|1-e^{i\lambda x}\right|\left|1-e^{-t\psi(\lambda)}\right|}{\psi(\lambda)}\,d\lambda.\end{aligned} \tag{2.51}$$

We obtain (2.50) by considering the last integral in (2.51) over the intervals $[0,1/x]$ and $[1/x,\infty]$ if $\int_0^1(\lambda\,d\lambda)/\psi(\lambda) < \infty$ and, if this is infinite, over the intervals $[0,\psi^{-1}(1/t)]$, $[\psi^{-1}(1/t),1/x]$ and $[1/x,\infty]$, for t sufficiently large. □

LEMMA 2.9. *Let X be a symmetric Lévy process for which $(1+\psi(\lambda))^{-1} \in L^1(R^+)$, and let $\sigma^2(x)$ be as defined in* (1.4). *Define*

$$h(x,y) = \sigma^2(y-x) - \sigma^2(y). \tag{2.52}$$

Then

$$\sup_y h(x,y) = \sigma^2(x).$$

PROOF. This follows from an interesting inequality for the α-potential of a symmetric Lévy process which seems to be rediscovered often, namely, that

$$u^\alpha(x) + u^\alpha(y) \le u^\alpha(0) + u^\alpha(y-x) \quad \forall\, x,y \in R.$$

(See, e.g., [17], Lemma 2.5.) □

3. Proofs of Theorems 1.1 and 1.2.

Let

$$\ell_2(t) = \log\log t, \qquad \phi(t) = \ell_2(t)\kappa\left(\frac{\ell_2(t)}{t}\right) \quad \text{and} \quad Y_t = \frac{L_t^0}{\phi(t)}. \tag{3.1}$$

To simplify matters, we will first consider the results pertaining to L_t^0.

LEMMA 3.1. *Assume that κ is regularly varying at zero with index $-1/\overline{\beta}$. Then, for any y, $\epsilon > 0$, there exists a t_0 such that, for all $t \ge t_0$ and all x,*

$$P^x(Y_t \ge y) \le \exp\big(-(1-\epsilon)c(\overline{\beta})y^{\beta}\ell_2(t)\big) \tag{3.2}$$

where

$$c(\overline{\beta}) = (\overline{\beta}-1)/\overline{\beta}^{\beta}. \tag{3.3}$$

Moreover, t_0 can be chosen uniformly in y, for y in any set bounded away from zero and infinity.

PROOF. By Theorem 2.2, Chebyshev's inequality and (2.43), for all $s \le s_0$, for some s_0 sufficiently small and for all $w \ge 0$,

$$P^x(L_t^0 > w) \le C \exp\left(\kappa^{-1}(1/s)t - sw\right) \quad \forall\, t \in [0, \infty). \tag{3.4}$$

For $a > 0$, define $\rho_a(t) = \kappa(a\ell_2(t)/t)$. Fix $y > 0$. Since κ is regularly varying at zero,

$$\rho_a(t) \le \left(1 + \frac{\epsilon}{2}\right) a^{-1/\overline{\beta}} \rho_1(t), \tag{3.5}$$

for all t sufficiently large, where $\epsilon \to 0$ as $t \to \infty$. Set $s = 1/\rho_a(t)$, where $t \ge t_0$ and t_0 is chosen large enough so that $s \le s_0$, and set $w = y\phi(t)$. Substituting these values for s and w in (3.4) and using (3.5), we see that

$$\begin{aligned} P^x(L_t^0 > w) &\le C \exp\left(- \left((1-\epsilon)a^{1/\overline{\beta}}y - a\right)\ell_2(t)\right) \\ &= C \exp\left(- (a^{1/\overline{\beta}}y - a)\ell_2(t) + \epsilon a^{1/\overline{\beta}} y \ell_2(t)\right). \end{aligned} \tag{3.6}$$

Setting $a = (y/\overline{\beta})^{\beta}$, the value that minimizes $a^{1/\overline{\beta}}y - a$, and using (3.6), we get (3.2). The final statement in this lemma follows because the estimate in (3.5) can be made uniformly for a in an interval bounded away from zero and infinity if t is taken to be sufficiently large. □

In the next lemma we will obtain a lower bound for the probability distribution of Y_t, but only in the narrow region needed to obtain the lower bound in (1.11).

LEMMA 3.2. *Assume that κ is regularly varying at zero with index $-1/\overline{\beta}$, and let*

$$a_\nu(x,t) = \frac{\int_0^{\nu t} p_v(x)\,dv}{\kappa\left(\ell_2(t)/t\right)}. \tag{3.7}$$

Then for all t sufficiently large we can find a $\nu_0 > 0$ such that, for all $0 < \nu < \nu_0$,

$$\begin{aligned} &P^x\left(Y_t \ge (1 - \eta_1(\nu))c(\overline{\beta})^{-1/\beta}\right) \\ &\quad \ge C\left(a_\nu(x,t)\exp\left[-\ell_2(t)\right] - \exp\left[-\left(1 + \eta_2(\nu)\right)\ell_2(t)\right]\right), \end{aligned} \tag{3.8}$$

where $\eta_i(\nu) > 0$ and $\lim_{\nu \to 0} \eta_i(\nu) = 0$, $i = 1, 2$.

PROOF. Following Davies [8] and using the fact that Y_t is continuous in t, we see that, for any $u,\ t > 0,\ 0 < \delta < 1$ and $0 < w < y$, we have

$$
\begin{aligned}
P^x(Y_t \geq w) &\geq \int_w^y dP^x(Y_t \leq z) \\
&\geq e^{-(1-\delta)uy} \int_w^y e^{(1-\delta)uz}\, dP^x(Y_t \leq z) \\
&= e^{-(1-\delta)uy} \left(\left(\int_0^\infty - \int_y^\infty - \int_0^w \right) \left(e^{(1-\delta)uz}\, dP^x(Y_t \leq z) \right) \right) \\
&\geq e^{-(1-\delta)uy} E^x\left(e^{(1-\delta)uY_t}\right) - e^{-uy} E^x(e^{uY_t}) \\
&\quad - e^{-(1-\delta)uy} \int_0^w e^{(1-\delta)uz}\, dP^x(Y_t \leq z) \\
&= J_1 - J_2 - J_3.
\end{aligned}
\tag{3.9}
$$

By (2.44) and Theorem 2.2 we see that, for all $0 < \nu < 1$ and all t sufficiently large,

$$
\begin{aligned}
J_1 \geq{} & \frac{C(1-\delta)u}{\phi(t)} \left(\int_0^{\nu t} p_v(x)\, dv \right) \\
& \times \exp\left(-(1-\delta)uy + \kappa^{-1}\left(\frac{\phi(t)}{(1-\delta)u} \right)(1-\nu)t \right).
\end{aligned}
\tag{3.10}
$$

Let $u = b\ell_2(t)$, where $b = (y/\overline{\beta})^{\beta-1}$, and let $\nu = \delta^4$. Then, taking into account the regular variation of κ and hence of κ^{-1} at zero, we see that, for all δ sufficiently small,

$$
\begin{aligned}
&(1-\delta)uy - \kappa^{-1}\left(\frac{\phi(t)}{(1-\delta)u} \right)(1-\nu)t \\
&= (1-\delta)yb\ell_2(t) - \kappa^{-1}\left(\frac{\kappa\left(\ell_2(t)/t\right)}{(1-\delta)b} \right)(1-\nu)t \\
&\leq (1+\delta^3)\left((1-\delta)yb - (1-\delta)^{\overline{\beta}} b^{\overline{\beta}} \right)\ell_2(t) \\
&\leq y^\beta \left(\frac{1}{\overline{\beta}} \right)^\beta (1+\delta^3)\left((1-\delta)\overline{\beta} - (1-\delta)^{\overline{\beta}}\right)\ell_2(t) \equiv v(\delta) y^\beta \ell_2(t),
\end{aligned}
\tag{3.11}
$$

where

$$
\begin{aligned}
v(\delta) &= \left(\frac{1}{\overline{\beta}} \right)^\beta (1+\delta^3)\left((1-\delta)\overline{\beta} - (1-\delta)^{\overline{\beta}} \right) \\
&= c(\overline{\beta})\left(1 - \frac{\overline{\beta}}{2}\delta^2 + O(\delta^3) \right).
\end{aligned}
\tag{3.12}
$$

We now set $y = v^{-1/\beta}(\delta)$ and see that, for all $0 < \nu < 1$ and all t sufficiently large,

$$J_1 \geq C a_\nu(x,t) e^{-\ell_2(t)}. \tag{3.13}$$

By Theorem 2.2 and (2.43) we see that

$$J_2 \leq \exp(-uy) E^0\big(\exp(uY_t)\big) \leq C \exp\left(-\left(uy - \kappa^{-1}\left(\frac{\phi(t)}{u}\right) t\right)\right). \tag{3.14}$$

Substituting for u and b and taking into account the regular variation of κ at zero, as we did in dealing with J_1, we see that, for all $\epsilon > 0$,

$$\begin{aligned} J_2 &\leq C \exp\Big(-(1-\epsilon) c(\overline{\beta}) y^\beta \ell_2(t)\Big) \\ &= C \exp\left(-(1-\epsilon)\frac{c(\overline{\beta})}{v(\delta)} \ell_2(t)\right), \end{aligned} \tag{3.15}$$

for all t sufficiently large. Taking $\delta^3 = \epsilon$ for δ as given in the beginning of this proof and substituting for $v(\delta)$, we get

$$\begin{aligned} J_2 &\leq C \exp\left(-\left(\frac{1-\delta^3}{1-(\overline{\beta}/2)\delta^2 + O(\delta^3)}\right)\ell_2(t)\right) \\ &\leq C \exp\left(-\left(1 + \frac{\overline{\beta}\delta^2}{4}\right)\ell_2(t)\right), \end{aligned} \tag{3.16}$$

since we can take ϵ and hence δ as small as we like.

We now obtain an upper bound for J_3. Let $\gamma > 0$. By integration by parts we see that

$$\begin{aligned} J_3 \leq\ & \exp\big[-(1-\delta)uy\big]\Big(\exp\big[(1-\delta)u\gamma\big] \\ &\qquad + \int_\gamma^w \exp\big[(1-\delta)uz\big]\, dP^x\big(Y_t \leq z\big)\Big) \\ \leq\ & 2\exp\big[-(1-\delta)u(y-\gamma)\big] \\ &+ u(1-\delta)\exp\big[-(1-\delta)uy\big] \int_\gamma^w \exp\big[(1-\delta)uz\big] P^x\big(Y_t \geq z\big)\, dz. \end{aligned} \tag{3.17}$$

Substituting for u and y and using the last line of (3.12), we see that

$$\begin{aligned} \exp\big(-(1-\delta)uy\big) &= \exp\left(-\frac{(1-\delta)\overline{\beta}\ell_2(t)}{(\overline{\beta})^\beta c(\overline{\beta})\Big(1 - (\overline{\beta}\delta^2)/2 + O(\delta^3)\Big)}\right) \\ &= \exp\left(-\frac{(1-\delta)\beta\ell_2(t)}{\Big(1-(\overline{\beta}\delta^2)/2 + O(\delta^3)\Big)}\right). \end{aligned} \tag{3.18}$$

Clearly, for all δ sufficiently small we can find a $\gamma > 0$, independent of δ, such that

$$\exp(-(1-\delta)uy) \le \exp\left(-\left(1+\frac{\beta-1}{2}\ell_2(t)\right)\right)$$

and

$$\exp(1-\delta)u\gamma \le \exp\left(\frac{(\beta-1)}{4}\ell_2(t)\right).$$

Thus for all δ sufficiently small we can find a $\gamma > 0$, independent of δ, such that, for some $\eta > 0$,

$$2\exp\left[-(1-\delta)u(y-\gamma)\right] \le \exp\left[-(1+2\eta)\ell_2(t)\right]. \tag{3.19}$$

Next let us consider the integrand in (3.17). By Lemma 3.1,

$$\begin{aligned}&\exp\left[(1-\delta)uz\right]P^x(Y_t \ge z)\\ &\quad\le \exp\left((1-\delta)uz - (1-\delta^3)c(\overline{\beta})z^{\beta}\ell_2(t)\right) \equiv R(z),\end{aligned} \tag{3.20}$$

for any $\delta > 0$ and z in a compact set bounded away from zero as long as t is sufficiently large. We have already chosen γ. We will take $w = (1-c\delta)y$. Note that $R(z)$ is increasing for $z \le z_0$, where z_0 is determined by

$$(1-\delta)u = (1-\delta^3)\beta z_0^{\beta-1}c(\overline{\beta})\ell_2(t).$$

Substituting for u and $c(\overline{\beta})$ this is

$$\frac{1-\delta}{1-\delta^3}\left(\frac{y}{\overline{\beta}}\right)^{\beta-1} = \left(\frac{z_0}{\overline{\beta}}\right)^{\beta-1}. \tag{3.21}$$

Writing $z_0 = (1-d\delta)y$, (3.21) becomes

$$\frac{1-\delta}{1-\delta^3} = 1-(\beta-1)d\delta + (\beta-1)(\beta-2)\frac{(d\delta)^2}{2} + O(\delta^3). \tag{3.22}$$

When $\beta = 2$, (3.22) is satisfied for $d = 1+O(\delta^2)$. Thus for $c = 2$, $R(z)$ is increasing for $z \le w$. Before considering the cases $1 < \beta < 2$ let us note that, in general, if $R(z)$ is increasing for $z \le w$, then by Lemma 3.1,

$$\begin{aligned}&u(1-\delta)\exp\left[-(1-\delta)uy\right]\int_{\gamma}^{w}\exp\left[(1-\delta)uz\right]P^x(Y_t \ge z)\,dz\\ &\quad\le u(w-\gamma)\exp\left[-(1-\delta)uy\right]R(w)\\ &\quad\le \exp\left(-\left((1-\delta)c\delta y + (1-\delta^3)c(\overline{\beta})(1-c\delta)^{\beta}y^{\beta}\ell_2(t)\right)\right)\\ &\quad= \exp\left(-\left((1-\delta)c\delta\overline{\beta} + (1-\delta^3)(\overline{\beta}-1)(1-c\delta)^{\beta}\right)\frac{y^{\beta}\ell_2(t)}{(\overline{\beta})^{\beta}}\right).\end{aligned}$$

In general we will choose c depending on β such that

$$((1-\delta)c\delta\overline{\beta}+(1-\delta^3)(\overline{\beta}-1)(1-c\delta)^{\beta})\frac{y^{\beta}}{(\overline{\beta})^{\beta}}\geq(1+2\eta), \tag{3.23}$$

for some $\eta > 0$ (which clearly goes to zero as δ goes to zero), and this and (3.19) will show that

$$J_3 \leq \exp\left[-(1+\eta)\ell_2(t)\right], \tag{3.24}$$

for some $\eta > 0$ for all t sufficiently large. Clearly, (3.23) holds if

$$(1-\delta)c\delta\overline{\beta}+(1-\delta^3)(\overline{\beta}-1)(1-c\delta)^{\beta}\geq(1-\delta)\overline{\beta}-(1-\delta)^{\overline{\beta}}. \tag{3.25}$$

Returning to the case $\beta = 2$ (in which case $c = 2$), (3.25) is simply

$$(1-\delta)4\delta+(1-\delta^3)(1-2\delta)^2\geq 1-\delta^2.$$

Thus (3.24) holds when $\beta = 2$.

Now let us consider (3.21) when $1 < \beta < 2$. It is easy to see that (3.22) is satisfied for some d for which $1-\delta < 1-(\beta-1)d\delta$, that is, for some $d < 1/(\beta-1)$. Take $c = (1-p)/(\beta-1)$, where $p > 0$ is small enough so that $c > d$. For this choice of c, $R(z)$ is increasing for $z \leq w$. Thus we can obtain (3.24) if we can verify (3.25). The left-hand side of (3.25) can be expanded as

$$\begin{aligned}&(1-\delta)c\delta\overline{\beta}+(1-\delta^3)(\overline{\beta}-1)(1-c\delta)^{\beta}\\&=(\overline{\beta}-1)+\frac{\overline{\beta}(\overline{\beta}-1)}{2}(-1+p^2)\delta^2+O(\delta^3),\end{aligned}$$

whereas, as in (3.12), the right-hand side of (3.25) is

$$(\overline{\beta}-1)-\frac{\overline{\beta}(\overline{\beta}-1)}{2}\delta^2+O(\delta^3).$$

Thus (3.25) does hold. Combining (3.9), (3.13), (3.16) and (3.24), substituting for y and making δ smaller if necessary, we get (3.8). This completes the proof of Lemma 3.2. □

PROOF OF (1.11) IN THEOREM 1.1. By (1.9), κ is regularly varying at zero. Let

$$d(\overline{\beta})=\left(c(\overline{\beta})\right)^{-1/\beta}=\beta^{1/\beta}\overline{\beta}^{1/\overline{\beta}}. \tag{3.26}$$

In Lemma 3.1 let $y^{\beta} = (1+3\epsilon)(d(\overline{\beta}))^{\beta}$ and $t_n = \theta^n$, where $\epsilon > 0$ and $\theta > 1$. This gives us

$$Y_{t_n}\leq(1+3\epsilon)d(\overline{\beta}),$$

for all n sufficiently large almost surely. Also note that, by the regular variation of κ,

$$\lim_{n\to\infty} \frac{\phi(t_{n+1})}{\phi(t_n)} = \theta^{1/\beta}. \tag{3.27}$$

Therefore, since L_t^0 is monotone in t and since ϵ can be taken to be arbitrarily small and θ can be taken arbitrarily close to 1, we get

$$\limsup_{n\to\infty} \frac{L_t^0}{\phi(t)} \le d(\overline{\beta}) \quad \text{a.s.} \tag{3.28}$$

the desired upper bound in (1.11). To obtain the lower bound needed in (1.11), set

$$H_n = L_{t_n}^0 - L_{t_{n-1}}^0. \tag{3.29}$$

We show that, for any $\epsilon < 0$,

$$\limsup_{n\to\infty} \frac{H_n}{\phi(t_n)} \ge (1-\epsilon)d(\overline{\beta}) \quad \text{a.s.} \tag{3.30}$$

which implies that, for all $\epsilon < 0$,

$$\limsup_{t\to\infty} \frac{L_{t_n}^0}{\phi(t_n)} \ge (1-\epsilon)d(\overline{\beta}) \quad \text{a.s.},$$

and this along with (3.28) will complete the proof of (1.11).

We proceed to obtain (3.30). Set $s_n = t_n - t_{n-1}$ and note that $H_n = L_{s_n}^0 \circ \theta_{t_{n-1}}$. By the Markov property we have that

$$\begin{aligned} &\sum_{n=1}^{\infty} P^0\left(\frac{H_n}{\phi(t_n)} \ge (1-\epsilon)d(\overline{\beta}) \,\middle|\, \mathcal{F}_{t_{n-1}}\right) \\ &\qquad = \sum_{n=1}^{\infty} P^{X_{t_{n-1}}}\left(\frac{L_{s_n}^0}{\phi(t_n)} \ge (1-\epsilon)d(\overline{\beta})\right), \end{aligned} \tag{3.31}$$

where $\mathcal{F}_{t_n}$ is the sigma field generated by $\{X(t),\ 0 \le t \le t_n\}$ and $\theta.$ is the shift operator. We will show that the sums in (3.31) are infinite almost surely, and this will give us (3.30) by Lévy's version of the Borel–Cantelli lemma (see, e.g., [4], Corollary 5.29). Since $s_n = (1 - 1/\theta)t_n$, taking θ sufficiently large, we see that the divergence of the sums in (3.31) follows if we show that

$$\sum_{n=1}^{\infty} P^{X_{t_{n-1}}}\left(\frac{L_{s_n}^0}{\phi(s_n)} \ge (1-\epsilon)d(\overline{\beta})\right) = \infty \quad \text{a.s.}, \tag{3.32}$$

for all $\epsilon > 0$. Now we see by Lemma 3.2 that (3.32) holds if

$$\sum_{n=1}^{\infty} a_\nu(X_{t_{n-1}, s_n}) \frac{1}{n} = \infty \quad \text{a.s.}, \tag{3.33}$$

for all ν sufficiently small. So, finally, to complete the proof of (1.11), we obtain (3.33). We first show that

$$E^0\left(\sum_{n=1}^{\infty} a_\nu\big(X_{t_{n-1}}, s_n\big)\frac{1}{n}\right) = \infty. \tag{3.34}$$

Note that

$$a_\nu\big(X_{t_{n-1}}, s_n\big) = \frac{\int_0^{\nu s_n} p_v\big(X_{t_{n-1}}\big)\,dv}{\kappa\big(\ell_2(s_n)/s_n\big)} > \frac{\int_0^{\nu s_n} p_v\big(X_{t_{n-1}}\big)\,dv}{\kappa\big(1/s_n\big)}, \tag{3.35}$$

for all n sufficiently large, because κ is monotonically decreasing. Since $\kappa(\cdot)$ is the Laplace transform of $p_\cdot(0)$ it follows from [3, Theorem 1.7.6] that p_v is regularly varying at infinity with index $-1/\beta$ and that

$$\kappa\left(\frac{1}{s_n}\right) \sim C s_n p_{s_n}(0) \quad \text{as } n \to \infty, \tag{3.36}$$

where we use the notation $f(x) \sim g(x)$ as $x \to \infty$ to mean $\lim_{x\to\infty} f(x)/g(x) = 1$ and similarly when $x \to 0$. We note that

$$E^0 \int_0^{\nu s_n} p_v(X_{t_{n-1}})\,dv = \int_0^{\nu s_n} p_{v+t_{n-1}}(0)\,dv = \int_{t_{n-1}}^{t_{n-1}+\nu s_n} p_v(0)\,dv. \tag{3.37}$$

Therefore, using this, the regular variation of p_v and (3.36), we have

$$\begin{aligned} \frac{E^0 \int_0^{\nu s_n} p_v(X_{t_{n-1}})\,dv}{\kappa(1/s_n)} &\sim C\frac{(t_{n-1}+\nu s_n)p_{t_{n-1}+\nu s_n}(0) - t_{n-1}p_{t_{n-1}}(0)}{s_n p_{s_n}(0)} \\ &\sim C\frac{\big(1+\nu(\theta-1)\big)^{1/\beta} - 1}{(\theta-1)^{1/\beta}} > 0 \qquad \text{as } n \to \infty. \end{aligned} \tag{3.38}$$

Using (3.35), (3.37) and (3.38), we obtain (3.34).

Note that, by (3.36) and the obvious fact that $p_y(x) \le p^v(0)$ for all x,

$$\frac{\int_0^{\nu s_n} p_v(X_{t_{n-1}})\,dv}{\kappa\big(1/s_n\big)} \le C\frac{\int_0^{\nu s_n} p_v(0)\,dv}{\kappa\big(1/s_n\big)} \sim C\nu^{1/\beta} \quad \text{as } n \to \infty. \tag{3.39}$$

Therefore, for $m \ge n_0$ for n_0 sufficiently large, we have

$$E^0\left(\sum_{n=n_0}^{m} a_\nu\big(X_{t_{n-1},s_n}\big)\frac{1}{n}\right)^2 \le C\left(\sum_{n=n_0}^{m} \frac{\kappa(1/s_n)}{n\kappa\big(\ell_2(s_n)/s_n\big)}\right)^2, \tag{3.40}$$

and as we have already seen

$$E^0\left(\sum_{n=n_0}^{m} a_\nu\big(X_{t_{n-1}}, s_n\big)\frac{1}{n}\right) \ge C'\left(\sum_{n=n_0}^{m} \frac{\kappa(1/s_n)}{n\kappa\big(\ell_2(s_n)/s_n\big)}\right). \tag{3.41}$$

Hence

$$\frac{E^0\left(\sum_{n=n_0}^m a_\nu(X_{t_{n-1}}, s_n)(1/n)\right)^2}{\left(E^0\left(\sum_{n=n_0}^m a_\nu(X_{t_{n-1}}, s_n)(1/n)\right)\right)^2} \leq C < \infty, \quad \forall\, m \geq n_0. \tag{3.42}$$

Thus by the Paley–Zygmund lemma (see, e.g., [14], inequality 2, page 8) the left-hand side of (3.33) is infinite with probability greater than zero. We now show that this probability is 1 by showing that the divergence of the left-hand side of (3.33) is a tail event. Fix r and let $n_0 = \inf\{n: t_{n-1} > r\}$. Then

$$\begin{aligned}\sum_{n=n_0}^{\infty} a_\nu\left(X_{t_{n-1}}, s_n\right)\frac{1}{n} &= \sum_{n=n_0}^{\infty} a_\nu\left(X_{t_{n-1}} - X_r, s_n\right)\frac{1}{n} \\ &\quad + \sum_{n=n_0}^{\infty}\left(a_\nu\left(X_{t_{n-1}}, s_n\right) - a_\nu\left(X_{t_{n-1}} - X_r, s_n\right)\right)\frac{1}{n}.\end{aligned} \tag{3.43}$$

The first sum to the right of the equality sign in (3.43) is independent of $\mathcal{F}_r$, and the second sum to the right of the equality sign in (3.43) is finite. This latter fact follows from Lemma 2.8, which gives us

$$\begin{aligned}a_\nu\left(X_{t_{n-1}}, s_n\right) - a_\nu\left(X_{t_{n-1}} - X_r, s_n\right) &\leq \frac{F(X_r)\log \nu s_n}{\kappa\left(\ell_2(s_n)/s_n\right)} \\ &\leq CF(X_r)\left(\theta^{-1/\overline{\beta}}\right)^n \log n,\end{aligned}$$

where $F(X_r)$ is finite almost surely. This shows that the divergence of the left-hand side of (3.33) is a tail event and hence we obtain (3.33), which completes the proof of (1.11). □

The proof of (1.12) of Theorem 1.1 is essentially the same as the proof of (1.11). However, there are a couple of places where the fact that $L_t^0 - L_t^x$ is not monotonic in t must be taken into consideration. We begin by noting the counterparts of Lemmas 3.1 and 3.2 for the difference of the local times. Let

$$\rho(t) = \ell_2(t)\kappa^{1/2}\left(\frac{\ell_2(t)}{t}\right) \quad \text{and} \quad Z_t = \varepsilon\frac{L_t^0 - L_t^x}{\sqrt{2}\sigma(x)\rho(t)}.$$

LEMMA 3.3. *Assume that κ is regularly varying at zero with index $-1/\overline{\beta}$. Then, for any y, $\epsilon > 0$, there exists a t_0 such that, for all $t \geq t_0$, and all v,*

$$P^v(Z_t \geq y) \leq \exp\left(-(1-\varepsilon)c(\overline{\alpha})y^\alpha \ell_2(t)\right) \tag{3.44}$$

where $\overline{\alpha} = 2\overline{\beta}$, $1/\alpha + 1/\overline{\alpha} = 1$ and

$$c(\overline{\alpha}) = \frac{\overline{\alpha} - 1}{\overline{\alpha}^\alpha}. \tag{3.45}$$

Moreover, t_0 can be chosen uniformly in y for y on any set bounded away from zero and infinity.

PROOF. We use Corollary 2.5 and follow the proof of Lemma 3.1. □

LEMMA 3.4. *Assume that* κ *is regularly varying at zero with index* $-1/\overline{\beta}$. *Then for all* t *sufficiently large we can find a* $\nu_0 > 0$ *such that, for all* $0 < \nu < \nu_0$,

$$\begin{aligned} P^s\Big(Z_t \geq \big(1 - \eta_1(\nu)\big)c(\overline{\alpha})^{-1/\alpha}\Big) \\ \geq C\Big(a_{\nu/2}(s,t)\exp\big[-\ell_2(t)\big] - \exp\big[-\big(1+\eta_2(\nu)\big)\ell_2(t)\big]\Big), \end{aligned} \tag{3.46}$$

where $a_{\nu/2}(s,t)$ *is given in* (3.7), $\eta_i(\nu) > 0$ *and* $\lim_{\nu\to 0}\eta_i(\nu) = 0$, $i = 1,2$.

PROOF. The proof of Lemma 3.2 goes through with only minor modifications. It is clear that we want to obtain the inequality in (3.8) with Y_t replaced by Z_t, $\overline{\beta}$ by $\overline{\alpha}$ and ν by $\nu/2$. In obtaining the lower bound for J_1 in (3.10) we used the inequality

$$\begin{aligned} E^s\Big(\exp\big[(1-\delta)uY_t\big]\Big) \\ \geq \frac{C(1-\delta)u}{\phi(t)}\bigg(\int_0^{\nu t} p_\nu(s)\,dv\bigg)\exp\bigg(\kappa^{-1}\bigg(\frac{\phi(t)}{(1-\delta)u}\bigg)(1-\nu)t\bigg). \end{aligned} \tag{3.47}$$

Using (2.42) and (2.46), we see that

$$\begin{aligned} E^s\big(\exp\big[(1-\delta)uZ_t\big]\big) \\ \geq \frac{C(1-\delta)^2u^2}{\rho^2(t)}\bigg(\int_0^{t\nu/2} p_v(s)\,dv\bigg)\exp\bigg((1-\epsilon)\kappa^{-1}\bigg(\frac{\rho^2(t)}{(1-\delta)^2u^2}\bigg)(1-\nu)t\bigg), \end{aligned}$$

where ϵ can be made arbitrarily small for t sufficiently large. Making the substitution $u = b\ell_2(t)$ as in the proof of Lemma 3.2, this is

$$\begin{aligned} E^s\big(\exp\big[(1-\delta)uZ_t\big]\big) \\ = Cb^2\bigg(\kappa\bigg(\frac{\ell_2(t)}{t}\bigg)\bigg)^{-1}\int_0^{t\nu/2} p_v(s)\,dv\exp\bigg((1-\epsilon)\kappa^{-1}\bigg(\frac{\kappa\big(\ell_2(t)/t\big)}{(1-\delta)^2u^2}\bigg)(1-\nu)t\bigg) \\ \geq Ca_{\nu/2}(s,t)\exp\Big((1-\delta^3)(1-\delta)^{2\overline{\beta}}u^{2\overline{\beta}}\ell_2(t)\Big). \end{aligned}$$

Thus in evaluating the lower bound for the term corresponding to J_1 in this lemma we are led to the same calculation as in (3.11) except that $\overline{\beta}$ is replaced by $\overline{\alpha}$ and, naturally, β by α. Note that since $\overline{\beta}$ is arbitrary in the proof of Lemma 3.2 and since $\overline{\alpha} > \overline{\beta}$, the calculations needed have already been done. Thus we see that paralleling (3.13) the term corresponding to J_1 is greater than or equal to $Ca_{\nu/2}(s,t)e^{-\ell_2(t)}$. The bounds for the terms corresponding to J_2 and J_3 are obtained similarly. □

PROOF OF (1.12) IN THEOREM 1.1. Following the proof of (1.11), we see that, for $t_n = \theta^n$, $\theta > 1$,

$$Z_{t_n} \leq \big(1+\epsilon(\theta)\big)d(\overline{\alpha}), \tag{3.48}$$

for all $n = n(\epsilon(\theta))$ sufficiently large, almost surely. Furthermore $\epsilon(\theta)$ can be taken to be arbitrarily small. In order to interpolate between the $\{t_n\}$, since Z_t is not monotonic in t, we introduce the martingale

$$M_t \equiv L_t^0 - L_t^x + \sigma^2\big(X(t)-x\big) - \sigma^2\big(X(t)\big) \equiv L_t^0 - L_t^x + h\big(x, X(t)\big)$$

with respect to $\mathcal{F}_t$, $0 \leq t < \infty$. [Note that $h(\cdot)$ is defined in (2.52).] Consider

$$M_{t_n}^* = \sup_{t_{n-1}\leq t \leq t_n} \big|M_t - M_{t_{n-1}}\big|.$$

By Lemma 2.9 and Doob's inequality for $k > 2$ and even (see, e.g., [20], Chapter 2, 1.7), along with the Markov property, we have

$$\begin{aligned} E^0(M_{t_n}^*)^k &\leq \left(\frac{k}{k-1}\right)^k E^0\big(M_{t_n} - M_{t_{n-1}}\big)^k \\ &\leq 4E^0\left(\Big|\big(L_{t_n}^0 - L_{t_{n-1}}^0\big) - \big(L_{t_n}^x - L_{t_{n-1}}^x\big)\Big| + \sigma^2(x)\right)^k \\ &\leq 4E^0E^{X(t_{n-1})}\left(\big|L_{s_n}^0 - L_{s_n}^x\big| + \sigma^2(x)\right)^k \\ &\leq 4E^0\left(\left(\big|L_{s_n}^0 - L_{s_n}^x\big| + \sigma^2(x)\right)\right)^k, \end{aligned}$$

where, as above, $s_n = t_n - t_{n-1}$. It follows that

$$E^0 \exp\big(sM_{t_n}^*\big) \leq 4\exp\big(s\sigma^2(x)\big)E^0 \exp\Big(\varepsilon s\big(L_{s_n}^0 - L_{s_n}^x\big)\Big),$$

and so, for $s \in [0, s_0]$ for s_0 sufficiently small,

$$E^0 \exp\big(sM_{t_n}^*\big) \leq CE^0 \exp\Big(\varepsilon s\big(L_{s_n}^0 - L_{s_n}^x\big)\Big). \tag{3.49}$$

Using the same argument as in the proof of Lemma 3.3, we see that (3.49) implies that

$$\limsup_{n\to\infty} \frac{M_{t_n}^*}{\rho(t_n)} = \limsup_{n\to\infty} \frac{M_{t_n}^*}{\rho(s_n)}\frac{\rho(s_n)}{\rho(t_n)} \leq C \limsup_{n\to\infty} \frac{\rho(s_n)}{\rho(t_n)}.$$

By taking $\theta > 1$ arbitrarily close to 1 we get

$$\limsup_{n\to\infty} \frac{M_{t_n}^*}{\rho(t_n)} = 0 \quad \text{a.s.} \tag{3.50}$$

Combining (3.48) with (3.50) we get the desired upper bound in (1.12) and the statement following (1.12).

The lower bound in (1.12) follows immediately from the corresponding result in the proof of (1.11). Analogously to (3.29), let

$$\widetilde{H}_n = (L^0_{t_n} - L^x_{t_n}) - (L^0_{t_{n-1}} - L^x_{t_{n-1}}).$$

We show that, for any $\epsilon > 0$,

$$\limsup_{n\to\infty} \frac{\widetilde{H}_n}{\sqrt{2}\sigma(x)\rho(t_n)} \geq (1-\epsilon)d(\overline{\alpha}) \quad \text{a.s.} \tag{3.51}$$

Following the proof of (1.11), with obvious modifications, we see that (3.51) holds if

$$\sum_{n=1}^{\infty} P^{X_{t_{n-1}}}\left(\frac{L^0_{s_n} - L^x_{s_n}}{\sqrt{2}\sigma(x)\rho(s_n)} \geq (1-\epsilon)d(\overline{\alpha})\right) = \infty \quad \text{a.s.}, \tag{3.52}$$

for all $\epsilon > 0$, where $s_n = t_n - t_{n-1}$. Now we see that by Lemma 3.4, that (3.52) holds if

$$\sum_{n=1}^{\infty} a_{\nu/2}(X_{t_{n-1}}, s_n)\frac{1}{n} = \infty \quad \text{a.s.},$$

for all ν sufficiently small. We showed this in the proof of Lemma 3.2. Hence we obtain (3.51), which gives us the lower bound in (1.12). □

PROOF OF THEOREM 1.2. The proof is almost exactly the same as the proof of Theorem 1.1. This is because the discrete Laplace transforms for the local times of symmetric random walks are the same as those given on the right-hand sides of (2.5) and (2.6) except that κ is as defined for random walks in (1.16) and (1.19). (See [17], Lemma 4.1.) Moreover, the functions $\kappa(\alpha)$ defined in (1.5) for Lévy processes and in (1.16) and (1.19) for symmetric random walks are asymptotically equivalent as α goes to zero. This implies that Theorem 2.2 and Corollary 2.5 also hold for the local times of symmetric random walks and this is all we need to extend the proof of Theorem 1.1 to this case. There is only one place where the procedure differs. When we invert the Laplace transform we use a rectangular contour with horizontal sides given by $y = \pi$ and $y = -\pi$. This is possible because $\phi(\lambda)$, given in (1.17), is periodic. This makes the inversion a little easier and also enables us to dispense with condition (1.7). □

4. Proofs of Theorems 1.3–1.5. In this section we give the proofs of the two theorems on the functional law of the iterated logarithm and the corollary dealing with the return times to zero of a symmetric random walk.

PROOF OF THEOREM 1.3. Fix k, and let $\{r_i\}_{i=1}^k$, $r_i > 0$, be such that $\sum_{i=1}^k r_i^{\overline{\beta}} = 1$, that is, $\{r_i\}_{i=1}^k$ is in the unit sphere of R^k equipped with the $\ell_{\overline{\beta}}$ norm. Set

$$Q_t = \frac{1}{\gamma(\overline{\beta})\phi(t)}\sum_{i=1}^{k} r_i(L^0_{it} - L^0_{(i-1)t}), \tag{4.1}$$

where $\phi(t)$ is given in (3.1). Let $\lambda = \beta b^{\beta-1}\ell_2(t)$, where $b > 0$, and set $a = \beta b^{\beta-1}$. Then using the Markov property and (2.43) we get that, for any $\epsilon > 0$,

$$\begin{aligned} P^0(Q_t \geq b) &\leq \exp(-\lambda b) E^0 \exp(\lambda Q_t) \\ &\leq \exp(-\lambda b) \prod_{i=1}^{k} E^0 \left(\exp\left(\frac{\lambda r_i L_t^0}{\gamma(\overline{\beta})\phi(t)} \right) \right) \\ &\leq C^k \exp\left[-ab\ell_2(t) \right] \exp\left(\sum_{i=1}^{k} \kappa^{-1} \left(\frac{\gamma(\overline{\beta})\kappa(\ell_2(t)/t)}{ar_i} \right) t \right) \end{aligned} \tag{4.2}$$

where, at the last step, we used (2.32). Using the regular variation of κ at zero we see that this last term is less than or equal to

$$\begin{aligned} &C^k \exp\left[-ab\ell_2(t) \right] \exp\left((1+\epsilon) \left(\sum_{i=1}^{k} r_i^{\overline{\beta}} \right) a^{\overline{\beta}} \gamma(\overline{\beta})^{-\overline{\beta}} \ell_2(t) \right) \\ &= C^k \exp\left(-\left(ab - a^{\overline{\beta}}\gamma(\overline{\beta})^{-\overline{\beta}} - O(\epsilon)\right)\ell_2(t) \right) \\ &= C^k \exp\left(-(1-\epsilon')b^{\beta}\ell_2(t) \right), \end{aligned} \tag{4.3}$$

for some $\epsilon' > 0$ which goes to zero as ϵ goes to zero. Take $b = 1 + \epsilon'$ and $t_n = \theta^n$, for $\theta > 1$, and note that k is fixed so C^k is only a constant. It follows from (4.2) and (4.3) that

$$\limsup_{t_n \to \infty} \frac{1}{\gamma(\overline{\beta})\phi(t_n)} \sum_{i=1}^{k} r_i \left(L_{it_n}^0 - L_{(i-1)t_n}^0 \right) \leq 1 \quad \text{a.s.} \tag{4.4}$$

Note that, for $t_{n-1} < t \leq t_n$, we have

$$\begin{aligned} L_{it}^0 - L_{(i-1)t}^0 &\leq L_{it_n}^0 - L_{(i-1)t_{n-1}}^0 \\ &= \left(L_{it_n}^0 - L_{(i-1)t_n}^0 \right) + \left(L_{(i-1)t_n}^0 - L_{(i-1)t_{n-1}}^0 \right). \end{aligned}$$

Using our original proof for the upper bound in (1.11), it is easy to see that, for any $1 \leq i \leq k$,

$$\limsup_{n \to \infty} \frac{L_{(i-1)t_n}^0 - L_{(i-1)t_{n-1}}^0}{\phi(t_n - t_{n-1})} \leq C \quad \text{a.s.},$$

and since

$$\limsup_{n \to \infty} \frac{\phi(t_n - t_{n-1})}{\phi(t_n)}$$

can be made arbitrarily small for θ sufficiently close to 1, we see that (4.4) implies that

$$\limsup_{t \to \infty} \frac{1}{\gamma(\overline{\beta})\phi(t)} \sum_{i=1}^{k} r_i \left(L_{it}^0 - L_{(i-1)t}^0 \right) \leq 1 \quad \text{a.s.} \tag{4.5}$$

To show that (4.5) holds with an equality sign we first show that

$$E^x \exp\left(\sum_{i=1}^{k} \lambda_i (L_{it}^0 - L_{(i-1)t}^0)\right) \tag{4.6}$$
$$\geq \left(\int_0^{\nu t} p_r(x)\, dr\right)\left(\int_t^{(1+\nu)t} p_r(0)\, dr\right)^{k-1} \prod_{i=1}^{k} \left(\lambda_i E^0 \exp\left(\lambda_i L_{(1-\nu)t}^0\right)\right).$$

To prove (4.6) we note that by (2.44) and the Markov property we have

$$E^x \exp\left(\sum_{i=1}^{k} \lambda_i (L_{it}^0 - L_{(i-1)t}^0)\right)$$
$$= E^x\left(\exp\left(\sum_{i=1}^{k-1} \lambda_i (L_{it}^0 - L_{(i-1)t}^0)\right) E^{X_{(k-1)t}} \exp\left(\lambda_k L_t^0\right)\right)$$
$$\geq E^x\left(\exp\left(\sum_{i=1}^{k-1} \lambda_i (L_{it}^0 - L_{(i-1)t}^0)\right) \int_0^{\nu t} p_r(X_{(k-1)t})\, dr\right) \lambda_k E^0 \exp\left(\lambda_k L_t^0\right)$$
$$= E^x\left(\exp\left(\sum_{i=1}^{k-2} \lambda_i (L_{it}^0 - L_{(i-1)t}^0)\right)\right.$$
$$\left. \times E^{X_{(k-2)t}}\left(\exp\left(\lambda_{k-1} L_t^0\right) \int_0^{\nu t} p_r(X_t)\, dr\right)\right) \lambda_k E^0 \exp\left(\lambda_k L_t^0\right),$$

where we write X_t instead of $X(t)$. Using this, the proof of (4.6) is completed inductively with the help of the following inequality:

$$E^x\left(e^{\lambda L_t^0} \int_0^{\nu t} p_r(X_t)\, dr\right) \tag{4.7}$$
$$\geq \int_0^{\nu t} p_r(x)\, dr \left(\int_t^{(1+\nu)t} p_s(0)\, ds\right) \lambda E^0 \left(e^{\lambda L_t^0}\right).$$

To obtain (4.7), let $0 = s_0 < s_1 < \cdots < s_n = t$ be a partition of the interval $[0, t]$. Using (2.44), the Markov property and fact that dL_s^0 is supported on $\{s \mid X_s = 0\}$, we have

$$E^x\left(\exp\left(\lambda L_t^0\right) \int_0^{\nu t} p_r(X_t)\, dr\right)$$
$$\geq E^x\left(\sum_{i=1}^{n} \left(\exp\left(\lambda L_{s_i}^0\right) - \exp\left(\lambda L_{s_{i-1}}^0\right)\right) \int_0^{\nu t} p_r(X_t)\, dr\right) \tag{4.8}$$
$$= \sum_{i=1}^{n} E^x\left(\left(\exp\left(\lambda L_{s_i}^0\right) - \exp\left(\lambda L_{s_{i-1}}^0\right)\right) E^{X_{s_i}}\left(\int_0^{\nu t} p_r(X_{t-s_i})\, dr\right)\right)$$
$$= \sum_{i=1}^{n} E^x\left(\left(\exp\left(\lambda L_{s_i}^0\right) - \exp(\lambda L_{s_{i-1}}^0)\right) \int_{t-s_i}^{(1+\nu)t-s_i} p_r(X_{s_i})\, dr\right).$$

Note that X is cadlag and therefore so is $\int_{t-s}^{(1+\nu)t-s} p_r(X_s)\,dr$. Also $p_r(0)$ is decreasing in r. Therefore, taking the limit as n goes to infinity in (4.8), we get

$$E^x\left(e^{\lambda L_t^0}\int_0^{\nu t} p_r(X_t)\,dr\right) \geq E^x\left(\int_0^t de^{\lambda L_s^0}\int_{t-s}^{(1+\nu)t-s} p_r(X_s)\,dr\right)$$
$$\geq E^x\left(e^{\lambda L_t^0}-1\right)\int_t^{(1+\nu)t} p_r(0)\,dr.$$

Using (2.44) again establishes (4.7) and hence (4.6).

Using (4.6) we can get the critical step for showing that 1 is also a lower bound for the left-hand side of (4.5). Recall that is obtaining the lower bound in (1.11) the key point is Lemma 3.2. The critical inequality used to obtain the lower bound was in the computation of J_1, where in (3.9)–(3.11) we used the inequality

$$\text{(4.9)}\qquad \begin{aligned} &E^x \exp\left(\frac{(1-\delta)bL_t^0}{\kappa\big(\ell_2(t)/t\big)}\right)\\ &\quad\geq \frac{C(1-\delta)^b}{\kappa\big(\ell_2(t)/t\big)}\left(\int_0^{\nu t} p_v(x)\,dv\right)\exp\big((1-\delta^3)(1-\delta)^{\overline{\beta}}b^{\overline{\beta}}\ell_2(t)\big).\end{aligned}$$

By (4.6) we have

$$\text{(4.10)}\qquad \begin{aligned} &E^x \exp\left(\frac{(1-\delta)b\sum_{i=1}^k r_i\big(L_{it}^0-L_{(i-1)t}^0\big)}{\kappa(\ell_2(t)/t)}\right)\\ &\quad\geq \frac{\big(b(1-\delta)\big)^k\Pi_{i=1}^k r_i}{\kappa\big(\ell_2(t)/t\big)}\left(\int_0^{\nu t} p_v(x)\,dv\right)\\ &\qquad\times D_t^{k-1}\left(\exp\left(\sum_{i=1}^k \kappa^{-1}\left(\frac{\kappa(\ell_2(t)/t)}{(1-\delta)b}\right)(1-\nu)t\right)\right),\end{aligned}$$

where

$$D_t = \frac{\int_t^{(1+\nu)t} p_v(0)\,dv}{\kappa\big(\ell_2(t)/t\big)}.$$

As in (3.11) set $\nu = \delta^4$. By calculations almost exactly like those used in (3.35)–(3.38) we see that $\liminf_{t\to\infty} D_t > 0$. Making use of the regular variation of κ at zero as we did in (3.11), we see that the left-hand side of (4.10) is greater than or equal to

$$\frac{C}{\kappa\big(\ell_2(t)/t\big)}\left(\int_0^{\nu t} p_v(x)\,dv\right)\exp\Big(\big(1-\delta^3\big)(1-\delta)^{\overline{\beta}}b^{\overline{\beta}}\ell_2(t)\Big),$$

the same as in (4.9). Thus we can proceed to get an analogue of Lemma 3.2 for this case. Given this analogue of Lemma 3.2, the rest of the proof to show that

1 is also a lower bound for the left-hand side of (4.5) is exactly as the proof of the lower bound in (1.11).

Theorem 1.3 follows easily now that we have shown that (4.5) holds with an equal sign. Since equality also holds for a countable dense set of R_+^k satisfying $\Sigma_{i=1}^k r_i^{\overline{\beta}} = 1$, we see that the set of limit points of

$$V_t^{(k)} = \frac{1}{\gamma(\overline{\beta})\phi(t)}(L_t^0,\, L_{2t}^0 - L_t^0, \dots, L_{kt}^0 - L_{(k-1)t}^0) \tag{4.11}$$

is contained in the unit ball of $\ell_\beta(R_+^k)$ and moreover contains the unit sphere in $\ell_\beta(R_+^k)$. Repeating the same argument for $V_t^{(k+1)}$ shows that the set of limit points for $V_t^{(k)}$ is the unit ball in $\ell_\beta(R_+^k)$.

Replace t by t/k in (4.11). Using the regular variation of $\phi(t)$ at infinity we now see that the set of limit points of

$$\frac{k^{1/\overline{\beta}}}{\gamma(\overline{\beta})\phi(t)}(L_{t/k}^0, L_{2t/k}^0 - L_{t/k}^0, \dots, L_t^0 - L_{(k-1)t/k}^0) \tag{4.12}$$

is the unit ball in $\ell_\beta(R_+^k)$.

We can reexpress this as follows: Recalling (1.21) and (3.1) let us define, for $0 \le x \le 1$,

$$L_t^{(k)}(x) = L_t\left(\frac{i}{k}\right) + \left(x - \frac{i}{k}\right)k\left(L_t\left(\frac{(i+1)}{k}\right) - L_t\left(\frac{i}{k}\right)\right), \quad \frac{i}{k} \le x \le \frac{i+1}{k}. \tag{4.13}$$

It follows from (4.12) that, for each k, $\{L_t^{(k)}(\cdot),\ 1 \le t < \infty\}$ is relatively compact in $C(0,1)$ and the set of its limit points coincides with the set of $f \in C(0,1)$, such that

$$f(x) = a_i + \left(x - \frac{i}{k}\right)k(a_{i+1} - a_i), \qquad \frac{i}{k} \le x \le \frac{i+1}{k}, \tag{4.14}$$

with $(a_i - a_{i-1}) \ge 0$ and

$$\sum_{i=1}^k \left(k(a_i - a_{i-1})\right)^\beta \frac{1}{k} \le 1. \tag{4.15}$$

For functions of the form of (4.14), the condition (4.15) is precisely the condition that $f \in B_M^\beta$.

Let $B_{M,k}^\beta$ be the functions in B_M^β of the form (4.14). It is clear that the increasing union $\cup_k B_{M,k}^\beta$ is dense in B_M^β in the topology on $C(0,1)$. Thus, to complete the proof of Theorem 1, we need only to verify that

$$\limsup_{t\to\infty} \|L_t(\cdot) - L_t^{(k)}(\cdot)\|_\infty \le 2/k^{1/\overline{\beta}} \quad \text{a.s.}, \tag{4.16}$$

and this will follow once we show that

$$\limsup_{t\to\infty} |L_t(x) - L_t(y)| \le 2/k^{1/\overline{\beta}} \quad \text{a.s.,} \tag{4.17}$$

for all x and y such that $|x-y| \le 1/k$. Finally, to get (4.17) we use the monotonicity of $L_t(x)$ and the fact that

$$\limsup_{t\to\infty} \left| L_t\left(\frac{i}{k}\right) - L_t\left(\frac{i-1}{k}\right)\right| \le \frac{1}{k^{1/\overline{\beta}}} \quad \text{a.s.,}$$

which itself follows from (1.11). This completes the proof of Theorem 1.3. □

PROOF OF THEOREM 1.4. Theorem 1.4 has the same proof as Theorem 1.3. □

PROOF OF THEOREM 1.5. We first note that by Theorem 1.4 we have that, for any positive function G of bounded variation,

$$\begin{aligned}\limsup_{n\to\infty} \int_0^1 G(x)\,dL_n(x) &= \limsup_{n\to\infty} \left(L_n(1)G(1) - \int_0^1 L_n(x)\,dG(x)\right)\\ &= \sup_{f\in B_M^\beta} \left(f(1)G(1) - \int_0^1 f(x)\,dG(x)\right)\\ &= \sup_{f\in B_M^\beta} \int_0^1 f'(x)G(x)\,dx = \|G\|_{\overline{\beta}}.\end{aligned} \tag{4.18}$$

By (1.23) we have that

$$dL_n(x) = \frac{n\left(L_k^0 - L_{k-1}^0\right)}{\gamma(\overline{\beta})\phi(n)}\,dx, \qquad \frac{k-1}{n} \le x < \frac{k}{n}.$$

Hence, we see that

$$\int_0^1 G(x)\,dL_n(x) = \frac{1}{\gamma(\overline{\beta})\phi(n)} \sum_{k=1}^n \left(n\int_{(k-1)/n}^{k/n} G(x)\,dx\right)\left(L_k^0 - L_{k-1}^0\right). \tag{4.19}$$

Let $G(x) = x^q$, for $q > 0$, and note that

$$\begin{aligned} n\int_{(k-1)/n}^{k/n} x^q\,dx &= \frac{n}{q+1}\left(\frac{k}{n}\right)^{q+1}\left(1 - \left(1 - \frac{1}{k}\right)^{q+1}\right)\\ &= \left(\frac{k}{n}\right)^q\left(1 + O\left(\frac{1}{k}\right)\right).\end{aligned} \tag{4.20}$$

Therefore, since $L_k^0 - L_{k-1}^0 = \delta_0(X_k)$, we can take the limit superior as n goes to infinity of both sides of (4.19), with $G(x) = x^q$, and use (4.18) and (4.20) to get (1.24). □

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