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ITÔ EXCURSION THEORY FOR SELF-SIMILAR MARKOV PROCESSES

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Let X_t be an α-self-similar Markov process on $(0, \infty)$ killed when hitting 0. α-self-similar extensions of $X(t)$ to $[0, \infty)$ are studied via Itô execusion theory (entrance laws). We give a condition that guarantees the existence of an extension, which either leaves 0 continuously (a.s.) or (a.s.) jumps from 0 to $(0, \infty)$ according to the "jumping in" measure $\eta(dx) = dx/x^{\beta+1}$. Two applications are given: the diffusion case and the "reflecting barrier process" of S. Watanabe.

0. Introduction. The class of α-self-similar Markov processes (α-ssmp) was introduced by Lamperti [7], who used the name semistable. Lamperti considered α-ssmp on $(0, \infty)$ and on $[0, \infty)$. The rotation-invariant self-similar Markov processes on $R^d \backslash \{0\}$ were characterized in [4] and [13].

The purpose of this paper is to study the following problem, which was stated by Lamperti in [7]: Let $(X(t), P^x)$ be an α-ssmp on $(0, \infty)$ which is killed at time T_0 on first hitting the state 0. Characterize all the possible extensions $(\widetilde{X}(t), \widetilde{P}^x)$ to $[0, \infty)$ which are α-self-similar, strong Markov and behave up to the time T_0 like $(X(t), P^x)$. Lamperti solved it in the special case where $(X(t), P^x)$ is a Brownian motion, killed at 0. He applied the well-known characterization by Itô and McKean [5] of *all* the extensions of Brownian motion after $t = T_0$ and showed that the class of those extensions which are self-similar consists of the reflecting and absorbing Brownian motions and the extensions which immediately after reaching 0 jump according to the measure $dx/x^{\beta+1}$, $0 < \beta < 1$ (see [7], Theorem 5.2). Lamperti's method was to study the boundary conditions, which determine the domain of the generator of the process (in this case, the Laplace operator).

Lamperti [7] suggested that Itô excursion theory could be used in the general case, and that is in fact our approach. We apply the results of Blumenthal [1] and Rogers [9] and characterize the entrance laws of $(X(t), P^x)$. We also show a kind of zero–one law for self-similar processes [this was already known by Lamperti [7, Theorem 5.2] in the case $(X(t), P^x)$ is a Brownian motion on $(0, \infty)$]: An α-ssmp [fulfilling our condition (A), which is formulated in Section 1] either has a jump from 0 a.s. or it leaves 0 continuously a.s. In Section 2 we prove that if condition (A) is valid, then $(X(t), P^x)$ can be extended to $[0, \infty)$ either so that the extension leaves 0 continuously (or, of course, stays there forever) or that it jumps from 0 according to the "jumping-in" measure $\eta(dx) = dx/x^{\beta+1}$ (compare Theorem 5.2 in [7]). In Section 3 we consider the diffusion case and

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show that (A) is valid for an α-self-similar diffusion iff 0 is a regular boundary point. If 0 is an exit boundary point, then it is not possible to extend $(X(t), P^x)$ to $[0, \infty)$ so that the extension leaves 0 continuously. We show that in this case one can construct an α-self-similar extension which jumps from 0 according to the jumping-in measure $\eta(dx) = dx/x^{\beta+1}, 0 < \beta < 1/\alpha$.

It should be remarked here that the Bessel processes form exactly the class of $\frac{1}{2}$-self-similar diffusions on $(0, \infty)$.

Finally, in the last section we have an example, originally due to Watanabe [14], of a self-similar process with only right-continuous paths for which (A) is valid. This so-called reflecting barrier process is constructed from a symmetric stable process in the same way as the reflecting Brownian motion is constructed from a Brownian motion. We shall see that it a.s. leaves 0 continuously (this is a special case of the result of Rogers [10], which states that a process which is constructed similarly from any real Lévy process a.s. leaves 0 continuously).

Comparison of Lamperti's and our assumptions and results. Lamperti's [7] assumptions were slightly different from ours; he assumed the following:

(*) The corresponding transition function $(\widetilde{P}_t)$ is uniformly stochastically continuous in some neighborhood of $x = 0$.

REMARK. This was one of the "usual" assumptions at the time when Lamperti wrote his paper (see [7] and references therein). The problem of characterizing those standard processes which fulfill (*) remains open.

Assumption (*) is used in [7, Lemma 2.1] to show that

$$(**) \qquad t \to \widetilde{P}_t f(x), \qquad f \in C_0[0, \infty),\ x \in (0, \infty), \qquad \text{is continuous,}$$

which is further used to establish the continuity of $x \to \widetilde{P}_t f(x)$, $f \in C_0[0, \infty)$, $t > 0$, where $C_0[0, \infty)$ is the class of all continuous functions on $[0, \infty)$ which approach 0 as $x \to \infty$.

Now, to prove (**), it is sufficient to assume the right-continuity and quasi-left-continuity of the paths.

Another result of Lamperti [7] which depends on (*) is the result in Theorem 4.1 that the time-changed process $Y_t = \widetilde{X}_{T_t}$ is a Feller process, where T_t is the right-continuous inverse of a continuous additive functional $\int_0^t \widetilde{X}_h^{-1/\alpha}$, $t < T_0$. It seems to us that the uniform stochastic continuity in general plays an important role in the preservation of the Feller property in random time-changes (see [7]).

We assume, as Lamperti also did, that $(\widetilde{X}_t)$ has an infinite lifetime. We do it because of simplicity, even if this assumption could be relaxed. If $(\widetilde{X}_t)$ does not have an infinite lifetime, then $(\log \widetilde{X}_{T_t})$, $t < T_0$, is a Lévy process which has been killed according to an independent, exponentially distributed random variable (see [4] and [13], or [7]).

The method we use to characterize all self-similar extensions can be applied also to the more general case, where the lifetime is allowed to be finite; then, however, one must notice an extra parameter δ, different from 0, in the resolvent

$$U^{\lambda}f(0) = \frac{n_{\lambda}(f)}{\lambda n_{\lambda}(1) + \delta}$$

(see beginning of Section 1 and [9]).

Lamperti's result about the connection between α-ssmp and Lévy processes is still valid in our situation, even if the assumption of infinite lifetime were relaxed (see [4], (2.2), page 153): If (X_t) is an α-ssmp on $(0,\infty)$, possibly with a finite lifetime, then $(\log X_{T_t})$ is a Lévy process on $(-\infty,+\infty)$, possibly killed according to an independent, exponentially distributed random variable, and T_t is the right-continuous inverse of an additive functional

$$\int_0^t X_h^{-1/\alpha}\, dh.$$

Even the reverse is true (see [4] and [12]): Starting from any Lévy process (Z) on $(-\infty,+\infty)$ one can construct an α-ssmp $(\acute{X}_t)$, by defining

$$\acute{X}_t = Y_{S_t}, \qquad \text{where } Y_t = \exp Z_t$$

and S_t is the right-continuous inverse of a continuous additive functional

$$\int_0^t Y_h^{1/\alpha}\, dh.$$

(Note that the quasi-left-continuity of $t \to Y_t$ (and $t \to Z_t$) is preserved in this random time-change because $\int_0^t Y_h^{1/\alpha}\, dh$ is a strictly increasing, continuous additive functional; see [2], page 212.) Moreover, if $Z_t = \log X_{T_t}$, then X_t is equivalent to $\acute{X}_t$.

We can conclude that all the results of Lamperti [7] which are needed in this paper are valid under our assumptions.

1. Preliminaries; entrance laws.

Definition. Let $(\widetilde{X}(t), \widetilde{P}^x)$ be a standard Markov process (see the definition in [2]) on $[0,\infty)$ with an infinite lifetime, and let $(\widetilde{P}_t)_{t\geq 0}$ and $(U^{\lambda})_{\lambda>0}$ be the corresponding transition function and resolvent.

The process $(\widetilde{X}(t), \widetilde{P}^x)$ is called an α-self-similar Markov process (α-ssmp) on $[0,\infty)$, $\alpha > 0$, if

$$\widetilde{P}_t(x, A) = \widetilde{P}_{at}(a^{\alpha}x, a^{\alpha}A), \tag{1.1}$$

or, equivalently,

$$U^{\lambda}(x,A) = a^{-1}U^{\lambda/a}(a^{\alpha}x, a^{\alpha}A),\ t \geq 0, \lambda > 0, a > 0, x \geq 0, A \in \mathcal{B}[0,\infty),$$

where $\mathcal{B}[0,\infty)$ is the collection of all Borel sets on $[0,\infty)$.

REMARK. We define α-self-similar Markov processes on $(0,\infty)$ similarly, except that *we allow here* a *finite lifetime*; a typical example is an α-ssmp on $[0,\infty)$ which has been killed just before hitting 0.

Let T_0 be a stopping time defined by $T_0 = \inf\{t > 0; \widetilde{X}(t) = 0\}$. As shown by Lamperti ([7], Lemma 2.5), either $T_0 < \infty$ a.s. $(\widetilde{P}^x), \forall\, x > 0$, or $T_0 = \infty$ a.s. $(\widetilde{P}^x)$, $\forall x > 0$.

We assume from now on that $T_0 < \infty$ a.s. $(\widetilde{P}^x)$, $\forall x > 0$.

Define further

$$
\begin{aligned}
&\psi_\lambda(x) = \widetilde{E}^x\big(\exp(-\lambda T_0)\big),\\
(1.2)\qquad &Q_t f(x) = \widetilde{E}^x\Big\{f\big(\widetilde{X}(t)\big); t < T_0\Big\},\\
&V^\lambda f(x) = \int_0^\infty e^{-\lambda t} Q_t f(x)\,dt, \qquad \lambda > 0, t > 0, x > 0, f \in \mathcal{B}(0,\infty),
\end{aligned}
$$

where $\mathcal{B}(0,\infty)$ is the collection of all the Borel sets on $(0,\infty)$.

Equation (1.1) is equivalent to

$$
(1.1')\qquad \widetilde{X}(t) \sim \widetilde{P}^x = a^{-\alpha}\widetilde{X}(at) \sim \widetilde{P}^{a^\alpha x}
$$

where "$\sim P$" means "finite-dimensional distributions under the measure P." Equation (1.1′) implies $T_0 \sim \widetilde{P}^x = a^{-1}T_0 \sim \widetilde{P}^{a^\alpha x}$, and thus

$$
\begin{aligned}
\psi_\lambda(x) &= \psi_{\lambda/a}(a^\alpha x),\\
(1.3)\qquad Q_t(x,A) &= Q_{at}(a^\alpha x, a^\alpha A),\\
V^\lambda(x,A) &= a^{-1}V^{\lambda/a}(a^\alpha x, a^\alpha A),\\
&\qquad t > 0, a > 0, \lambda > 0, x > 0, A \in \mathcal{B}(0,\infty).
\end{aligned}
$$

We consider the stopped process

$$
X(t)(\omega) = \begin{cases} \widetilde{X}(t)(\omega), & t < T_0(\omega),\\ \Delta, & t \geq T_0(\omega). \end{cases}
$$

[Δ is the "graveyard" for $X(t)$.]

Blumenthal [1] studied the extension problem for standard processes which have the following additional properties:

(a) If $f \in C_0(0,\infty)$, then $Q_t \in C_0(0,\infty)$ and $Q_t f \to f$ uniformly as $t \to 0$ [$C_0(0,\infty)$ is the class of all continuous functions on $(0,\infty)$, vanishing at 0 and ∞].

(b) The mapping $x \to \psi_\lambda(x)$ is continuous on $(0,\infty)$, for all $\lambda > 0$, and $\psi_1(x) \to 1$ as $x \to 0$, $\psi_1(x) \to 0$ as $x \to \infty$.

We will show that both properties (a) and (b) are valid in our case. From (1.3) we have

$$\psi_\lambda(x) = \psi_{\lambda x^{1/\alpha}}(1) = \widetilde{E}^1\Big(\exp\big(-\lambda x^{1/\alpha} T_0\big)\Big)$$

and thus $x \to \psi_\lambda(x)$ is continuous on $(0,\infty)$,

$$\lim_{x\to 0} \psi_\lambda(x) = 1 \quad \text{and} \quad \lim_{x\to\infty} \psi_\lambda(x) = 0.$$

Because $\widetilde{P}^1\{T_0 < \infty\} = 1$, we have

$$\widetilde{P}^x\{T_0 > t\} = \widetilde{P}^1\{x^{1/\alpha} T_0 > t\} = \widetilde{P}^1\{T_0 > t/x^{1/\alpha}\} \to 0 \quad \text{when } x \to 0.$$

This implies that, for any bounded Borel function f, we have

$$\lim_{x\to 0} Q_t f(x) = 0 \quad \text{and} \quad \lim_{x\to 0} V^\lambda f(x) = 0.$$

Because

$$\begin{aligned} \widetilde{P}^x\{\widetilde{X}(t) \geq b, t < T_0\} &= \widetilde{P}^1\{x\widetilde{X}(t/x^{1/\alpha}) \geq b, t < x^{1/\alpha} T_0\} \\ &= \widetilde{P}^1\{\widetilde{X}(t/x^{1/\alpha}) \geq b/x, t < x^{1/\alpha} T_0\} \to 1 \\ &\qquad \text{as } x \to \infty, \text{ for any } b > 0, \end{aligned}$$

then, for any bounded Borel function with $\lim_{x\to\infty} f(x) = 0$, we have $\lim_{x\to\infty} Q_t f(x) = 0$ and $\lim_{x\to\infty} V^\lambda f(x) = 0$. By the same method as [4, Lemma 2.1] for rotation-invariant processes, we can prove that $x \to V^\lambda f(x)$ is continuous on $(0,\infty)$ for $f \in C_0(0,\infty)$. Thus $(Q_t)_{t\geq 0}$ and $(V^\lambda)_{\lambda>0}$ are a Feller–Dynkin semigroup and resolvent and $(X(t))$ is a standard process (in fact, a Feller process) so that both conditions (a) and (b) in Blumenthal's paper [1] are satisfied. It follows immediately from (1.3) that $(X(t))$ is an α-self-similar Markov process on $(0,\infty)$ and has the lifetime T_0.

Using the strong Markov property of $(\widetilde{X}(t), \widetilde{P}^x)$ at T_0, we get

$$U^\lambda f(x) = V^\lambda f(x) + \psi_\lambda(x) U^\lambda f(0),$$

for any bounded Borel function f on $[0,\infty)$, $f \in \mathcal{B}_b[0,\infty)$. According to the result of Rogers [9], there exist nonnegative constants γ and δ and finite measures $(n_\lambda)_{\lambda>0}$ on $(0,\infty)$ such that

$$U^\lambda f(0) = \frac{n_\lambda(f) + \gamma f(0)}{\delta + \lambda n_\lambda(1) + \lambda\gamma} \quad \text{for all } f \in \mathcal{B}_b[0,\infty), \lambda > 0$$

and

$$n_\lambda V^\mu = \frac{n_\lambda - n_\mu}{\mu - \lambda} \quad \text{for all } \lambda > 0, \mu > 0, \lambda \neq \mu. \tag{1.4}$$

Further, $\widetilde{P}^0\{\widetilde{X}(t) = 0\} = 0 \; \forall \, t > 0$ except in the absorbing case (see [7], page 220), which means $\gamma = 0$. If, as we have assumed, $\widetilde{X}(t)$ has infinite lifetime a.s., we also have $\delta = 0$. So we have

$$U^\lambda f(0) = \frac{n_\lambda(f)}{\lambda n_\lambda(1)} \quad \text{for all } f \in \mathcal{B}_b[0,\infty) \text{ and for all } \lambda > 0.$$

Using α-self-similarity, we obtain

$$\frac{n_\lambda(A)}{\lambda n_\lambda(1)} = a \frac{n_{\lambda a}(a^{-\alpha}A)}{\lambda a n_{\lambda a}(1)} \quad \text{for all } \lambda > 0, a > 0, A \in \mathcal{B}(0,\infty).$$

which is equivalent to

$$\frac{n_\lambda(A)}{n_\lambda(1)} = \frac{n_{\lambda a}(a^{-\alpha}A)}{n_{\lambda a}(1)}.$$

This and the α-self-similarity of $X(t)$ imply

$$n_\lambda(V^\mu 1) = a \int V^{\mu a} 1(x) \frac{n_\lambda(1)}{n_{\lambda a}(1)} n_{\lambda a}(dx).$$

Because of (1.4) we have

$$\frac{n_\lambda(1) - n_\mu(1)}{\mu - \lambda} = a \frac{n_\lambda(1)}{n_{\lambda a}(1)} \frac{n_{\lambda a}(1) - n_{\mu a}(1)}{\mu a - \lambda a},$$

which implies

$$n_\mu(1) n_{\lambda a}(1) = n_\lambda(1) n_{\mu a}(1).$$

Taking $\lambda = 1$, we obtain

$$n_\mu(1) n_a(1) = n_1(1) n_{\mu a}(1).$$

However, this is possible only if $n_a(1)$ has a representation $n_a(1) = ka^v$, for some $k > 0, v \in R$. According to Rogers [9, Lemma 2, page 244], $n_a(1) \to 0$, when $a \to \infty$, which means $v < 0$. Thus we have $n_\lambda(A) = a^{-v} n_{\lambda a}(a^{-\alpha}A)$, or, equivalently,

(1.5) $$n_\lambda(A) = a^v n_{\lambda/a}(a^\alpha A) \quad \text{for some } v < 0, \text{ for all } a > 0, \lambda > 0.$$

It was shown by Getoor and Sharpe [3] that there exists a family $(\eta_t)_{t>0}$ of entrance laws [that is, (η_t) are finite measures having the properties

(1.6)
(i) $\eta_t Q_s = \eta_{t+s}$ for all $s, t > 0$;
(ii) $\int \big(1 - \psi_1(x)\big) \eta_s(dx)$ remains bounded as $s \to 0$]

such that

(1.7) $$n_\lambda(A) = \int_0^\infty e^{-\lambda t} \eta_t(A)\, dt \quad \text{for all } \lambda > 0, A \in \mathcal{B}(0,\infty).$$

Equations (1.5) and (1.7) imply now

(1.8) $\eta_t(A) = a^{v+1}\eta_{at}(a^\alpha A)$ for all $a > 0$, $A \in \mathcal{B}(0,\infty)$ for some $v < 0$.

Blumenthal [1] showed that any entrance law (η_t) has a representation

$$\eta_t(A) = \theta_t(A) + \int Q_t(x, A)\eta(dx), \tag{1.9}$$

where (θ_t) is an entrance law with the property $\int g(x)\theta_t(dx) \to 0$ as $t \to 0$, for any $g \in C_c(0,\infty)$ [$C_c(0,\infty)$ is the class of continuous functions $(0,\infty)$ with a compact support], and η is a σ-finite measure on $(0,\infty)$ with the additional property

$$\int (1 - \psi_1(x))\eta(dx) < +\infty.$$

REMARK 1. There always exists a trivial extension (corresponding to the entrance law $\eta_t \equiv 0\ \forall\, t > 0$), which stays at 0 after T_0 (the absorbing case). This is clearly α-self-similar and we shall ignore it in the rest of this paper.

REMARK 2. Representation (1.9) is easily seen to be unique.

REMARK 3. The measure η is the one mentioned by Meyer [8, Theorem 6, page 189]. It corresponds to the case when the process leaves 0 by jumping while (θ_t) corresponds to the case $\widetilde{X}(t)$ leaves 0 continuously. We shall later see that, for an α-ssmp [with a further condition (A), which is formulated later], we either have $\eta \equiv 0$ or $\theta_t \equiv 0\ \forall\, t > 0$.

Let us now assume (η_t) is an entrance law for

$$X(t)(\omega) = \begin{cases} \widetilde{X}(t)(\omega), & t < T_0(\omega), \\ \Delta, & t \geq T_0, \end{cases}$$

such that (η_t) satisfies (1.8) and that in (1.9) η is not identically zero. Now, according to Blumenthal [1],

$$\lim_{t\to 0} \int g(x)\eta_t(dx) = \int g(x)\eta(dx) \quad \text{for } g \in C_c(0,\infty).$$

Equation (1.8) implies

$$\begin{aligned} \lim_{t\to 0} \int g(x)\,\eta_t(dx) &= a^{v+1} \lim_{t\to 0} \int g(x)\eta_{at}\big(d(a^\alpha x)\big) \\ &= a^{v+1} \lim_{t\to 0} \int g(a^{-\alpha}x)\,\eta_t(dx) \\ &= a^{v+1} \int g(a^{-\alpha}x)\,\eta\,(dx) \\ &= a^{v+1} \int g(x)\,\eta\,\big(d(a^\alpha x)\big). \end{aligned}$$

This means $\eta(dx) = a^{v+1}\eta(d(a^\alpha x))$. According to the well-known result for Haar measures on $(0,\infty)$, we obtain

$$\eta(dx) = \frac{m}{x^{\beta+1}}dx \quad \text{for some } m > 0, \beta = \frac{v+1}{\alpha}\left(v < 0 \text{ iff } \beta < \frac{1}{\alpha}\right).$$

We shall now make an assumption, which we will keep in force the rest of this section.

(A) There exists a constant $k > 0$ such that the following hold:

(a) the limit $\lim_{x\to 0}[E(1-e^{-T_0})/x^k]$ exists and is strictly positive;

(b) the limit $\lim_{x\to 0}[V^\lambda f(x)/x^k]$ exists for all $f \in C_0(0,\infty)$ and is strictly positive for some such f; $(V^\lambda)_{\lambda>0}$ is the resolvent corresponding to $(X(t))$.

REMARK. It will be shown in the last section that (A) is valid if $X(t)$ has continuous paths and 0 is a regular boundary point. Also, in the case of a symmetric stable process killed when hitting the negative half-axis, (A) holds (see Section 4).

LEMMA 1.1. *Let us assume* (A). *The Laplace transform,* $(n_\lambda)_{\lambda>0}$, *of an entrance law* $(\eta_t)_{t>0}$ *has a unique representation*

$$n_\lambda(f) = p\lim_{x\to 0}\frac{V^\lambda f(x)}{1-\psi_1(x)} + m\int_0^\infty V^\lambda f(x)\frac{dx}{x^{\beta+1}} \quad \textit{for some } p \ge 0, m \ge 0, \beta < 1/\alpha \tag{1.10}$$

$(p, m$ and β are independent of $\lambda)$ and for all $f \in C_0(0,\infty)$.

PROOF. Let $f \in C_0(0,\infty)$. Define

$$g(x) = \frac{V^\lambda f(x)}{1-\psi_1(x)}, \quad x > 0, \quad g(0) = \lim_{x\to 0}\frac{V^\lambda f(x)}{1-\psi_1(x)}.$$

[The existence of this limit is a consequence of (A).] Now $g \in C_b[0,\infty)$ [where $C_b[0,\infty)$ is the class of all bounded, continuous real functions on $[0,\infty)$]. The measures

$$k_t(dx) = \big(1-\psi_1(x)\big)\eta_t(dx)$$

remain bounded, when $t \to 0$, and so we have a weak limit k [if necessary, we can take a subsequence of (k_t)] such that

$$\begin{aligned}\lim_{t\to 0}\int g(x)\big(1-\psi_1(x)\big)\eta_t(dx) &= k(0)g(0) + \int_0^\infty g(x)k(dx)\\ &= k(0)\lim_{x\to 0}\frac{V^\lambda f(x)}{1-\psi_1(x)} + \int_0^\infty \frac{V^\lambda f(x)}{1-\psi_1(x)}k(dx).\end{aligned}$$

Now also

$$\begin{aligned}\lim_{t\to 0}\int g(x)(1-\psi_1(x))\eta_t(dx) &= \lim_{t\to 0}\int V^\lambda f(x)\eta_t(dx)\\ &= \lim_{t\to 0}\int \eta_t(dx)\left\{\int_0^\infty e^{-\lambda s}Q_s f(x)ds\right\}\\ &= \lim_{t\to 0}\int_0^\infty e^{-\lambda s}\eta_{s+t}(f)\,ds\\ &= \int_0^\infty e^{-\lambda s}\eta_s(f)ds = n_\lambda(f),\end{aligned}$$

which follows from (1.6).

Thus we have proved

$$n_\lambda(f) = k(0)\lim_{x\to 0}\frac{V^\lambda f(x)}{1-\psi_1(x)} + \int_0^\infty \frac{V^\lambda f(x)}{1-\psi_1(x)}k(dx).$$

Writing $k\{0\} = p$ and $k(dx)/[1-\psi_1(x)] = \eta(dx)$, we obtain

$$n_\lambda(f) = p\lim_{x\to 0}\frac{V^\lambda f(x)}{1-\psi_1(x)} + \int_0^\infty V^\lambda f(x)\eta(dx) \quad \text{for all } f\in C_0(0,\infty).$$

Obviously the measure η is the same as that in (1.9), which gives (1.10). The uniqueness of the Laplace transform and the uniqueness of the representation (1.9) together imply that (1.10) is uniquely determined. □

We will use Lemma 1.1 to obtain a kind of 0–1 law for α-ssmp.

THEOREM 1.2. *Let $(\widetilde{X}_t, \widetilde{P}^x)$ be an α-ssmp on $[0,\infty)$, with $X(t)$ defined by*

$$X(t) = \begin{cases}\widetilde{X}(t), & t < T_0,\\ \Delta, & t\geq T_0.\end{cases}$$

Let $(V^\lambda)_{\lambda>0}$ be the resolvent corresponding to $(X(t))$ and let (A) *be valid. Then in the representation* (1.10) *either $p=0$ or $m=0$.*

REMARK. Theorem 1.2 says that an α-ssmp has precisely two possible ways to leave 0: Either (when $p=0$) it jumps, immediately after hitting 0, into $(0,\infty)$ a.s. $(\widetilde{P}^x)$ or (when $m=0$) it leaves 0 continuously a.s. $(\widetilde{P}^x)$. Compare with Remark 3.

PROOF OF THEOREM 1.2. Assume $p>0$ and $m>0$. According to Lemma 1.1, (1.10) is valid. The α-self-similarity implies

$$\int V^\lambda f(x)\frac{dx}{x^{\beta+1}} = a^{\alpha\beta-1}\int V^{\lambda/a}(f\circ a^{-\alpha})(x)\frac{dx}{x^{\beta+1}}.$$

Uniqueness of (1.10) implies that, to have (1.5) fulfilled, we must have

$$\lim_{x\to 0}\frac{V^\lambda f(x)}{1-\psi_1(x)} = a^{\alpha\beta-1}\lim_{x\to 0}\frac{V^{\lambda/a}(f\circ a^{-\alpha})(x)}{1-\psi_1(x)} \quad \text{and} \quad \alpha\beta<1.$$

Assumption (A) gives

$$\begin{aligned}\lim_{x\to 0}\frac{V^\lambda f(x)}{1-\psi_1(x)} &= \frac{\lim_{x\to 0}V^\lambda f(x)/x^k}{\lim_{x\to 0}\big(1-\psi_1(x)\big)/x^k}\\ &= \frac{\lim_{x\to 0}\big[(a^{\alpha k}/a)V^{\lambda/a}(f\circ a^{-\alpha})(a^\alpha x)\big]/(a^\alpha x)^k}{\lim_{x\to 0}\big(1-\psi_1(x)\big)/x^k}\\ &= a^{\alpha k-1}\frac{\lim_{x\to 0}V^{\lambda/a}(f\circ a^{-\alpha})(x)/x^k}{\lim_{x\to 0}\big(1-\psi_1(x)\big)/x^k}\\ &= a^{\alpha k-1}\lim_{x\to 0}\frac{V^{\lambda/a}(f\circ a^{-\alpha})(x)}{1-\psi_1(x)},\end{aligned}$$

and so we must have $\beta = k$.

According to (A), $\lim_{x\to 0}[(1-\psi_1(x))/x^\beta]$ exists and is greater than 0. Expression (1.6) (ii) implies

$$\int_0^\infty \big(1-\psi_1(x)\big)\frac{dx}{x^{\beta+1}} < \infty$$

and thus also

$$\int_0^\delta \big(1-\psi_1(x)\big)\frac{dx}{x^{\beta+1}} < \infty \quad \forall\,\delta>0.$$

Because $\lim_{x\to 0}[(1-\psi_1(x))/x^\beta] > 0$ there exist $\varepsilon > 0$ and $\delta > 0$ such that $(1-\psi_1(x))/x^\beta \ge \varepsilon$, for all $x\in(0,\delta)$.

However, then

$$\int_0^\delta \big(1-\psi_1(x)\big)\frac{dx}{x^{\beta+1}} \ge \int_0^\delta \frac{\varepsilon}{x}\,dx = +\infty,$$

and we get a contradiction.

Thus we have shown that in (1.10) either $p = 0$ or $m = 0$. □

REMARK. Different $p \neq 0$ (or $m \neq 0$) correspond to the same extension. We shall now on write $p = 1\,(m = 1)$ if $p\,(m)$ is not equal to 0.

2. The existence of an α-self-similar extension. We shall assume (A) throughout this section. Let $(X(t), P^x)$ be, as in Section 1, an α-ssmp on $(0,\infty)$ killed at T_0, $T_0 < \infty$ a.s (p^x). It was shown by Blumenthal [1] that, corresponding to any entrance law (η_t), there exists a standard process $(\widetilde{X}(t), \widetilde{P}^x)$ on $[0,\infty)$ such that

$$\widetilde{P}^x\{\widetilde{X}(t)\in A;\ t<T_0\} = P^x\{X(t)\in A\} \quad \text{for all } A\in\mathcal{B}(0,\infty).$$

It is evident from the representation

$$U^\lambda f(0) = \frac{n_\lambda(f)}{\lambda n_\lambda(1)}, \qquad \lambda > 0, f \in \mathcal{B}_b[0,\infty),$$

that, to any entrance law (η_t) fulfilling (1.8), there corresponds an α-self-similar extension. In this section we shall study the existence of such (η_t).

First let η be a measure on $(0,\infty)$ such that

$$\eta(dx) = \frac{dx}{x^{\beta+1}} \quad \text{for some } \beta < \frac{1}{\alpha}.$$

Then, if we define

$$\eta_t(A) = \int Q_t(x, A)\eta(dx), \qquad t > 0, A \in \mathcal{B}(0,\infty),$$

the property (1.6)(i) and the property (1.8) are obviously valid. So we only have to check (1.6)(ii). It is valid iff

$$\int (1 - \psi_1(x))\eta(dx) < +\infty. \tag{2.1}$$

We show that (A) implies (2.1). Let $\delta > 0$. Now $\int_\delta^\infty (1 - \psi_1(x))\, dx/x^{\beta+1}$ is easily seen to be finite iff $\beta > 0$. Because $\lim_{x\to 0}[(1 - \psi(x))/x^k]$ exists and is strictly positive, $(1 - \psi_1(x))/x^k$ must be bounded (away from 0 and $+\infty$) on $(0,\delta)$. Thus

$$\int_0^\delta \frac{1 - \psi_1(x)}{x^k} \frac{dx}{x^{\beta-k+1}}$$

is finite iff $\beta < k$. However, we also have $\beta < 1/\alpha$ and thus (2.1) holds iff $\beta \in (0, \min\{1/\alpha, k\})$.

So we have proved the following.

THEOREM 2.1. *Let $(X(t), P^x)$ be an α-ssmp on $(0,\infty)$ which has been killed at 0, and let* (A) *be valid. Then the totality of α-self-similar extensions which leave* 0 *by jumping are those corresponding to the jumping-in measures $\eta(dx) = dx/x^{\beta+1}$, where $\beta \in (0, \min\{1/\alpha, k\})$ and k is the strictly positive constant mentioned in* (A).

REMARK. In Theorem 2.1 it indeed suffices to assume that only the latter limit, $\lim_{x\to 0}[(1 - \psi_1(x))/x^k]$, exists (in fact, $0 \le \limsup_{x\to 0}[(1 - \psi_1(x))/x^k] < \infty$ is sufficient; see the end of Section 3).

Let us now consider the case when $\widetilde{X}(t)$ leaves 0 continuously, that is, $m = 0$. Define

$$n_\lambda(f) = \lim_{x\to 0} \frac{V^\lambda f(x)}{1 - \psi_1(x)} \quad \text{for } \lambda > 0, f \in C_0(0,\infty). \tag{2.2}$$

Easy calculations show that

$$n_\lambda V^\mu f = \frac{n_\lambda - n_\mu}{\mu - \lambda} f \quad \text{for } \lambda, \mu > 0, \lambda \neq \mu, f \in C_0(0,\infty).$$

According to the Riesz representation theorem, there exists a uniquely determined, finite Borel measure n_λ on $(0,\infty)$ such that (2.2) is valid for all $f \in C_0(0,\infty)$. It is easily seen that also

$$n_\lambda V^\mu f = \frac{n_\lambda - n_\mu}{\mu - \lambda} f \quad \text{for all } f \in \mathcal{B}(0,\infty).$$

Because

$$n_\lambda(f) = \lim_{x\to 0} \frac{V^\lambda f(x)}{1 - \psi_1(x)} = \frac{\lim_{x\to 0}\big(V^\lambda f(x)\big)/x^k}{\lim_{x\to 0}\big(1-\psi_1(x)\big)/x^k} \quad \forall f \in C_0(0,\infty)$$

[this is a consequence of (A)], condition (1.5) is fulfilled.

According to the result of Getoor and Sharpe [3], there exists an entrance law (η_t) such that

$$n_\lambda(A) = \int_0^\infty e^{-\lambda t} \eta_t(A)\, dt \quad \text{for all } \lambda > 0, A \in \mathcal{B}(0,\infty).$$

Equation (1.8), which ensures the self-similarity of the extension, is obviously equivalent to (1.5).

We have a theorem.

THEOREM 2.2. *Let $(X(t), P^x)$ be an α-ssmp on $(0,\infty)$, which has been killed at 0, and let (A) be valid. Then there is precisely one α-self-similar extension $(\widetilde{X}(t), \widetilde{P}^x)$ of $(X(t), P^x)$ on $[0,\infty)$ such that $\widetilde{X}(t)$ leaves 0 continuously. In that case, if $(V^\lambda)_{\lambda>0}$ is the resolvent corresponding to $(X(t), P^x)$,*

$$n_\lambda(f) = \lim_{x\to 0} \frac{V^\lambda f(x)}{1-\psi_1(x)} \quad \textit{for } f \in C_0(0,\infty),$$

is the Laplace transform of the entrance law (θ_t). In addition (θ_t) has the property,

$$\theta_t(f) \to 0 \quad \textit{as } t \to 0, f \in C_c(0,\infty).$$

REMARK 1. In Theorem 2.2 it is essential that both limits in (A) [(a) and (b)] exist and are strictly positive.

REMARK 2. The fact that $n_\lambda(f) = \int_0^\infty e^{-\lambda t}\theta_t(f) dt$ such that $\theta_t(f) \to 0$, when $t \to 0$, $f \in C_c(0,\infty)$, is a consequence of the uniqueness of (1.9) and (1.10).

REMARK 3. Theorem 2.2 shows that in the diffusion case, if 0 is an exit boundary point, (A) does not hold because otherwise it would be possible for the process to leave 0 continuously.

3. The diffusion case. In this section we shall study the validity of (A) in the case when $X(t)$ is a diffusion, that is, all the paths are continuous. Let $(X(t), P^x)$ be an α-self-similar diffusion on $(0, \infty)$, killed at $T_0, T_0 < \infty$ a.s., and let $h(x) = x^{\nu/\alpha}$ be the scale function for $(X(t), P^x), \nu > 0$. Then $h(x)$ is excessive and we can define the h-transform $(\widehat{X}(t), \widehat{P}^x)$ such that

$$\widehat{Q}_t f(x) = \frac{1}{h(x)} Q_t \left[h(x) f(x)\right], \quad f \in \mathcal{B}(0, \infty),$$

where $\widehat{Q}_t$ is the transition function of $(\widehat{X}(t), \widehat{P}^x)$.

According to [13, Theorem 3], $(\widehat{X}(t), \widehat{P}^x)$ is an α-self-similar diffusion on $(0, \infty)$ [in fact, a weak dual to $(X(t), P^x)$ with respect to the measure $x^{1/\alpha - 1}dx$] with a scale function $s(x) = x^{-\nu/\alpha}$. For $(\widehat{X}(t), \widehat{P}^x)$ we have $\widehat{T}_0 = +\infty$ a.s. ($\widehat{T}_0$ is the first hitting time of 0), $\lim_{x \to 0} \widehat{X}(t) = +\infty$ a.s. and $s(x)$ is excessive.

Now let $f \in C_b([0, \infty))$. Then we have

$$\frac{Q_t f(x)}{h(x)} = \widehat{Q}_t \left[\frac{f(x)}{h(x)}\right] = \widehat{Q}_t \left[S(x) f(x)\right].$$

As was remarked by Lamperti [7], $\widehat{X}(t)$ can always be started at 0, such that it leaves 0 immediately and never returns, and the extended process is an α-self-similar diffusion on $[0, \infty)$. Let $\widehat{Q}_t(0, \cdot)$ be the transition function at 0 corresponding to this extension. We will show that

$$\begin{aligned} \lim_{x \to 0} \widehat{Q}_t \left[s(x) f(x)\right] &= \lim_{x \to 0} \widehat{E}^x \left[s\big(\widehat{X}(t)\big) f\big(\widehat{X}(t)\big)\right] \\ &= \widehat{E}^0 \left[s\big(\widehat{X}(t)\big) f\big(\widehat{X}(t)\big)\right] = \widehat{Q}_t \left[s(0) f(0)\right], \end{aligned} \tag{3.1}$$

for $f \in C_b([0, \infty))$.

The continuity of the paths and the strong Markov property imply

$$\widehat{E}^x \left[s\big(\widehat{X}(t)\big) f\big(\widehat{X}(t)\big)\right] = \widehat{E}^0 \left[s\big(\widehat{X}(t + \widehat{T}_x)\big) f\big(\widehat{X}(t + \widehat{T}_x)\big)\right],$$

where $\widehat{T}_x$ is the first hitting time to $\{x\}$ for $\widehat{X}(t)$. Because s and f are continuous on $(0, \infty)$, we have

$$\begin{aligned} \lim_{x \to 0} \widehat{E}^x \left[s\big(\widehat{X}(t)\big) f\big(\widehat{X}(t)\big)\right] &= \lim_{x \to 0} \widehat{E}^0 \left[s\big(\widehat{X}(t + \widehat{T}_x)\big) f\big(\widehat{X}(t + \widehat{T}_x)\big)\right] \\ &= \widehat{E}^0 \left[s\big(\widehat{X}(t)\big) f\big(\widehat{X}(t)\big)\right]. \end{aligned}$$

Now $\widehat{E}^x s(\widehat{X}(t))$ is excessive on $[0, \infty)$. This and the boundedness of f imply that

$$\begin{aligned} \widehat{E}^x \left[s\big(\widehat{X}(t)\big) f\big(\widehat{X}(t)\big)\right] &= \widehat{E}^0 \left[s\big(\widehat{X}(t + \widehat{T}_x)\big) f\big(\widehat{X}(t + \widehat{T}_x)\big)\right] \\ &\le \sup_y f(y) \widehat{E}^0 s\big(\widehat{X}(t)\big). \end{aligned}$$

We shall now consider $\widehat{E}^0 s(\widehat{X}(t))$. The α-self-similarity of $\widehat{X}(t)$ implies

$$\widehat{E}^0 s\big(\widehat{X}(t)\big) = t^{-\nu}\widehat{E}^0 s\big(\widehat{X}(1)\big),$$

which shows that either $\widehat{E}^0 s(\widehat{X}(t)) < +\infty\ \forall\, t > 0$ or $\widehat{E}^0 s(\widehat{X}(t)) = +\infty\ \forall\, t > 0$. Let us suppose $\widehat{E}^0 s(\widehat{X}(t)) = +\infty\ \forall\, t > 0$. Then also $\widehat{E}^0[s(\widehat{X}(t)); \widehat{X}(t) \le r] = +\infty\ \forall\, r > 0$, $\forall\, t > 0$; that is,

$$\int_0^r s(y)\widehat{Q}_t(0, dy) = +\infty \quad \forall\, r > 0, \forall\, t > 0.$$

Integration with respect to t gives $\int_0^r s(y)\widehat{U}(0, dy) = +\infty$. Because of the α-self-similarity,

$$\widehat{U}(0, dy) = \frac{1}{a}\widehat{U}\big(0, d(a^\alpha y)\big) \quad \forall\, a > 0.$$

Because $\lim_{t\to\infty}\widehat{X}(t)(\omega) = +\infty$ a.s. $(\widehat{P}^x)$, $\forall\, x \ge 0$, there must be $A \in \mathcal{B}(0,\infty)$ such that $0 < \widehat{U}(0, A) < \infty$. Well-known results about Haar measure give

$$\widehat{U}(0, dy) = My^{1/\alpha - 1}\, dy \quad \text{for some } M > 0.$$

This means

$$M\int_0^r s(y)y^{1/\alpha-1}\, dy = M\int_0^r y^{(1-\nu)/\alpha - 1}\, dy = +\infty,$$

which is false when $\nu < 1$. Thus $\widehat{E}^0 s(\widehat{X}(t)) < +\infty\ \forall\, t > 0$ when $\nu < 1$.

Now, according to the theorem of dominated convergence, we obtain

$$\begin{aligned}
\lim_{x\to 0}\frac{V^\lambda f(x)}{x^{\nu/\alpha}} &= \lim_{x\to 0}\widehat{V}^\lambda\big[s(x)f(x)\big] \\
&= \lim_{x\to 0}\int_0^\infty e^{-\lambda t}\widehat{E}^x\Big[s\big(\widehat{X}(t)\big)f\big(\widehat{X}(t)\big)\Big]\,dt \\
&= \int_0^\infty e^{-\lambda t}\widehat{E}^0\Big[s\big(\widehat{X}(t)\big)f\big(\widehat{X}(t)\big)\Big]\,dt.
\end{aligned} \tag{3.2}$$

For the right-hand side of (3.2) we have

$$\int_0^\infty e^{-\lambda t}\widehat{E}^0\Big[s\big(\widehat{X}(t)\big)f\big(\widehat{X}(t)\big)\Big]\,dt \le \sup_y f(y)\widehat{E}^0 s\big(\widehat{X}(1)\big)\int_0^\infty e^{-\lambda t}t^{-\nu}\, dt.$$

It is well known that $\int_0^\infty e^{-\lambda t}t^{-\nu}\, dt$ is finite if $\nu \in (0,1)$ and infinite if $\nu \ge 1$.

Taking $f \equiv 1$ and $\lambda = 1$, we get

$$\lim_{x\to 0}\frac{1-\psi_1(x)}{x^{\nu/\alpha}} = \widehat{E}^0\Big[\big(\widehat{X}(1)\big)^{-\nu/\alpha}\Big]\int_0^\infty e^{-t}t^{-\nu}\, dt. \tag{3.3}$$

The quantity $E^0[(X(1))^{-\nu/\alpha}]$ is strictly positive. It is easily seen that

$$\widehat{E}^0\Big[\big(\widehat{X}(t)\big)^{-\nu/\alpha}\Big] \geq \widehat{E}^0\Big[\big(\widehat{X}(t)\big)^{-\nu/\alpha};\ \widehat{X}(t) \leq r\Big] \geq r^{-\nu/\alpha}\widehat{P}^0\big\{\widehat{X}(t) \leq r\big\} \text{ for all } t > 0.$$

Because of the continuity of the paths, $\widehat{P}^0\{\widehat{X}(t) \leq r\} \to 1$, when $t \to 0$, and thus there exists $t_0 > 0$ such that $\widehat{P}^0\{\widehat{X}(t_0) \leq r\} > 0$. So we have

$$\widehat{E}^0\Big[\big(\widehat{X}(1)\big)^{-\nu/\alpha}\Big] = t_0^{\nu}\,\widehat{E}^0\Big[\big(\widehat{X}(t_0)\big)^{-\nu/\alpha}\Big] > 0.$$

Thus we have shown that the right-hand side of (3.3) is strictly positive and finite if $0 < \nu < 1$.

The right-hand side of (3.2) is strictly positive for some $f \in C_c(0,\infty)$. Obviously, $f \in C_c(0,\infty)$ iff $g = fs \in C_c(0,\infty)$ and so it is enough to prove that there exists $g \in C_c(0,\infty)$, $g > 0$, such that

$$\int_0^\infty e^{-\lambda t}\widehat{E}^0 g\big(\widehat{X}(t)\big)\,dt > 0.$$

This is valid if $\widehat{E}^0 g(\widehat{X}(t)) \geq m$, for some $m > 0$, for all t in some time interval.

It is easily seen that there exist $g \in C_c(0,\infty)$, $g > 0$ and $t_0 \in (0,\infty)$ such that $\widehat{E}^0 g(\widehat{X}(t_0)) > 0$. As proved by Lamperti [7], $t \to \widehat{E}^0 g(\widehat{X}(t))$ is continuous on $(0,\infty)$ and thus there exists some interval $[a,b]$, which includes t_0, such that $\widehat{E}^0 g(\widehat{X}(t)) \geq m$ for all $t \in [a,b]$, for some $m > 0$.

Thus we have shown that (A) is valid for an α-self-similar diffusion with $k = \nu/\alpha$, $\nu \in (0,1)$, if $h(x) = x^{\nu/\alpha}$ is a scale function.

REMARK. The existence of a completely reflecting extension $(\widetilde{X}(t),\widetilde{P}^x)$ with continuous paths was remarked by Lamperti [7]. It was shown by Itô and McKean [6] that if $(\widetilde{X}(t),\widetilde{P}^x)$ is any diffusion on $[0,\infty)$ such that 0 is a regular boundary point with the boundary condition of instantaneous reflection, and if $h(x)$ is a scale function with $h(0) = 0$ and $h(+\infty) = +\infty$, then

$$\lim_{x\to 0}\frac{E^x(1-e^{-\lambda T_0})}{h(x)} = \int_0^\infty (1-e^{-\lambda l})n(dl) \quad \forall\, \lambda > 0,$$

where $n(dl)$ is the Lévy measure for the inverse of the local time at 0 and $(X(t), P^x)$ is $(\widetilde{X}(t),\widetilde{P}^x)$ killed at T_0. For more about this see Itô and McKean [6, pages 214–216]. Another proof, which uses the idea of h-transform of $(X(t), P^x)$, can be found in Salminen [11].

According to [7], an α-self-similar diffusion on $(0,\infty)$ is governed by the generator

$$L = \frac{1}{2}\sigma^2 x^{2-1/\alpha}\frac{d^2}{dx^2} + \mu x^{1-1/\alpha}\frac{d}{dx}, \qquad \sigma^2 > 0,\ \mu \in R.$$

Our case $T_0 < +\infty$ a.s. corresponds to σ and μ such that $\mu < \frac{1}{2}\sigma^2$. The scale function h is $h(x) = x^{1-2\mu/\sigma^2}$. This means $\nu/\alpha = 1-2\mu/\sigma^2$ and, because $\nu \in (0,1)$, we have $0 < 1 - 2\mu/\sigma^2 < 1/\alpha$, which gives $\frac{1}{2}(1-1/\alpha)\sigma^2 < \mu < \frac{1}{2}\sigma^2$.

Thus we have the following theorem.

THEOREM 3.1. *Let* $(X(t), P^x)$ *be an* α*-self-similar diffusion on* $(0, \infty)$*, killed at* T_0 *, with a generator*

$$L = \frac{1}{2}\sigma^2 x^{2-1/\alpha}\frac{d^2}{dx^2} + \mu x^{1-1/\alpha}\frac{d}{dx}, \qquad \sigma^2 > 0,\ \mu \in R \text{ such that } \mu < \frac{1}{2}\sigma^2.$$

Then condition (A) *is valid iff* $\frac{1}{2}(1 - 1/\alpha)\sigma^2 < \mu < \frac{1}{2}\sigma^2$ *, that is,* 0 *is a regular boundary point. In that case* $X(t)$ *can be extended to be an* α*-self-similar standard process on* $[0, \infty)$ *either such that it leaves* 0 *continuously a.s.* (P^x) *or that it jumps into* $(0, \infty)$ *according to the jumping-in measure*

$$\eta(dx) = \frac{dx}{x^{\beta+1}}, \qquad \beta \in \left(0, \min\left\{1 - \frac{2\mu}{\sigma^2}, \frac{1}{\alpha}\right\}\right).$$

If 0 is an exit boundary point for $X(t)$, that is, $\mu \leq \frac{1}{2}(1-1/\alpha)\sigma^2$, it is not possible to extend $X(t)$ to $[0, \infty)$ such that the extension has everywhere continuous paths (except the absorbing case). Consequently (see Theorem 2.2), (A) fails in this case. It is, however, possible to find an α-self-similar extension $\widetilde{X}(t)$ with a jumping-in measure

$$\eta(dx) = \frac{dx}{x^{\beta+1}}, \qquad \beta \in \left(0, \frac{1}{\alpha}\right).$$

We have a proposition.

PROPOSITION 3.2. *Let* $(X(t), P^x)$ *be as in Theorem* 3.1*, except that* 0 *is an exit boundary point, that is,* $\mu \leq \frac{1}{2}(1 - 1/\alpha)\sigma^2$*. Then* $X(t)$ *can be extended to be an* α*-self-similar standard process on* $[0, \infty)$ *such that it, immediately after reaching* 0*, jumps into* $(0, \infty)$ *according to the jumping-in measure*

$$\eta(dx) = \frac{dx}{x^{\beta+1}}, \qquad \beta \in (0, 1/\alpha).$$

PROOF. We only have to show that $\int_0^\infty (1 - \psi_1(x))\eta(dx) < +\infty$. It is obvious that for any α-ssmp on $(0, \infty)$ we have

$$T_0(\omega) = \int_0^\infty \left(Y(t)(\omega)\right)^{1/\alpha} dt, \tag{3.4}$$

where $Y(t)(\omega) = X_{T(t)}(\omega)$, $T(t)(\omega)$ is the inverse of an additive functional

$$A(t)(\omega) = \int_0^t \left(X(h)(\omega)\right)^{-1/\alpha} dh.$$

See more about this time change in [4], [7] and [12]. According to the proof of Theorem 5.1 in [7], we have $Y(t)(\omega) = \exp\{\sigma B(t)(\omega) + \log x + (\mu - \frac{1}{2}\sigma^2)t\}$, where $B(t)$ is a standard Brownian motion, starting at 0, on $(-\infty, +\infty)$.

Let us first assume $\mu < \frac{1}{2}(1 - 1/\alpha)\sigma^2$. It is well known that

$$\exp\{cB(t)(\omega) - \tfrac{1}{2}c^2 t\} \quad \text{is a } Q^0\text{-martingale for all } c > 0,$$

where Q^0 is the Wiener measure. This implies

$$Q^0\left[\exp\{cB(t) - \tfrac{1}{2}c^2 t\}\right] = 1 \quad \forall\, c > 0,$$

and thus

$$\begin{aligned} E^1 T_0 = E^1\left\{\int_0^\infty (Y(t))^{1/\alpha}\,dt\right\} &= Q^0\left\{\int_0^\infty \exp\frac{1}{\alpha}\left[\sigma B(t) - \frac{1}{2}\sigma^2 t + \mu t\right]dt\right\} \\ &= \int_0^\infty \exp\frac{1}{\alpha}\left[\mu t - \frac{1}{2}\sigma^2 t + \frac{1}{2}\frac{\sigma^2}{\alpha}t\right] Q^0\left[\exp\left(\frac{1}{\alpha}\sigma B(t) - \frac{1}{2}\frac{\sigma^2}{\alpha^2}t\right)\right]dt \\ &= \int_0^\infty \exp\left[\frac{t}{\alpha}\left(\mu - \frac{1}{2}\left(1 - \frac{1}{\alpha}\right)\sigma^2\right)\right]dt < +\infty \quad \text{iff } \mu < \frac{1}{2}\left(1 - \frac{1}{\alpha}\right)\sigma^2. \end{aligned}$$

Now,

$$\begin{aligned} \lim_{x\to 0}\frac{E^x(1 - \exp(-T_0))}{x^{1/\alpha}} &= \lim_{x\to 0}\frac{E^1(1 - \exp(x^{-1/\alpha}T_0))}{x^{1/\alpha}} \\ &= \lim_{x\to 0}\frac{E^1(1 - \exp(-xT_0))}{x} \\ &= -\frac{d}{dx}\left[E^1(\exp(-xT_0))\right]_{x=0} = E^1 T_0 < +\infty, \end{aligned}$$

and thus

$$\int_0^\infty (1 - \psi_1(x))\frac{dx}{x^{\beta+1}} < +\infty \quad \text{when } \beta \in \left(0, \frac{1}{\alpha}\right).$$

If $\mu = \frac{1}{2}(1 - 1/\alpha)\sigma^2$, 0 is still an exit boundary point, but $E^1 T_0 = +\infty$. In this case,

$$\begin{aligned} T_0(\omega) &= \int_0^\infty \exp\frac{1}{\alpha}\left[\sigma B(t)(\omega) + \left(\mu - \frac{1}{2}\sigma^2\right)t\right]dt \\ &\le \int_0^\infty \exp\frac{1}{\alpha}\left[\sigma B(t)(\omega) + \left(\mu' - \frac{1}{2}\sigma^2\right)t\right]dt = T_0^{\mu',\sigma^2}(\omega), \end{aligned}$$

where T_0^{μ',σ^2} is the first hitting time of 0 for an α-self-similar diffusion $(\acute{X}(t), \acute{P}^x)$

with generator

$$\frac{1}{2}\sigma^2 x^{2-1/\alpha}\frac{d^2}{dx^2} + \mu' x^{1-1/\alpha}\frac{d}{dx}, \quad \frac{1}{2}\left(1-\frac{1}{\alpha}\right)\sigma^2 < \mu' < \frac{1}{2}\sigma^2.$$

This implies

$$\frac{E^x\left(1-\exp\left[-T_0(\omega)\right]\right)}{x^k} \le \frac{\acute{E}^x\left(1-\exp\left[-T_0^{\mu',\sigma^2}(\omega)\right]\right)}{x^k} \quad \forall k > 0.$$

When $x \to 0$, the limit of the right-hand side, as we have seen, exists and is finite for $k = 1 - 2\mu'/\sigma^2$. Because this is true for all $\mu' \in (\frac{1}{2}(1-1/\alpha)\sigma^2, \frac{1}{2}\sigma^2)$, we also have

$$\limsup_{x\to 0} \frac{E^x(1-e^{-T_0})}{x^k} \le +\infty \quad \text{for all } k = 1 - \frac{2\mu'}{\sigma^2},\ \mu' \in \left(\frac{1}{2}\left(1-\frac{1}{\alpha}\right)\sigma^2, \frac{1}{2}\sigma^2\right),$$

that is, $k \in (0, 1/\alpha)$. However, that gives $\int_0^\infty (1-\psi_1(x))\,dx/x^{\beta+1} < +\infty$, for all $\beta \in (0, 1/\alpha)$. □

4. An example of Watanabe. In [14] Watanabe studied an example of a self-similar Markov process on $(0,\infty)$ with only right-continuous paths with left limits. Assume $X(t)$ is a symmetric α-stable process, $\alpha \in (0,2)$, starting from the positive real axis. Let $T = \inf\{t > 0;\ X(t) \le 0\}$. Define a new process by setting

$$\widetilde{X}(t) = \begin{cases} X(t), & t < T, \\ X(t) - \inf\limits_{T\le s\le t} X(s), & t \ge T. \end{cases} \tag{4.1}$$

Obviously $\widetilde{X}(t)$ is a $1/\alpha$-ssmp on $[0,\infty)$. Watanabe calls $\widetilde{X}(t)$ a reflecting barrier process. When $\alpha = 2$, (4.1) becomes the reflecting Brownian motion (see also the introduction in Blumenthal [1]).

Now $T = \widetilde{T}_0 = \inf\{t > 0;\ \widetilde{X}(t) = 0\}$. Define further

$$Y(t) = \begin{cases} \widetilde{X}(t), & t < \widetilde{T}_0, \\ \Delta, & t \ge \widetilde{T}_0. \end{cases}$$

All the conditions in Blumenthal's paper [1] are satisfied for $Y(t)$ and $Y(t)$ is a $1/\alpha$-ssmp on $(0,\infty)$. Watanabe's computations in [14] show that (A) is valid with $k = \alpha/2$.

PROPOSITION 4.1. *The process $\widetilde{X}(t)$ leaves 0 continuously a.s., that is,*

$$\widetilde{n}_\lambda(f) = \lim_{x\to 0} \frac{V^\lambda f(x)}{1-\psi_1(x)} \quad \forall f \in C_0(0,\infty),$$

where $\widetilde{n}_\lambda$ is the Laplace transform of the entrance law corresponding to $\widetilde{X}(t)$.

PROOF. According to Watanabe [14, page 190], if $(\widetilde{U}^\lambda)_{\lambda>0}$ is a resolvent of $\widetilde{X}(t)$, then we have

$$\widetilde{U}^\lambda f(x) = V^\lambda f(x) + \frac{\psi_\lambda(x)}{\lim_{\varepsilon\to 0}\left[\left(1-\psi_\lambda(\varepsilon)\right)/\varepsilon^{\alpha/2}\right]} \lim_{\varepsilon\to 0} \frac{V^\lambda f(\varepsilon)}{\varepsilon^{\alpha/2}}$$

$$= V^\lambda f(x) + \psi_\lambda(x) \lim_{\varepsilon\to 0} \frac{V^\lambda f(\varepsilon)}{1-\psi_\lambda(\varepsilon)}, \quad \forall f \in C_0(0,\infty),$$

where $(V^\lambda)_{\lambda>0}$ is a resolvent of $Y(t)$. Calculations based on (1.5), the proof of Theorem 1.2, assumption (A) and the fact that $\widetilde{U}^\lambda f(0) = \widetilde{n}_\lambda(f)/[\lambda\widetilde{n}_\lambda(1)]$ show that this is equivalent to

$$\widetilde{n}_\lambda(f) = \lim_{x\to 0} \frac{V^\lambda f(x)}{1-\psi_1(x)}. \qquad \square$$

REMARK 1. Rogers [10] has shown that $\widetilde{X}(t)$, constructed similarly from any real Lévy process, a.s. leaves 0 continuously. Proposition 4.1 is a special case of his result.

REMARK 2. Theorem 2.1 gives those $1/\alpha$-self-similar extensions that jump into $(0,\infty)$ immediately after hitting 0.

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