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VARIATIONAL INEQUALITIES WITH EXAMPLES AND AN APPLICATION TO THE CENTRAL LIMIT THEOREM

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Upper bounds for the distance in variation between an arbitrary probability measure and the standard normal one are established via some integrodifferential functionals including information. The results are illustrated by gamma- and t-distributions. Moreover, as a by-product, another proof of the central limit theorem is obtained.

1. Introduction and summary. Let X be an r.v. with $EX = 0$, $\mathrm{Var}(X) = 1$ and $\rho(F_X, \phi)$ the usual distance in variation between the distribution of X with d.f. F_X and the standard normal with d.f. Φ, namely,

$$\rho(F_X, \Phi) = \sup_A |F_X(A) - \Phi(A)|, \tag{1.1}$$

where the supremum is taken over the class of Borel sets A. Utev (1989) obtained the upper bound $\rho(F_X, \Phi) \le 3\sqrt{U_X - 1}$, in relation to the functional [Borovkov and Utev (1983)]

$$U_X = \sup_g \frac{\mathrm{Var}\big[g(X)\big]}{\mathrm{Var}(X)E\big[g'(X)^2\big]}, \tag{1.2}$$

where the supremum is taken over the class H_1 of absolutely continuous functions g with $0 < E[g'(X)^2] < \infty$; $U_X \ge 1$ characterizes normality ($U_X = 1$). The proof is based on the equation [see Bolthausen (1984) and Stein (1972)]

$$\psi'(x) = x\psi(x) + \big(I_A(x) - \Phi(A)\big), \tag{1.3}$$

where

$$\psi(x) = e^{x^2/2} \int_{-\infty}^{x} \big(I_A(t) - \Phi(A)\big)e^{-t^2/2}\, dt. \tag{1.4}$$

This relation as well as Stein's method hinges on Stein's identity for a standard normal r.v. η, namely,

$$E\big[\eta g(\eta)\big] = E\big[g'(\eta)\big]. \tag{1.5}$$

The same identity turned up explicitly in Cacoullos (1982) in a different context, namely, the derivation of variance bounds for any r.v. X, in the spirit of

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the Chernoff (1981) inequality. In fact, if X has density f with interval support, mean μ and variance σ^2, then, from Cacoullos and Papathanasiou (1989),

$$\operatorname{Cov}\big(X, g(X)\big) = \sigma^2 E\big[w(X)g'(X)\big], \tag{1.6}$$

where the covariance kernel $w(x)$ is defined by

$$\sigma^2 w(x)f(x) = \int_{-\infty}^{x} (\mu - t)f(t)\,dt, \qquad x \in R. \tag{1.7}$$

If X is $N(\mu, \sigma^2)$, then $w(x) \equiv 1$, and, of course, (1.6) implies (1.5). Most importantly, the w-function characterizes the corresponding f, as shown by Cacoullos and Papathanasiou (1989) and Cacoullos (1989). In particular, X is normal iff

$$w(x) \equiv 1, \qquad x \in R. \tag{1.8}$$

In the present paper the stability of the preceding characterization with respect to the convergence in variation is established along with the corresponding rate of convergence. This is achieved by making use of the basic identity (1.5) and the next result, of independent interest, stated here as follows.

THEOREM 1.1. *Let X be an r.v. with $EX = 0$, $\operatorname{Var}(X) = a$ and an absolutely continuous distribution F with an interval support. Then*

$$\rho(F, \Phi) \le 2\big(E|w(X) - 1| + |1 - a|\big),$$

where w is the w-function associated with X.

The theorem is used in Section 3 to show the following result.

THEOREM 1.2. *Let $X_1, \dots, X_n, \dots$ be a sequence of continuous r.v.'s with means* 0, *variances* 1 *and absolutely continuous distributions each with an interval support. Let $f_1, \dots, f_n, \dots$ be the corresponding density functions and $w_1, \dots, w_n, \dots$ the corresponding w-functions. Then*

$$w_n(X_n) \xrightarrow{P} 1 \quad \text{as } n \to \infty \text{ iff } \int_{-\infty}^{\infty} |f_n(t) - \phi(t)|\,dt \to 0 \text{ as } n \to \infty. \tag{1.9}$$

Furthermore, motivated by (1.6) and the more general inequality obtained from it, namely,

$$\operatorname{Var}\big[g(X)\big] \ge \sigma^2 E^2\big[w(X)g'(X)\big],$$

we consider the functional [cf. (1.2)]

$$J_X = \inf_{g \in H_1} \frac{\operatorname{Var}\big[g(X)\big]}{\sigma^2 E^2\big[g'(X)\big]}. \tag{1.10}$$

Clearly, $J_X \leq 1$ and equality holds iff X is normal [Cacoullos and Papathanasiou (1989)]. Here the stability of this characterization, with respect to the convergence in variation and the rate of convergence, is established in the following theorem.

THEOREM 1.3. *Let X be an r.v. with $EX = 0$, $\mathrm{Var}(X) = 1$. Then*

$$\rho(F_X, \Phi) \leq \left(2 + \sqrt{\frac{2}{\pi}}\right)\sqrt{1 - J_X}.$$

An interesting by-product of this is obtained through the information

$$I(X) = \int \left(\frac{f'(x)}{f(x)}\right)^2 f(x)\,dx$$

and the result [cf. Huber (1981)] that if X has a continuously differentiable density on the whole real line, then

$$\sup_{g \in H_1} \frac{E^2[g'(X)]}{\mathrm{Var}[g(X)]} = I(X).$$

It follows therefore that $J_X^{-1} = I(X)\,\mathrm{Var}(X)$ and Theorem 1.2 may be restated as follows.

COROLLARY 1.1. *Let X be as in Theorem 1.3 and, furthermore, with a continuously differentiable density on the whole real line. Then*

$$\rho(F_X, \Phi) \leq \left(2 + \sqrt{\frac{\pi}{2}}\right)\sqrt{1 - \frac{1}{I(\xi)}}.$$

A similar result was shown by Mayer-Wolf (1990).

The proofs of these theorems are given in Section 3 and auxiliary results in Section 2. In Section 4 the results are illustrated by some examples, including the t- and gamma-distributions. An application to the central limit theorem (CLT) is presented in Section 5.

For other applications of these and similar characterizations to the clt, see Chen and Lou (1987), Chen (1988) and Cacoullos, Papathanasiou and Utev (1992). However, Chen's approach differs from ours, based on the convergence properties of the w-function.

2. Preliminaries.

For our purposes we require the following results.

LEMMA 2.1 [Bolthausen (1984)]. *Let $\psi(x)$ be defined by (1.4). Then $\psi'(x)$ is given by (1.3) and*

$$\sup_x |\psi(x)| \leq \sqrt{\frac{\pi}{2}}, \tag{2.1}$$

$$\sup_x |\psi'(x)| \leq 2. \tag{2.2}$$

The following lemma summarizes the basic properties of the w-functions. Further properties are given in Cacoullos, Papathanasiou and Utev (1992).

LEMMA 2.2. *Let X be an r.v. as in* (1.6) *and w_X its associated w-function* [*see* (1.7)]. *Then*

(a) *The f^*-transform of f defined by*

$$f^*(x) = w_X(x) f_X(x), \qquad x \in R, \tag{2.3}$$

is also a density:

$$E[w_X(X)] = 1. \tag{2.4}$$

(b) *For any constants $a \neq 0, b$,*

$$w_{aX+b}(t) = w_X\left(\frac{t-b}{a}\right), \tag{2.5}$$

$$\operatorname{Cov}(X, g(X)) = \operatorname{Var}(X) E[w_X(X) g'(X)]. \tag{2.6}$$

(c) *For independent r.v.'s $X_1, \dots, X_n$ and $S_n = X_1 + \cdots + X_n$,*

$$\operatorname{Cov}(X_i, g(S_n)) = \operatorname{Var}(X_i) E[w_{X_i}(X_i) g'(S_n)], \tag{2.7}$$

$$\operatorname{Var}(S_n) E[w_{S_n}(S_n) g'(S_n)] = \sum_{i=1}^{n} \operatorname{Var}(X_i) E[w_{X_i}(X_i) g'(S_n)]. \tag{2.8}$$

PROOF. For a proof of (2.3) to (2.6), see Cacoullos and Papathanasiou (1989). It suffices to show (2.7) for $n = 2$. We have, by (2.6),

$$\begin{aligned} \operatorname{Cov}(X_1, g(X_1 + X_2)) &= E\left[\operatorname{Cov}(X_1, g(X_1 + X_2)) \mid X_2\right] \\ &= E\left\{\operatorname{Var}(X_1) E[w_{X_1}(X_1) g'(X_1 + X_2) \mid X_2]\right\} \\ &= \operatorname{Var}(X_1) E[w_{X_1}(X_1) g'(X_1 + X_2)]. \end{aligned}$$

Hence, (2.7) holds for any $n > 2$.

As regards (2.8), observe that, by (2.7), we have

$$\begin{aligned} \sum_{i=1}^{n} \operatorname{Var}(X_i) E[w_{X_i}(X_i) g'(S_n)] &= \sum_{i=1}^{n} \operatorname{Cov}(X_i, g(S_n)) \\ &= \operatorname{Cov}(S_n, g(S_n)) = \operatorname{Var}(S_n) E[w_{S_n}(S_n) g'(S_n)]. \end{aligned}$$

This completes the proof of the lemma. □

REMARK. The condition of an interval support is necessary for identity (2.6); otherwise, the definition of $w(x)$ becomes unnecessarily too involved for our purposes.

3. Proofs of the main results.

PROOF OF THEOREM 1.1. By identity (2.6) of Lemma 2.2,

$$E\big[Xg(X)\big] = aE\big[w(X)g'(X)\big], \tag{3.1}$$

from which, by (2.4), we have

$$\begin{aligned} |EXg(X) - Eg'(X)| \le{}& \Big|E\Big[\big(w(X)-1\big)g'(X)\Big]\Big| \\ &+ |1-a|\,\big|E\big[w(X)g'(X)\big]\big|. \end{aligned} \tag{3.2}$$

Let us fix a Borel set A and take

$$g(x) \equiv \psi(x) = e^{x^2/2}\int_{-\infty}^{x}\big(I(t\in A) - \Phi(A)\big)e^{-t^2/2}\,dt.$$

Applying Lemma 2.1, we conclude

$$\begin{aligned} |P(X\in A) - \Phi(A)| &= |EX\psi(X) - E\psi'(X)| \\ &\le \big\{E|w(X)-1| + |1-a|\big\}\operatorname{ess\,sup}|\psi'(X)| \\ &\le 2\big(E|w(X)-1| + |1-a|\big). \end{aligned}$$

Thus Theorem 1.1 is proved. □

PROOF OF THEOREM 1.2. Assume that

$$w_n(X_n) \xrightarrow{P} 1 \quad \text{as } n\to\infty.$$

By definition (1.7) of the w-function and (2.4),

$$\begin{aligned} Ew_n(X_n) &= 1 \quad \text{for all } n, \\ 0 &\le w_n(x) \quad \text{for all } x. \end{aligned}$$

Hence, by Scheffé's theorem,

$$E|w_n(X_n) - 1| \to 0 \quad \text{as } n\to\infty.$$

Applying Theorem 1.1, we obtain

$$\int_{-\infty}^{\infty}|f_n(t) - \phi(t)|\,dt = 2\rho(F_n,\Phi) \le 4E|w_n(X_n)-1| \to 0.$$

Suppose that

$$I_n \equiv \int_{-\infty}^{\infty}|f_n(t)-\phi(t)|\,dt \to 0 \quad \text{as } n\to\infty. \tag{3.3}$$

Let $\{n'\}$ be a sequence of positive integer numbers. It suffices to prove that there exists a subsequence $\{n''\}\subset\{n'\}$ such that

$$E|w_{n''}(X_{n''}) - 1| \to 0 \quad \text{as } n''\to\infty.$$

Taking $\{n''\} \subset \{n'\}$ such that

$$0 \leq f_{n''}(x) \to \phi(x) \quad \text{a.e. as } n'' \to \infty,$$

we obtain

$$1 = \int_{-\infty}^{\infty} x^2 f_{n''}(x)\,dx \to \int_{-\infty}^{\infty} x^2 \phi(x)\,dx = 1.$$

Hence, by Scheffé's theorem,

$$\int_{-\infty}^{\infty} |f_{n''}(x) - \phi(x)| x^2\,dx \to 0,$$

and applying (3.3), we obtain

$$\int_{-\infty}^{\infty} |f_{n''}(x) - \phi(x)|(1+x^2)\,dx \to 0 \quad \text{as } n'' \to \infty.$$

Calculate

$$\begin{aligned}
E|w_{n''}(X_{n''}) - 1| &= \int_{-\infty}^{\infty} \left| -\int_{-\infty}^{x} t f_{n''}(t)\,dt - f_{n''}(x) \right| dx \\
&\leq \int_{-\infty}^{\infty} \left| \int_{-\infty}^{x} t\big((f_{n''}(t) - \phi(t)\big)dt \right| dx + \int_{-\infty}^{\infty} |f_{n''}(x) - \phi(x)|\,dx \\
&\leq \int_{0}^{\infty} \left(\int_{x}^{\infty} t|f_{n''}(t) - \phi(t)|\,dt \right) dx + \int_{-\infty}^{0} \left(\int_{-\infty}^{x} |t|\,|f_{n''}(t) - \phi(t)| \right) dx + I_{n''} \\
&= \int_{0}^{\infty} t^2 |f_{n''}(t) - \phi(t)|\,dt + \int_{-\infty}^{0} t^2 |f_{n''}(t) - \phi(t)|\,dt + I_{n''} \\
&= \int_{-\infty}^{\infty} (1+t^2)|f_{n''}(t) - \phi(t)|\,dt \to 0 \quad \text{as } n'' \to \infty.
\end{aligned}$$

Thus Theorem 1.2 is proved. □

PROOF OF THEOREM 1.3. By the definition (1.10) of the functional J_X [cf. Utev (1989)],

$$J_X E^2\big[g'(X)\big] \leq \operatorname{Var}\big[g(X)\big]$$

for all functions $g \in H_1$.

Setting $g(x) = x + \lambda h(x)$, we obtain

$$\begin{aligned}
J_X E^2\big[g'(X)\big] &= J_X\Big(1 + 2\lambda E\big[h'(X)\big] + \lambda^2 E^2\big[h'(X)\big]\Big) \leq \operatorname{Var} g(X) \\
&= \Big(1 + 2\lambda E\big[Xh(X)\big] + \lambda^2 \operatorname{Var}\big[h(X)\big]\Big).
\end{aligned}$$

Hence we find

$$\begin{aligned}
&2\lambda\Big(J_X E\big[h'(X)\big] - E\big[Xh(X)\big]\Big) + \lambda^2\Big(J_X E^2\big[h'(X)\big] - \operatorname{Var}\big[h(X)\big]\Big) \\
&\quad \equiv 2\lambda a - \lambda^2 \beta^2 \leq 1 - J_X
\end{aligned}$$

for all λ, so that

$$\begin{aligned}|a| &= \big|J_X E[h'(X)] - E[Xh(X)]\big|\\ &\le \sqrt{1-J_X}|\beta| = \sqrt{1-J_X}\sqrt{\operatorname{Var} h(X) - J_X E^2[h'(X)]}\\ &\le \sqrt{1-J_X}\sqrt{Eh^2(X)}.\end{aligned}$$

Consequently,

$$\big|E[h'(X)] - E[Xh(X)]\big| \le |1-J_X|\big|E[h'(X)]\big| + \sqrt{1-J_X}\,\sqrt{Eh^2(X)}.$$

As in the proof of Theorem 1.1, setting

$$h(x) \equiv \psi(x) = e^{x^2/2}\int_{-\infty}^{x} \big(I(t\in A) - \Phi(A)\big)e^{-t^2/2}\,dt$$

and applying Lemma 2.1, we complete the proof of Theorem 1.3. □

4. Examples. To illustrate Theorems 1.1 and 1.3, we consider the following examples.

EXAMPLE 1. Let T_n be an r.v. with t-distribution with $n > 2$ degrees of freedom and density function

$$f_n(t) = \frac{\Gamma\big((n+1)/2\big)}{\sqrt{\pi n}\,\Gamma\big(n/2\big)\big(1+x^2/n\big)^{(n+1)/2}}.$$

One may derive the associated w-function $w_{T_n}(t)$ from the following identity [see Cacoullos and Papathanasiou (1985)]:

$$-\int_{-\infty}^{x} tf_n(t)\,dt = \frac{n}{n-1}\left(1+\frac{x^2}{n}\right)f_n(x).$$

Since $E[T_n] = 0$, $\operatorname{Var}(T_n) = n/(n-2)$, we find

$$w_{T_n}(t) = \frac{n-2}{n-1}\left(1+\frac{t^2}{n}\right). \tag{4.1}$$

Define

$$X_n = \sqrt{\frac{n-2}{n}}T_n, \qquad F_n \equiv P(X_n \in A).$$

Using (2.5) of Lemma 2.2, we obtain, by (4.1),

$$w_n(t) = \frac{n-2}{n-1}\left(1+\frac{x^2}{n-2}\right),$$

where w_n is the w-function associated with X_n. Hence, by Theorem 1.1, we find

$$\rho(F_{X_n},\Phi) \le 2E\left|w_{X_n}(X_n)-1\right| \le 2\left(\frac{1}{n-1}+\frac{EX_n^2}{n-1}\right)=\frac{4}{n-1},$$
$$\rho(F_{T_n},\Phi) \le 2\left(E\left|w_{T_n}(T_n)-1\right|+\left|ET_n^2-1\right|\right) \le \frac{8}{n-2}.$$

This estimate is sharp [see Shimizu, (1987)].

The information of T_n is

$$I(T_n)=E\left[-\frac{\partial^2 \ln f_n(t)}{\partial t^2}\right]=\frac{n+1}{n+3}.$$

Hence

$$I(X_n)=I\left(\sqrt{\frac{n-2}{n}}T_n\right)=\frac{n}{n-2}\frac{n+1}{n+3},$$

and it follows from Theorem 1.3 or Corollary 1.1 that

$$\rho(F_n,\Phi)\le\left(2+\sqrt{\frac{\pi}{2}}\right)\sqrt{1-\frac{1}{I(X_n)}}=O\left(\frac{1}{\sqrt{n}}\right).$$

EXAMPLE 2. Let Z be an r.v. with $EZ^2<\infty$ and independent of the standard normal variable η. Without loss of generality, we may take $Z\ge 0$, since $Z\eta \stackrel{d}{=} |Z|\eta$.

By definition, $X=Z\eta$ has a strictly positive density on the whole real line, and

$$EX=0,\qquad EX^2=EZ^2\equiv a\quad\text{(say)}.$$

First, we consider the situation when

$$E\left(\frac{1}{|Z|}\right)<\infty. \tag{4.2}$$

By (1.8) and Theorem 1.1, we can derive that

$$\begin{aligned}\rho(F_X,\Phi)&\le 2\left(E|w(X)-1|+|1-a|\right)\\ &=\frac{2}{a}\int_{-\infty}^{\infty}\left|\int_0^{\infty}\left(u-\frac{a}{u}\right)\Phi\left(\frac{x}{u}\right)F_{|z|}(du)\right|dx+2|1-a|\\ &\le\frac{2}{a}\int_0^{\infty}|u^2-a|\int_{-\infty}^{\infty}\frac{1}{|u|}\Phi\left(\frac{x}{u}\right)dx\,F_{|z|}(du)+2|1-a|\\ &=\frac{2}{a}E\left|Z^2-a\right|+2|1-a|.\end{aligned}$$

Hence, if $E(Z^2)=1=a$, then we obtain the following bound:

$$\rho(F_X,\Phi)\le 2E\left|Z^2-1\right|. \tag{4.3}$$

Let us now show that, if

$$P(Z=0)=0,$$

then the estimate (4.3) is valid without the restriction (4.2). Define

$$P(Z_\varepsilon \in A) = P\big(Z \in A \mid |Z| > \varepsilon\big).$$

Then

$$|P(Z\eta \in A) - P(\eta \in A)| \le P\big(|Z| \le \varepsilon\big) + P\big(|Z| > \varepsilon\big)\big|P(Z_\varepsilon \eta \in A) - P(\eta \in A)\big|.$$

Hence, by Theorem 1.1, we conclude that

$$\begin{aligned}\rho\big(F_{Z_\eta}, \Phi\big) &\le \limsup_{\varepsilon \to 0} \Big\{ P\big(|Z| \le \varepsilon\big) \\ &\qquad + 2P\big(|Z| > \varepsilon\big)\Big(\frac{1}{EZ_\varepsilon^2} E\big|Z_\varepsilon^2 - EZ_\varepsilon^2\big| + \big|EZ_\varepsilon^2 - 1\big|\Big)\Big\} \\ &= 3P(Z=0) + 2P(Z \ne 0)E\Big[\big|Z^2 - 1\big| I(Z \ne 0)\Big] = 2E\big|Z^2 - 1\big|,\end{aligned}$$

because $EZ_\varepsilon^2 \to EZ^2/P(Z\ne 0)$, $P(|Z| > \varepsilon) \to P(Z \ne 0)$.

In general, we obtain

$$\rho(F_{Z\eta}, \Phi) \le 3E\big|Z^2 - 1\big|,$$

since $P[Z=0] = E[|Z^2 - 1| I(Z=0)]$.

We formulate our result in the following lemma [cf. Shimizu (1987)].

LEMMA 4.1. *Let Z be an r.v. with $EZ^2 = 1$ and independent of the standard normal random variable η. Then*

$$\rho(F_{Z\eta}, \Phi) \le 3E\big|Z^2 - 1\big|.$$

EXAMPLE 3. Let Y have a gamma-distribution with density

$$f(x) = \frac{1}{\Gamma(k)} x^{k-1} e^{-x}, \qquad x \ge 0.$$

We have $E(Y) = k$, $\mathrm{Var}[Y] = k$ and, by (1.11) [cf. Cacoullos and Papathanasiou (1985)],

$$\begin{aligned} f(x)w(x) &= \frac{1}{k\Gamma(k)} \int_x^\infty (t-k) e^{-t} t^{k-1}\, dt \\ &= -\frac{1}{k\Gamma(k)} \int_x^\infty \big(t^k - kt^{k-1}\big)\, de^{-t} = f(x)\frac{x}{k}, \end{aligned} \tag{4.4}$$

$$I(Y) = -E\left(\frac{\partial^2 \ln f(Y)}{\partial Y^2}\right) = (k-1)E\left(\frac{1}{Y^2}\right) = \frac{1}{k-2}. \tag{4.5}$$

Let w_k denote the w-function of $Y_k = (Y - k)/\sqrt{k}$ and let F_k denote its d.f. Then, by (2.5), the w-function of Y_k is, by virtue of (4.4),

$$w_k(t) = w_{(Y-k)/\sqrt{k}}(t) = w\left(\sqrt{k}(t + \sqrt{k})\right) = \frac{t + \sqrt{k}}{\sqrt{k}} = \frac{t}{\sqrt{k}} + 1,$$

where the information, by (4.5), is

$$I(Y_k) = I\left(\frac{Y - k}{\sqrt{k}}\right) = kI(Y) = \frac{k}{k - 2}, \qquad k > 2.$$

Hence, from Theorem 1.1,

$$\rho(F_k, \Phi) \leq 2E|w_k(Y) - 1| \leq 2\sqrt{\operatorname{Var} w_k(Y_k)} = \frac{2}{\sqrt{k}}.$$

Analogously, it follows from Theorem 1.3 or Corollary 1.1 that

$$\rho(F_k, \Phi) \leq \left(2 + \sqrt{\frac{\pi}{2}}\right)\sqrt{1 - \frac{1}{I(X_k)}} = \left(2 + \sqrt{\frac{\pi}{2}}\right)\sqrt{1 - \frac{k-2}{k}} = \frac{(2\sqrt{2} + \sqrt{\pi})}{\sqrt{k}}.$$

It was shown by Chen and Lou (1987) that

$$U_{Y_k} = \frac{(k+1)^2}{k^2}.$$

Hence, from (1.4), we obtain

$$\rho(F_k, \Phi) \leq 3\sqrt{U_{Y_k} - 1} = \frac{3}{\sqrt{k}}\sqrt{2 + \frac{1}{k}}.$$

These estimates have right behavior in k due to the Edgeworth expansion, since

$$E\left[\exp(itY_n)\right] = E^n\left[\exp\left(i\frac{t}{\sqrt{n}}Y_1\right)\right]$$

and

$$EY_1 = 0, \qquad \operatorname{Var} Y_1 = 1, \qquad EY_1^3 = 2.$$

5. Application to the central limit theorem.

THEOREM 5.1. *Let $X, X_1, X_2, \dots$ be independent random variables with a common absolutely continuous distribution with an interval support and* $EX = 0$, $\operatorname{Var} X = 1$. *Define* $S_n = X_1 + \cdots + X_n$, $F_n(A) \equiv P(S_n/\sqrt{n} \in A)$. *Then*

$$\rho(F_n, \Phi) \to 0 \quad \text{as } n \to \infty.$$

PROOF. Applying identity (2.8) of Lemma 2.2, we obtain

$$E\left[w_{S_n}(S_n)G(S_n)\right] = E\left(\frac{1}{n}\sum_{i=1}^{n} w_{X_i}(X_i)G(S_n)\right).$$

By the law of large numbers and property (2.4) of Lemma 2.2,

$$\frac{1}{n}\sum_{i=1}^{n} w_{X_i}(X_i) \xrightarrow{P} Ew_X(X) = 1 \quad \text{as } n \to \infty.$$

Hence, by Scheffé's theorem, for every bounded function $G(X)$,

$$E\left[\left(w_{S_n}(S_n) - 1\right)G(S_n)\right] \to 0 \quad \text{as } n \to \infty.$$

Setting $G(x) = \operatorname{sign}(w(x) - 1)$, we obtain

$$E\left|w_{S_n}(S_n) - 1\right| \to 0 \quad \text{as } n \to \infty.$$

Hence, by Theorem 1.1 and property (2.5) of Lemma 2.2,

$$\begin{aligned}\rho(F_n, \Phi) &\le 2E\left|w_{S_n/\sqrt{n}}\left(\frac{S_n}{\sqrt{n}}\right) - 1\right| \\ &= 2E\left|w_{S_n}(S_n) - 1\right| \to 0 \quad \text{as } n \to \infty.\end{aligned}$$

Thus the lemma is proved. □

As regards the CLT for the general case of iid r.v.'s $X_1, X_2, \dots$, not necessarily with an interval support, it should be observed that, by considering another sequence $Y, Y_1, \dots$ of iid standard normal r.v.'s, one can apply Theorem 5.1 again to the sequence of sums $X_i + Y_i$, so that

$$\begin{aligned}E\left[\exp\left(it\frac{1}{\sqrt{n}}\sum_{k=1}^{n} X_k\right)\right] &= e^{t^2/2} E\exp\left(it\frac{1}{\sqrt{n}}\sum_{k=1}^{n}(X_k + Y_k)\right) \\ &\to e^{t^2/2}e^{-t^2} \equiv e^{-t^2/2},\end{aligned}$$

which implies that the usual CLT holds for the X_i.

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