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ON CONCENTRATION FUNCTIONS ON DISCRETE GROUPS

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Let $(\xi_n)_{n \geq 0}$ be a random walk on a countable group G. Sufficient and necessary conditions for the existence of a finite set $A \subseteq G$ and a sequence $g_n \in G$ such that for all natural n we have $P(\xi_n \in A | \xi_0 = g_n) = 1$ are presented. This provides a complete solution to the problem of behavior of concentration functions on discrete groups.

Let ξ_n be a random walk generated by a probability measure μ on a countable group G [i.e., its transition probabilities are $P(\xi_{n+1} = g | \xi_n = h) = \mu(h^{-1}g)$]. By $S(\mu) = \{g \in G: \mu(g) > 0\}$ we denote the support of μ. The probability measure μ is called adapted if $S(\mu)$ generates the whole group G. The smallest normal subgroup containing $S(\mu)^{-1}S(\mu)$ is denoted by $H(\mu)$. We say that the random walk ξ_n is *concentrated* (or the measure μ is concentrated) if there exist a finite set $A \subseteq G$ and a sequence $g_n \in G$ such that $P(\xi_n \in g_n A) = 1$ for all natural n. If for any finite $A \subseteq G$ we have $\sup_{g \in G} P(\xi_n \in gA) = \mathbf{k}_{\mu^{*n}}(A) \to 0$, then the process ξ_n (or the measure μ) is called *scattered*. The function $\mathbf{k}_{\mu^{*n}}$ is usually called the concentration function of a random variable ξ_n.

The aim of this article is to provide a full characterization of concentrated random walks on countable groups regardless of their algebraic structures. So far, this characterization was known only for Abelian groups.

Let us begin with the notation and auxiliary results. The convolution operator P_μ corresponding to a measure μ on G is defined as $P_\mu f(g) = \sum_{h \in G} f(gh)\mu(h)$. We notice that for a probability measure μ, the operator P_μ is a linear positive contraction on $(l^p(G), \| \cdot \|_p)$, where $1 \leq p \leq \infty$. The inner product in $l^2(G)$ is denoted by $\langle , \rangle$, and # stands for cardinality of sets. It is easy to check that $P_\mu \circ P_v = P_{\mu * \nu}$ for any pair of measures μ, ν on G. The adjoint P_μ^* to P_μ is the convolution operator $P_{\check{\mu}}$ corresponding to $\check{\mu}(A) = \mu(A^{-1})$. By $\mathbf{1}_A$ we denote the characteristic function of a set $A \subseteq G$.

Proofs of the following two lemmas may be found in [1], Propositions 1.1 and 3.1.

LEMMA 1. *For any finiteset $A \subseteq G$ and a probability measure μ on G, the following three conditions are equivalent*:

(a) $$\lim_{n \to \infty} \|P_\mu^n \mathbf{1}_A\|_\infty = 0,$$

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(b) $$\lim_{n \to \infty} \|P_\mu^n \mathbf{1}_A\|_2 = 0,$$

(c) $$\lim_{n \to \infty} \mathbf{k}_{\mu^{*n}}(A) = 0.$$

LEMMA 2. *If μ is an adapted probability measure, then $H(\mu)$ is the subgroup generated by $\cup_{n=1}^{\infty}[S(\check{\mu}^{*n} * \mu^{*n}) \bigcup S(\mu^{*n} * \check{\mu}^{*n})]$.*

The next lemma is an easy adaptation of a well-known result of operator theory (see [2], page 40, for instance).

LEMMA 3. *Let μ be a probability measure on G. Then there exists a measure ρ on G such that*

$$\lim_{n \to \infty} P_{\check{\mu}^{*n} * \mu^{*n}} = P_\rho$$

in the strong operator topology on $l^2(G)$. The measure ρ satisfies $\check{\mu} * \rho * \mu = \rho$.

Now we are in a position to formulate the main result of the paper.

THEOREM. *Let μ be an adapted probability measure on G. Then the following conditions are equivalent:*

(i) *μ is concentrated.*
(ii) *μ is not scattered.*
(iii) *There exists a function $f \in l^2(G)$ such that $\lim_{n \to \infty} \|P_\mu^n f\|_2 > 0$.*
(iv) *There exists a probability measure $\widetilde{\rho}$ on G such that $\check{\mu} * \widetilde{\rho} * \mu = \widetilde{\rho}$.*
(v) $\lim_{n \to \infty} \#S(\mu^{*n}) = d_\mu < \infty$.
(vi) *$H(\mu)$ is finite.*

Moreover, if the above hold, then there is a natural N_0 such that for all $n \geq N_0$ we have $H(\mu) = S(\check{\mu}^{*n} * \mu^{*n}) \cup S(\mu^{*n} * \check{\mu}^{*n}) = S(\check{\mu}^{*n} * \mu^{*n}) = S(\mu^{*n} * \check{\mu}^{*n})$.

PROOF. (i) $\Rightarrow$ (ii) is obvious and (ii) $\Rightarrow$ (iii) follows from Lemma 1.

(iii) $\Rightarrow$ (iv). If there exists a function $f \in l^2(G)$ such that

$$\lim_{n \to \infty} \|P_\mu^n f\|_2^2 = \lim_{n \to \infty} \langle P_{\check{\mu}^{*n} * \mu^{*n}} f, f \rangle > 0,$$

then the measure ρ is Lemma 3 is nonzero. Now $\widetilde{\rho} = \rho / \rho(G)$ satisfies (iv).

(iv) $\Rightarrow$ (v). Let $\check{\mu} * \widetilde{\rho} * \mu = \widetilde{\rho}$ hold for a probability measure $\widetilde{\rho}$ on G. Then for some $g_0 \in G$ we have $\widetilde{\rho}(g_0) = \sup_{g \in G} \widetilde{\rho}(g) > 0$. Since for any natural n we have

$$\widetilde{\rho}(g_0) = \sum_{h_1, h_2} \widetilde{\rho}(h_1 g_0 h_2) \mu^{*n}(h_1) \check{\mu}^{*n}(h_2),$$

thus $\widetilde{\rho}(h_1 g_0 h_2) = \widetilde{\rho}(g_0)$ for all $h_1 \in S(\mu^{*n})$ and $h_2 \in S(\check{\mu}^{*n})$. It follows that for a fixed $h_2 \in S(\check{\mu}^{*n})$ and all natural n we have

$$1 \geq \sum_{h_1 \in S(\mu^{*n})} \tilde{\rho}(h_1 g_0 h_2) = \#\big(S(\mu^{*n})\big)\tilde{\rho}(g_0).$$

The above gives the following estimation:

$$d_\mu = \lim_{n \to \infty} \#\big(S(\mu^{*n})\big) \leq \frac{1}{\tilde{\rho}(g_0)}.$$

(v) $\Rightarrow$ (vi) Since $\#(S(\mu^{*n}))$ is nondecreasing, then there exists a natural number N_1 such that for all $n \geq N_1$ we have $\#(S(\mu^{*n})) =$ const. $= d_\mu$. This implies that for every $g \in S(\mu)$ the sets $gS(\mu^{*n})$, $S(\mu^{*n})g$ and $S(\mu^{*(n+1)})$ are equal ($n \geq N_1$). Consequently, for all $n \geq N_1$ and all natural k we get

$$S\big(\mu^{*(n+k)}\big)S\big(\check{\mu}^{*k}\big) = S\big(\check{\mu}^{*k}\big)S\big(\mu^{*(n+k)}\big) = S(\mu^{*n})$$

and

$$S\big(\check{\mu}^{*(n+k)}\big)S\big(\mu^{*k}\big) = S\big(\mu^{*k}\big)S\big(\check{\mu}^{*(n+k)}\big) = S(\check{\mu}^{*n}).$$

Now let us consider the following sequences of finite sets: $R_{n,1} = S(\check{\mu}^{*n} * \mu^{*n})$ and $R_{n,2} = S(\mu^{*n} * \check{\mu}^{*n})$. Clearly, both of them are nondecreasing, symmetric and $\#(R_{n,j}) \leq d_\mu^2$ (where $j = 1, 2$). Hence $R_{n,1} = R_{N_0,1} = R_1$ and $R_{N_0,2} = R_{N_0,2} = R_2$ for all $n \geq N_0$, where $N_0 \geq N_1$ is sufficiently large. Then

$$\begin{aligned} R_1R_2 &= S\big(\check{\mu}^{*n} * \mu^{*n}\big)S\big(\mu^{*n} * \check{\mu}^{*n}\big) = S\big(\check{\mu}^{*n}\big)S\big(\mu^{*2n}\big)S\big(\check{\mu}^{*n}\big) \\ &= S\big(\check{\mu}^{*n}\big)S\big(\mu^{*n}\big) = S\big(\mu^{*n}\big)S\big(\check{\mu}^{*n}\big) = R_1 = R_2. \end{aligned}$$

Since $R_{n,i}^{-1} = R_{n,i}$, the relation $R_1R_2 = R_1 = R_2$ implies that $R = R_1 = R_2$ is a subgroup of G. Similarly, if $g \in S(\mu)$, then $g^{-1}Rg = g^{-1}R_{n,1}g = R_{n+1,1} = R$ for $n \geq N_0$. Thus, since $S(\mu)$ generates G, R is a normal subgroup of G. Finally, since $R = S(\check{\mu}^{*n} * \mu^{*n})$ we have $H(\mu) = R$ by Lemma 2.

(vi) $\Rightarrow$ (i) Consider the finite set $A = H(\mu)$. For any $g \in S(\mu)$ we have

$$S\big(\mu^{*n}\big) \subseteq gH(\mu)gH(\mu)\dots gH(\mu) = g^nH(\mu),$$

so $\mu^{*n}(g^nA) = 1$ for all natural n. It follows the measure μ is concentrated.

Finally, it follows from the proof of (v) $\Rightarrow$ (vi) that $H(\mu) = R = R_1 = S(\check{\mu}^{*n} * \mu^{*n}) = R_2 = S(\mu^{*n} * \check{\mu}^{*n}) = R_1 \cup R_2$ for $n \geq N_0$. □

REMARK 1. We notice that any adapted and *strictly aperiodic* [i.e., $H(\mu) = G$] probability measure μ on an infinite G is scattered. Among others, this result was established in [1]. It was also noticed there that for an adapted probability measure μ the essentially weaker condition $G/_{H_{(\mu)}} \neq \mathbb{Z}$ (integers) is still sufficient for scattering. However, the following simple example shows that $G/_{H_\mu} \neq \mathbb{Z}$ is not necessary. In fact, let us consider the group $G = \mathbb{Z}^2$ with ordinary group operations and adapted measure $\mu = \frac{1}{2}\delta_{(0,1)} + \frac{1}{2}\delta_{(1,1)}$. Clearly, $H(\mu) = \{(k, 0) : k \in \mathbb{Z}\}$

and $G/_{H_{(\mu)}} \cong \mathbb{Z}$. For all natural n the nth convolution

$$\mu^{*n} = \sum_{k=0}^{n} \binom{n}{k} \delta_{(k,n)},$$

so $\#(S(\mu^{*n})) = n + 1 \to \infty$, and by our theorem μ is scattered.

REMARK 2. Similarly, as in step (iv) $\Rightarrow$ (v) of the proof of the theorem, we may prove that, if the measure μ is concentrated, then the measure ρ appearing in Lemma 3 is exactly the normalized Haar measure on the subgroup $H(\mu)$. On the other hand, if μ is scattered, then the strong operator topology limit of $P_{\check{\mu}^{*n} * \mu^{*n}}$ is the zero operator.

It may be easily inferred from the above theorem that, if $\#(S(\mu)) = \infty$, then μ is scattered. The next obvious observation is that, if an infinite group G does not possess nontrivial, finite and normal subgroups, then any adapted probability measure μ on G is scattered unless $G = \mathbb{Z}$ and $\mu = \delta_1$ or $\mu = \delta_{-1}$. We end this article with the following result which seems to be another good example of possible applications of our theorem.

COROLLARY. *Let μ be an adapted probability measure on infinite G. If there exists in $S(\mu)$ an element of finite index, or $S(\mu) \cap S(\check{\mu}) \neq \emptyset$, then μ is scattered.*

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