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MWI REPRESENTATION OF THE NUMBER OF CURVE-CROSSINGS BY A DIFFERENTIABLE GAUSSIAN PROCESS, WITH APPLICATIONS[1]

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Let $\mathbf{X} = (X_t, t \geq 0)$ be a stationary Gaussian process with zero mean, continuous spectral distribution and twice-differentiable correlation function. An explicit representation is given for the number $N_\psi(T)$ of crossings of a C^1 curve ψ by $\mathbf{X}$ on the bounded interval [0, T], in a multiple Wiener–Itô integral expansion. This continues work of the author in which the result was given for $\psi \equiv 0$. The representation is applied to prove new central and noncentral limit theorems for numbers of crossings of constant levels, and some consequences for asymptotic variances are given in mixed-spectrum settings.

1. Introduction. Level-crossings by stationary Gaussian processes have been studied systematically for almost 50 years. The pioneering contributions of S. O. Rice in 1945 were motivated by signal-processing in communication theory, as were the variance formulas developed in the 1950s by Steinberg, et al. (1955). Rigorous development of the notion of "crossing" in continuous time was supplied by various authors [in particular, Ylvisaker (1965)] leading up to a definitive treatment in the book by Cramér and Leadbetter (1967). The Cramér–Leadbetter book summarized the available techniques and results up to 1967, which were based on careful asymptotics involving joint densities of values and derivatives of the underlying (Gaussian) process. Further developments of Ylvisaker (1966) and Marcus (1977) allowed moment formulas for crossing counts to be generalized to a large class of nonstationary and non-Gaussian processes to which the same joint-density asymptotics applied. In addition, central limit theorems for crossing counts of the mean level, in discrete and continuous time, were proved by Malevich (1969) and Cuzick (1976), facilitating statistical signal-processing applications. Within the last two decades, statistics of zero-crossing counts has been primarily an applied subject [Lomnicki and Zaremba (1955)], finding particular favor in acoustical and speech processing [viz. Niederjohn and Castelaz (1978)]. Other references and statistical techniques related to zero-crossings are given in the book by Kedem (1980) and subsequent papers by Kedem and co-workers [Kedem and Slud (1982), (1994), Kedem (1986) and He and Kedem (1989)], the most recent of which concern the detection of hidden periodicities by zero-crossing analysis.

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The purpose of the present paper is to apply a powerful theoretical tool, the theory of multiple Wiener–Itô integrals, to study crossings of general continuously differentiable curves by stationary (continuous-time) Gaussian processes with twice-differentiable correlation functions. Such a study was initiated, for counts of crossings of the mean level, in Slud (1991). The main result of this paper is an explicit representation of the number of curve-crossings in a bounded interval, from which useful limit theorems and (asymptotic) variance formulas can be deduced.

In the remainder of this section, we sketch the necessary background concerning multiple Wiener–Itô integrals (MWI's) and previous work on level-crossings by Gaussian processes. The representation theorem is stated in Section 2. Section 3 applies the MWI representation together with a general central limit theorem of Chambers and Slud (1989) for MWI expansions to deduce central limit theorems for level-crossing counts. For very special correlation functions with "long-range dependence," Section 3 provides a novel noncentral limit theorem. Section 4 shows how asymptotic-variance formulas for the level-crossing counts, which extend to mixed-spectrum Gaussian processes via the MWI representation, yield new information in some special cases of statistical interest. Finally, Section 5 proves Theorem 2.1.

1.1. *Multiple Wiener–Itô integral notation and basic facts.* Multiple Wiener–Itô integrals are a tool for representing non-linear functionals of a stationary Gaussian process $\mathbf{X}$, generalizing the Hermite polynomial expansions for functions which depend upon only a single coordinate X_0 of the process. Our general reference for MWI's is the excellent monograph of Major (1981).

Assume that $\mathbf{X} = (X_t, t \in \mathbb{R}^+)$ is a stationary mean-0, variance-1 Gaussian process on the probability space $(\Omega, \mathcal{F}, P)$. The correlation function $r(t) \equiv E(X_t X_0)$, being positive definite, has the form $\int_{\mathbb{R}} e^{it\lambda} \sigma(d\lambda)$ for some symmetric Borel probability measure σ on $\mathbb{R}$ which we assume for now is nonatomic. [However, Major (1981a) shows that the following MWI theory holds also if σ has atoms.] It is well known [e.g., see Cramér and Leadbetter (1967), Chapter 7] that X_t has the spectral representation $\int_{\mathbb{R}} e^{it\lambda} \beta(d\lambda)$, a stochastic Wiener-type integral with respect to the complex-Gaussian process β which has the properties that the real and imaginary parts of $(\beta(\lambda), \lambda \geq 0)$ are independent-increment real Gaussian processes with $\mathrm{Var}(\mathrm{Re}(\beta(\lambda)) = \mathrm{Var}(\mathrm{Im}(\beta(\lambda)) = \sigma([0, \lambda])/2$, and that $\beta(-\lambda) \equiv \overline{\beta(\lambda)}$ a.s. as a random element of the space of complex-valued continuous functions on $\mathbb{R}^+$, with the sup-norm, where the overbar denotes the complex conjugate. The MWI theory represents elements of $\mathcal{H} \equiv L^2(\Omega, \sigma(\mathbf{X}), P)$ as orthogonal sums of multiple stochastic integrals with respect to the same integrator β.

For each $n \geq 1$, let $\sigma^n \equiv \sigma \times \cdots \times \sigma$ denote the product Borel measure on $\mathbb{R}^n$, and define the complex Hilbert space

$$L^2_{\mathrm{sym}}(\mathbb{R}^n, \sigma^n) \equiv \Big\{ f_n \in L^2(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), \sigma^n) \colon f_n(\boldsymbol{\lambda}) = \overline{f_n(-\boldsymbol{\lambda})},$$
$$f_n(\lambda_1, \ldots, \lambda_n) = f_n(\lambda_{p(1)}, \ldots, \lambda_{p(n)}) \,\forall\, p \in \mathcal{S}_n \Big\},$$

where $\mathcal{S}_n$ denotes the *symmetric group* of permutations of $\{1,\dots,n\}$. By convention, let $L^2_{\mathrm{sym}}(\mathbb{R}^n,\sigma^n)\equiv\mathbb{R}$ when $n=0$. Then there exists an orthogonal decomposition

$$\mathcal{H}\equiv L^2(\Omega,\sigma(\mathbf{X}),P)=\bigoplus_{n=0}^{\infty}\mathcal{H}_n$$

of the square-integrable functions Y of $\mathbf{X}$ into sums of *homogeneous multiple Wiener–Itô integrals* $\sum_{n=0}^{\infty}h_n\equiv\sum_{n=0}^{\infty}I_n(f_n)$, where $h_0=f_0\equiv E(Y)$ and the following properties hold:

(1.i) The integral operator $\sqrt{n!}I_n{:}L^2_{\mathrm{sym}}(\mathbb{R}^n,\sigma^n)\to\mathcal{H}_n$ is an isometry, where I_0 is the identity operator on $\mathbb{R}\equiv\mathcal{H}_0$.

(1.ii) If for each $s\in\mathbb{R}$, U^s: $L^2(\Omega,\sigma(\mathbf{X}),P)\to L^2(\Omega,\sigma(\mathbf{X}),P)$ denotes the extension of the time-shift isometry which maps $g(X_1,\dots,X_k)$ to $g(X_{s+1},\dots,X_{s+k})$ for each bounded Borel-measurable function of k arguments, and if $e_{s,n}(\boldsymbol{\lambda})\equiv e_s(\boldsymbol{\lambda}_n)$ denotes the $L^2_{\mathrm{sym}}(\mathbb{R}^n,\sigma^n)$ function $\exp(is(\lambda_1+\cdots+\lambda_n))$, then, for all $n\ge 1$,

$$U^sI_n(f_n)=I_n(e_sf_n),\qquad f_n\in L^2_{\mathrm{sym}}(\mathbb{R}^n,\sigma^n).$$

(1.iii) For all $n\ge 0$, $I_n(1)=H_n(X_0)/n!$, where the Hermite polynomials with leading coefficient 1 are defined by

$$H_n(x)=(-1)^ne^{x^2/2}\frac{d^n}{dx^n}\left(e^{-x^2/2}\right),\qquad n\ge 0. \tag{1.0}$$

(1.iv) (*Multiplication rule.*) If $f_p\in L^2_{\mathrm{sym}}(\mathbb{R}^p,\sigma^p)$ and $g_q\in L^2_{\mathrm{sym}}(\mathbb{R}^q,\sigma^q)$, then

$$I_p(f_p)I_q(g_q)=\sum_{k=0}^{p\wedge q}k!\frac{(p+q-2k)!}{p!\,q!}\binom{p}{k}\binom{q}{k}I_{p+q-2k}\left(f_p\widetilde{\otimes}_kg_q\right),$$

where $p\wedge q\equiv\min(p,q)$ and $f_p\widetilde{\otimes}_kg_q$ denotes the average over all permutations of λ-arguments of the function

$$\int_{\mathbb{R}^k}f_p(\lambda_1,\dots,\lambda_{p-k},x_1,\dots,x_k)g(\lambda_{p-k+1},\dots,\lambda_{p+q-2k},-x_1,\dots,-x_k)\sigma^k(d\mathbf{x}).$$

1.2. *Previous work on crossings.* The continuous-time numbers of crossings by a stationary Gaussian process $(X_t,0\le t\le T)$ of a differentiable curve ψ can be approximated by the numbers of crossings of continuous polygonal curves agreeing with ψ at points kh, $k\in\mathbb{N}$. The method is elaborated in the book by Cramér and Leadbetter [(1967), Sections 10.3-4 and 13.1-2], but the main idea is very simple. Let $\{X_t,t\ge 0\}$ be a stationary Gaussian process as above, with mean 0 and variance 1, and correlation function $r(t)=\int_{\mathbb{R}}e^{i\lambda t}\sigma(d\lambda)$, with σ a symmetric nonatomic Borel probability measure on $\mathbb{R}$, and assume further that $r(t)$ is twice differentiable. Let $\psi(\cdot)$ be a continuously differentiable function. Then the number $N_\psi(T)$ of crossings of the curve ψ by the process X_t on the time-interval [0, T] is a well-defined random variable with finite expectation. [An explicit integral formula for $EN_\psi(T)$, proved as (5.8) below, is given by Cramér and Leadbetter (1967), (13.2.1), page 285]. This generalizes the famous formula $(T/\pi)e^{-c^2/2}\sqrt{(-r''(0))}$ of Rice (1945) for the expected number of crossings of the constant level c. Indeed, for each T, as $m \to \infty$ the discrete-time number of crossings based on time steps $h = T \cdot 2^{-m}$,

$$N_\psi^*(T, h) \equiv \sum_{j=0^2}^{2^m-1} I_{[\{X_{jh} - \psi(jh)\}\{X_{(j+1)h} - \psi((j+1)h)\}<0]} \tag{1.1}$$

increases monotonically to the limit $N_\psi(T)$. Moreover, Cramér and Leadbetter [(1967), pages 204 ff.], and Ylvisaker (1966) show that if the joint Gaussian densities $p_t(\cdot)$ of $(X(t_1), X(t_2), X'(t_1), X'(t_2))$ are nonsingular, that is, have nonsingular covariance for all $t_1 \neq t_2$, then the variance of $N_\psi(T)$ whether finite or not is given by

$$\int_0^T \int_0^T dt_1 dt_2 \int_0^\infty \int_0^\infty y_1 y_2 p_t\big(-\psi(t_1), -\psi(t_2), y_1, y_2\big)\, dy_1\, dy_2. \tag{1.2}$$

Moreover, the condition

$$\int_0^\varepsilon (r''(t) - r''(0))\frac{dt}{t} < \infty \quad \text{for some } \varepsilon > 0 \tag{1.3}$$

is both sufficient [Cramér and Leadbetter (1967)] and necessary [Geman (1972)] for the variance of $N_\psi(T)$ to be finite for each $T < \infty$ when $\psi \equiv 0$, and in this case a double-integral formula suitable for numerical integration is given by Steinberg et al. (1955) and is rigorously justified by Cramér and Leadbetter (1967). Apparently no such criterion or formula for finite variance simpler than (1.2) is known for counts of crossings of a level other than the mean.

Several authors have proved limit theorems for crossing counts $N_0(T)$ of the mean level by a stationary Gaussian process X_t as T gets large. The best results, due to Cuzick (1976), derive from the work of Malevich (1969) approximating the underlying Gaussian process by an m-dependent process.

2. The MWI representation. The theorems given here extend those of Slud (1991), where the representation of the number of crossings of the mean level relied upon properties of the MWI integrands as generalized hypergeometric functions. The main new technique here is the Hermite-function summation formula of Lemma 5.1 and the associated identity (5.12).

THEOREM 2.1. *Let $(X_t, t \geq 0)$ be a mean 0, variance 1, stationary Gaussian process with continuous spectral measure σ and twice-differentiable correlation* *function $r(t)$, and let $\psi(t)$ be an arbitrary continuously differentiable function. Then the number $N_\psi(T)$ of crossings of the curve ψ by the process X_t on $[0, T]$, $T < \infty$, is a well-defined random variable with finite expectation*

$$E[N_\psi(T)] = \eta \int_0^T \varphi(\psi(x)) \left[2\varphi\left(\frac{\psi'(x)}{\eta}\right) + \frac{\psi'(x)}{\eta}\left(2\Phi\left(\frac{\psi'(x)}{\eta}\right) - 1\right)\right] dx, \tag{2.1}$$

where $\eta^2 \equiv -r''(0)$ and φ is the density and Φ the d.f. for $\mathcal{N}(0,1)$. If $\mathcal{G}_n^T(\boldsymbol{\lambda}) \equiv \mathcal{G}_n(\boldsymbol{\lambda}) \in L^2_{\mathrm{sym}}(\mathbb{R}^n, \sigma^n)$ is defined for $n \geq 1$ by

$$\begin{aligned} \mathcal{G}_n(\boldsymbol{\lambda}) \equiv \int_0^T e^{is(\lambda_1 + \cdots + \lambda_n)} \Bigg\{ & \frac{\eta}{\pi} H_n(u) e^{-u^2/2} - \frac{e^{-u^2/2}}{\pi} \sum_{j=1}^n H_{n-j}(u) \frac{i^j}{j!} \\ & \times \sum_{1 \leq m_1 < \cdots < m_j \leq n} \lambda_{m_1} \dots \lambda_{m_j} \\ & \times \int_0^{1/\eta} e^{-z^2 y^2/2} H_j(-zy) y^{j-2}\, dy \Bigg\}_{u = \psi(s), z = \psi'(s)} ds \end{aligned} \tag{2.2}$$

and if the variance of $N_\psi(T)$ is finite, then

$$\sum_{n=1}^\infty \frac{1}{n!} \int_{\mathbb{R}^n} |\mathcal{G}_n(\boldsymbol{\lambda})|^2 \sigma^n(d\boldsymbol{\lambda}) < \infty, \tag{2.3}$$

$$N_\psi(T) - E[N_\psi(T)] = \sum_{n=1}^\infty I_n(\mathcal{G}_n). \tag{2.4}$$

THEOREM 2.2. *Let X_t be as in Theorem 2.1, and let $c \in \mathbb{R}$ be arbitrary. If the number $N_c(T)$ of crossings of the level c by the process X_t on a bounded interval $[0, T]$ has finite variance, then (2.3) holds with $\psi(x) \equiv c$ and $\psi'(x) \equiv 0$ in (2.2) and*

$$\begin{aligned} N_c(T) = T\frac{\eta}{\pi} e^{-c^2/2} - \sum_{n=1}^\infty \int_0^T U^s I_n \Bigg(& \frac{e^{-c^2/2}}{\pi} \sum_{\substack{j=0 \\ \text{even}}}^n \eta^{1-j} H_{n-j}(c) \frac{H_j(0)}{j-1} \frac{i^j}{j!} \\ & \times \sum_{1 \leq m_1 < \cdots < m_j \leq n} \lambda_{m_1} \dots \lambda_{m_j} \Bigg) ds. \end{aligned} \tag{2.5}$$

Since

$$H_j(0) = (-1)^{j/2} \frac{j!}{(j/2)!}$$

if j is even, and is equal to 0 for j odd, it is easy to check that the expansion (2.5) coincides in the case $c = 0$ with that of Theorem 2 of Slud (1991), except that the latter [together with expressions (2.2) and (2.3) on the same page] should have an extra factor $(-1)^m$ within the summations $\sum_{m=1}^{\infty}$ [The proofs of Slud (1991) establish the correct expansions with the factor $(-1)^m$ included.]

In the preceding theorems, the MWI integrands $\mathcal{G}_n^T(\boldsymbol{\lambda})$ of (2.2) for the expansion of $N_\psi(T)$ have the special form

$$\mathcal{G}_n^T(\lambda) = \int_0^T e^{is(\lambda_1+\cdots+\lambda_n)} \mathcal{g}_n(\boldsymbol{\lambda}, s)\,ds,$$

so that $I_n(\mathcal{G}_n^T(\boldsymbol{\lambda})) = \int_0^T U^s I_n(\mathcal{g}_n(\cdot, s))ds$, and when $\psi \equiv c$ the functions $\mathcal{g}_n$ do not depend on s. In other words, the functional $N_\psi(T)$—an increasing pure-jump function of T—is expressed as the integral on $[0, T]$ with respect to Lebesgue measure of $e^{is(\lambda_1+\cdots+\lambda_n)}$ multiplied by the *formal* MWI expansion

$$\sum_n I_n(\mathcal{g}_n(\cdot, s)) \equiv \frac{e^{-\psi^2(s)/2}}{\pi} \sum_{n=1}^{\infty} I_n\Bigg(\eta H_n(u) - \sum_{j=1}^{n} H_{n-j}(u)\frac{i^j}{j!} \sum_{m_1<\cdots<m_j} \lambda_{m_1}\cdots\lambda_{m_j} \times \int_0^{1/\eta} e^{-z^2y^2/2} H_j(-zy) y^{j-2}\,dy\Bigg)_{u=\psi(s),\, z=\psi'(s)} \tag{2.6}$$

The intuitive interpretation of this is that the expansion (2.6) represents the *generalized random functional* $\delta_{X_s}(\psi(s))$, where δ_x denotes the Dirac delta function. The same limiting procedure which led to Theorem 2.1 leads via expressions

$$\sum_{k=0}^{2^m-1} \gamma(kh) I_{[(X_{kh}-\psi(kh))(X_{(k+1)h}-\psi((k+1)h))<0]}$$

to the following MWI expansions for Stieltjes integrals $\int_0^T \gamma\, dN_\psi$.

THEOREM 2.3. *Let X_t and ψ be as in Theorem* 2.1, *and assume that* $E(N_\psi^2(T)) < \infty$. *Let $\gamma(\cdot)$ be an arbitrary C^1 function. Then for each $T < \infty$, the a.s. Stieltjes integral $\int_0^T \gamma(s)\, dN_\psi(s)$ is a well-defined square-integrable random variable with MWI expansion*

$$\sum_{n=1}^{\infty} I_n\left(\int_0^T \gamma(s) \mathcal{g}_n(\boldsymbol{\lambda}, s) e^{is(\lambda_1+\cdots+\lambda_n)}\, ds\right), \tag{2.7}$$

where $\mathcal{g}_n(\cdot, s)$ is defined as the MWI integrand in (2.6). *Thus the generalized Gaussian random field $\overline{N}_\psi(\gamma)$ which maps compactly supported C^∞ functions γ on $\mathbb{R}^+$ to $\int_{\mathbb{R}^+} \gamma(s)\, dN_\psi(s)$ has the MWI representation* (2.7) *with T replaced by ∞ [cf. Major* (1981a), *Theorems* 3B *and* 3.1].

Observe that the integral $\int_{\mathbb{R}^+} \gamma(s)\, dN_\psi(s)$ has a clear meaning as the sum $\sum_s I_{[\Delta N_\psi(s)>0]} \gamma(s)$ of γ-function values at the finitely many random times when

X_t crosses ψ (on the bounded interval of support of γ). Further discussion of the generalized random field $\overline{N}_\psi$ can be found in Slud (1994). Note that the expansions in Corollaries 2 and 3 of Slud (1991) for the case $\psi \equiv 0$ are formal sums and make precise sense only in the generalized Gaussian random field setting of Theorem 2.3.

REMARK 2.1. Two further extensions of Theorems 2.1 and 2.2 deserve comment here. First, the number of *upcrossings* by $\{X_t : 0 \le t \le T\}$ of a curve ψ or level c is related simply to the number of crossings through the formula $\frac{1}{2}N_\psi(T) + \frac{1}{2}(I_{[X_T \ge \psi(T)]} - I_{[X_0 \ge \psi(0)]})$. The upcrossing representation analogous to (2.5) has all terms in the integrand of $I_n(\cdot)$ divided by 2, plus the new term

$$\frac{1}{2}(\lambda_1 + \cdots + \lambda_n)H_{n-1}(c)\frac{e^{-c^2/2}}{\sqrt{2\pi}}.$$

The second interesting extension of the theorems is to drop the requirement that the spectral measure σ of X_t be nonatomic, that is, to drop the restriction that $\{X_t\}$ be ergodic. The definition of MWI's can then be modified by the trick of "splitting the atoms" as in Major (1981a), and representation formulas like (2.1) to (2.5) would still hold. The only use which we make of these results for spectral measures with atoms is in formulas for mean and variance, which could be proved directly as in Kedem and Slud (1994) by approximating X_t and $N_c(T)$ by a continuous-spectrum stationary Gaussian process X_t^* and its crossing counts $N_c^*(T)$.

COROLLARY 2.4. *Let X_t be a stationary Gaussian process with mean* 0, *variance* 1, *twice-differentiable correlation function and spectral measure σ which may have atoms. Then, for each $c \in \mathbb{R}$,*

$$E(N_c(T)) = \pi^{-1}T\left[\int_{\mathbb{R}} \lambda^2\sigma(d\lambda)\right]^{1/2} \exp\left(\frac{-c^2}{2}\right) \tag{2.8}$$

and if $\mathrm{Var}(N_c(T))$ *is finite it is given by*

$$\begin{aligned}\mathrm{Var}(N_c(T)) = \sum_{n=1}^{\infty} \frac{2}{n!}e^{-c^2} \int_{\mathbb{R}^n} \frac{1-\cos\big(T(\lambda_1+\cdots+\lambda_n)\big)}{\pi^2(\lambda_1+\cdots+\lambda_n)^2}\\ \times\left(\sum_{\substack{j=0\\ \text{even}}}^{n} \eta^{1-j}H_{n-j}(c)\frac{H_j(0)}{j-1}\frac{i^j}{j!}\sum_{1\le m_1<\cdots<m_j\le n}\lambda_{m_1}\ldots\lambda_{m_j}\right)^2 \sigma^n(d\boldsymbol{\lambda}).\end{aligned} \tag{2.9}$$

3. Central and noncentral limit theorems. We now turn to the application of MWI representations (2.4) and (2.5) in establishing general limit theorems for suitably standardized random variables $N_c(T)$ as $T \to \infty$. Such limit theorems are of two types: *central* if the correlation function $r(\cdot)$ exhibits sufficient *mixing* or *weak dependence* for the underlying process X; *noncentral* in certain cases where r has a property of *regular long-range dependence.* As mentioned in Section 1, the central limit case has been studied for $\psi \equiv 0$ by Malevich (1969), with a strong theorem proved by Cuzick (1976). The central and noncentral limit theorems below for $\psi \equiv c \neq 0$ are new; the central limit theorem for $\psi \equiv 0$ is the slight extension of Cuzick's (1976) given in Slud (1991). A further extension to nonconstant periodic ψ can be found in Slud (1994).

3.1. *Central limit theorem.* For square-integrable random variables Y_0 expressed as a MWI expansion $\sum_{n=0}^{\infty} I_n(f_n)$ with respect to a fixed stationary Gaussian process $\{X_t, t \geq 0\}$ with continuous spectral measure σ, Chambers and Slud (1989) together with Slud [(1991), pages 356 and 357], established verifiable sufficient conditions for the stationary (discrete-time) sequence $Y_t \equiv U^t Y_0 \equiv \sum_{n=0}^{\infty} I_n(e_t(\boldsymbol{\lambda}_n) f_n(\boldsymbol{\lambda}_n))$ to satisfy the central limit theorem (CLT) in the sense that, as $T \to \infty$,

$$\sum_{t=0}^{T-1} \big(Y_t - E(Y_0)\big)/\sqrt{T} \to_{\mathcal{D}} \mathcal{N}(0, \alpha^2) \quad \text{for some } \alpha > 0. \tag{3.1}$$

If we denote by $m \equiv \min\{n \geq 1 : f_n \not\equiv 0\}$ the so-called *Hermite rank* of Y_0, then the following conditions imply (3.1) [by Chambers and Slud (1989), Lemma 2.3 and Theorem 2, and Slud (1991), pages 356 and 357]:

(a1) $\int_{\mathbb{R}} |r(t)|^m dt < \infty$.

(a2) For each $n \geq m$, $f_n(\cdot)$ is σ^n a.e. continuous on a neighborhood of $\mathbf{0} \in \mathbb{R}^n$, and for some $n \geq m$, $\int_{\mathbb{R}} r^n(t)\, dt \neq 0$ and

$$E_{\sigma^n}\big\{|f_n(\boldsymbol{\Lambda})|^2 \mid \Lambda_1 + \cdots + \Lambda_n = 0\big\} > 0.$$

(a3) For some constant $M < \infty$ and all T, n,

$$\operatorname{Var}\left\{ I_n\left(\frac{e_T(\boldsymbol{\lambda}_n) - 1}{e_1(\boldsymbol{\lambda}_n) - 1} f_n(\boldsymbol{\lambda}) \right) \right\} \leq MT.$$

Fix $\psi \equiv c$; let $(X_t, t \geq 0)$ be as in Theorem 2.2, satisfying (2.3); and let $Y_0 \equiv N_c(1)$. Then, according to (2.5), continuity in condition (a2) is immediate, with

$$\begin{aligned} f_n(\lambda_n) \equiv {} & \frac{e_1(\boldsymbol{\lambda}_n) - 1}{\lambda_1 + \cdots + \lambda_n} \frac{e^{-c^2/2}}{\pi} \sum_{\substack{j=0 \\ \text{even}}}^{n} \eta^{1-j} H_{n-j}(c) \\ & \times \frac{H_j(0)}{j-1} \frac{i^{j+1}}{j!} \sum_{1 \leq m_1 < \cdots < m_j \leq n} \lambda_{m_1} \cdots \lambda_{m_j}. \end{aligned} \tag{3.2}$$

Moreover, (a3) follows from an argument of Ho and Sun [(1987), (2.3) and (2.6) to (2.9)], exactly as in Slud [(1991), bottom of page 363], using the fact that

$N_c(T)$ is the limit of sums (1.1) of time-shifted functionals of X_t, each of which depend on only two coordinates of X_t.

Using the specific form (3.2) for $f_n, n \geq 1$, we check that $m = 1$ when $c \neq 0$ and $m = 2$ when $c = 0$. In either case, $\int_{\mathbb{R}} r^2\,dt > 0$, and the conditional law of Λ_1 given $\Lambda_1 + \Lambda_2 = 0$ is nonatomic, where Λ_i are independent σ-distributed random variables. Thus

$$\begin{aligned} &E_{\sigma^2}\{|f_2(\Lambda)|^2 \mid \Lambda_1 + \Lambda_2 = 0\} \\ &\quad = E\left\{\pi^{-2}e^{-c^2}\left| -\eta H_2(c) - \eta^{-1}\frac{1}{2}H_2(0)\Lambda_1\Lambda_2\right|^2 \,\middle|\, \Lambda_2 = -\Lambda_1\right\} \\ &\quad = E\left\{\pi^{-2}e^{-c^2}\left|\eta(c^2-1) + \frac{1}{2\eta}\Lambda_1^2\right|^2 \,\middle|\, \Lambda_1\Lambda_2 = 0\right\} > 0. \end{aligned}$$

By our verification of conditions (a2) and (a3) and the result that (a1) to (a3) imply (3.1), we have proved the following CLT, which includes Theorem 3 of Slud (1991) and the CLT of Cuzick (1976) in the case $c = 0$. Formula (3.3) below for asymptotic variance follows directly from Chambers and Slud (1989), Lemma 2.3, as in Slud (1991), page 357.

THEOREM 3.1. *Let $c \in \mathbb{R}$ be arbitrary, and let $(X_t, t \geq 0)$ be as in Theorem 2.2, with* $\operatorname{Var}(N_c(T)) < \infty$. *Assume that $\int_{\mathbb{R}} r^2(t)\,dt < \infty$ if $c = 0$ and that $\int_{\mathbb{R}} r(t)\,dt < \infty$ if $c \neq 0$. Then, as $T \to \infty$,*

$$\left\{N_c(T) - e^{-c^2/2}\frac{T}{\pi}\eta\right\} \Big/ \sqrt{T} \to_{\mathcal{D}} \mathcal{N}(0, \alpha^2),$$

where $\alpha^2 > 0$ is given in terms of f_n in (3.2) *by the expansion*

$$\alpha^2 = \sum_{n=1}^{\infty} \frac{1}{n!} E_{\sigma^n}\{|f_n(\Lambda)|^2 | \Lambda_1 + \cdots + \Lambda_n = 0\} \int_{\mathbb{R}} r^n(t)\,dt. \tag{3.3}$$

3.2. *Noncentral limit theorem.* The best available noncentral limit theorems concerning functionals of Gaussian processes seem to be those of Dobrushin and Major (1979) and Major (1981a, b) for a special class of covariances. We say that a correlation function $r(t)$ exhibits *regular long-range dependence* if it has the form $r(t) = |t|^{-\alpha}L(t)$ where $0 < \alpha < 1$ and $L(\cdot)$ is an even function, bounded on bounded intervals, which is *slowly varying* at ∞ in the sense that

$$\frac{L(tx)}{L(t)} \to 1 \quad \text{as } t \to \infty, \text{ for all } x > 0.$$

Such correlation functions can also be twice-differentiable, as, for example, when $r(t) \equiv r_1(t) = E(\exp(it(Z_1 - Z_2)))$, where Z_1 are independent $\Gamma(\alpha/2, \beta)$ random variables. The following theorem, which follows from the ideas but not the results of Major [(1981a), Chapter 8], provides a noncentral limit for $N_0(T)$ for a large class of correlation functions including this example $r_1(t)$.

THEOREM 3.2. *Suppose that X_t is a stationary Gaussian process with mean* 0, *variance* 1, *continuous spectrum and twice-differentiable correlation with regular long-range dependence*, $r(t) = (1+|t|)^{-\alpha}L(t)$, *with* $0 < \alpha < \frac{1}{2}$. *Assume also that for some* $\delta \in (-\infty, \alpha)$ *and constant* $C \le \infty$, *for* $k = 1, 2$ *and all* $x \ge 0$,

$$\frac{\left|(d^k/dx^k)r(x)\right|}{|r(x)|} \le C(1+|x|)^{\delta}. \tag{3.4}$$

Let $c \neq 0$ *be fixed arbitrarily. Then, as* $T \to \infty$,

$$\left(T^{2-\alpha}L(T)\right)^{-1/2}\left(N_c(T) - \frac{T}{\pi}\eta e^{-c^2/2}\right) \to_{\mathcal{D}} \mathcal{N}\left(0, \frac{2c^2\eta^2e^{-c^2}}{(1-\alpha)(2-\alpha)\pi^2}\right), \tag{3.5}$$

where $\eta^2 \equiv -r''(0)$, *while*

$$\frac{T^{\alpha-1}}{L(T)}\left(N_0(T) - \frac{T}{\pi}\eta\right) \to_{\mathcal{D}} \frac{\eta}{\pi}\tilde{I}_2\left(\frac{e^{i(\lambda_1+\lambda_2)}-1}{i(\lambda_1+\lambda_2)}\right), \tag{3.6}$$

where $\tilde{I}_2$ *denotes the second-order multiple Wiener–Itô integral operator for a stationary Gaussian process* $\tilde{X}$ *with correlation function* $r_0(t) \equiv \int_{\mathbb{R}} e^{itx}\sigma_0(dx)$ *uniquely determined by*

$$\int_{\mathbb{R}} e^{itx}\left(1-\cos(x)\right)^2 x^{-2}\sigma_0(dx) = \int_0^1 (1-x)|x+t|^{-\alpha}\,dx, \qquad t > 0. \tag{3.7}$$

PROOF. Consider the terms arising in expanding the square in the variance formula (2.9). The general term of the resulting expansion is a constant (not depending on k, m or n) multiplied by

$$\int_{\mathbb{R}^n}\left|\frac{\exp\left(iT(\lambda_1+\cdots+\lambda_n)\right)-1}{i(\lambda_1+\cdots+\lambda_n)}\right|^2 \lambda_1^2\cdots\lambda_k^2\lambda_{k+1}\ldots\lambda_{k+m}\sigma^n(d\boldsymbol{\lambda}) \tag{3.8}$$

for $n \ge 1$, $k \ge 0$, $m \ge 0$ even and $k+m \le n$. But this term is equal to

$$\begin{aligned} &\int_{\mathbb{R}^n}\int_0^T\int_0^T e^{i(t-s)(\lambda_1+\cdots+\lambda_n)}\,dt\,ds\,\lambda_1^2\ldots\lambda_k^2\lambda_{k+1}\ldots\lambda_{k+m}\sigma^n(d\boldsymbol{\lambda}) \\ &\quad = \int_{-T}^T (T-|t|)\left(r(t)\right)^{n-k-m}\left(-ir'(t)\right)^m\left(-r''(t)\right)^k dt. \end{aligned} \tag{3.9}$$

It follows from (3.4) that (3.8) is bounded by

$$2C^{k+m}T\int_0^T r^n(t)(1+t)^{(k+m)\delta}dt.$$

Now change variables by $u = t/T$, and use the representation of Karamata [Major (1981a), Theorem 8.A] to bound $L(t)$, for $\le t \le T$, uniformly by $C'(1+T)^{\varepsilon}$, where $\varepsilon > 0$ is selected arbitrarily small. The result is that (3.8) is, for some constant C_*, less than or equal to

$$2C^{k+m}T^2 \int_0^1 (1+uT)^{-n\alpha+(k+m)\delta} L^n(uT)\, du \le C_*^n T^{2-n\alpha+(k+m)\delta+n\varepsilon}.$$

Define the *Hermite rank* m of $N_c(T)$ by $m \equiv 2$ if $c = 0$ and $m \equiv 1$ if $c \neq 0$. By means of the previous estimate, (2.5) and (2.9), it is straightforward to check that, as $T \to \infty$,

$$T^{-1}\left(\frac{T^{m\alpha}}{L(T)}\right)^{1/2}\Bigg[N_c(T) - \frac{T}{\pi}\eta e^{-c^2/2} - \frac{\eta}{\pi}e^{-c^2/2} I_m\left(-H_m(c)\frac{e_T(\lambda_m)-1}{i(\lambda_1+\cdots+\lambda_m)}\right)\Bigg] \to_P 0. \tag{3.10}$$

[The method is to expand the variance of (3.10) and bound the absolute values of all terms in the infinite-series expansion analogous to (2.9), using the estimate of the previous paragraph. The resulting infinite series is convergent for all large T and is easily seen to converge to 0 as $T \to \infty$.]

The theorem now follows from (3.10) by a very slight modification of Theorem 8.2 of Major (1981a). In particular, (3.7) coincides with Major's (8.4), and the expression for variance in (3.5) when $c \neq 0$ follows directly from (3.10) and (3.7). □

Theorem 3.2 allows both central and noncentral limiting behavior for crossing counts $N_c(T)$ as T gets large. There is only an apparent discontinuity in c in the possible limiting behavior: the point is that, for $c \neq 0$, $N_c(T) - E(N_c(T)$ has stochastic order of magnitude $T^{1-\alpha/2}\sqrt{L(T)}$, but (3.5) shows that this top-order term becomes degenerate as c gets close to 0; for $c = 0$, (3.6) establishes for $N_0(T) - E(N_0(T))$ the smaller stochastic order of magnitude $T^{1-\alpha}L(T)$.

The proof of Theorem 3.2 uses in an essential way the particular form of MWI integrands established in Theorem 2.2. The proof shows that (3.4) and the special long-range-dependent form of r already imply that, for all c, the variance of $N_c(T)$ is finite (for sufficiently large T).

4. Application to time series. One way in which the representation (2.4) and formulas (2.8) and (2.9) may be applied, with reference to engineering signal-processing, is in calculating variance and asymptotic variances for level-crossings in mixed-spectrum settings, where formulas like (1.2) are not available.

In this section let $X_t \equiv B\cos(\omega_0 t+\phi)+\xi_t$, where $B \sim \mathcal{N}(0, b^2)$ and $\phi \sim$ Uniform $[0, 2\pi]$ are independent and ξ_t is a stationary Gaussian continuous-spectrum process with mean 0 and variance D^2 which is independent of (B, Φ). In this setting the ratio $s \equiv b^2/(b^2 + 2D^2)$ is called the *signal-to-noise ratio* for X_t. We show here that the infinite series (2.9) for $\mathrm{Var}(N_c(T))$ simplifies markedly in the case of Gaussian sinusoid-plus-noise processes. In particular, we learn from (2.9) the useful fact, not easily seen without MWI's, that the asymptotic variance of $N_c(T)/T$ for large T depends only on the frequency $\omega_0 \in (0, \pi]$ of the sinusoid, the signal-to-noise ratio s and $\eta^2 \equiv -r''(0)$. A discrete-time analogue was noted first by Kedem and Slud (1993) for $c = 0$.

PROPOSITION 4.1. *Let X_t be a stationary Gaussian mean 0 sinusoid, as in the previous paragraph, for which* $\mathrm{Var}(N_c(1)) < \infty$, $c \in \mathbb{R}$. *Then* $\lim_{T\to\infty} \mathrm{Var}(N_c(T)/T)$ *exists and depends on the distribution of $\{X_t\}$ only through the parameters ω_0, s and η.*

PROOF. By expression (2.9) and the dominated convergence theorem,

$$\lim_{T\to\infty} \mathrm{Var}\left(\frac{N_c(T)}{T}\right) = \frac{1}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n!} e^{-c^2}$$

$$\times E_{\sigma^n} \left\{ I_{[\Lambda_1 + \cdots + \Lambda_n = 0]} \left(\sum_{\substack{j=0 \\ \text{even}}}^{n} \eta^{1-j} H_{n-j}(c) \frac{H_j(0)}{j-1} \frac{i^j}{j!} \right. \right. \tag{4.1}$$

$$\left. \left. \times \sum_{1 \le m_1 < \cdots < m_j \le n} \Lambda_{m_1} \cdots \Lambda_{m_j} \right)^2 \right\}.$$

In the present mixed-spectrum setting, $\sigma^n([\Lambda_1 + \cdots + \Lambda_n = 0]) = \sigma^n([\Lambda_j = \pm\omega_0, 1 \le j \le n; \Lambda_1 + \cdots + \Lambda_n = 0])$ is 0 unless n is even, and in that case $= \binom{n}{n/2}(s/2)^n$. Conditionally given $[\Lambda_1 + \cdots + \Lambda_n = 0]$, the variables $\{\lambda_j/\omega_0\}_{j=1}^n$ have the same joint distribution as the conditional distribution of $\{\varepsilon_j\}_{n=1}^n$ given $\varepsilon_1 + \cdots + \varepsilon_n = 0$, where ε_j are independent and identically distributed with $P(\varepsilon_1 = 1) = P(\varepsilon_1 = -1) = \frac{1}{2}$. The proposition follows immediately. □

Unlike the analogous discrete-time formula developed by Kedem and Slud (1994), the limiting formula (4.1) does not appear to be useful for numerical computations. For each c, one can check as in Kedem and Slud (1994) (in the case $c = 0$) or Slud (1994) first that the expression (4.1) is 0 for $s = 1$ and $\eta = \omega_0$ and then conclude that it must also be identically 0 for general s when $\eta = \omega_0$.

5. Proof of Theorem 2.1. The starting point for representation of the crossings indicator $I_{[(X_t - \psi(t))(X_{t+h} - \psi(t+h)) < 0]}$ is the Hermite polynomial expansion of the indicator $I_{[X_0 \ge c]}$ for an arbitrary level c. Since the normalized Hermite polynomials $H_n(X_0)/\sqrt{n!}$, $n \ge 0$, form an orthonormal basis of $L^2(\Omega, \sigma(X_0), P)$,

$$\begin{aligned} I_{[X_0 \ge c]} &= \sum_{n=0}^{\infty} \frac{H_n(X_0)}{n!} (-1)^n \int_c^\infty \frac{d^n}{dx^n} \left(e^{-x^2/2}\right) \frac{dx}{\sqrt{2\pi}} \\ &= 1 - \Phi(c) + \sum_{n=1}^{\infty} \frac{H_{n-1}(c) e^{-c^2/2}}{\sqrt{2\pi}} I_n(\mathbf{1}), \end{aligned} \tag{5.1}$$

where we have used (1.iii) in the last line.

The next step is the identity

$$I_{[(X_0-a)(X_h-b)<0]} = I_{[X_0 \geq a]} + I_{[X_h \geq b]} - 2I_{[X_0 \geq a]}I_{[X_h \geq b]}.$$

Each of the indicators on the right-hand side has the representation (5.1), and the multiplication rule (1.iv) leads immediately to the following MWI expansion:

$$\begin{aligned} I_{[(X_0-a)(X_h-b)<0]} &= \big(\Phi(a) + \Phi(b) - 2\Phi(a)\Phi(b)\big) \\ &\quad + \frac{1}{\sqrt{2\pi}} \sum_{n=1}^{\infty} \Big\{ \big(2\Phi(b)-1\big) H_{n-1}(a) e^{-a^2/2} I_n(\mathbf{1}) \\ &\qquad\qquad + \big(2\Phi(a)-1\big) H_{n-1}(b) e^{-b^2/2} I_n(e_h) \Big\} \\ &\quad - \frac{1}{\pi} e^{-(a^2+b^2)/2} \sum_{k,m \geq 1} H_{k-1}(a) H_{m-1}(b) \\ &\quad \times \sum_{j=0}^{k \wedge m} \binom{k}{j}\binom{m}{j} j!\, r(h)^j \frac{(k+m-2j)!}{k!\, m!} I_{k+m-2j}\big(e_h(\boldsymbol{\lambda}_{k+m-2j})\big). \end{aligned} \tag{5.2}$$

The termwise application of the multiplication rule (1.iv) to yield (5.2) is justified as in Slud (1991) by orthogonally projecting both sides of (5.2) to $\bigoplus_{m=0}^{M} \mathcal{H}_m$ (the orthogonal direct sum of the range spaces of the operators I_m) and observing that the multiplication rule then yields the finite sums corresponding to $k, m \leq M$ on the right-hand side of (5.2). This also justifies collecting terms on the right-hand side of (5.2) according to the order of MWI being applied in the last summations, leading to

$$\begin{aligned} &I_{[(X_0-a)(X_h-b)<0]} \\ &\quad = \left(\Phi(a) + \phi(b) - 2\Phi(a)\Phi(b) - \frac{e^{-(a^2+b^2)/2}}{\pi} \sum_{n=0}^{\infty} \frac{H_n(a)H_n(b) r(h)^{n+1}}{(n+1)!} \right) \\ &\quad + \sum_{n=1}^{\infty} \left\{ \frac{2\Phi(b)-1}{\sqrt{2\pi}} H_{n-1}(a) e^{-a^2/2} I_n(\mathbf{1}) + \frac{2\Phi(a)-1}{\sqrt{2\pi}} H_{n-1}(b) e^{-b^2/2} I_n(e_h) \right\} \\ &\quad - \frac{1}{\pi} e^{-(a^2+b^2)/2} \sum_{n=1}^{\infty} \Bigg[\sum_{m=1}^{n-1} H_{m-1}(a) H_{n-m-1}(b) I_n(S_n^m(h)) \\ &\qquad\qquad + \sum_{m=0}^{n} I_n(S_n^m(h)) \sum_{j=1}^{\infty} \frac{r(h)^j}{j!} H_{m+j-1}(a) H_{n-m+j-1}(b) \Bigg], \end{aligned} \tag{5.3}$$

where $S_n^0 \equiv 1$ and, for $m \geq 1$,

$$S_n^m(h) = \sum_{1 \leq k_1 < \cdots < k_m \leq n} e_h\big(\lambda_{k_1}, \ldots, \lambda_{k_m}\big).$$

The next step, for shifting in time by an amount $k \cdot h$ and then summing k from 0 to $[1/h] - 1$, is to simplify the infinite summations in the second and last lines of (5.3). To that end, we have the following generalization of the Hermite-polynomial expansion for the bivariate-normal density.

LEMMA 5.1. *For all* $x, y \in \mathbb{R}$, $k, m = 0, 1, 2, \ldots$ *and* $|t| < 1$,

$$\sum_{j=0}^{\infty} \frac{t^j}{j!} H_{k+j}(x) H_{m+j}(y) = e^{(x^2+y^2)/2} \frac{(-1)^{k+m}}{\sqrt{1-t^2}} \frac{\partial^{k+m}}{\partial x^k \partial y^m} \exp\left(-\frac{x^2+y^2-2xyt}{2(1-t^2)}\right). \tag{5.4}$$

PROOF. Substitute (1.0) and take the Fourier transform, both with respect to x and to y, of $(1/2\pi)e^{-(x^2+y^2)/2}$ multiplied by each term of the left-hand side of (5.4):

$$\begin{aligned}
&\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{t^j}{j!} H_{k+j}(x) H_{m+j}(y) e^{-(x^2+y^2)/2} e^{i(xz+yw)} \frac{dx\,dy}{2\pi} \\
&\quad = \int_{\mathbb{R}} \frac{t^j}{j!} H_{m+j}(y) e^{iyw - y^2/2} \int_{\mathbb{R}} (-1)^{k+j} e^{ixz} \frac{d^{k+j}}{dx^{k+j}} e^{-x^2/2} \frac{dx}{\sqrt{2\pi}} \frac{dy}{\sqrt{2\pi}} \\
&\quad = \int_{\mathbb{R}} \frac{t^j}{j!} H_{m+j}(y) e^{iyw - y^2/2} (iz)^{k+j} e^{-z^2/2} \frac{dy}{\sqrt{2\pi}} \\
&\quad = \frac{t^j}{j!} (iz)^{k+j} (iw)^{m+j} e^{-(z^2+w^2)/2},
\end{aligned}$$

where between the second and third lines we have integrated by parts $k+j$ times with respect to x, and between the third and fourth we have integrated by parts $m+j$ times with respect to y. Since the last expression is uniformly absolutely summable in $j \geq 0$ for each fixed z, w, and since the terms $t^j H_{k+j}(x) H_{m+j}(y)/j!$ are mutually orthogonal within $L^2(\mathbb{R}^2, e^{-(x^2+y^2)/2} dx\,dy)$ for distinct values of j, we conclude that the Fourier transform of left-hand side of (5.4) multiplied by $e^{-(x^2+y^2)/2}/(2\pi)$ is

$$\sum_{j=0}^{\infty} \frac{t^j}{j!} (iz)^{k+j} (iw)^{m+j} e^{-(z^2+w^2)/2} = i^{k+m} z^k w^m e^{-tzw - (z^2+w^2)/2}.$$

On the other hand, the Fourier transform of $e^{-(x^2+y^2)/2}/(2\pi)$ multiplied by the right-hand side of (5.4) is, after k integrations by parts in x and m in y, equal to

$$\frac{1}{2\pi\sqrt{1-t^2}} (iz)^k (iw)^m \int_{\mathbb{R}}\int_{\mathbb{R}} \exp\left(i(xz+yw) - \frac{x^2+y^2-2xyt}{2(1-t^2)}\right) dx\,xy$$

which is equal to $i^{k+m} z^k w^m\, e^{-tzw - (z^2+w^2)/2}$. By uniqueness of the (inverse) Fourier transform, we have proved (5.4). □

The substitution of (5.4) within (5.3) simplifies first the expectation term of (5.3) and then the general term.

LEMMA 5.2. *For the stationary Gaussian process X_t of* (5.3),

$$\lim_{h\to 0+} h^{-1}P\big((X_0-a)(X_h-a-zh)<0\big) = \frac{1}{\pi}e^{-a^2/2}\left[\eta e^{-z^2/(2\eta^2)} + z\sqrt{2\pi}\left(\Phi\left(\frac{z}{\eta}\right)-\frac{1}{2}\right)\right], \tag{5.5}$$

where $\eta^2 \equiv -r''(0)$. If the quantity zh on the left-hand side of (5.5) *is replaced by a differentiable function $\gamma(h)$ of h for which $\gamma'(0)=z$, then the limit exists and has the same value.*

PROOF. Take expectations within (5.3) and substitute (5.4) with $k=m$ and $j=0$ to obtain

$$P\big((X_0-a)(X_h-b)<0\big) = \left(\Phi(a)+\Phi(b)-2\Phi(a)\Phi(b)-\frac{1}{\pi}\int_0^{r(h)} e^{-(a^2+b^2-2abt)/(2(1-t^2))}\frac{dt}{\sqrt{1-t^2}}\right). \tag{5.6}$$

Now observe that this expression must tend to 0 when a and b are replaced by the same value c and h decreases to 0, that is,

$$2\Phi(c)\big(1-\Phi(c)\big) = \frac{1}{\pi}\int_0^1 e^{-c^2/(1+t)}\frac{dt}{\sqrt{1-t^2}}.$$

[This fact can also be verified directly via the change of variable $s/c = (1-t)/(1+t)$ on the right-hand side.] Substituting the last identity into (5.6) with $b=a+zh$ and $c=(a+b)/2$ gives

$$\begin{aligned} h^{-1}&P\big((X_0-a)(X_h-a-zh)<0\big)\\ &= h^{-1}\Bigg(\Phi(a)+\Phi(a+zh)-2\Phi(a)\Phi(a+zh)\\ &\qquad -2\Phi\left(a+\frac{zh}{2}\right)\left(1-\Phi\left(a+\frac{zh}{2}\right)\right)\Bigg)\\ &\quad+\frac{1}{\pi h}\int_{r(h)}^1 e^{-c^2/(1+t)}\frac{dt}{\sqrt{1-t^2}}\\ &\quad+\frac{1}{\pi h}\int_0^{r(h)}\big[e^{-c^2/(1+t)}-e^{-(a^2+b^2-2abt)/(2(1-t^2))}\big]\frac{dt}{\sqrt{1-t^2}}. \end{aligned} \tag{5.7}$$

Now fix a and z in (5.7) (with $b=a+zh, c=a+zh/2$ as above) and let h be positive and tend to 0. Elementary calculations show that on the right-hand side of (5.7), the first term tends to 0, and the second term to

$$\lim_{h\to 0+} \frac{1}{\pi h} e^{-a^2/2} \int_{r(h)}^{1} \frac{t\,dt}{\sqrt{1-t^2}} = \frac{\eta}{\pi} e^{-a^2/2},$$

since $r'(0) = 0$ implies $(1-r^2(h))^{1/2}/h \to \sqrt{(-r''(0))} \equiv \eta$. Finally, factor $\exp(-c^2/(1+t))$ in the integrand and substitute the definitions of b and c to reduce the last line of (5.7) to

$$\frac{1}{\pi h} \int_0^{r(h)} e^{-c^2/(1+t)} \left[1 - e^{-z^2h^2/(4(1-t))}\right] \frac{dt}{\sqrt{1-t^2}}.$$

In the limit as $h \to 0$, this expression is unchanged if the lower limit of integration is changed to $1-\varepsilon$, for any fixed $\varepsilon > 0$. Thus the limit of the last line of (5.7) is

$$\lim_{h\to 0+} \frac{1}{\pi h} \int_{1-\varepsilon}^{r(h)} e^{-c^2/(1+t)} \left[1 - e^{-z^2h^2/(4(1-t))}\right] \frac{dt}{\sqrt{1-t^2}}.$$

Replacing $(1/\sqrt{1+t})e^{-c^2/(1+t)}$ in the last expression by $(1/\sqrt{2})e^{-a^2/2}$ does not change the limit, and the change of variable $s = h/\sqrt{2(1-t)}$ then leads to the expression

$$\frac{1}{\pi} e^{-a^2/2} \int_0^{1/\eta} \left[1 - e^{-z^2s^2/2}\right] s^{-2}\, ds,$$

where we have replaced the limits $h/(2\varepsilon)^{1/2}$ and $h/(2(1-r(h))^{1/2}$ by their limits as $h \to 0$ since the integrand no longer depends on h. An integration by parts in the final integral (treating ds/s^2 as the differential term), then leads to

$$\frac{1}{\pi} e^{-a^2/2} \left[\left(e^{-z^2/(2\eta^2)} - 1\right)\eta + z\sqrt{2\pi} \left(\Phi\left(\frac{z}{\eta}\right) - \frac{1}{2} \right) \right].$$

Collecting terms, we have proved (5.5). The final assertion, that the limit is unchanged in zh is replaced by a quantity asymptotically equal to it as $h \to 0$, is obvious from the method of proof. □

Recall by (1.ii) that if U^s denotes the time-shift operator on $L^2(\Omega, \sigma(X), P)$, then, for $f_n \in L^2(\mathbb{R}^n, \sigma^n)$, $U^s I_n(f_n(\boldsymbol{\lambda})) = I_n(e_s(\boldsymbol{\lambda}) f_n(\boldsymbol{\lambda}))$, where $e_s(\boldsymbol{\lambda}_n) \equiv \exp(is(\lambda_1 + \cdots + \lambda_n))$. We sum (5.3) time-shifted by U^{jh}, substituting the result (5.4), to obtain the following result.

COROLLARY 5.3. *Let the stationary Gaussian process X_t be as above, and let $\psi(t)$ denote a continuously differentiable function on $[0, \infty)$. Then the discrete-time crossing count $N_\psi^*(1, 2^{-m})$ defined in* (1.1) *with $h \equiv 2^{-m}$ and $a_j \equiv \psi(jh)$ is* *given by*

$$
\begin{aligned}
N^*_\psi(1, 2^{-m}) = \sum_{j=0}^{2^m-1} \Bigg\{ & P\big((X_0 - a_j)(X_h - a_{j+1}) < 0\big) \\
& + \sum_{n=1}^{\infty} \Bigg\{ \frac{2\Phi(a_{j+1}) - 1}{\sqrt{2\pi}} H_{n-1}(a_j) e^{-a_j^2/2} I_n(e_{jh}) \\
& \qquad + \frac{2\Phi(a_j) - 1}{\sqrt{2\pi}} H_{n-1}(a_{j+1}) e^{-a_{j+1}^2/2} I_n(e_{(j+1)h}) \Bigg\} \\
& - \frac{1}{\pi} e^{-(a_j^2 + a_{j+1}^2)/2} \sum_{n=1}^{\infty} \sum_{k=1}^{n-1} H_{k-1}(a_j) H_{n-k-1}(a_{j+1}) I_n\big(S_n^k(h) e_{jh}\big) \\
& - \sum_{n=1}^{\infty} \frac{(-1)^n}{\pi} \sum_{k=0}^{n} I_n\big(S_n^k(h) e_{jh}\big) \\
& \times \int_0^{r(h)} \frac{\partial^n}{\partial x^k \, \partial y^{n-k}} \exp\left(- \frac{x^2 + y^2 - 2xyt}{2(1-t^2)} \right) \frac{dt}{\sqrt{1-t^2}} \Bigg|_{x=a_j, y=a_{j+1}} \Bigg\}.
\end{aligned}
$$

COROLLARY 5.4. *Under the assumption that the stationary mean* 0, *variance* 1 *Gaussian process* X *has continuous spectrum and twice-differentiable covariance, and that the function* ψ *on* $[0, \infty)$ *is continuously differentiable, the expected number of crossings is*

$$
(5.8) \quad E\big(N_\psi(T)\big) = \eta \int_0^T \varphi\big(\psi(x)\big) \left[2\varphi\left(\frac{\psi'(x)}{\eta} \right) + \frac{\psi'(x)}{\eta} \left(2\Phi\left(\frac{\psi'(x)}{\eta} \right) - 1 \right) \right] dx,
$$

where we denote the $\mathcal{N}(0, 1)$ *density by* $\varphi(x) \equiv \exp(-x^2/2)/\sqrt{2\pi}$ *and* $\eta^2 \equiv -r''(0)$.

PROOF. The variables $N^*_\psi(1, 2^{-m})$ are nonnegative and increase as $m \to \infty$ to $N_\psi(T)$, so that also $E(N^*_\psi(1, 2^{-m})) \to E(N_\psi(T))$. Putting $h \equiv 2^{-m}$ and applying Corollary 5.3, we have

$$
E\big(N_\psi(T)\big) = \lim_{h \to 0} \sum_{j=0}^{[T/h]} P\Big(\big(X_0 - \psi(jh)\big)\Big(X_h - \psi\big((j+1)h\big)\Big) < 0 \Big).
$$

Since the limit in Lemma 5.2 is uniform for $a \equiv \psi(jh)$ lying in compact sets, as $jh \to x$ with $z = \psi'(x)$, the last summand divided by h converges to the integrand in (5.8), and in the usual way the summation converges to the Riemann integral (5.8). □

The formula (5.8) is given by Cramér and Leadbetter [(1967), (13.2.1), page 285]. In the special case where $\psi(\cdot) \equiv c$ is constant, the integrand of (5.8) is constant, and (5.8) becomes $E(N_c(T)) = \pi^{-1}T\eta\exp(-c^2/2)$, a famous formula of Rice (1945).

The main result of the paper, Theorem 2.1, is proved by examining the nonconstant terms in the expansion for $N_\psi^*(1, 2^{-m})$ given in Corollary 5.3 and finding the σ^n a.e. limits of the MWI integrands. Since $N_\psi^*(1, 2^{-m})$ increases to $N_\psi(1)$, we will learn from this via monotone convergence that the variance of $N_\psi(1)$ is finite if and only if the limiting variance of $N_\psi^*(1, 2^{-m})$ is. When variances are finite, the orthogonal decomposition of $L^2(\Omega, \sigma(X), P)$ into the range spaces of the integral operators $I_n(\cdot)$ implies that the MWI integrands for $N_\psi(1)$ are the (L^2 and a.e.) limits of the corresponding integrands for $N_\psi^*(1, 2^{-m})$.

PROPOSITION 5.5. *For each $n \geq 1$, $x \in \mathbb{R}$ and σ^n a.e. λ_n,*

$$\begin{aligned}
\lim_{h\to 0+}\frac{1}{h}\Bigg\{&\frac{2\Phi(x+zh)-1}{\sqrt{2\pi}}H_{n-1}(x)e^{-x^2/2}e_x(\lambda_n)\\
&+\frac{2\Phi(x)-1}{\sqrt{2\pi}}H_{n-1}(x+zh)e^{-(x+zh)^2/2}e_{x+h}(\lambda_n)\\
&-\frac{1}{\pi}e^{-(x^2+(x+zh)^2)/2}\sum_{k=1}^{n-1}H_{k-1}(x)H_{n-k-1}(x+zh)S_n^k(h)e_x(\lambda_n)\\
&-\frac{(-1)^n}{\pi}e_x(\lambda_n)\sum_{k=0}^{n}S_n^k(h)\int_0^{r(h)}\frac{\partial^n}{\partial x^k\partial y^{n-k}}\\
&\times\exp\left(-\frac{x^2+y^2-2xyt}{2(1-t^2)}\right)\frac{dt}{\sqrt{1-t^2}}\bigg|_{y=x+zh}\Bigg\}\\
=e_x(\lambda_n)\Bigg\{&\frac{\eta}{\pi}H_n(x)e^{-x^2/2}-\frac{e^{-x^2/2}}{\pi}\sum_{j=1}^{n}H_{n-j}(x)\frac{i^j}{j!}\\
&\times\sum_{1\leq m_1<\cdots<m_j\leq n}\lambda_{m_1}\cdots\lambda_{m_j}\int_0^{1/\eta}e^{-z^2y^2/2}H_j(-zy)y^{j-2}\,dy\Bigg\}.
\end{aligned}\tag{5.9}$$

Moreover, the limit is uniform over x lying in compact sets and is equally valid if zh on the left-hand side is replaced by a differentiable function $\gamma(h)$ such that $\gamma'(0) = z$.

PROOF. As in Lemma 5.2, observe first that the bracketed quantity on the left is an analytic function of x and must tend to 0 when h decreases to 0. This (nonprobabilistic) fact, which does not depend on ψ, could again be verified directly, but must also follow from the fact that $N_\psi(0+) = 0$ whenever $\psi \equiv 0$ and the correlation function r is such that (1.2) is finite. Subtracting from the bracket on the left-hand side of (5.9) the same quantity with $h = 0+$, we find the left-hand side of (5.9) equal to

$$\lim_{h\to 0+}\frac{1}{h}\Bigg\{2\frac{\Phi(x+zh)-\Phi(x)}{\sqrt{2\pi}}H_{n-1}(x)e^{-x^2/2}e_x(\boldsymbol{\lambda}_n)$$
$$+\frac{2\Phi(x)-1}{\sqrt{2\pi}}\Big[H_{n-1}(x+zh)e^{-(x+zh)^2/2}e_{x+zh}(\boldsymbol{\lambda}_n)-H_{n-1}(x)e^{-x^2/2}e_x(\boldsymbol{\lambda}_n)\Big]$$
$$-\frac{1}{\pi}e^{-x^2/2}\sum_{k=1}^{n-1}H_{k-1}(x)e_x(\boldsymbol{\lambda}_n)\Bigg[e^{-(x+zh)^2/2}H_{n-k-1}(x+zh)S_n^k(h)$$
$$-e^{-x^2/2}H_{n-k-1}(x)\binom{n}{k}\Bigg]-\frac{(-1)^n}{\pi}e_x(\boldsymbol{\lambda}_n)$$
$$\times\Bigg[\int_0^{r(h)}\sum_{k=0}^{n}S_n^k(h)\frac{\partial^n}{\partial x^k\partial y^{n-k}}\exp\bigg(-\frac{x^2+y^2-2xyt}{2(1-t^2)}\bigg)\frac{dt}{\sqrt{1-t^2}}\bigg|_{y=x+zh}$$
$$-\int_0^{1-}\sum_{k=0}^{n}\binom{n}{k}\frac{\partial^n}{\partial x^k\partial y^{n-k}}\exp\bigg(-\frac{x^2+y^2-2xyt}{2(1-t^2)}\bigg)\frac{dt}{\sqrt{1-t^2}}\bigg|_{y=x}\Bigg]\Bigg\}.$$

Taking difference-quotient limits and making use of the fact that

$$\frac{d}{dh}S_n^k(h)\bigg|_{h=0}=\sum_{1\le j_1<\cdots<j_k\le n}i(\lambda_{j_1}+\cdots+\lambda_{j_k})=\binom{n-1}{k-1}(\lambda_1+\cdots+\lambda_n)i$$

for $k\ge 1$, together with (1.0), we obtain the left-hand side of (5.9) equal to the sum of $\mathfrak{J}_1e_x(\boldsymbol{\lambda}_n)\Phi'(x)$, and $\mathfrak{J}_2e_x(\boldsymbol{\lambda}_n)$, where

$$\begin{aligned}\mathfrak{J}_1&\equiv 2z\Phi'(x)H_{n-1}(x)\\&\quad+\big(2\Phi(x)-1\big)\big[-zH_n(x)+i(\lambda_1+\cdots+\lambda_n)H_{n-1}(x)\big]\\&\quad+2z\Phi'(x)\sum_{k=1}^{n-1}\binom{n}{k}H_{k-1}(x)H_{n-k}(x)\\&\quad-2\Phi'(x)\sum_{k=1}^{n-1}H_{k-1}(x)H_{n-k-1}(x)i\binom{n-1}{k-1}(\lambda_1+\cdots+\lambda_n)\\&=z\Bigg[2\Phi'(x)\sum_{k=1}^{n}\binom{n}{k}H_{k-1}(x)H_{n-k}(x)-\big(2\Phi(x)-1\big)H_n(x)\Bigg]\\&\quad+i(\lambda_1+\cdots+\lambda_n)\Bigg[\big(2\Phi(x)-1\big)H_{n-1}(x)-2\Phi'(x)\\&\qquad\times\sum_{k=1}^{n-1}\binom{n-1}{k-1}H_{k-1}(x)H_{n-k-1}(x)\Bigg]\end{aligned}\tag{5.10}$$

and

$$
\begin{aligned}
\mathfrak{I}_2 \equiv -\lim_{h\to 0+}\frac{1}{h}\Biggl\{\frac{(-1)^n}{\pi}\Biggl[&\int_0^{r(h)}\sum_{k=0}^{n} S_n^k(h)\frac{\partial^n}{\partial x^k\partial y^{n-k}}\\
&\times \exp\biggl(-\frac{x^2+y^2-2xyt}{2(1-t^2)}\biggr)\frac{dt}{\sqrt{1-t^2}}\bigg|_{y=x+zh}\\
&-\int_0^1\sum_{k=0}^{n}\binom{n}{k}\frac{\partial^n}{\partial x^k\partial y^{n-k}}\exp\biggl(-\frac{x^2+y^2-2xyt}{2(1-t^2)}\biggr)\frac{dt}{\sqrt{1-t^2}}\bigg|_{y=x}\Biggr]\Biggr\}.
\end{aligned}
\tag{5.11}
$$

Now the identity

$$
\left\{\left(-\frac{\partial}{\partial x}-\frac{\partial}{\partial y}\right)^n-\left(\frac{2}{1+t}\right)^{n/2}H_n\left(\frac{x+y}{\sqrt{2+2t}}\right)\right\}\exp\left(-\frac{x^2+y^2-2xyt}{2(1-t^2)}\right)=0
\tag{5.12}
$$

(which is easily proved by induction) implies that the second integral in (5.11) is well defined [and indeed, that the bracketed quantity in (5.9) is real analytic in both x and h]. Moreover, the identity that the bracket on the left-hand side of (5.9) is 0 when $h = 0$ can now be reexpressed using (5.12) as

$$
\begin{aligned}
&2\bigl(2\Phi(x)-1\bigr)\Phi'(x)H_{n-1}(x)-2\bigl(\Phi'(x)\bigr)^2\sum_{k=1}^{n-1}\binom{n}{k}H_{k-1}(x)H_{n-k-1}(x)\\
&\qquad -\frac{1}{\pi}\int_0^1\left(\frac{2}{1+t}\right)^{n/2}H_n\left(\frac{x\sqrt{2}}{\sqrt{1+t}}\right)e^{-x^2/(1+t)}\frac{dt}{\sqrt{1-t^2}}\equiv 0.
\end{aligned}
\tag{5.13}
$$

Substituting (5.12) also into (5.11), we obtain

$$
\begin{aligned}
\pi\mathfrak{I}_2 &= \lim_{h\to 0}\frac{1}{h}\int_{r(h)}^{1-}\left(\frac{2}{1+t}\right)^{n/2}H_n\left(\frac{x\sqrt{2}}{\sqrt{1+t}}\right)e^{-x^2/(1+t)}\frac{dt}{\sqrt{1-t^2}}\\
&\quad-\lim_{h\to 0}\frac{1}{h}\int_0^{r(h)}\left(\frac{2}{1+t}\right)^{n/2}e^{-x^2/(1+t)}\\
&\quad\times\left[H_n\left(\frac{2x+zh}{2+2t}\right)\exp\left(-\frac{2xzh(1-t)+z^2h^2}{2(1-t^2)}\right)\right.\\
&\qquad\qquad\left.-H_n\left(\frac{x\sqrt{2}}{\sqrt{1+t}}\right)\right]\frac{dt}{\sqrt{1-t^2}}-\pi\mathfrak{I}_3
\end{aligned}
\tag{5.14}
$$

$$
\begin{aligned}
&= \eta H_n(x)e^{-x^2/2}+\int_0^1\left(\frac{2}{1+t}\right)^{n/2}e^{-x^2/(1+t)}\\
&\quad\times H_{n+1}\left(\frac{x\sqrt{2}}{\sqrt{1+t}}\right)\frac{z}{\sqrt{2+2t}}\frac{dt}{\sqrt{1+t^2}}-\pi\mathfrak{I}_3,
\end{aligned}
$$

where

$$\mathfrak{J}_3 \equiv \frac{(-1)^n}{\pi} \lim_{h\to 0} \int_0^{r(h)} \sum_{k=0}^{n} \frac{S_n^k(h) - \binom{n}{k}}{h} \times \frac{\partial^n}{\partial x^k \partial y^{n-k}} \exp\left(-\frac{x^2+y^2-2xyt}{2(1-t^2)}\right)\bigg|_{y=x+zh} \frac{dt}{\sqrt{1-t^2}}.$$

Collecting terms from (5.11), (5.12) and (5.13) with n replaced by $n+1$, and (5.14), and using the identity

$$\binom{n}{k} + \binom{n}{n-k+1} = \binom{n+1}{k},$$

we find that the left-hand side of (5.9) equals

$$e_x(\lambda_n)\left\{ \tfrac{n}{\pi} H_n(x)e^{-x^2/2} + i(\lambda_1 + \cdots + \lambda_n)\Phi'(x)\left[\big(2\Phi(x)-1\big)H_{n-1}(x) - \Phi'(x)\sum_{k=1}^{n-1}\binom{n}{k} H_{k-1}(x)H_{n-k-1}(x)\right] - \mathfrak{J}_3 \right\}. \tag{5.15}$$

It remains now to evaluate the limit $\mathfrak{J}_3$. The method is first to observe that after the n differentiations in the integrand, the power of $1-t^2$ appearing in the denominator is at most $(n+1)/2$. Thus, since the integrand is meromorphic in t, and $1-t^2 \le \eta^2h^2 + O(h^3)$ on the range of integration, the limit $\mathfrak{J}_3$ can be evaluated by developing $(S_n(h) - \binom{n}{k})/h$ in a Taylor series in h up to terms of order h^{n-1}, with remainder $o(h^{n-1})$. To this end, we expand

$$\frac{S_n^k(h) - \binom{n}{k}}{h} = \sum_{j=1}^{n} \frac{h^{j-1}}{j!} \sum_{1\le m_1 < \cdots < m_k \le n} i^j(\lambda_{m_1} + \cdots + \lambda_{m_k})^j + o(h^{n-1}).$$

When the inner k-fold sum is expanded into monomials $\lambda_{m_1}^{q_1}\ldots\lambda_{m_s}^{q_s}$ with $q_i \ge 1$ for $1 \le i \le s$ and $q_1 + \cdots + q_s = j$, we find $(S_n(h) - \binom{n}{k})/h$ to have the form [up to $o(h^{n-1})$] of a sum of terms

$$\frac{h^{j-1}}{j!} i^j \lambda_{m_1}^{q_1}\cdots\lambda_{m_s}^{q_s}\binom{j}{q_1,\ldots,q_z}\binom{n-s}{k-s}.$$

Therefore $\mathfrak{J}_3$ will be a sum (over $j = 1,\ldots,n$, $s = 1,\ldots,n$, indices $m_1 < \cdots < m_s$ and positive integers $q_1 + \ldots + q_s$ with $q_1 + \cdots + q_s = j$) of terms

$$(-1)^n \frac{i^j}{\pi} \lambda_{m_1}^{q_1}\cdots\lambda_{m_s}^{q_s}\binom{j}{q_1,\ldots,q_s}$$

with coefficients

$$\begin{aligned}
&\lim_{h\to 0}\frac{h^{j-1}}{j!}\int_0^{r(h)}\sum_{k=s}^{n}\binom{n-s}{k-s}\frac{\partial^n}{\partial x^k\,\partial y^{n-k}}\\
&\qquad\times\exp\left(-\frac{x^2+y^2-2xyt}{2(1-t^2)}\right)\bigg|_{y=x+zh}\frac{dt}{\sqrt{1-t^2}}\\
&=\lim_{h\to 0}\frac{h^{j-1}}{j!}\int_0^{r(h)}\frac{\partial^s}{\partial x^s}\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)^{n-s}\\
&\qquad\times\exp\left(-\frac{x^2+y^2-2xyt}{2(1-t^2)}\right)\frac{dt}{\sqrt{1-t^2}}\bigg|_{y=x+zh}\\
&=(-1)^{n-s}\lim_{h\to 0}\frac{h^{j-1}}{j!}\int_0^{r(h)}\left(\frac{2}{1+t}\right)^{(n-s)/2}\\
&\qquad\times\frac{\partial^s}{\partial x^s}\left\{H_{n-s}\left(\frac{x+y}{\sqrt{2+2t}}\right)\exp\left(-\frac{x^2+y^2-2xyt}{2(1-t^2)}\right)\right\}_{y=x+zh}\frac{dt}{\sqrt{1-t^2}},
\end{aligned}\tag{5.16}$$

where we have used (5.12) in the last equality. It is easy to see that for $j>1$, the s x-differentiations in (5.16) lead to terms of largest order of magnitude [for t near $r(h)$] only if they are all applied to the exponential. Thus, when $j>1$, (5.16) is equal to

$$\begin{aligned}
&(-1)^{n-s}\lim_{h\to 0}\frac{h^{j-1}}{j!}\int_0^{r(h)}\left(\frac{2}{1+t}\right)^{(n-s)/2}\\
&\qquad\times\left\{H_{n-s}\left(\frac{x+y}{\sqrt{2+2t}}\right)\frac{\partial^s}{\partial x^s}\exp\left(-\frac{x^2+y^2-2xyt}{2(1-t^2)}\right)\right\}_{y=x+zh}\frac{dt}{\sqrt{1-t^2}}\\
&=(-1)^n\lim_{h\to 0}\frac{h^{j-1}}{j!}\int_0^{r(h)}\left(\frac{2}{1+t}\right)^{(n-s)/2}H_{n-s}\left(\frac{2x+zh}{\sqrt{2+2t}}\right)(1-t^2)^{-s/2}\\
&\qquad\times H_s\left(\frac{x-xt-zht}{\sqrt{1-t^2}}\right)e^{-(x^2+xzh)/(1+t)}e^{-z^2h^2/(2(1-t^2))}\frac{dt}{\sqrt{1-t^2}}.
\end{aligned}$$

Now the contribution to the last limit of the part of the integral up to $1-\varepsilon$, for any fixed $\varepsilon>0$, is obviously 0. Moreover, since $s\le j$, the limit does exist and can be nonzero only in the case where $s=j$ and $q_1=\cdots=q_s=1$. In this case the last expression is equal to

$$(-1)^nH_{n-j}(x)e^{-x^2/2}\lim_{h\to 0}\frac{h^{j-1}}{j!}\int_{1-\varepsilon}^{r(h)}e^{-z^2h^2/\left(2(1-t^2)\right)}H_j\left(\frac{-zh}{\sqrt{1-t^2}}\right)\frac{t\,dt}{(\sqrt{1-t^2})^{j+1}}.$$

The change of variable $y = h/\sqrt{1-t^2}$ makes this equal to

$$\frac{(-1)^n}{j!} H_{n-j}(x) e^{-x^2/2} \int_0^{1/\eta} e^{-z^2 y^2/2} H_j(-zy) y^{j-2}\, dy. \tag{5.17}$$

Therefore, accumulating the terms with coefficients (5.16) for $j = s = 1$ together with the reduced form (5.17) established for coefficients of terms with $j > 1$, we have

$$\begin{aligned}
\mathfrak{J}_3 &= -\frac{i}{\pi}(\lambda_1 + \cdots + \lambda_n) \lim_{h\to 0} \int_0^{r(h)} \left(\frac{2}{1+t}\right)^{(n-1)/2} \\
&\quad \times \frac{\partial}{\partial x}\left\{ H_{n-1}\left(\frac{x+y}{\sqrt{2+2t}}\right) \exp\left(-\frac{x^2+y^2-2xyt}{2(1-t^2)}\right)\right\}_{y=x+zh} \frac{dt}{\sqrt{1-t^2}} \\
&\quad + \frac{e^{-x^2/2}}{\pi} \sum_{j=2}^{n} H_{n-j}(x) \frac{i^j}{j!} \sum_{1\le m_1<\cdots<m_j\le n} \lambda_{m_1}\cdots\lambda_{m_j} \\
&\qquad \times \int_0^{1/\eta} e^{-z^2y^2/2} H_j(-zy) y^{j-2}\, dy \\
&= \frac{i}{2\pi}(\lambda_1 + \cdots + \lambda_n) \int_0^1 \left(\frac{2}{1+t}\right)^{n/2} e^{-x^2/(1+t)} H_n\left(\frac{x\sqrt{2}}{\sqrt{1+t}}\right) \frac{dt}{\sqrt{1-t^2}} \\
&\quad + \frac{e^{-x^2/2}}{\pi} \sum_{j=1}^{n} H_{n-j}(x) \frac{i^j}{j!} \sum_{1\le m_1<\cdots<m_j\le n} \lambda_{m_1}\cdots\lambda_{m_j} \\
&\qquad \times \int_0^{1/\eta} e^{-z^2y^2/2} H_j(-zy) y^{j-2} dy.
\end{aligned}$$

Finally, by (5.15) and another application of (5.13), we have shown that the left-hand side of (5.9) has a limit equal to

$$\begin{aligned}
e_x(\lambda_n) \Bigg\{ &\frac{\eta}{\pi} H_n(x) e^{-x^2/2} - \frac{e^{-x^2/2}}{\pi} \sum_{j=1}^{n} H_{n-j}(x) \frac{i^j}{j!} \\
&\times \sum_{1\le m_1<\cdots<m_j\le n} \lambda_{m_1}\ldots\lambda_{m_j} \int_0^{1/\eta} e^{-z^2y^2/2} H_j(-zy) y^{j-2}\, dy \Bigg\},
\end{aligned}$$

which is equal to the right-hand side of (5.9). □

REMARK. The integral $\int_0^{1/\eta} e^{-z^2y^2/2} H_j(-zy) y^{j-2} dy$ appearing on the right-hand side of (5.9) is equal to

$$(5.18)\qquad \begin{cases} -\sqrt{2\pi}\left(\Phi\left(\frac{z}{\eta}\right)-\frac{1}{2}\right), & \text{if } j=1,\\ 0, & \text{if } z=0,\ j \text{ odd},\\ \eta^{1-j}H_j(0)\frac{1}{j-1}, & \text{if } z\neq 0,\ j\geq 2 \text{ even},\\ -e^{-z^2/(2\eta^2)}\sum_{k=0}^{j-2}\frac{(j-2)!}{(j-k-2)!} & \\ \quad\times (-z)^{-k-1}\eta^{2-j+k}H_{j-1-k}\left(-\frac{z}{\eta}\right), & \text{if } z\neq 0,\ j\neq 2. \end{cases}$$

The previous proposition is the main ingredient in the proof of Theorem 2.1, which now follows from Corollary 5.3 in the same way that Corollary 5.4 did.

PROOF OF THEOREM 2.1. Since each of the random variables $N^*_\psi(1, 2^{-m})$ has finite variance, the MWI expansion for it makes sense and is given explicitly by Corollary 5.3 in the form

$$N^*_\psi(1,2^{-m}) - E[N^*_\psi(1,2^{-m})] = \sum_{k=0}^{2^m-1}\sum_{n=1}^{\infty} I_n\big(g_n(x,y,h)(\lambda_n)|_{x=\psi(kh),y=\psi(kh+h)}\big)$$

for functions $g_n(x,y,h)(\cdot) \in L^2_{\text{sym}}(\mathbb{R}^n,\sigma^n)$, where $h \equiv 2^{-m}$. Then Proposition 5.5 says that, for each $n \geq 1$ and for $k \equiv k(m)$ and $h \equiv 2^{-m}$ behaving as $m \to \infty$ in such a way that $kh \to x$,

$$\begin{aligned} &h^{-1}g_n(\psi(kh),\psi(kh+h),h)(\lambda_n)\\ &\quad\to e_u(\lambda_n)\Bigg\{\frac{\eta}{\pi}H_n(u)e^{-u^2/2} - \frac{e^{-u^2/2}}{\pi}\sum_{j=1}^{n}H_{n-j}(u)\frac{i^j}{j!}\\ &\qquad\times\sum_{i\leq m_1<\cdots<m_j\leq n}\lambda_{m_1}\cdots\lambda_{m_j}\int_0^{1/\eta}e^{-z^2y^2/2}H_j(-zy)y^{j-2}\,dy\Bigg\}_{u=\psi(x),\,z=\psi'(x)}. \end{aligned}$$

By the usual expression of Riemann integrals as Riemann sums, we have for each $n \geq 1$, as $m \to \infty$,

$$\sum_{k=0}^{2^m-1} I_n\big(g_n(u,v,h)(\lambda_n)|_{u=\psi(kh),\,v=\psi(kh+h)}\big) \to I_n\big(\mathcal{G}_n(\lambda_n)\big),$$

where

$$\mathcal{G}_n(\boldsymbol{\lambda}_n) \equiv \int_0^1 e^{i\psi(x)(\lambda_1+\cdots+\lambda_n)} \Bigg\{ \frac{\eta}{\pi} H_n(u) e^{-u^2/2} - \frac{e^{-u^2/2}}{\pi} \sum_{j=1}^{n} H_{n-j}(u) \times \frac{i^j}{j!} \sum_{1 \le m_1 < \cdots < m_j \le n} \lambda_{m_1} \ldots \lambda_{m_j} \times \int_0^{1/\eta} e^{-z^2 y^2/2} H_j(-zy) y^{j-2}\, dy \Bigg\}_{u=\psi(x),\, z=\psi'(x)} dx.$$

Now, if $\mathrm{Var}(N_\psi(1)) \le \infty$, then the square-integrable random variable $N_\psi(1) - E[N_\psi(1)]$ is the a.s. and mean-square limit as $m \to \infty$ of $N_\psi^*(1, 2^{-m}) - E[N_\psi^*(1, 2^{-m})]$. By the isometry property (1.i) of $(1/\sqrt{n!}) I_n(\cdot)$, the integrands in the MWI expansion of $N_\psi(1) - E[N_\psi(1)]$ must coincide with the σ^n a.e. limits $\mathcal{G}_n(\boldsymbol{\lambda}_n)$ of the integrands in the MWI expansion of $N_\psi^*(1, 2^{-m}) - E[N_\psi^*(1, 2^{-m})]$. In this case the assertion of the theorem is that $N_\psi(1) - E[N_\psi(1)] = \sum_{n=1}^\infty I_n(\mathcal{G}_n)$, and the variance has the expansion

$$\mathrm{Var}\big(N_\psi(1)\big) = \sum_{n=1}^\infty \frac{1}{n!} \int_{\mathbb{R}^n} \left|\mathcal{G}_n(\boldsymbol{\lambda})\right|^2 \sigma^n(d\boldsymbol{\lambda}).$$

The proof of the theorem is now complete. □

In order to specialize Theorem 2.1 to the setting of Theorem 2.2, apply (5.1) with $z = 0$, implying that only the terms with even j will enter the expression (2.5).

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