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THE FULL MARTIN BOUNDARY OF THE BI-TREE

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We determine the Martin boundary for aperiodic simple random walk on a bi-tree, that is, the Cartesian product of two homogeneous trees. This is obtained by first deriving a "renewal theorem," giving an asymptotic estimate of the Green kernel as the space variable tends to infinity. The basic tool is a result of Lalley that gives a uniform estimate of transition probabilities of nearest neighbour random walks on trees.

1. Introduction. Let T_1 and T_2 be two homogeneous trees with degrees l_1 and $l_2 \geq 3$, respectively. On the bi-tree, that is, their Cartesian product $T_1 \times T_2$, consider aperiodic simple random walk or, more generally, a random walk obtained by taking a convex combination of aperiodic simple random walks on each of the T_j. Let ρ be the critical eigenvalue of the transition operator. We are interested in the Martin compactification of the bi-tree with respect to each eigenvalue $t \geq \rho$ of the transition operator acting on nonnegative functions.

The *minimal* Martin boundary (i.e., set of minimal t-harmonic functions) is well understood: see [20] and also [22] for Cartesian products of Markov chains in general.

On the other hand, the description of the full Martin compactification has remained an open problem. Some evidence has been obtained by Picardello and Sjögren [21], who studied boundary behaviour of harmonic functions on the bi-tree.

The situation is similar for the (hyperbolic) bi-disk, equipped with the Laplace (–Beltrami) operator: In this case, the minimal Martin boundary is known via the work of Karpelevič [13] and Guivarc'h [10] (see also the general result of Freire [7] and Taylor [26] concerning Cartesian products of manifolds). The problem of determining the full Martin boundary was proposed by Guivarc'h and Taylor [11], who gave the answer only for the critical eigenvalue (where the behaviour is different).

In the present paper, we describe the full Martin compactification of the bi-tree with respect to all eigenvalues $t \geq \rho$. This is possible thanks to recent work of Lalley [14], who has given a local limit theorem for aperiodic, nearest-neighbour random walks on homogeneous trees, that is, an asymptotic estimate of the n-step transition probabilities, which is *uniform* both in time and space (a considerable improvement of previous work of Sawyer [24], Gerl and Woess [8] and others). We use this uniform estimate to derive a kind of "renewal theorem" for our random walk on $T_1 \times T_2$, that is, to describe the asymptotic

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behaviour in space of the Green function (resolvent) of the random walk for $t > \rho$. Our method is a bi-dimensional discrete variant of Laplace's method for determining the asymptotics of parametrized integrals. For a nice description of the one-dimensional, continuous version in the context of Lalley's work, see [25].

As an immediate corollary, we find the directions of convergence of the Martin kernels. The full Martin boundary $\mathcal{M}(t)$ is then what has been suggested by previous evidence: If Ω_j denotes the space of *ends* of T_j, then

$$\mathcal{M}(t) = (\Omega_1 \times T_2) \cup (\Omega_1 \times \Omega_2 \times S_t) \cup (T_1 \times \Omega_2),$$

where S_t is a closed line segment in $\mathbb{R}^2$ that collapses to a single point when $t = \rho$. In particular, the Martin boundary of the product is considerably larger than the product of the boundaries. We remark that a similar approach was used by Ney and Spitzer [19]: they determined the Martin boundary for random walks on the Euclidean lattices $\mathbb{Z}^d$ via computations of the Green kernel based on a uniform estimate of the n-step transition probabilities. (The technical difference is that their estimate gives an additive error term, whereas here we use asymptotic equivalence in terms of quotients tending to 1.)

We also remark that in a similar way, one can find the full Martin compactification of $\mathbb{H}_m \times \mathbb{H}_n$ for positive eigenfunctions of the Laplace–Beltrami operator (here, $\mathbb{H}_n$ denotes the n-dimensional hyperbolic sphere): this is shown in a note by Giulini and Woess [9]. It is noteworthy that the full understanding of the Martin boundary in the discrete environment (that of trees) provided inspiration for similar results in the continuous setup. Comparing continuous and discrete settings, it should be observed that the natural analogue of taking the pointwise product of the heat kernels (continuous time) is the Cartesian product of Markov chains (discrete time).

We mention that the minimal Martin boundary for *direct* products of Markov chains has (in part) been studied by Molchanov [17, 18]. Our method easily adapts to the direct product of aperiodic simple random walks on two trees. However, the simple random walk on $T_1 \times T_2$ does not arise as a direct, but as a Cartesian product of simple random walks on the factors. See [22] for a discussion of Cartesian versus direct products.

The structure of this paper is as follows. In Section 2, we present the necessary ingredients concerning simple random walk on a tree and the end compactification, taken from [14] and others. In Section 3, we state and prove our "renewal theorem." Finally, in Section 4, we explain the Martin compactification of the bi-tree.

2. Simple random walk on a homogeneous tree. This section is a recasting of known results on trees. We consider a homogeneous tree T with degree $l \geq 3$. Recall that T arises naturally as the (right) Cayley graph of the group $\mathbb{Z}_2 * \cdots * \mathbb{Z}_2$ (free product of l two-element groups). Writing $x^{-1}y$ for $x,y \in T$ refers to operations in this group. The distance $d(x,y)$ between two elements is the number of edges on the unique shortest path $\overline{xy}$ in T. We select a reference vertex o, corresponding to the group identity. Setting $|x| = d(x,o)$, we have $d(x,y) = |x^{-1}y|$. A *ray* is a one-sided infinite sequence of successively adjacent vertices without repetitions. Two rays are *equivalent* if they differ only by finite initial pieces. An *end* is an equivalence class of rays. The set of all ends is denoted by Ω. If $x \in T$ and $\omega \in \Omega$, then there is a unique ray $\overline{x\omega}$ that starts at x and represents ω. If $y,z \in T \cup \Omega$, then their *confluent* $c(y,z)$ is the last common element of $\overline{oy}$ and $\overline{oz}$. This is a vertex unless $y = z \in \Omega$; in this case, we set $c(y,z) = z$. On $T \cup \Omega$, we define an ultrametric:

$$\theta(y,z) = \begin{cases} \exp\bigl(-|c(y,z)|\bigr), & y \neq z, \\ 0, & y = z. \end{cases}$$

Thus, $T \cup \Omega$ becomes compact, totally disconnected and T is open-dense. The *horocycle index* of $x \in T$ with respect to $z \in T \cup \Omega$ is

$$\mathfrak{h}(x,z) = d(x,c) - d(o,c) \quad \text{where } c = c(x,z).$$

The *aperiodic simple random walk* on T is the Markov chain with state space T and transition operator P given by

$$p(x,y) = \begin{cases} \dfrac{1}{2}, & x = y, \\ \dfrac{1}{2l}, & d(x,y) = 1, \\ 0, & \text{otherwise}. \end{cases} \tag{2.1}$$

Here and in the sequel, the assumption $p(x,x) = 1/2$ serves only for technical convenience; if instead one chooses the usual simple random walk [$p(x,y) = 1/l$ if $d(x,y) = 1$ and $= 0$ otherwise], then one has to specify parities on various occasions. Our results apply to the ordinary simple random walk by elementary operations.

The *Green kernel* or *resolvent* is

$$G(x,y \mid t) = \sum_{n=0}^{\infty} \frac{1}{t^n} p^{(n)}(x,y), \qquad x,y \in T, t \in \mathbb{C}.$$

(Thus, the complex variable z of [14] is replaced by $1/t$.) Here, $p^{(n)}(x,y)$ is the probability that the random walk is in y after n steps, having started in x. The series converges for $t \geq \rho$, where

$$\rho = \rho(P) = \lim_{n\to\infty} p^{(n)}(x,y)^{1/n} = \frac{1}{2} + \frac{\sqrt{l-1}}{l}, \tag{2.2}$$

Setting

$$\begin{aligned} F(t) &= \frac{l}{l-1}\left(t - \frac{1}{2} - \sqrt{\left(t-\frac{1}{2}\right)^2 - \frac{l-1}{l^2}}\right), \\ G(t) &= \frac{t}{t - (1/2)\bigl(1+F(t)\bigr)}, \end{aligned} \tag{2.3}$$

we have

$$G(x,y \mid t) = G(t)F(t)^{|x^{-1}y|}; \tag{2.4}$$

compare, for example, with [24] and [14].

A real-valued function h on T is called *t-harmonic* if

$$Ph = th \quad \text{where } Ph(x) = \sum_y p(x,y)h(y).$$

We are interested in the cone $\mathcal{H}^+(P,t)$ of positive t-harmonic functions: it is nonempty if and only if $t \geq \rho(P)$; see Pruitt [23]. The general tool for describing $\mathcal{H}^+(P,t)$ is provided by the *Martin boundary*: set

$$K(x,y \mid t) = \frac{G(x,y \mid t)}{G(o,y \mid t)}, \tag{2.5}$$

the *Martin kernel*. The *Martin compactification* is the unique minimal compactification of T such that for every $x \in T$, $K(x,\cdot \mid t)$ extends continuously in the second variable. The Martin boundary $\mathcal{M}(t) = \mathcal{M}(P,t)$ is the set of points added to T in this compactification. Every $h \in \mathcal{H}^+(P,t)$ has an integral representation

$$h(x) = \int_{\mathcal{M}(P,t)} K(x,\cdot \mid t)\,d\nu^h, \tag{2.6}$$

where ν^h is a positive Borel measure on $\mathcal{M}(P,t)$. For the general theory, see [3], [12] or [4]. For nearest neighbour random walks on trees, the Martin boundary has been determined by Cartier [1]; see also [5] and [16].

THEOREM 2.1 [1]. *For every eigenvalue $t \geq \rho(P)$, the Martin compactification of (T,P) is given by $T \cup \Omega$. The continuous extension of the Martin kernel is given by*

$$K(x,z \mid t) = F(t)^{\mathfrak{h}(x,z)}, \qquad x \in T, z \in T \cup \Omega.$$

The measure ν^h in the integral representation of $h \in \mathcal{H}^+(P,t)$ is unique.

The last statement follows from the fact that each $K(\cdot,\omega \mid t)$, $\omega \in \Omega$, is a minimal t-harmonic function. Recall that $h \in \mathcal{H}^+(P,t)$ is *minimal* if $h(o) = 1$ and, whenever $h \geq h' \in \mathcal{H}^+(P,t)$, then h'/h is constant.

We shall need another characterization of the function $x \mapsto K(x,\omega \mid \rho)$, $\omega \in \Omega$, which is given in the next lemma. Following [1], for $k \in \mathbb{Z}$ we define the *horocycles*

$$H_k(\omega) = \{x \in T\colon\ \mathfrak{h}(x,\omega) = k\}.$$

LEMMA 2.2. *If h is a positive function on T that satisfies $Ph = \rho h$ and is constant on each horocycle $H_k(\omega)$, where $\omega \in \Omega$, then $h = h(o)K(\cdot,\omega \mid \rho)$.*

PROOF. Because it is constant on the horocycles $H_k(\omega)$, h may be identified with a function $\widetilde{h}$ of the variable k, which satisfies the following identity: For every $k \in \mathbb{Z}$,

$$\frac{1}{2(l+1)}\widetilde{h}(k-1) + \frac{l}{2(l+1)}\widetilde{h}(k+1) + \frac{1}{2}\widetilde{h}(k) = \rho\widetilde{h}(k).$$

For $k \in \mathbb{Z}$, let now $g(k) = l^{k/2}\widetilde{h}(k)$. Then $g(k) = (g(k-1) + g(k+1))/2$ for all k. Therefore, g is a positive affine function. Hence g is constant and $\widetilde{h}(k) = \widetilde{h}(0)l^{-k/2}$. Now it follows from (2.3) and Theorem 2.1 that $h(x) = h(o)K(x, \omega \mid \rho)$ for all $x \in T$. □

As mentioned by the referee, Lemma 2.2 may also be deduced from the maximum principle for harmonic functions.

For every $t \geq \rho(P)$, there is precisely one function in $\mathcal{H}^+(P, t)$ that has value 1 at o and depends only on $|x|$: this is a *spherical function.* Of particular interest will be the one when $t = \rho(P)$. It can be calculated explicitly

$$\Psi(x) = \psi(|x|) \quad \text{where } \psi(r) = \left(\frac{l-2}{l}r + 1\right)\frac{1}{\sqrt{l-1}^{\,r}}, \; r \geq 0.$$

We shall need a normalized translate of $\Psi(x)$:

$$\Psi(x,y) = \frac{\Psi(x^{-1}y)}{\Psi(y)}. \tag{2.7}$$

Next, we present Lalley's uniform estimates; [14] gives these for arbitrary aperiodic nearest neighbour random walks, but here we only study the "radial" case (2.1). Consider the function

$$\varphi(\xi) = \inf_{t \geq \rho}\left(\xi \log F(t) + \log t\right), \quad 0 \leq \xi \leq 1. \tag{2.8}$$

φ is continuous, negative and decreasing, $\varphi(0) = \log \rho$ and $\varphi(1) = -\log 2l$. If $0 < \xi < 1$, then the infimum is attained for $t = t(\xi)$, which is the unique solution in (ρ, ∞) of

$$\frac{tF'(t)}{F(t)} + \frac{1}{\xi} = 0, \quad \text{that is,} \quad \frac{1}{t(\xi)}\sqrt{\left(t(\xi) - \frac{1}{2}\right)^2 - \frac{l-1}{l^2}} = \xi. \tag{2.9}$$

If $\xi = 0$, then the infimum is attained for $t(0) = \rho$, whereas for $\xi = 1$ it is attained when $t \to \infty$. Furthermore, φ is differentiable on [0, 1),

$$\varphi'(\xi) = \log F\big(t(\xi)\big), \qquad \varphi'(1-) = -\infty. \tag{2.10}$$

The function $t(\xi)$ is increasing and differentiable on [0, 1). Both $\varphi(\xi)$ and $t(\xi)$ can be computed explicitly, but we shall not need this. However, we shall need

$$u(\xi) = \frac{\xi\, t(\xi)}{t'(\xi)} = \frac{1-\xi^2}{1 + 1/\sqrt{1 - ((l-2)/l)^2(1-\xi^2)}}, \qquad 0 < \xi < 1, \tag{2.11}$$

which is a positive, continuous, decreasing function. (It corresponds to $\psi_i''(s(\xi))\cdot\xi_i^3$ in [14].)

From now on, keep in mind that $p^{(n)}(o,x) = 0$ when $n < |x|$ and that $\sim$ denotes asymptotic equivalence (i.e., quotients tending to 1).

THEOREM 2.3 [14]. *There is a sequence of functions β_n: $[0,1] \to \mathbb{R}^+$ such that*

$$\lim_{n\to\infty} \sup_{\xi\in[0,1]} \frac{1}{n}|\log\beta_n(\xi)| = 0$$

and

$$p^{(n)}(o,x) = \beta_n\left(\frac{|x|}{n}\right)\exp\left(n\varphi\left(\frac{|x|}{n}\right)\right)$$

for all $x \in T$ and $n \geq x$. Furthermore, $\beta_n(\xi) \sim \overline{\beta}_n(\xi)$ uniformly in $\xi \in [0,1]$ as $n \to \infty$, where

$$\overline{\beta}_n(\xi) = \begin{cases} \dfrac{G(t(\xi))}{\sqrt{2\pi u(\xi)}}\, n^{-3/2}\left(n\xi + \dfrac{l}{l-2}\right), & \xi \leq 1 - n^{-3/4}, \\ \left(\dfrac{n(1-\xi)}{e}\right)^{n(1-\xi)} \Big/ (n(1-\xi))!, & \xi > 1 - n^{-3/4}. \end{cases}$$

We remark that in the formula for $\overline{\beta}_n$, [14] has a further subdivision according to whether $\xi < n^{-1/4}$ or not. At least in the radial case, this is not necessary: elementary and lengthy computations lead to the foregoing formula.

3. A renewal theorem for the bi-tree. We now consider two homogeneous trees T_1, T_2 with degrees $l_1, l_2 \geq 3$, respectively. Their *Cartesian product* has vertex set $T_1 \times T_2$ and neighbourhood $\leftrightarrow$ described by

$$x_1x_2 \leftrightarrow y_1y_2 \iff \begin{cases} x_1 \leftrightarrow y_1 \text{ and } x_2 = y_2 \text{ or} \\ x_1 = y_1 \text{ and } x_2 \leftrightarrow y_2. \end{cases}$$

For $j = 1,2$, we denote by Ω_j the space of ends of T_j, by P_j the aperiodic simple random walk on T_j and by $\rho_j = \rho(P_j)$, $F_j(t)$, $G_j(t)$, $\beta_n^j(\xi)$ and so forth, the quantities and functions associated with P_j according to Section 2. Also, I_j denotes the identity operator on T_j. By a *Cartesian product of P_1 and P_2*, we mean a transition operator on $T_1 \times T_2$ of the form

$$Q = Q_a = aP_1 \otimes I_2 + (1-a)I_1 \otimes P_2, \qquad 0 < a < 1.$$

Here, $\otimes$ denotes tensor product. These are the kinds of transition operators that are naturally associated with taking Cartesian products of graphs. In particular, the aperiodic simple random walk on $T_1 \times T_2$, given by

$$q(x_1x_2, y_1y_2) = \begin{cases} \frac{1}{2}, & x_1x_2 = y_1y_2, \\ \frac{1}{2(l_1 + l_2)}, & x_1x_2 \leftrightarrow y_1y_2, \\ 0, & \text{otherwise}, \end{cases}$$

arises as Q_a with $a = l_1/(l_1 + l_2)$.

We now study the asymptotic behaviour of the Green kernel of Q_a. By group invariance, it is enough to consider $G(o_1o_2, y_1y_2 \mid t)$ as $y_1y_2 \to \infty$. Note that

$$q^{(n)}(x_1x_2, y_1y_2) = \sum_{k=0}^{n} \binom{n}{k} a^k p_1^{(k)}(x_1, y_1)(1-a)^{n-k} p_2^{(n-k)}(x_2, y_2).$$

Thus $\rho = \rho(Q)$ is obtained as $\rho = a \cdot \rho_1 + (1-a) \cdot \rho_2$, and as $p_j^{(k)}(o_j, y_j) = 0$ if $k < |y_j|$, we get

$$\begin{aligned} &G(o_1o_2, y_1y_2 \mid t) \\ &= \sum_{k=|y_1|}^{\infty} \sum_{m=|y_2|}^{\infty} \frac{1}{t^{k+m}} \binom{k+m}{k} a^k p_1^{(k)}(o_1, y_1)(1-a)^m p_2^{(m)}(o_2, y_2). \end{aligned} \tag{3.1}$$

Set

$$\lambda = |y_2|/|y_1|, \qquad \xi_k = |y_1|/k \quad \text{and} \quad \eta_m = |y_1|/m.$$

Suppose for the moment that both $|y_1|, |y_2| \to \infty$. We may then use Stirling's formula to approximate the binomial coefficients and then apply Theorem 2.3:

$$\begin{aligned} &G(o_1o_2, y_1y_2 \mid t) \\ &\sim \frac{1}{\sqrt{2\pi}} \sum_{k=|y_1|}^{\infty} \sum_{m=|y_2|}^{\infty} \frac{(k+m)^{k+m}}{k^k m^m} \sqrt{\frac{k+m}{km}} \beta_k^1\left(\frac{|y_1|}{k}\right) \beta_m^2\left(\frac{|y_2|}{m}\right) \\ &\quad \times \left(\frac{a}{t}\right)^k \left(\frac{1-a}{t}\right)^m \exp\left(k\varphi_1\left(\frac{|y_1|}{k}\right)\right) \exp\left(m\varphi_2\left(\frac{|y_2|}{m}\right)\right) \\ &= \frac{1}{\sqrt{2\pi|y_1|}} \sum_{k=|y_1|}^{\infty} \sum_{m=|y_2|}^{\infty} \beta_k^1(\xi_k)\beta_m^2(\lambda\eta_m)\sqrt{\xi_k + \eta_m} \exp\big(|y_1|\Phi_{\lambda,t}(\xi_k, \eta_m)\big), \end{aligned} \tag{3.2}$$

where

$$\begin{aligned} \Phi_{\lambda,t}(\xi, \eta) &= \frac{1}{\xi}\left(\log\left(1 + \frac{\xi}{\eta}\right) + \log\frac{a}{t} + \varphi_1(\xi)\right) \\ &\quad + \frac{1}{\eta}\left(\log\left(1 + \frac{\eta}{\xi}\right) + \log\frac{1-a}{t} + \varphi_2(\lambda\eta)\right), \quad 0 < \xi \le 1,\ 0 < \eta \le \frac{1}{\lambda}. \end{aligned} \tag{3.3}$$

Our strategy will be to show that the principal contribution to the last sum in (3.2) comes from the point where $\Phi_{\lambda,t}$ is maximal. For this purpose, we now analyze $\Phi_{\lambda,t}$ for $0 \le \lambda < \infty$ and $t > \rho$. First of all, observe that

$$(3.4)\quad \Phi_{\lambda,t}(\xi,\eta) - \Phi_{\lambda_0,t}(\xi,\eta) = \frac{1}{\eta}\big(\varphi_2(\lambda\eta) - \varphi_2(\lambda_0\eta)\big) = (\lambda - \lambda_0)\log F_2\big(t_2(\widetilde{\lambda}\eta)\big)$$

($\widetilde{\lambda}$ between λ and λ_0) if $0 < \xi \le 1$ and $0 < \eta \le \min\{1/\lambda, 1/\lambda_0\}$. In particular, if $\lambda \to \lambda_0 \in [0, \infty)$, then $\Phi_{\lambda,t} \to \Phi_{\lambda_0,t}$ uniformly for $0 < \xi \le 1$ and η in bounded intervals. By (2.8), we see that the gradient is given by

$$(3.5)\quad \begin{aligned} \frac{\partial\Phi_{\lambda,t}}{\partial\xi} &= -\frac{1}{\xi^2}\left(\log\left(1+\frac{\xi}{\eta}\right) + \log\frac{a}{t} + \log t_1(\xi)\right),\\ \frac{\partial\Phi_{\lambda,t}}{\partial\eta} &= -\frac{1}{\eta^2}\left(\log\left(1+\frac{\eta}{\xi}\right) + \log\frac{1-a}{t} + \log t_2(\lambda\eta)\right). \end{aligned}$$

Elementary computations show that the absolute maximum of $\Phi_{\lambda,t}$ is attained in its only stationary point $(\xi(\lambda),\eta(\lambda))$, which satisfies

$$t_1(\xi)\left(1+\frac{\xi}{\eta}\right)\frac{a}{t} = 1 \quad\text{and}\quad t_2(\lambda\eta)\left(1+\frac{\eta}{\xi}\right)\frac{1-a}{t} = 1.$$

Thus $at_1 + (1-a)t_2 = t$ and

$$(3.6)\quad \frac{t_2(\lambda\eta)}{t_1(\xi)}\frac{\eta}{\xi}\frac{1-a}{a} = 1.$$

By (2.9), we have the system of equations

$$(3.7)\quad \begin{aligned} &at_1 + (1-a)t_2 = t,\\ &\sqrt{\left(t_2-\frac{1}{2}\right)^2 - \frac{l_2-1}{l_2^2}} = \frac{\lambda a}{1-a}\sqrt{\left(t_1-\frac{1}{2}\right)^2 - \frac{l_1-1}{l_1^2}}. \end{aligned}$$

(3.7) has a unique solution in $[\rho_1, \infty) \times [\rho_2, \infty)$: this is $(t_1(\xi(\lambda)), t_2(\lambda\eta(\lambda)))$. Via (2.9) we can now compute $\xi(\lambda)$, and $\eta(\lambda)$ is then obtained from (3.6). Note that if $\lambda = 0$, then $t_2 = \rho_2$ and $t_1 = (t-(1-a)\rho_2)/a$. [If $\lambda \to \infty$, then for the solutions of (3.7) we get $t_1 \to \rho_1$ and $t_2 \to (t-a\rho_1)/(1-a)$.] Also note that, by (2.9), $(\xi(\lambda),\eta(\lambda))$ lies in the interior of the domain of $\Phi_{\lambda,t}$ for every real $\lambda \ge 0$. By (3.5), (3.3) and (2.8), the value of the maximum of $\Phi_{\lambda,t}$ is

$$(3.8)\quad M(\lambda) = \log F_1(t_1) + \lambda\log F_2(t_2),$$

where (t_1,t_2) is the solution of (3.7). Next, let

$$(3.9)\quad H_{\lambda,t} = \begin{pmatrix} -a(\lambda) & b(\lambda)\\ b(\lambda) & -c(\lambda)\end{pmatrix}$$

be the Hessian of $\Phi_{\lambda,t}$ at $(\xi(\lambda),\eta(\lambda))$:

$$a(\lambda) = \frac{1}{\xi(\lambda)^2}\left(\frac{1}{\xi(\lambda)+\eta(\lambda)} + \frac{t_1'(\xi(\lambda))}{t_1(\xi(\lambda))}\right),$$
$$b(\lambda) = \frac{1}{\xi(\lambda)\eta(\lambda)}\frac{1}{\xi(\lambda)+\eta(\lambda)},$$
$$c(\lambda) = \frac{1}{\eta(\lambda)^2}\left(\frac{1}{\xi(\lambda)+\eta(\lambda)} + \frac{\lambda t_2'(\lambda\eta(\lambda))}{t_2(\lambda\eta(\lambda))}\right).$$

It is negative definite and depends continuously on $\lambda \in [0, \infty)$. We shall write $H_{\lambda,t}(\xi,\eta) = (\xi,\eta)\cdot H_{\lambda,t}\cdot(\xi,\eta)^T$ for the associated quadratic form.

THEOREM 3.1. *Suppose that $t > \rho(Q)$, $|y_1| \to \infty$ and $\lambda = |y_2|/|y_1| \to \lambda_0 \in [0, \infty)$. Let (t_1,t_2) be the solution of* (3.7), *depending on λ. Then*

$$G(o_1o_2, y_1y_2 \mid t) \sim G_1(o_1, y_1 \mid t_1)G_2(o_2, y_2 \mid t_2)\left(|y_1| + \frac{l_1}{l_1-2}\right)$$
$$\times\left(|y_2| + \frac{l_2}{l_2-2}\right)\frac{1}{\sqrt{2|y_1|^5\pi}}C(\lambda),$$

where $C(\lambda)$ is a continuous, strictly positive function of λ.

PROOF. We split the sum of (3.1) into two parts:

$$G(o_1o_2, y_1y_2 \mid t) = S_1(y_1,y_2) + S_2(y_1,y_2),$$

where

$$S_1(y_1,y_2) = \sum_{k:|\xi_k-\xi(\lambda)|\le\delta}\ \sum_{m:|\eta_m-\eta(\lambda)|\le\delta} \frac{1}{t^{k+m}}\binom{k+m}{k} a^k p_1^{(k)}(o_1,y_1)(1-a)^m p_2^{(m)}(o_2,y_2)$$

and $S_2(y_1,y_2)$ is the rest. Here, δ is chosen such that the points (ξ_k,η_m) of $S_1(y_1,y_2)$ are in the interior of the domain of $\Phi_{\lambda,t}$ when λ is close to λ_0. This is possible by (3.4) and the facts that $(\xi(\lambda_0),\eta(\lambda_0))$ lies in the respective interior and that $\xi(\lambda)$ and $\eta(\lambda)$ depend continuously on λ. We first analyse S_1. Observe that this sum ranges over all k, m with

$$\frac{|y_1|}{\xi(\lambda)+\delta} \le k \le \frac{|y_1|}{\xi(\lambda)-\delta} \quad\text{and}\quad \frac{|y_1|}{\eta(\lambda)+\delta} \le m \le \frac{|y_1|}{\eta(\lambda)-\delta}.$$

As $|y_1| \to \infty$, we may use the same approximation as in (3.2), plus the asymptotic equivalents of Theorem 2.3 for β_k and β_m, respectively, to obtain

$$\begin{aligned}
&S_1(y_1,y_2)\\
&\sim \frac{1}{\sqrt{2\pi|y_1|}}\left(|y_1|+\frac{l_1}{l_1-2}\right)\left(|y_2|+\frac{l_2}{l_2-2}\right)\frac{1}{2\pi|y_1|^3}\exp(|y_1|M(\lambda))\\
&\times \sum_{|\xi_k-\xi(\lambda)|\le\delta}\ \sum_{|\eta_m-\eta(\lambda)|\le\delta} \frac{G_1\big(t_1(\xi_k)\big)\xi_k^{3/2}}{\sqrt{u_1(\xi_k)}}\frac{G_2\big(t_2(\lambda\eta_m)\big)\eta_m^{3/2}}{\sqrt{u_2(\lambda\eta_m)}}\\
&\times\sqrt{\xi_k+\eta_m}\exp\Big(|y_1|\big(\Phi_{\lambda,t}(\xi_k,\eta_m)-M(\lambda)\big)\Big).
\end{aligned}\tag{3.10}$$

Straightforward computations show that

$$\frac{G_2\big(t_2(\lambda\eta)\big)}{\sqrt{u_2(\lambda\eta)}}\sim\frac{G_2\big(t_2(\lambda_0\eta)\big)}{\sqrt{u_2(\lambda_0\eta)}}\tag{3.11}$$

uniformly for η in bounded intervals, as $\lambda\to\lambda_0$. Also,

$$\begin{aligned}
\Phi_{\lambda,t}(\xi,\eta)-M(\lambda)&=\tfrac{1}{2}H_{\lambda_0,t}\big(\xi-\xi(\lambda),\eta-\eta(\lambda)\big)\\
&\quad+o\Big(\big(\xi-\xi(\lambda)\big)^2+\big(\eta-\eta(\lambda)\big)^2\Big)
\end{aligned}\tag{3.12}$$

and the $o(\cdot)$ for the argument tending to zero is uniform in λ, as $\lambda\to\lambda_0$. In (3.10), we substitute the terms corresponding to the left-hand sides of (3.11) and (3.12) with the respective right-hand sides. Also, we perform a "change of variables," setting

$$\sigma_k=\big(\xi_k-\xi(\lambda)\big)\sqrt{|y_1|}\quad\text{and}\quad\tau_m=\big(\eta_m-\eta(\lambda)\big)\sqrt{|y_1|}.$$

Write $\Delta\sigma_k=\sigma_k-\sigma_{k+1}$ and $\Delta\tau_m=\tau_m-\tau_{m+1}$. As $|y_1|\to\infty$, we have in the range of summation of S_1

$$\Delta\sigma_k\sim\frac{|y_1|^{3/2}}{k^2}=\frac{\xi_k^2}{\sqrt{|y_1|}}\quad\text{and}\quad\Delta\tau_m\sim\frac{|y_1|^{3/2}}{m^2}=\frac{\eta_m^2}{\sqrt{|y_1|}},$$

which tend to zero uniformly. Consequently,

$$\begin{aligned}
&S_1(y_1,y_2)\\
&\sim\frac{1}{(2\pi)^{3/2}|y_1|^{5/2}}\left(|y_1|+\frac{l_1}{l_1-2}\right)\left(|y_2|+\frac{l_2}{l_2-2}\right)\exp(|y_1|M(\lambda))\\
&\times\sum_{|\sigma_k|\le\delta\sqrt{|y_1|}}\ \sum_{|\tau_m|\le\delta\sqrt{|y_1|}}\frac{G_1\Big(t_1\big(\xi(\lambda_0)+\sigma_k/\sqrt{|y_1|}\big)\Big)}{\sqrt{u_1\big(\xi(\lambda_0)+\sigma_k/\sqrt{|y_1|}\big)}}\\
&\times\frac{G_2\Big(t_2\big(\lambda_0\eta(\lambda_0)+\lambda_0\tau_m/\sqrt{|y_1|}\big)\Big)}{\sqrt{u_2\big(\lambda_0\eta(\lambda_0)+\lambda_0\tau_m/\sqrt{|y_1|}\big)}}
\end{aligned}\tag{3.13}$$

$$\times\sqrt{\frac{1}{\xi(\lambda_0)+\sigma_k/\sqrt{|y_1|}}+\frac{1}{\eta(\lambda_0)+\tau_m/\sqrt{|y_1|}}}$$
$$\times\exp\left(\frac{1}{2}H_{\lambda_0,t}(\sigma_k,\tau_m)+o\left(\sigma_k^2+\tau_m^2\right)\right)\Delta\sigma_k\,\Delta\tau_m,$$

where $o(\sigma^2+\tau^2)\to 0$ as $(\sigma^2+\tau^2)/|y_1|\to 0$, uniformly with respect to $\lambda\sim\lambda_0$. In (3.13), the term in the double summation before the exponential is bounded by some constant $D=D(\lambda_0)$. Also, the exponent is

$$|y_1|\left(\Phi_{\lambda,t}\left(\xi(\lambda_0)+\frac{\sigma_k}{\sqrt{|y_1|}},\eta(\lambda_0)+\frac{\tau_m}{\sqrt{|y_1|}}\right)-M(\lambda)\right)\leq-\varepsilon_0\left(\sigma_k^2+\tau_m^2\right)$$

by (3.9), where $\varepsilon_0>0$. As $\int\int_{\mathbb{R}^2}\exp(-\varepsilon_0(\sigma^2+\tau^2))\,d\sigma\,d\tau<\infty$, we may apply Lebsegue's theorem and see that the double sum in (3.13) converges to

$$\frac{G_1\left(t_1(\xi(\lambda_0))\right)}{\sqrt{u_1(\xi(\lambda_0))}}\frac{G_2\left(t_2(\lambda_0\eta(\lambda_0))\right)}{\sqrt{u_2(\lambda_0\eta(\lambda_0))}}\sqrt{\frac{1}{\xi(\lambda_0)}+\frac{1}{\eta(\lambda_0)}}\times\int\int_{\mathbb{R}^2}\exp\left(\frac{1}{2}H_{\lambda_0,t}(\sigma,\tau)\right)d\sigma\,d\tau. \tag{3.14}$$

Because all the functions in (3.14) depend continuously on λ, we may replace λ_0 with λ. It follows from (3.8) that

$$\exp(|y_1|M(\lambda))=F_1\left(t_1(\xi(\lambda))\right)^{|y_1|}F_2\left(t_2(\lambda\eta(\lambda))\right)^{|y_2|}.$$

This and (2.4) yield that $S_1(y_1,y_2)$ has the asymptotic behaviour proposed in the statement of the theorem.

Next, we show that $S_2(y_1,y_2)/S_1(y_1,y_2)\to 0$, that is,

$$S_2(y_1,y_2)\exp(-|y_1|M(\lambda))=o\left(\left(|y_2|+\frac{l_2}{l_2-2}\right)|y_1|^{-3/2}\right) \tag{3.15}$$

as $|y_1|\to\infty$ and either $|y_2|\to\infty$ or y_2 remains fixed.

By (3.4) there is $\varepsilon_1>0$ such that for λ close to λ_0,

$$\Phi_{\lambda,t}(\xi,\eta)-M(\lambda)\leq-\varepsilon_1$$

for all (ξ,η) with $|\xi-\xi(\lambda)|>\delta$ or $|\eta-\eta(\lambda)|>\delta$.

We assume for the moment that $y_2\neq o_2$. For $k,m\geq 1$, the error term in Stirling's formula yields

$$\binom{k+m}{k}=\frac{1}{\sqrt{2\pi}}\frac{(k+m)^{k+m+1/2}}{k^{k+1/2}m^{m+1/2}}\exp\left(\frac{\theta_{k+m}}{k+m}-\frac{\theta_k}{k}-\frac{\theta_m}{m}\right),$$

where $0 < \theta_l < 1$. Thus, the exponential term is bounded by $c_0 = \exp(1/4)$. Furthermore, $(k+m)^{1/2}/(km)^{1/2} \le 1/\sqrt{2}$. We get

$$\sqrt{2\pi}\exp\bigl(-|y_1|M(\lambda)\bigr)S_2(y_1,y_2) \le \frac{c_0}{\sqrt{2}}\sum_{k=|y_1|}^{\infty}\sum_{m=|y_2|}^{\infty}\beta_k^1(\xi_k)\beta_m^2(\lambda\eta_m)\exp\bigl(-\varepsilon_1|y_1|\bigr).$$

We split this sum into four parts according to whether ξ_k and $(1+\lambda)\eta_m > 1/2$ or $\le 1/2$, that is, whether $k \ge 2|y_1|$ or $|y_1| \le k < 2|y_1|$, and $m \ge 2(1+\lambda)|y_1|$ or $|y_2| \le m < 2(1+\lambda)|y_1|$, respectively. Given $\varepsilon > 0$, by Theorem 2.3,

$$\beta_k^1(\xi_k) \le \exp(\varepsilon k), \qquad \beta_m^2(\eta_m) \le \exp(\varepsilon m)$$

if $k \ge k_\varepsilon, m \ge m_\varepsilon$. If $|y_2|$ remains fixed, then let

$$c_\varepsilon = \max\bigl\{1,\ \beta_m^2(\lambda\eta_m)\exp(-\varepsilon m) : m = |y_2|, \dots m_\varepsilon - 1\bigr\}.$$

If $|y_2| \to \infty$, then set $c_\varepsilon = 1$. We obtain

$$\beta_m^2(\lambda\eta_m) \le c_\varepsilon \exp(\varepsilon m)$$

for all $m \ge |y_2|$. Thus,

$$\begin{aligned}
&\sum_{\xi_k > \frac12}\ \sum_{(1+\lambda)\eta_m > \frac12} \beta_k^1(\xi_k)\beta_m^2(\lambda\eta_m)\exp\bigl(-\varepsilon_1|y_1|\bigr)\\
&\quad\le \sum_{k=|y_1|}^{2|y_1|-1}\ \sum_{m=|y_2|}^{2(1+\lambda)|y_1|-1} \exp(\varepsilon k)\, c_\varepsilon \exp(\varepsilon m)\exp\bigl(-\varepsilon_1|y_1|\bigr)\\
&\quad\le c_\varepsilon|y_1|\bigl(2|y_1|+|y_2|\bigr)\exp\Bigl(-|y_1|\bigl(\varepsilon_1 - 2\varepsilon(2+\lambda)\bigr)\Bigr).
\end{aligned} \tag{3.16}$$

If we choose $\varepsilon < \varepsilon_1/(4+2\lambda_0)$ and if λ is close to λ_0, then the last term tends to zero faster than $(|y_2| + l_2/(l_2-2))|y_1|^{-3/2}$, as proposed in (3.15).

Recalling once more that $k\xi_k = |y_1| \to \infty$, we get from Theorem 2.3 that

$$\begin{aligned}
&\beta_k^1(\xi_k) \le C_1|y_1|k^{-3/2}, && k \ge 2|y_1|,\\
&\beta_m^2(\lambda\eta_m) \le C_2\Bigl(|y_2| + \frac{l_2}{l_2-2}\Bigr)m^{-3/2}, && m \ge 2(1+\lambda)|y_1|,
\end{aligned}$$

where $C_1, C_2 < \infty$. Indeed, the lower bound for m gives $m \to \infty$ and $\lambda\eta_m \le 1/2$. We get

$$\begin{aligned}
&\sum_{\xi_k \le \frac12}\ \sum_{(1+\lambda)\eta_m \le \frac12} \beta_k^1(\xi_k)\beta_m^2(\lambda\eta_m)\exp\bigl(-\varepsilon_1|y_1|\bigr)\\
&\quad\le C_1|y_1|C_2\Bigl(|y_2| + \frac{l_2}{l_2-2}\Bigr)\exp\bigl(-\varepsilon_1|y_1|\bigr)\\
&\quad\quad\times \sum_{k=2|y_1|}^{\infty}\ \sum_{m=2(1+\lambda)|y_1|}^{\infty} k^{-3/2}m^{-3/2},
\end{aligned} \tag{3.17}$$

which also tends to zero faster than $(|y_2| + l_2/(l_2-2))|y_1|^{-3/2}$.

Next,

$$\sum_{\xi_k \le \frac{1}{2}} \sum_{(1+\lambda)\eta_m > \frac{1}{2}} \beta_k^1(\xi_k)\beta_m^2(\lambda\eta_m)\exp(-\varepsilon_1|y_1|)$$

$$\tag{3.18} \le C_1|y_1|\exp(-\varepsilon_1|y_1|)\sum_{k=2|y_1|}^{\infty}\sum_{m=|y_2|}^{2(1+\lambda)|y_1|-1} k^{-3/2}c_\varepsilon\exp(\varepsilon m)$$

$$\le C_1|y_1|c_\varepsilon(2|y_1|+|y_2|)\exp\Big(-|y_1|(\varepsilon_1 - 2\varepsilon(1+\lambda))\Big)\sum_{k=2|y_1|}^{\infty} k^{-3/2},$$

which with the same choice of ε as above is also an $o((|y_2| + l_2/(l_2-2))|y_1|^{-3/2})$.

Analogously,

$$\sum_{\xi_k > \frac{1}{2}} \sum_{(1+\lambda)\eta_m \le \frac{1}{2}} \beta_k^1(\xi_k)\beta_m^2(\lambda\eta_m)\exp(-\varepsilon_1|y_1|)$$

$$\tag{3.19} \le C_2\left(|y_2| + \frac{l_2}{l_2-2}\right)\exp(-\varepsilon_1|y_1|)\sum_{k=|y_1|}^{2|y_1|-1}\sum_{m=2(1+\lambda)|y_1|}^{\infty} \exp(\varepsilon k)m^{-3/2}$$

$$\le |y_1|C_2\left(|y_2| + \frac{l_2}{l_2-2}\right)\exp(-|y_1|(\varepsilon_1 - 2\varepsilon))\sum_{m=2(1+\lambda)|y_1|}^{\infty} m^{-3/2},$$

which with the same choice of ε as previously is once more an $o((|y_2| + l_2/(l_2 - 2))|y_1|^{-3/2})$.

If $y_2 = o_2$ then we first split $S_2(y_1, o_2)$ into two parts according to whether in the summation $m \ge 1$ or $m = 0$. The part with $m \ge 1$ is treated exactly as in (3.16)–(3.19). From (3.1), we see that the part with $m = 0$ is

$$\sum_{k=|y_1|}^{\infty} \frac{1}{t^k} a^k p_1^{(k)}(o_1, y_1) = G_1\left(o_1, y_1 \,\Big|\, \frac{t}{a}\right).$$

Observe that $\lambda = 0$ and hence $t_2 = \rho_2$ and $t_1 = (t - (1-a)\rho_2)/a < t/a$; see (3.7). Using the asymptotic equivalent for $S_1(y_1, o_2)$, we get

$$\frac{G_1(o_1, y_1 \mid t/a)}{S_1(y_1, o_2)} \sim \frac{G_1(o_1, y_1 \mid t/a)}{G_1(o_1, y_1 \mid t_1)} G_2(\rho_2)^{-1}$$

$$\times\left(|y_1| + \frac{l_1}{l_1-2}\right)^{-1}\frac{l_2-2}{l_2}\sqrt{2|y_1|^5\pi}\,C(0)^{-1}$$

$$\sim \text{Const}\,|y_1|^{-3/2}\left(\frac{F_1(t/a)}{F_1(t_1)}\right)^{|y_1|};$$

see (2.4). As $F_1(t)$ is decreasing, the last term tends to zero as $|y_1| \to \infty$. This concludes the proof. □

REMARK. By symmetry, exchanging the role of y_1 and y_2, one also obtains an asymptotic estimate when $|y_2|/|y_1| \to \infty$.

We also remark that for $t = \rho(Q)$, the renewal theorem becomes more complicated. This is due to the fact that the maximum of $\Phi_{\lambda,\rho}$ is no longer attained in the interior of its domain. Indeed, the solutions of (3.7) become $t_1 = \rho_1$, $t_2 = \rho_2$ and $\xi(\lambda) = \eta(\lambda) = 0$ for all λ. However, it is already known [11] how to determine the Martin boundary in this case (see Theorem 4.2).

4. The Martin boundary. On the basis of Theorem 3.1, we can determine the Martin boundary for eigenvalues $t > \rho(Q)$. With an abuse of the notation introduced in the preceding sections, we write $t_1(\lambda) = t_1(\lambda, t)$ and $t_2(\lambda) = t_2(\lambda, t)$ for the solution of the system of equations (3.7), which we repeat here for the convenience of the reader:

$$at_1(\lambda) + (1-a)t_2(\lambda) = t,$$

$$\sqrt{\left(t_2(\lambda) - \frac{1}{2}\right)^2 - \frac{l_2 - 1}{l_2^2}} = \frac{\lambda a}{1-a}\sqrt{\left(t_1(\lambda) - \frac{1}{2}\right)^2 - \frac{l_1 - 1}{l_1^2}}. \tag{4.1}$$

Here, we include also $\lambda = \infty$. The solutions are then $t_1(\infty) = \rho_1$ and $t_2(\infty) = (t - a\rho_1)/(1-a)$. By

$$K(x_1x_2, y_1y_2 \mid t) = \frac{G(x_1x_2, y_1y_2 \mid t)}{G(o_1o_2, y_1y_2 \mid t)}$$

we denote the Martin kernel of $T_1 \times T_2$, whereas $K_j(\cdot,\cdot \mid t_j)$ denotes the Martin kernel of T_j, $j = 1, 2$. We also need the normalized spherical functions $\Psi_j(\cdot,\cdot)$; see (2.7). The directions of convergence of the Martin kernel on $T_1 \times T_2$ are the following.

THEOREM 4.1. *Let* $t > \rho = \rho(Q)$.

(a) *Suppose that* $y_1 \to \omega_1 \in \Omega_1$, $y_2 \to \omega_2 \in \Omega_2$ *and* $|y_2|/|y_1| \to \lambda_0 \in [0, \infty]$. *Let* $t_j = t_j(\lambda_0)$, $j = 1, 2$, *as determined by* (4.1). *Then*

$$K(x_1x_2, y_1y_2 \mid t) \to K_1(x_1, \omega_1 \mid t_1)K_2(x_2, \omega_2 \mid t_2).$$

(b) *Suppose that* $y_1 \to \omega_1 \in \Omega_1$ *and that* $y_2 \in T_2$ *remains fixed. Let* $t_j = t_j(0)$, $j = 1, 2$, *as determined by* (4.1). *Then*

$$K(x_1x_2, y_1y_2 \mid t) \to K_1(x_1, \omega_1 \mid t_1)\Psi_2(x_2, y_2).$$

(c) *Suppose that* $y_1 \in T_1$ *remains fixed and that* $y_2 \to \omega_2 \in \Omega_2$. *Let* $t_j = t_j(\infty)$, $j = 1, 2$, *as determined by* (4.1). *Then*

$$K(x_1x_2, y_1y_2 \mid t) \to \Psi_1(x_1, y_1)K_2(x_2, \omega_2 \mid t_2).$$

PROOF. (a) We assume without loss of generality that $\lambda_0 < \infty$. (For $\lambda_0 = \infty$, the result then follows by symmetry; see the remark after the proof of Theorem 3.1.) Set $\lambda = |y_2|/|y_1|$ and $\overline{\lambda} = |x_2^{-1}y_2|/|x_1^{-1}y_1|$. Both λ and $\overline{\lambda}$ tend to λ_0. Theorem 3.1 yields

$$\begin{aligned} K(x_1x_2, y_1y_2 \mid t) &\sim \frac{F_1(t_1(\overline{\lambda}))^{|x_1^{-1}y_1|}}{F_1(t_1(\lambda))^{|y_1|}} \frac{F_2(t_2(\overline{\lambda}))^{|x_2^{-1}y_2|}}{F_2(t_2(\lambda))^{|y_2|}} \\ &= F_1(t_1(\overline{\lambda}))^{\mathfrak{h}(x_1,y_1)} F_2(t_2(\overline{\lambda}))^{\mathfrak{h}(x_2,y_2)} \\ &\quad \times \left(\frac{F_1(t_1(\overline{\lambda}))}{F_1(t_1(\lambda))}\right)^{|y_1|} \left(\frac{F_2(t_2(\overline{\lambda}))}{F_2(t_2(\lambda))}\right)^{|y_2|} \end{aligned} \tag{4.2}$$

As $y_j \to \omega_j$, $\mathfrak{h}(x_j, y_j)$ stabilizes at the value $\mathfrak{h}(x_j, \omega_j)$, $j = 1, 2$. Hence, the first two factors of the last term in (4.2) converge to the proposed limit. It remains to prove that

$$\begin{aligned} &\left(\frac{F_1(t_1(\overline{\lambda}))}{F_1(t_1(\lambda))}\right)^{|y_1|} \left(\frac{F_2(t_2(\overline{\lambda}))}{F_2(t_2(\lambda))}\right)^{|y_2|} \\ &\quad = \exp\Big(|y_1| \big(\log F_1(t_1(\overline{\lambda})) - \log F_1(t_1(\lambda))\big) \\ &\qquad\quad + |y_2| \big(\log F_2(t_2(\overline{\lambda})) - \log F_2(t_2(\lambda))\big)\Big) \\ &\quad \to 1. \end{aligned} \tag{4.3}$$

Observe that by easy computations, (4.1) yields

$$\frac{d}{d\lambda} \log F_1(t_1(\lambda)) + \lambda \frac{d}{d\lambda} \log F_2(t_2(\lambda)) = 0 \tag{4.4}$$

for all $\lambda \in (0, \infty)$.

If $\overline{\lambda}$ and λ happen to coincide, then there is nothing to prove; otherwise, at least one of the two is not equal to 0, and we may rewrite the exponent in (4.3) as

$$\begin{aligned} &|y_1| \Big(\big(\log F_1(t_1(\overline{\lambda})) + \overline{\lambda} \log F_2(t_2(\overline{\lambda}))\big) - \big(\log F_1(t_1(\lambda)) + \lambda \log F_2(t_2(\lambda))\big)\Big) \\ &\quad - |y_1|(\overline{\lambda} - \lambda) \log F_2(t_2(\overline{\lambda})) \\ &\quad = |y_1|(\overline{\lambda} - \lambda) \left(\frac{F_1'(t_1(\widetilde{\lambda}))}{F_1(t_1(\widetilde{\lambda}))} t_1'(\widetilde{\lambda}) + \widetilde{\lambda} \frac{F_2'(t_2(\widetilde{\lambda}))}{F_2(t_2(\widetilde{\lambda}))} t_2'(\widetilde{\lambda}) + \log F_2(t_2(\widetilde{\lambda})) \right. \\ &\qquad\qquad \left. - \log F_2(t_2(\overline{\lambda})) \right) \\ &\quad = |y_1|(\overline{\lambda} - \lambda)\big(\log F_2(t_2(\widetilde{\lambda})) - \log F_2(t_2(\overline{\lambda}))\big), \end{aligned}$$

where $\widetilde{\lambda}$ is between λ and $\overline{\lambda}$. [We have used (4.4) in the last step.] As $|y_1|, |y_2| \to \infty$, $\widetilde{\lambda} \to \lambda_0$ and $\lambda \to \lambda_0$, that is,

$$\log F_2\big(t_2(\widetilde{\lambda})\big) - \log F_2\big(t_2(\overline{\lambda})\big) \to 0 .$$

On the other hand,

$$\begin{aligned}|y_1|(\overline{\lambda} - \lambda) &= \frac{|y_1|}{|x_1^{-1}y_1|}\mathfrak{h}(x_2,y_2) - \frac{|y_2|}{|x_1^{-1}y_1|}\mathfrak{h}(x_1,y_1) \\ &\to \mathfrak{h}(x_2,\omega_2) - \lambda_0\mathfrak{h}(x_1,\omega_1),\end{aligned}$$

a constant. Hence, the term in (4.3) tends to 1, as desired.

(b) With the same choice of λ and $\overline{\lambda}$, which tend to zero in this case, we now get

$$\begin{aligned}K(x_1x_2,y_1y_2 \mid t) &\sim \frac{F_1\big(t_1(\overline{\lambda})\big)^{|x_1^{-1}y_1|}}{F_1\big(t_1(\lambda)\big)^{|y_1|}}\frac{F_2(\rho_2)^{|x_2^{-1}y_2|}}{F_2(\rho_2)^{|y_2|}}\frac{\big(|x_2^{-1}y_2| + l_2/(l_2-2)\big)}{\big(|y_2| + l_2/(l_2-2)\big)} \\ &\sim F_1\big(t_1(\overline{\lambda})\big)^{\mathfrak{h}(x_1,y_1)}\Psi_2(x_2,y_2)\left(\frac{F_1\big(t_1(\overline{\lambda})\big)}{F_1\big(t_1(\lambda)\big)}\right)^{|y_1|}\end{aligned}$$

Now,

$$\left(\frac{F_1\big(t_1(\overline{\lambda})\big)}{F_1\big(t_1(\lambda)\big)}\right)^{|y_1|} \sim \left(\frac{F_1\big(t_1(\overline{\lambda})\big)}{F_1\big(t_1(\lambda)\big)}\right)^{|y_1|}\left(\frac{F_2\big(t_2(\overline{\lambda})\big)}{F_2\big(t_2(\lambda)\big)}\right)^{|y_2|}$$

and the same argument as before shows that this "error" term tends to 1.

By exchanging the two coordinates, (c) is symmetrical to (b). □

THEOREM 4.2. *Let* $t = \rho = \rho(Q)$.

(a) *Suppose that* $y_1 \to \omega_1 \in \Omega_1$ *and* $y_2 \to \omega_2 \in \Omega_2$. *Then*

$$K(x_1x_2,y_1y_2 \mid \rho) \to K_1(x_1,\omega_1 \mid \rho_1)K_2(x_2,\omega_2 \mid \rho_2).$$

(b) *Suppose that* $y_1 \to \omega_1 \in \Omega_1$ *and that* $y_2 \in T_2$ *remains fixed. Then*

$$K(x_1x_2,y_1y_2 \mid \rho) \to K_1(x_1,\omega_1 \mid \rho_1)\Psi_2(x_2,y_2).$$

(c) *Suppose that* $y_1 \in T_1$ *remains fixed and that* $y_2 \to \omega_2 \in \Omega_2$. *Then*

$$K(x_1x_2,y_1y_2 \mid \rho) \to \Psi_1(x_1,y_1)K_2(x_2,\omega_2 \mid \rho_2).$$

PROOF. This result is proved as in the continuous case [11], by making use of results of [20] (see also [22]). Choose a sequence $(y_{1,n}y_{2,n})_n$ that tends to infinity as in statement (a), (b) or (c) and, in addition, converges in the Martin topology:

$$\lim_{n\to\infty} K(x_1x_2,y_{1,n}y_{2,n} \mid \rho) = h(x_1x_2).$$

Then h is a positive ρ-harmonic function on $T_1 \times T_2$. Now the minimal ρ-harmonic functions are given by

$$x_1x_2 \mapsto K_1(x_1, \overline{\omega}_1 \mid \rho_1)K_2(x_2, \overline{\omega}_2 \mid \rho_2), \qquad \overline{\omega}_1 \in \Omega_1, \overline{\omega}_2 \in \Omega_2;$$

see [20] or [22]. By the Poisson–Martin representation theorem, there is a measure ν^h on $\Omega_1 \times \Omega_2$ such that

$$h(x_1x_2) = \int_{\Omega_1 \times \Omega_2} K_1(x_1, \cdot \mid \rho_1)K_2(x_2, \cdot \mid \rho_2)\, d\nu^h.$$

Therefore h is separately ρ_j-harmonic on T_j, $j = 1, 2$.

In (a) and (b), we are assuming that $y_{1,n} \to \omega_1 \in \Omega_1$. Hence for fixed $x_2 \in T_2$, $h(\cdot\, x_2)$ is constant (in the first variable) on the horocycles $H_k(\omega_1)$. To see this, it is enough to observe that, for each $u_1, v_1 \in H_k(\omega_1)$ and for $y_{1,n}$ sufficiently far from o_1 [namely, farther than the confluent $c(u_1, v_1)$], there exists an automorphism (self-isometry) of T_1 that interchanges u_1 and v_1 and fixes y_1; compare with [20]. Then we can apply Lemma 2.2 to the function $h(\cdot\, x_2)$ on T_1 and obtain

$$h(x_1x_2) = K_1(x_1, \omega_1 \mid \rho_1)h(o_1x_2)$$

for all x_1, x_2.

If now $y_{2,n} \to \omega_2 \in \Omega_2$, the same argument shows that

$$h(o_1x_2) = K_2(x_2, \omega_2 \mid \rho_2)$$

for all x_2. Therefore, we have the desired limit, independently of the particular choice of the sequence $(y_{1,n}y_{2,n})_n$: by compactness, this proves (a).

To prove (b), first suppose that $y_{2,n} = o_2$. Observe that the function

$$x_2 \mapsto K(x_1x_2, y_{1,n}o_2 \mid \rho) = \frac{G(x_1x_2, y_{1,n}o_2 \mid \rho)}{G(o_1o_2, y_{1,n}o_2 \mid \rho)}$$

depends only on $|x_2|$. Hence the same is true of the limit function $x_2 \mapsto h(o_1x_2)$. Moreover, this function is an eigenfunction of P_2 with eigenvalue ρ_2, and $h(o_1o_2) = 1$. By [6] and the references therein, $h(o_1\cdot)$ is the spherical function on T_2 with eigenvalue ρ_2: that is, $h(o_1x_2) = \Psi_2(x_2)$ for all x_2. Once more by compactness, this proves (b) in the case $y_2 = o_2$.

To conclude the proof of (b), suppose that $y_{2,n} = y_2 \neq o_2$. We argue as in [11], Proposition 2.4: Recall that multiplication in T_2 refers to group operation in $\mathbb{Z}_2 * \cdots * \mathbb{Z}_2$. Then

$$\begin{aligned}\frac{G(x_1x_2, y_{1,n}y_2 \mid \rho)}{G(o_1o_2, y_{1,n}y_2 \mid \rho)} &= \frac{G\big(x_1(y_2^{-1}x_2), y_{1,n}o_2 \mid \rho\big)}{G(o_1o_2, y_{1,n}o_2 \mid \rho)} \frac{G(o_1o_2, y_{1,n}o_2 \mid \rho)}{G(o_1y_2^{-1}, y_{1,n}o_2 \mid \rho)} \\ &= \frac{K\big(x_1(y_2^{-1}x_2), y_{1,n}o_2 \mid \rho\big)}{K(o_1y_2^{-1}, y_{1,n}o_2 \mid \rho)}.\end{aligned}$$

By the previous argument, the right-hand side converges to

$$K_1(x_1,\omega_1 \mid \rho_1)\frac{\Psi_2(y_2^{-1}x_2)}{\Psi_2(y_2^{-1})} = K_1(x_1,\omega_1 \mid \rho_1)\Psi_2(x_2,y_2)$$

[notation as in (2.7)], independently of the particular choice of $(y_{1,n})_n$ tending to ω_1.

Now (c) follows by switching the first and second variables in (b). □

COROLLARY 4.3. *For $t \geq \rho = \rho(Q)$, set*

$$S_t = \{(t_1,t_2) \mid t_1 \geq \rho_1,\ t_2 \geq \rho_2,\ at_1 + (1-a)t_2 = t\}.$$

Then the Martin boundary of $Q = Q_a$ on $T_1 \times T_2$ with respect to eigenvalue t is

$$\mathcal{M}(t) = (\Omega_1 \times T_2) \cup (\Omega_1 \times \Omega_2 \times S_t) \cup (T_1 \times \Omega_2).$$

The minimal Martin boundary $\Omega_1 \times \Omega_2 \times S_t$ carries the product topology relative to the three factors. The other two pieces consist of nonminimal points and also carry the relative product topology, with T_j discrete. The two nonminimal parts are attached to the minimal one as follows.

If $\omega_1 \to \overline{\omega}_1 \in \Omega_1$ and $y_2 \to \overline{\omega}_2 \in \Omega_2$, then

$$(\omega_1,y_2) \to (\overline{\omega}_1,\overline{\omega}_2,(t_1,\rho_2)),$$

where $t_1 = (t-(1-a)\rho_2)/a$.

Analogously, if $y_1 \to \overline{\omega}_1 \in \Omega_1$ and $\omega_2 \to \overline{\omega}_2 \in \Omega_2$, then

$$(y_1,\omega_2) \to (\overline{\omega}_1,\overline{\omega}_2,(\rho_1,t_2)),$$

where $t_2 = (t-a\rho_1)/(1-a)$.

In particular, when $t = \rho$, then the Martin boundary is the boundary of $T_1 \times T_2$ in $(T_1 \cup \Omega_1) \times (T_2 \cup \Omega_2)$, the latter equipped with the product topology.

5. Final comments. The uniform estimate of [14] does not only hold for simple random walk, but for arbitrary nearest neighbour random walks, which are invariant under the group $\mathbb{Z}_2 * \cdots * \mathbb{Z}_2$ (l factors). Therefore, our method for determining the Martin boundary of $T_1 \times T_2$ will also work for Cartesian products of arbitrary random walks of this type (at least for eigenvalues $t > \rho$). However, whereas the method remains the same, writing down the details will become much more space consuming, requiring a more involved notation. Also, the results extend obviously to the Cartesian product of a tree with a straight line (see [2]) and to Cartesian products of more than two trees.

More recently, Lalley [15] has also given (partial) uniform estimates for the case of group-invariant random walks on T with (arbitrary) finite range. Taking Cartesian products of two such walks, one should be able to determine on this basis the asymptotic behaviour of the Green kernel, when $|y_1|/|y_2| \to \lambda_0 \in (0,\infty)$.

Another possible approach regards Cartesian products of arbitrary *isotropic* random walks on T_1 and T_2. One may try to refine the methods of harmonic analysis used by [24] to obtain uniform estimates for the transition probabilities on the factors.

In any case, our result should be considered as a "prototype," waiting for extensions and refined methods. Note that besides the integer lattices studied in [19], the result presented here is the only example where one has full knowledge of the Martin compactification with respect to a natural "discrete Laplace operator" of a Cartesian product of two infinite graphs.

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