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WEAK CONVERGENCE OF MARKOV PROCESSES WITH EXTENDED GENERATORS

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In this article, we give some sufficient conditions for the weak convergence of Markov processes in terms of their extended generators. For the convergence of Markov processes with generators defined by Dynkin, necessary and sufficient conditions are obtained. As an application, we will discuss the convergence of diffusion processes with jumps.

1. Introduction. The weak convergence of Markov processes can be described by their characteristics [see, e.g., Jacod and Shiryaev (1987)], semigroups and generators [see, e.g., Ethier and Kurtz (1986)]. Although the semigroup method gives an explicit and sharp description for the convergence of the processes, in practice, what we can get are the generators. The famous Hille–Yosida theorem gives theoretical relations between some generators and their semigroups. Unfortunately, it is not always possible to explicitly solve the semigroups from the corresponding generators. The characteristics method gives very general results [see Jacod and Shiryaev (1987)], but for Markov processes with known generators, it seems unnecessary and may be impossible to solve for their characteristics. Thus, it is interesting to use the properties of generators to control the properties of the corresponding processes directly. The generator method given in this paper is based on these ideas.

In Ethier and Kurtz (1986), it is shown that for Feller processes, the convergence in some sense of their semigroups or their generators can imply the weak convergence of the processes. Jacod and Shiryaev (1987) discussed the convergence of diffusion processes with jumps. Markov chains are an important class of Markov processes. It seems that the rigorous convergence results involving generators in Ethier and Kurtz (1986) cannot be applied to the following example directly.

1.1. EXAMPLE [Ethier and Kurtz (1986), page 262]. Let $\mathcal{E} = \{0,1,2,\dots\}$. Let $q_{ij} \geq 0$, $i \neq j$, and let $\sum_{j\neq i} q_{ij} = -q_{ii} < \infty$. Suppose for each probability measure τ on $\mathcal{E}$ that there exists an $\mathcal{E}$-valued càdlàg Markov process X such that $\mathbf{P} \circ X(0)^{-1} = \tau$ and

$$\lim_{\varepsilon > 0,\, \varepsilon \downarrow 0} \varepsilon^{-1}\Big(\mathbf{P}\{X(t+\varepsilon) = j \mid X(t)\} - 1_{X(t)=j}\Big) = q_{X(t)j}, \qquad j \in \mathcal{E},\ t \geq 0.$$

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For $n = 1, 2, \ldots$ and a probability measure τ_n on $\mathcal{E}$, let X_n be an $\mathcal{E}$-valued càdlàg Markov process satisfying $\mathbf{P} \circ X_n(0)^{-1} = \tau_n$ and

$$\lim_{\varepsilon > 0, \varepsilon \downarrow 0} \varepsilon^{-1}\Big(\mathbf{P}\{X_n(t+\varepsilon) = j \mid X_n(t)\} - 1_{X_n(t)=j}\Big) = q^n_{X_n(t)j}, \qquad j \in \mathcal{E}, t \geq 0.$$

Then the following statements are equivalent:

(a) $\forall i, j$, $q^n_{ij} \to q_{ij}$.
(b) $\forall \tau_n \Rightarrow \tau$, $X_n \Rightarrow X$.
(c) $\forall i$, if $\tau_n = \tau = \delta_{\{i\}}$, $X_n \Rightarrow X$.

Here and in the sequel, we use the notation $\Rightarrow$ to stand for "converges weakly to."

Thus, it seems desirable to generalize the results on convergence of generators in Ethier and Kurtz (1986). It is reasonable to require that at least the new results should include the case of Markov chains. In Section 4 of this paper, we give some conditions that are different from those in Ethier and Kurtz (1986) and Jacod and Shiryaev (1987). On the other hand, how much further can these results be generalized? For the convergence of Markov processes with generators, the necessary and sufficient conditions are given in Section 5. In particular, if the processes are continuous, then equivalent conditions involving semigroups, resolvents and generators are obtained (see Corollary 5.4). As a preliminary, we will extend the concept of generator and discuss its properties in Section 3.

2. Preliminaries. We assume in this paper that $(\mathcal{E}, d)$ is a locally compact Polish space. It is well known that a locally compact separable metric space is σ-compact, and hence a Borel subset of a Polish space. So, at the cost of greater complexity, the results in this paper can be extended to locally compact separable metric spaces without any difficulties. [see Parthasarathy (1967)].

Let $\mathcal{B}(\mathcal{E})$ and $\mathcal{U}(\mathcal{E})$ be the collections of the Borel subsets and compact subsets of $(\mathcal{E}, d)$, respectively. For $\Gamma \in \mathcal{B}(\mathcal{E})$, let $\overline{\Gamma}$, Γ^c, $\partial\Gamma$ and Γ^o stand for the closure, the complement, the boundary and the interior of Γ, respectively. For Γ, $G \subset \mathcal{E}$, $x \in \mathcal{E}$, set

$$d(x, \Gamma) := \inf\{d(x, y) \colon y \in \Gamma\},$$
$$d(G, \Gamma) := \inf\{d(y, z) \colon y \in G,\ z \in \Gamma\},$$

where in both cases the infimum of the empty set is understood to be $+\infty$.

An important fact we will use in this paper is that if $G \in \mathcal{U}(\mathcal{E})$, $\Gamma \in \mathcal{B}(\mathcal{E})$ and $G \subset \Gamma^o$, then $d(G, \overline{\Gamma^c}) > 0$. In fact, if $d(G, \overline{\Gamma^c}) = 0$, then we can find two sequences $\{x_n\} \subset G$ and $\{y_n\} \subset \overline{\Gamma^c}$ such that $d(x_n, y_n) \to 0$. Because G is compact, without loss of generality we may assume $x_n \to x \in G$. Then $y_n \to x$ and it follows that $x \in G \cap \overline{\Gamma^c}$, which contradicts $G \subset \Gamma^o$.

2.1. LEMMA. *For any $G \in \mathcal{B}(\mathcal{E})$, define $B_{G,r} = \{x \in \mathcal{E} \colon d(x, \overline{G^c}) \geq r\}$. Then:*

(a) $(B_{G,r})^o \supset \{x \in \mathcal{E}: d(x, \overline{G^c}) > r\}$.
(b) $\partial B_{G,r} \subset \{x \in \mathcal{E}: d(x, \overline{G^c}) = r\}$.
(c) *For* $t < r$, $B_{G,r} \subset (B_{G,t})^o$.
(d) *For* $t < r$, $B_{G,r}(t) := \{x \in \mathcal{E}: d(x, B_{G,r}) \le t\} \subset B_{G,r-t}$.

PROOF. (a), (b) and (c) are obvious. To prove (d), let $x \in B_{G,r}(t)$. Then $\exists y_n \in B_{G,r}$, such that $d(x,y_n) \le d(x, B_{G,r}) + 1/n \le t + 1/n$. Thus, $d(x,\overline{G^c}) \ge d(y_n, \overline{G^c}) - d(x,y_n) \ge r - t - 1/n$. Letting $n \uparrow \infty$ gives (d). □

Let $D_{[0,\infty)}$ be the space of all càdlàg functions defined on $\mathbf{R}_+$ with values in $(\mathcal{E}, d)$, and let ρ be the Skorokhod distance [see Ethier and Kurtz (1986)] so that $(D_{[0,\infty)}, \rho)$ is a Polish space. Let $\mathcal{B}(D_{[0,\infty)})$ be the Borel σ-field of $(D_{[0,\infty)}, \rho)$.

For $\alpha \in D_{[0,\infty)}$ and $\Gamma \in \mathcal{B}(\mathcal{E})$, let $S_\Gamma(\alpha) = \inf\{t: \alpha(t) \in \overline{\Gamma^c}$ or $\alpha(t^-) \in \overline{\Gamma^c}\}$, and $S_{\Gamma^+}(\alpha) = \lim_{t>0, t\downarrow 0} S_{\Gamma(t)}(\alpha)$, where $\alpha(t^-) := \lim_{s<t, s\uparrow t} \alpha(s)$ and $\Gamma(t) := \{x \in \mathcal{E}: d(x,\Gamma) \le t\}$. Thus, $S_\Gamma(\alpha)$ is the first hitting time of $\overline{\Gamma^c}$ and $S_{\Gamma^+}(\alpha)$ is the first entry time of $(\Gamma^c)^o$. It is clear that $\forall t < S_\Gamma(\alpha)$, $\alpha(t) \in \Gamma^o$. We will frequently use the $S_\Gamma(\alpha)$ to confine processes to compact sets.

2.2. LEMMA. (a) *If* $S_\Gamma(\alpha) = S_{\Gamma^+}(\alpha)$, *then* $\alpha(S_\Gamma(\alpha)) \in \overline{\Gamma^c}$.
(b) *With respect to* r, $S_{B_{\Gamma,r}}(\alpha)$ *is a decreasing and right continuous function.*
(c) $S_{B_{\Gamma,r}}(\alpha) \le S_{B_{\Gamma,r}+}(\alpha) \le \lim_{t<r, t\uparrow r} S_{B_{\Gamma,t}}(\alpha)$.

PROOF. (a) In fact, if $\alpha(S_\Gamma(\alpha)) \in \Gamma^o$, then $\exists \delta > 0$, such that $\alpha(t) \in \Gamma^o$, $\forall t \in [S_\Gamma(\alpha), S_\Gamma(\alpha) + \delta]$; hence, $\forall t \le S_\Gamma(\alpha) + \delta$. Thus, $\forall s > 0$, $S_{\Gamma(s)}(\alpha) \ge S_\Gamma(\alpha) + \delta$, which contradicts $S_\Gamma(\alpha) = S_\Gamma + (\alpha)$.

(b) $\forall s < S_{B_{\Gamma,r}}(\alpha)$, it is clear that $C_s := \overline{\{\alpha(u): 0 \le u \le s\}} \subset (B_{\Gamma,r})^o$ is compact; hence $C_s \subset (B_{\Gamma,t})^o$ for $t (> r)$ near r enough, which implies $s \le S_{B_{\Gamma,t}}(\alpha) \le S_{B_{\Gamma,r}}(\alpha)$. Letting $t \downarrow r$ and then $s \uparrow S_{B_{\Gamma,r}}(\alpha)$ gives (b).

(c) By Lemma 2.1(d), we have $B_{\Gamma,r}(u) \subset B_{\Gamma,r-u}$ for $0 < u < r$. Thus the claim is true because $S_{B_{\Gamma,r}}(\alpha) \le S_{B_{\Gamma,r}(u)}(\alpha) \le S_{B_{\Gamma,r-u}}(\alpha)$. □

Let $\mathbf{B}_{\mathrm{loc}}(\mathcal{E}), \mathbf{C}(\mathcal{E}), \mathbf{B}(\mathcal{E}), \mathbf{C}^b(\mathcal{E}), \overline{\mathbf{C}}(\mathcal{E})$, and $\widehat{\mathbf{C}}(\mathcal{E})$ be the spaces of all Borel measurable functions bounded on compact sets, all continuous functions, all bounded Borel measurable functions, all bounded continuous functions, all bounded uniformly continuous functions and continuous functions vanishing at infinity on $\mathcal{E}$ with values in $\mathbf{R}$, respectively. Let $\mathcal{P}(\mathcal{E})$ denote the space of all probability measures on $\mathcal{E}$.

Let LIM denote the convergence of a sequence $f_n \in \mathbf{B}_{\mathrm{loc}}(\mathcal{E})$, $n \ge 1$, to $f \in \mathbf{B}_{\mathrm{loc}}(\mathcal{E})$ uniformly on any compact set, and let $\mathrm{LIM}_n^* f_n = f$ stand for $\mathrm{LIM}_{n\to\infty} f_n = f$ and $(\bigvee_n \|f_n\|) \vee \|f\| < \infty$, where $\|\cdot\|$ is the supremum norm.

2.3. LEMMA. *To each* $\beta \in D_{[0,\infty)}$ *and* $\Gamma \in \mathcal{B}(\mathcal{E})$, *we associate the stopped function* β_Γ *defined by* $\beta_\Gamma(t) = \beta(t \wedge S_\Gamma(\beta))$. *For* α_n, $n \ge 1$, $\alpha \in D_{[0,\infty)}$, *assume* $\rho(\alpha_n, \alpha) \to 0$ *and* $S_\Gamma(\alpha) = S_{\Gamma^+}(\alpha)$.

(i) $S_\Gamma(\alpha_n) \to S_\Gamma(\alpha)$.

(ii) *Suppose either α is continuous at $S_\Gamma(\alpha)$ or α is discontinuous at $S_\Gamma(\alpha)$ and $\alpha(S_\Gamma(\alpha)^-) \in \Gamma^o$. Then $\rho(\alpha_{n,\Gamma}, \alpha_\Gamma) \to 0$ [hence $d(\alpha_n(S_\Gamma(\alpha_n)), \alpha(S_\Gamma(\alpha))) \to 0$].*

(iii) *If $\Gamma \in \mathfrak{U}(\mathcal{E})$, $f_n \in \mathbf{B}_{\text{loc}}(\mathcal{E})$, $n \geq 1$, $f \in \mathbf{C}(\mathcal{E})$ and $\text{LIM}_n f_n = f$, then $\forall t_1 < t_2$,*

$$\int_{t_1 \wedge S_\Gamma(\alpha_n)}^{t_2 \wedge S_\Gamma(\alpha_n)} f_n(\alpha_n(s))\,ds \to \int_{t_1 \wedge S_\Gamma(\alpha)}^{t_2 \wedge S_\Gamma(\alpha)} f(\alpha(s))\,ds.$$

PROOF. Note that $\rho(\alpha_n, \alpha) \to 0$ if any only if there exists a sequence of time changes $(\lambda_n)_{n\geq 1}$ (a time change is a continuous and strictly increasing function mapping $[0, \infty)$ onto $[0,\infty)$) such that $\forall N > 0$, $\sup_{t\leq N} d(\alpha_n(\lambda_n(t)), \alpha(t)) \to 0$ and $\sup_{t\leq N} |\lambda_n(t) - t| \to 0$ [see Ethier and Kurtz (1986), page 119].

(i) If $t < S_\Gamma(\alpha)$, then $C_t = \overline{\{\alpha(s)\colon s \leq t\}} \subset \Gamma^o$. Note that because C_t is compact [see Ethier and Kurtz (1986)], we have $d(C_t, (\Gamma^o)^c) > 0$. Hence

$$\overline{\left\{\alpha_n(\lambda_n(s))\colon s \leq t\right\}} \subset \Gamma^o$$

for sufficiently large n, which implies $\lambda_n(t) \leq S_\Gamma(\alpha_n)$. Let $n \uparrow \infty$ and then $t \uparrow S_\Gamma(\alpha)$, so that $\liminf_n S_\Gamma(\alpha_n) \geq S_\Gamma(\alpha)$. On the other hand, if $t > S_\Gamma(\alpha) = S_{\Gamma+}(\alpha)$, then $t > S_{\Gamma(r)}(\alpha)$ for some $r > 0$. Thus $\exists s \in [S_\Gamma(\alpha), t]$, such that $d(\alpha(s), \Gamma) > 0$. Hence, $d(\alpha_n(\lambda_n(s)), \Gamma) > 0$ for sufficiently large n, which yields $S_\Gamma(\alpha_n) \leq \lambda_n(s)$. Thus, we get $\limsup_n S_\Gamma(\alpha_n) \leq S_\Gamma(\alpha)$ be letting $n \uparrow \infty$ first and $t \downarrow S_\Gamma(\alpha)$ second.

(ii) It suffices to show that $\forall N > 0$,

$$\begin{aligned} a_n &:= \sup_{t\leq N} d\Big(\alpha_n(\lambda_n(t) \wedge S_\Gamma(\alpha_n)), \alpha\big(t \wedge S_\Gamma(\alpha)\big)\Big) \\ &\leq \sup_{t\leq N} d\bigg(\alpha_n\Big(\lambda_n\big(t \wedge \lambda_n^{-1} \circ S_\Gamma(\alpha_n)\big)\Big), \alpha\big(t \wedge \lambda_n^{-1} \circ S_\Gamma(\alpha_n)\big)\bigg) \\ &\quad + \sup_{t\leq N} d\Big(\alpha\big(t \wedge \lambda_n^{-1} \circ S_\Gamma(\alpha_n)\big), \alpha\big(t \wedge S_\Gamma(\alpha)\big)\Big) \\ &\to 0. \end{aligned} \tag{2.1}$$

If α is continuous at $S_\Gamma(\alpha)$, we get (2.1) easily. If α is discontinuous at $S_\Gamma(\alpha)$, then $\alpha(S_\Gamma(\alpha)^-) \in \Gamma^o$, so $U := \overline{\{\alpha(t)\colon 0 \leq t < S_\Gamma(\alpha)\}} \subset \Gamma^o$, and the compactness of U implies $\overline{\{\alpha_n(t)\colon 0 \leq t < \lambda_n(S_\Gamma(\alpha))\}} \subset \Gamma^o$ as well for sufficiently large n. Thus, we have $S_\Gamma(\alpha_n) \geq \lambda_n(S_\Gamma(\alpha))$ and (2.1) follows from

$$\begin{aligned} &\sup_{t\leq N} d\Big(\alpha\big(t \wedge \lambda_n^{-1} \circ S_\Gamma(\alpha_n)\big), \alpha\big(t \wedge S_\Gamma(\alpha)\big)\Big) \\ &\qquad = \sup_{S_\Gamma(\alpha) < t \leq \lambda_n^{-1} \circ S_\Gamma(\alpha_n)} d\Big(\alpha(t), \alpha\big(S_\Gamma(\alpha)\big)\Big) \to 0. \end{aligned}$$

(iii) For each $s < S_\Gamma(\alpha_n)$, we have $\alpha_n(s) \in \Gamma^o$ and

$$\sup_{s < S_\Gamma(\alpha_n)} \big|f_n(\alpha_n(s))\big| \leq \sup_{x\in\Gamma} |f_n(x)| \to \sup_{x\in\Gamma} |f(x)| < \infty. \tag{2.2}$$

If α is continuous at s, then $\alpha_n(s) \to \alpha(s)$ and $f_n(\alpha_n(s)) \to f(\alpha(s))$. Because the discontinuities of α are at most countable, applying (i), (2.2) and the dominated convergence theorem gives (iii). □

For any $\mathcal{E}$-valued adapted càdlàg process Z on a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$ satisfying the usual conditions [see Jacod and Shiryaev (1987), page 2], let $\mathcal{N}$ be the collection of all $\mathbf{P}$-null sets of $\mathcal{F}$. Define

$$\mathcal{F}_t^{Z,o} = \sigma\{Z(s),\ s \leq t\}$$

and

$$\mathcal{F}_t^Z = \left[\bigcap_{s>t} \mathcal{F}_s^{Z,o}\right] \vee \mathcal{N}.$$

Then for all $G \in \mathcal{B}(\mathcal{E})$, $S_G(Z)$ is an $(\mathcal{F}_t^Z)_{t\geq 0}$ stopping time. Now, we set

$$V_\Gamma^1(Z) := \left\{r > 0\colon \mathbf{P}\left[\omega\colon S_{B_{\Gamma,r}}(Z(\omega)) < S_{B_{\Gamma,r+}}(Z(\omega))\right] > 0\right\},$$

$$V_\Gamma^2(Z) := \left\{r > 0\colon \mathbf{P}\left[\omega\colon Z(\omega) \text{ is discontinuous at } S_{B_{\Gamma,r}}(Z(\omega)) \text{ and } Z\left(S_{B_{\Gamma,r}}(Z(\omega))^-\right) \in \partial B_{\Gamma,r}\right] > 0\right\},$$

$$V_\Gamma(Z) := V_\Gamma^1(Z) \cup V_\Gamma^2(Z).$$

2.4. LEMMA. *$V_\Gamma(Z)$ is at most countable.*

PROOF. It is sufficient to show that both V_Γ^1 and V_Γ^2 are at most countable.

First, let $Y_r(\omega) := S_{B_{\Gamma,r}}(Z(\omega))$. Then Y is a decreasing and right continuous process and $\{r\colon \mathbf{P}[\Delta Y_r < 0] > 0\}$ is obviously at most countable. However, by Lemma 2.2(c), $0 \leq S_{B_{\Gamma,r+}}(Z(\omega)) - S_{B_{\Gamma,r}}(Z(\omega)) \leq -\Delta Y_r(\omega)$, so we obtain that $V_\Gamma^1(Z)$ is at most countable.

Second, for $u > 0$, define

$$t^0(Z(\omega), u) = 0,$$

$$t^{p+1}(Z(\omega), u) = \inf\left\{t > t^p(Z(\omega), u)\colon d(Z_{t-}(\omega), Z_t(\omega)) > u\right\}, \qquad \forall p \geq 0.$$

Then

$$V_\Gamma^2(Z) \subset \bigcup_{n,p,k=1}^{\infty} \left\{r\colon \mathbf{P}\left[\omega\colon t^p\left(Z(\omega), \frac{1}{n}\right) < \infty, \ Z\left(t^p\left(Z(\omega), \frac{1}{n}\right)^-\right) \in \partial B_{\Gamma,r}\right] > \frac{1}{k}\right\}.$$

The set on the right is clearly at most countable. □

For each $n \geq 1$, let X_n be an $\mathcal{E}$-valued adapted càdlàg process on a probability space $\mathcal{B}^n := (\Omega^n, \mathcal{F}^n, (\mathcal{F}_t^n)_{t\geq 0}, \mathbf{P}^n)$ satisfying the usual conditions. We use $\mathcal{L}(X_n)$ to denote the induced probability measure of X_n on $D_{[0,\infty)}$. The sequence $(X_n)_{n\geq 1}$ is called tight if $\{\mathcal{L}(X_n)\}_{n\geq 1}$ is tight. We say that $(X_n)_{n\geq 1}$ satisfies *the Compact Containment Condition* (or CCC for abbreviation) if for every $\varepsilon > 0$, $N > 0$, there exists a compact set $\Gamma_{\varepsilon,N} \in \mathcal{U}(\mathcal{E})$ such that

$$\liminf_{n\uparrow\infty} \mathbf{P}^n\big[X_n(t) \in \Gamma_{\varepsilon,N},\ \forall\, 0 \leq t \leq N\big] \geq 1 - \varepsilon, \tag{2.3}$$

or equivalently that there exists a sequence of compact sets $\{\Gamma_k\} \subset \mathcal{U}(\mathcal{E})$ such that

$$\lim_{k\uparrow\infty} \liminf_{n} \mathbf{P}^n\big[X_n(s) \in \Gamma_k,\ \forall\, 0 \leq s \leq N\big] = 1, \qquad \forall N > 0. \tag{2.4}$$

In fact, (2.4) is stronger in appearance than (2.3). If we take $\Gamma_k = \Gamma_{1/k,k}$, then (2.4) follows from (2.3) easily.

For $\alpha \in D_{[0,\infty)}$, $I \in \mathcal{B}(\mathbf{R}_+)$, let $W(\alpha, I) = \sup_{s,t\in I} d(\alpha(s), \alpha(t))$ and $W_N'(\alpha,\delta) = \inf\{\max_{1\leq i\leq r} W(\alpha, [t_{i-1}, t_i)): 0 = t_0 < t_1 < \cdots < t_r = N, \inf_{i\leq r-1}(t_i - t_{i-1}) > \delta\}$. The following criterion for tightness is established in Ethier and Kurtz [(1986), pages 128–130].

2.5. LEMMA. *The sequence $(X_n)_{n\geq 1}$ is tight if and only if it satisfies the* CCC *and* $\forall \varepsilon > 0$, $N > 0$,

$$\lim_{\delta\downarrow 0} \limsup_{n\uparrow\infty} \mathbf{P}^n\big(W_N'(X_n, \delta) > \varepsilon\big) = 0. \tag{2.5}$$

The following lemma plays an important role in the proofs of the main theorems.

2.6. LEMMA. *The sequence $(X_n)_{n\geq 1}$ converges weakly to Z if there exists a sequence $\{\Gamma_k\}_{k\geq 1}$ of compact subsets of $\mathcal{E}$, such that*:

(i) $S_{\Gamma_k}(Z) \Rightarrow \infty$ as $k \uparrow \infty$.
(ii) $\forall k$, $S_{\Gamma_k}(X_n) \Rightarrow S_{\Gamma_k}(Z)$.
(iii) $\forall k$, $X_{n,\Gamma_k} \Rightarrow Z_{\Gamma_k}$.

PROOF. $\forall N > 0$, $\varepsilon > 0$,

$$\begin{aligned}
&\lim_{k\uparrow\infty} \liminf_{n} \mathbf{P}^n\big[X_n(s) \in \Gamma_k,\ \forall\, 0 \leq s \leq N\big] \\
&\quad \geq \lim_{k\uparrow\infty} \liminf_{n} \mathbf{P}^n\big[S_{\Gamma_k}(X_n) > N \text{ and } X_n(s) \in \Gamma_k,\ \forall\, 0 \leq s \leq N\big] \qquad (2.6)\\
&\quad = \lim_{k\uparrow\infty} \liminf_{n} \mathbf{P}^n\big(S_{\Gamma_k}(X_n) > N\big) = 1,
\end{aligned}$$

$$\lim_{\delta\downarrow 0} \limsup_{n} \mathbf{P}^n\big(W_N'(X_n, \delta) > \varepsilon\big)$$

$$\text{(2.7)} \qquad \begin{aligned} &\le \limsup_n \mathbf{P}^n\big(S_{\Gamma_k}(X_n) \le N\big) + \lim_{\delta \downarrow 0} \limsup_n \mathbf{P}^n\big(W'_N(X_{n,\Gamma_k}, \delta) > \varepsilon\big) \\ &= \limsup_n \mathbf{P}^n\big(S_{\Gamma_k}(X_n) \le N\big). \end{aligned}$$

Let $k \uparrow \infty$ in (2.7), we obtain the tightness of $(X_n)_{n \ge 1}$ by Lemma 2.5.

Similar arguments give that $(X_n)_{n \ge 1}$ converges finite dimensionally to Z. □

2.7. LEMMA [Ethier and Kurtz (1986), page 142]. *Suppose that the sequence $(X_n)_{n \ge 1}$ satisfies the* CCC *and let H be a dense subset of $\mathbf{C}^b(\mathcal{E})$ with respect to* LIM*. *Then $(X_n)_{n \ge 1}$ is tight if and only if $(f \circ X_n)_{n \ge 1}$ is tight for each $f \in H$.*

The following lemma is a corollary of Theorem 9.4 and the proof of Ethier and Kurtz [(1986), page 145].

2.8. LEMMA. *Let $\mathcal{L}_n$ be the Banach space of real-valued $(\mathcal{F}_t^n)_{t \ge 0}$-progressive processes whose norm $\|Y\| = \sup_{t \ge 0} \mathbf{E}(|Y(t)|) < \infty$. Let*

$$\widehat{\mathcal{A}}_n = \left\{ (Y, Z) \in \mathcal{L}_n \times \mathcal{L}_n \colon Y(t) - \int_0^t Z(s)\,ds \text{ is an } \left(\mathcal{F}_t^n\right)_{t \ge 0} \text{ martingale} \right\}.$$

Let $f \in \mathbf{C}^b(\mathcal{E})$. If $\forall N > 0$, $\varepsilon > 0$, there exist (Y_n, Z_n) and $(\widetilde{Y}_n, \widetilde{Z}_n) \in \widehat{\mathcal{A}}_n$, such that:

(i) $\limsup_n \mathbf{E}^n[\sup_{t \in [0,N]} |Y_n(t) - f(X_n(t))|] \le \varepsilon$.

(ii) $\limsup_n \mathbf{E}^n[\sup_{t \in [0,N]} |\widetilde{Y}_n(t) - f^2(X_n(t))|] \le \varepsilon$.

(iii) $\limsup_n \mathbf{E}^n\left[\sqrt{\int_0^N (Z_n(s))^2\,ds}\right] < \infty$.

(iv) $\limsup_n \mathbf{E}^n\left[\sqrt{\int_0^N (\widetilde{Z}_n(s))^2\,ds}\right] < \infty$.

Then $(f \circ X_n)_{n \ge 1}$ is tight.

3. Extended generators. The terminologies of extended infinitesimal generator and extended generator have been used in Revuz and Yor (1991) and Jacod (1979), respectively. In this paper, we use the latter one and modify it slightly. Note that there are actually few cases in which a generator and its domain are completely known. Here some particular subspaces of the domain are considered.

3.1. DEFINITION. We say that A is an *extended generator* on $\mathbf{C}^b(\mathcal{E})$ if A is a linear operator defined on $\mathcal{D}(A) \subset \mathbf{C}^b(\mathcal{E})$ with values in $\mathbf{B}_{\text{loc}}(\mathcal{E})$.

By the optional sampling theorem [see Ethier and Kurtz (1986)], it is necessary to restrict the domain $\mathcal{D}(A)$ in $\mathbf{C}^b(\mathcal{E})$ for our localization method. The following definitions are similar to that in Ethier and kurtz (1986).

3.2. DEFINITION. Suppose that A is an extended generator.

(i) Let $\mu \in \mathcal{P}(\mathcal{E})$. An $\mathcal{E}$-valued càdlàg Markov process X is said to be a *solution of the local martingale problem* for (A, μ) on the probability space $(\Omega\ \mathcal{F}, \mathbf{P})$, if

$\mathbf{P}, \circ X(0)^{-1} = \mu$ and for each $f \in \mathcal{D}(A)$, $f(X(t)) - f(X(0)) - \int_0^t Af(X(s))\,ds$ is an $(\mathcal{F}^X)_{t\geq 0}$ local martingale.

(ii) We say that *uniqueness* holds for solution of the local martingale problem for (A, μ) if any two solutions of (A, μ) have the same finite-dimensional distributions.

(iii) We say that the local martingale problem for (A, μ) is *well posed* if there exists a solution of the local martingale problem for (A, μ) and uniqueness holds.

(iv) The local martingale problem for A is said to be well posed if $\forall \mu \in \mathcal{P}(\mathcal{E})$, the local martingale problem for (A, μ) is well posed.

3.3. REMARK. (i) Definition 3.1 permits much extension of the domain of a generator defined in Dynkin (1965) or Ethier and Kurtz (1986). For example, the domains of the density matrices in Example 1.1 are difficult to describe. However, by Definition 3.1, they can be easily extended to $\mathbf{C}^b(\overline{\mathbf{N}}) = \mathbf{B}(\overline{\mathbf{N}})$, where $\overline{\mathbf{N}} = \{0, 1, 2, \ldots\}$. The extension is very convenient for applications.

(ii) From now on, all processes are assumed to be càdlàg.

Note $S_\Gamma(X)$ is a stopping time with respect to $(\mathcal{F}_t^X)_{t\geq 0}, \forall \Gamma \in \mathcal{B}(\mathcal{E})$. The following lemma is standard.

3.4. LEMMA. *Let A be an extended generator. For $\nu \in \mathcal{P}(\mathcal{E})$, assume that X is a solution of the local martingale problem for (A, ν). For all $\Gamma \in \mathcal{B}(\mathcal{E})$, if Af is bounded on Γ, then*

$$M_t := f\big(X_\Gamma(t)\big) - f\big(X_\Gamma(0)\big) - \int_0^{t\wedge S_\Gamma(X)} Af\big(X(s)\big)\,ds$$

$$= f\big(X_\Gamma(t)\big) - f\big(X_\Gamma(0)\big) - \int_0^t Af\big(X(s)\big) 1_{[s < S_\Gamma(X)]}\,ds \tag{3.1}$$

is an $(\mathcal{F}_t^X)_{t\geq 0}$ martingale.

Lemma 2.6 is a key lemma in the proofs of the main theorems. To use it in the proofs, we need a kind of local uniqueness that is weaker than that in Jacod and Shiryaev (1987) (at least in appearance), but generally stronger than the uniqueness defined in this paper.

3.5. DEFINITION. *Suppose that A is an extended generator. $\Gamma \in \mathcal{U}(\mathcal{E})$ is fixed.*

(i) Let $\mu \in \mathcal{P}(\mathcal{E})$. An $\mathcal{E}$-valued càdlàg process X is said to be a *solution of the stopped martingale problem for* (A, μ, Γ) on the probability space $(\Omega, \mathcal{F}, \mathcal{P})$, if $\mathbf{P} \circ X(0)^{-1} = \mu$ and for each $f \in \mathcal{D}(A)$, $f(X(t \wedge S_\Gamma(X))) - f(X(0)) - \int_0^{t\wedge S_\Gamma(X)} Af(X(s))\,ds$ is an $(\mathcal{F}_t^X)_{t\geq 0}$ martingale.

(ii) We say that *local uniqueness* holds for a solution of the stopped martingale problem for (A, μ, Γ) if any two solutions Y_1, Y_2 of (A, μ, Γ) satisfy $\mathcal{L}(Y_{1,\Gamma}) = \mathcal{L}(Y_{2,\Gamma})$.

If $A \subset \mathbf{C}^b(\mathcal{E}) \times \mathbf{B}_{\mathrm{loc}}(\mathcal{E})$, then by Lemma 3.4, it is easy to see that a solution of the local martingale problem for (A, μ) is a solution of the stopped martingale problem for (A, μ, Γ), $\forall \Gamma \in \mathcal{U}(\mathcal{E})$, and hence the local uniqueness for $(A, \mu, \Gamma), \forall \Gamma \in \mathcal{U}(\mathcal{E})$, implies the uniqueness for (A, μ). As pointed out in Jacod and Shiryaev (1987), the local uniqueness is not so easy to check. Fortunately, because of the Markov property, we can prove that, under a mild condition, it is implied by uniqueness. Before proving this, we need the following lemma, which is a corollary of Lemma 5.15 of Ethier and Kurtz [(1986), page 204].

3.6. LEMMA. *Let $\mathcal{S}$ be a Polish space and let $\mathbf{P}_1, \mathbf{P}_2$ be two probability measures on $\mathcal{S}$. Suppose that $\xi_1, \xi_2 \colon \mathcal{S} \mapsto \mathcal{E}$ are Borel measurable and that $\mu = \mathbf{P}_1 \circ \xi_1^{-1} = \mathbf{P}_2 \circ \xi_2^{-1} \in \mathcal{P}(\mathcal{E})$. Then there exists a $\mathbf{P} \in \mathcal{P}(\mathcal{S}^2)$ satisfying*

$$\mathbf{P}(B_1 \times B_2) = \int \mathbf{P}_1\big(B_1 \mid \xi_1 = x\big)\mathbf{P}_2\big(B_2 \mid \xi_2 = x\big)\mu(dx),$$

for $B_1, B_2 \in \mathcal{B}(\mathcal{S})$.

3.7. THEOREM. *Let A be an extended generator. Assume that $A \subset \mathbf{C}^b(\mathcal{E}) \times \mathbf{B}_{\mathrm{loc}}(\mathcal{E})$.*

(i) *Suppose that for each $\mu \in \mathcal{P}(\mathcal{E})$, there exists a solution of the local martingale problem for (A, μ). Let $\Gamma \in \mathcal{U}(\mathcal{E})$ be fixed. For each $\nu \in \mathcal{P}(\mathcal{E})$, if Z is a solution of the stopped martingale problem for (A, ν, Γ), then there exists a solution X of the local martingale problem for (A, ν), such that $\mathcal{L}(X_\Gamma) = \mathcal{L}(Z_\Gamma)$.*

(ii) *If the local martingale problem for A is well posed, then for all compact sets Γ and all $\mu \in \mathcal{P}(\mathcal{E})$, local uniqueness holds for solution of the stopped martingale problem for (A, μ, Γ).*

PROOF. We use a well known technique in the theory of Markov processes to prove this theorem. The idea is as follows. Let $\mathbf{P}_1$ be a solution of the stopped martingale problem and let $\mathbf{P}_2$ be a solution of the local martingale problem for $(A, \mathbf{P}_1 \circ X(S_\Gamma(X))^{-1})$. Then define a probability measure $\mathbf{Q}$ as the law of the process Y obtained by pasting together at time $S_\Gamma(X)$ the process X_Γ under $\mathbf{P}_1$, and the process X under $\mathbf{P}_2$, and prove that Y is the required solution.

(i) Let $\mathbf{P}_1 = \mathcal{L}(Z_\Gamma), \mu = \mathbf{P}_1 \circ Z(S_\Gamma(Z))^{-1}$ and $\mathbf{P}_2$ be a solution of the local martingale problem for (A, μ). By Lemma 3.6, there exists a probability measure $\mathbf{Q}$ on $D_{[0,\infty)} \times D_{[0,\infty)}$, such that $\forall B, C \in \mathcal{B}(D_{[0,\infty)})$,

$$\mathbf{Q}(B \times C) = \int \mathbf{P}_1\Big[X \in B \mid X\big(S_\Gamma(X)\big) = x\Big]\mathbf{P}_2\big[X \in C \mid X(0) = x\big]\mu(dx),$$

where X denotes the coordinate random variable on $D_{[0,\infty)}$. Let (X_1, X_2) denote the coordinate random variable on $\Omega := D_{[0,\infty)} \times D_{[0,\infty)}$ and define

$$Y_1 = \begin{cases} X_1(t), & \text{if } t < S_\Gamma(X_1), \\ X_2\big(t - S_\Gamma(X_1)\big), & \text{if } t \geq S_\Gamma(X_1). \end{cases}$$

Then on $(\Omega, \mathcal{B}(\Omega), \mathbf{Q})$, $Y_\Gamma = X_{1,\Gamma}$ has the same distribution as Z_Γ. Now it remains to show that Y is a solution of the local martingale problem for (A, ν), that is

$\forall f \in \mathcal{D}(A), f(Y_t)-f(Y_0)-\int_0^t Af(Y_s)\,ds$ is an $(\mathcal{F}_t^Y)_{t\geq 0}$-local martingale. In fact, for any fixed $f \in \mathcal{D}(A)$, we have $a := \sup_{x\in\Gamma}|Af(x)| < \infty$ because of the compactness of Γ. Let $\Theta_b := \{x: |Af(x)| \leq b\}$. Then these conclusions are easily established:

$$S_{\Theta_b}(Y) = S_\Gamma(X_1) + S_{\Theta_b}(X_2) \qquad \forall b > a, \tag{3.2}$$

and

$$\lim_{b\to\infty} S_{\Theta_b}(Y) = \infty \quad \text{a.s.} \tag{3.3}$$

For any fixed $b > a$, write $\Theta := \Theta_b$ and set $\widetilde{Y}_s := Y_{s\wedge S_\Theta(Y)}$. By (3.3) and Definition 3.2, it suffices to prove that $\forall h_i \in \mathbf{C}^b(\mathcal{E}), i = 1,\dots,n, t_1 < t_2 < \cdots < t_{n+1}, \mathbf{E}^Q G = 0$ holds, where

$$G := \left(f(\widetilde{Y}_{t_{n+1}}) - f(\widetilde{Y}_{t_n}) - \int_{t_n\wedge S_\Theta(Y)}^{t_{n+1}\wedge S_\Theta(Y)} Af(\widetilde{Y}_s)\,ds\right)\prod_{i=1}^n h_i(\widetilde{Y}_{t_i}).$$

In order to simplify the typography, let $T = S_\Gamma(X_1)$. Note that

$$\begin{aligned} G &= \left(f(\widetilde{Y}_{t_{n+1}\wedge T}) - f(\widetilde{Y}_{t_n\wedge T}) - \int_{t_n\wedge T}^{t_{n+1}\wedge T} Af(\widetilde{Y}_s)\,ds\right)\prod_{i=1}^n h_i(\widetilde{Y}_{t_i}) \\ &\quad + \left(f(\widetilde{Y}_{t_{n+1}\vee T}) - f(\widetilde{Y}_{t_n\vee T}) - \int_{(t_{n+1}\vee T)\wedge S_\Theta(Y)}^{(t_{n+1}\vee T)\wedge S_\Theta(Y)} Af(\widetilde{Y}_s)\,ds\right)\prod_{i=1}^n h_i(\widetilde{Y}_{t_i}) \\ &:= G_1 + G_2. \end{aligned}$$

Because $G_1 = 0$ if $t_n \geq T$, it follows that

$$\begin{aligned} \mathbf{E}^{\mathbf{Q}}(G_1) &= \mathbf{E}^{\mathbf{Q}}\left[\left(f(\widetilde{Y}_{t_{n+1}\wedge T}) - f(\widetilde{Y}_{t_n\wedge T}) - \int_{t_n\wedge T}^{t_{n+1}\wedge T} Af(\widetilde{Y}_s)\,ds\right)\prod_{i=1}^n h_i(\widetilde{Y}_{t_i\wedge T})\right] \\ &= \mathbf{E}^{\mathbf{P}_1}\left[\left(f(\widetilde{X}_{t_{n+1}}) - f(\widetilde{X}_{t_n}) - \int_{t_n\wedge S_\Gamma(X)}^{t_{n+1}\wedge S_\Gamma(X)} Af(\widetilde{X}_s)\,ds\right)\prod_{i=1}^n h_i(\widetilde{X}_{t_i})\right] \\ &= \mathbf{E}\left[\left(f(Z_{T_0\wedge t_{n+1}}) - f(Z_{T_0\wedge t_n}) - \int_{t_n\wedge T_0}^{t_{n+1}\wedge T_0} Af(Z_s)\,ds\right)\prod_{i=1}^n h_i(Z_{T_0\wedge t_i})\right] \\ &= 0, \end{aligned}$$

where $T_0 = S_\Gamma(Z)$ and $\widetilde{X} = X_\Gamma$. Now, it remains to show

$$\mathbf{E}^{\mathbf{Q}}[G_2] = 0. \tag{3.4}$$

Let

$$T_j = \sum_{k=0}^{\infty} \frac{k}{j} 1_{[k/j\leq T<(k+1)/j]} + \infty 1_{[T=\infty]}$$

and

$$G_2^j = \left\{ f\left[X_2\big((t_{n+1} \vee T_j - T_j) \wedge S_\Theta(X_2)\big)\right] - f\left[X_2\big((t_n \vee T_j - T_j) \wedge S_\Theta(X_2)\big)\right] - \int_{(t_n \vee T_j - T_j) \wedge S_\Theta(X_2)}^{(t_{n+1} \vee T_j - T_j) \wedge S_\Theta(X_2)} Af(X_2(s))\, ds \right\} \prod_{t_i \geq T_j} h_i\big(X_2(t_i - T_j)\big) \prod_{t_i < T_j} h_i\big(X_1(t_i)\big).$$

Because X_2 is right continuous and f is continuous, (3.2) yields $G_2^j \to G_2$, a.s. as $j \to \infty$. It is obvious that $G_2^j = 0$ if $T_j \geq t_{n+1}$. Thus,

$$\begin{aligned}
\mathbf{E}^{\mathbf{Q}}[G_2^j] &= \sum_{l < j t_{n+1}} \mathbf{E}^{\mathbf{Q}}\big[G_2^j 1_{\{T_j = l/j\}}\big] \\
&= \sum_{l < j t_{n+1}} \int \mathbf{E}^{\mathbf{P}_1}\left[1_{T_j = l/j} \prod_{t_i < l/j} h_i\big(X(t_i)\big) \,\Big|\, X\big(S_\Gamma(X)\big) = x\right] \\
&\quad \times \mathbf{E}^{\mathbf{P}_2}\Bigg[\Bigg(f\left(X\left(\left(t_{n+1} - \frac{l}{j}\right) \wedge S_\Theta(X)\right)\right) \\
&\qquad - f\left(X\left(\left(t_n \vee \frac{l}{j} - \frac{l}{j}\right) \wedge S_\Theta(X)\right)\right) \\
&\qquad - \int_{(t_n \vee l/j - l/j) \wedge S_\Theta(X)}^{(t_{n+1} - l/j) \wedge S_\Theta(X)} Af\big(X(s)\big)\, ds\Bigg) \\
&\qquad \times \prod_{t_i \geq l/j} h_i\left(X\left(t_i - \frac{l}{j}\right)\right) \,\Bigg|\, X(0) = x\Bigg] \mu(dx) = 0,
\end{aligned}$$

which implies (3.4) by letting $j \to \infty$.

(ii) For all $\Gamma \in \mathcal{U}(\mathcal{E})$, suppose that Z_1, Z_2 are two solutions of the stopped martingale problem for (A, ν, Γ). Then there exist two solutions Y_1, Y_2 of the local martingale problem for (A, ν), such that $\mathcal{L}(Y_{i,\Gamma}) = \mathcal{L}(Z_{i,\Gamma})$, $i = 1, 2$. Because the local martingale problem for (A, ν) is well posed, we have $\mathcal{L}(Y_1) = \mathcal{L}(Y_2)$. Thus $\mathcal{L}(Z_{1,\Gamma}) = \mathcal{L}(Y_{1,\Gamma}) = \mathcal{L}(Y_{2,\Gamma}) = \mathcal{L}(Z_{2,\Gamma})$. □

4. Weak convergence of Markov processes with extended generators.

4.1. THEOREM. *Let A, $(A_n)_{n \geq 1}$ be extended generators and $\{\mu, \mu_n, n \geq 1\}$ be contained in $\mathcal{P}(\mathcal{E})$. Assume, $\forall n$, that the local martingale problem for (A_n, μ_n) has a solution X_n, and X is a solution of the local martingale problem for (A, μ). Suppose:*

(4.1) $$A \subset \mathbf{C}^b(\mathcal{E}) \times \mathbf{C}(\mathcal{E}).$$

(4.2) *For all compact sets* Γ, *local uniqueness holds for solutions of the stopped martingale problem for* (A, μ, Γ).

(4.3) *For all* $f \in \mathcal{D}(A)$, $\exists f_n \in \mathcal{D}(A_n)$, *such that* $\mathrm{LIM}_n^* f_n = f$, $\mathrm{LIM}_n A_n f_n = Af$.

(4.4) *The initial distributions satisfy* $\mu_n \Rightarrow \mu$.

If there exists a sequence $\{\Gamma_k\} \subset \mathcal{U}(\mathcal{E})$ *such that* $\Gamma_k \subset (\Gamma_{k+1})^o$, $\{X_{n,\Gamma_k}\}_{n\geq 1}$ *is tight*, $\forall k$, *and*

$$\lim_{k\uparrow\infty} \mathbf{P}(X_t \in \Gamma_k,\ \forall t \leq N) = 1 \qquad \forall N > 0,$$

then $X_n \Rightarrow X$.

PROOF. For $k \in N$, suppose that $X_{n,\Gamma_k} \Rightarrow Y$ and Y is defined on $(\Omega', \mathcal{F}', (\mathcal{F}'_t)_{t\geq 0}, \mathbf{P}')$ satisfying the usual conditions. Without loss of generality, we may assume $\Omega^n = \Omega'$, $\mathcal{F}^n = \mathcal{F}'$, $\mathcal{F}_t^n = \mathcal{F}'_t$, $\mathbf{P}^n = \mathbf{P}'$ and $X_{n,\Gamma_k} \to Y$ $\mathbf{P}'$-a.s. Because $\Gamma_{k-1} \subset (\Gamma_k)^o$, we have $d(\Gamma_{k-1}, \overline{(\Gamma_k)^c}) > 0$. Thus we can choose $t_k \in \mathbf{R}_+ \backslash V_{\Gamma_k}(Y)$, such that $B_k := B_{\Gamma_k, t_k} \supset \Gamma_{k-1}$ (see Lemma 2.4). Then applying Lemma 2.3 gives $X_{n,B_k} \to Y_{B_k}$ $\mathbf{P}'$-a.s. and $S_{B_k}(X_n) \to S_{B_k}(Y)$ $\mathbf{P}'$-a.s. For each $f \in \mathcal{D}(A)$, take $f_n \in \mathcal{D}(A_n)$ satisfying the condition (4.3); $\forall h_1, \dots, h_m \in \mathbf{C}^b(\mathcal{E})$, $0 \leq s_1 < s_2 < \cdots < s_m \leq t < t+s$ with $s_i, t, t+s \notin U(Y) := \{r > 0\colon \mathbf{P}'[\omega\colon Y(\omega, r) \neq Y(\omega, r^-)] > 0\}$. Applying Lemma 2.3(iii) yields

$$\begin{aligned}
\xi^n &:= \left[f_n\big(X_{n,B_k}(t+s)\big) - f_n\big(X_{n,B_k}(t)\big) - \int_{t\wedge S_{B_k}(X_n)}^{(t+s)\wedge S_{B_k}(X_n)} A_n f_n\big(X_n(u)\big)\, du \right] \\
&\quad \times \prod_{i=1}^m h_i\big(X_{n,B_k}(s_i)\big) \\
\to \quad \xi &:= \left[f\big(Y_{B_k}(t+s)\big) - f\big(Y_{B_k}(t)\big) - \int_{t\wedge S_{B_k}(Y)}^{(t+s)\wedge S_{B_k}(Y)} Af\big(Y(u)\big)\, du \right] \\
&\quad \times \prod_{i=1}^m h_i\big(Y_{B_k}(s_i)\big) \quad \text{a.s.}
\end{aligned}$$

It is clear that $\sup_{n\geq n_0} \sup_{x\in B_k} |A_n f_n(x)| < \infty$ for n_0 large enough, and $E^n \xi^n = 0$, $\forall n \geq n_0$, so $\{\xi^n\}_{n\geq n_0}$ is uniformly integrable and $\mathbf{E}'\xi = 0$. Note that $U(Y)$ is at most countable; hence Y is a solution of the stopped martingale problem for (A, μ, B_k). By (4.2), we get $\mathcal{L}(Y_{B_k}) = \mathcal{L}(X_{B_k})$, $\forall k \geq 1$, and it follows that $X_{n,B_k} \Rightarrow X_{B_k}$ and $S_{B_k}(X_n) \Rightarrow S_{B_k}(X)$.

On the other hand, note that $B_k \in \mathcal{U}(\mathcal{E})$ and $B_k \subset (B_{k+1})^o$, we have $d(B_k, \overline{(B_{k+1})^c}) > 0$, $\forall k$, which implies

$$\mathbf{P}\big(S_{B_{k+1}}(X) > N\big) \geq \mathbf{P}(X_t \in B_k,\ \forall t \leq N) \geq \mathbf{P}(X_t \in \Gamma_{k-1},\ \forall t \leq N).$$

Let $k \uparrow \infty$ and then $N \uparrow \infty$. We get $S_{B_k}(X) \Rightarrow \infty$ as $k \to \infty$. Therefore, this theorem follows from Lemma 2.6. □

4.2. REMARK. In order to get weak convergence of the processes from their localized properties, the condition $\lim_{k\uparrow\infty} \mathbf{P}(X_t \in \Gamma_k, \forall t \le N) = 1$, $\forall N > 0$, is reasonable. The telescopic condition $\Gamma_k \subset (\Gamma_{k+1})^o$, $\forall k$, although it may seem unnatural, is important here, because it implies $d(\Gamma_k, \overline{(\Gamma_{k+1})^c}) > 0$, $\forall k \ge 1$. These inequalities ensure the existence of $(B_k)_{k\ge 1}$ and $S_{B_k}(X) \Rightarrow \infty$ as $k \to \infty$. A counterexample without the telescopic condition can be constructed as follows. Let $\mathcal{E} = [0, \infty)$ and define

$$Ag(x) = g(0) - g(x),$$
$$A_n g(x) = \begin{cases} g(x+n) - g(x), & \text{for } x \le n-1, \\ 0, & \text{for } x > n-1, \end{cases}$$

$\forall g \in \mathcal{D}(A_n) = \mathcal{D}(A) := \mathbf{C}^b([0, \infty))$. For each $g \in \mathcal{D}(A)$, choose

$$g_n(x) = \begin{cases} g(x), & \text{for } x \le n-1, \\ g\big((n-1)(n-x)\big), & \text{for } n-1 < x < n, \\ g(0), & \text{for } x \ge n. \end{cases}$$

Then

$$A_n g_n(x) = \begin{cases} g(0) - g(x), & \text{for } x \le n-1, \\ 0, & \text{for } x > n-1, \end{cases}$$

so (4.3) holds. If we take $\mu_n = \mu = \delta_{\{0\}}$, then the unique solution of the local martingale problem for (A_n, μ_n) is $X_n(t) = n.1_{T\le t}$, and the unique solution of the local martingale problem for (A, μ) is $X(t) = 0$ $\forall t$, where T is an exponential random variable with parameter 1. Take $\Gamma_k = \{0\}$. Then $\overline{(\Gamma_k)^c} = \mathcal{E}$; hence, $S_{\Gamma_k}(X_n) = S_{\Gamma_k}(X) = 0$. It is easy to check that this example satisfies all conditions of Theorem 4.1 except the telescopic one, but X_n does not converge to X.

4.3. THEOREM. *Let A_n, μ_n and X_n be the same as in Theorem* 4.1. *Let H be a dense subset of* $\mathbf{C}^b(\mathcal{E})$ *with respect to* LIM*. *For $\Gamma \in \mathcal{U}(\mathcal{E})$, suppose $\forall f \in H$, $\exists f_n \in \mathcal{D}(A_n)$, such that* $\text{LIM}^*_n f_n = f$, $\limsup_n \sup_{x\in\Gamma} |A_n f_n(x)| < \infty$. *If $\{X_{n,\Gamma}\}_{n\ge1}$ satisfies the* CCC (2.4), *then it is tight.*

PROOF. For any fixed $f \in H$, take $f_n \in \mathcal{D}(A_n)$, such that $\text{LIM}^*_n f_n = f$ and

$$\limsup_n \sup_{x\in\Gamma} |A_n f_n(x)| < \infty.$$

Because $f^2 \in C^b(\mathcal{E})$, we can find $\{g^m\} \subset H$ satisfying $\text{LIM}^*_m g^m = f^2$. For each m, let $g^m_n \in \mathcal{D}(A_n)$, such that $\text{LIM}^*_n g^m_n = g^m$ and

$$\limsup_n \sup_{x\in\Gamma} \left|A_n g^m_n(x)\right| < \infty.$$

The condition (2.4) of $\{X_{n,\Gamma}\}$ is equivalent to

$$\lim_{k\uparrow\infty} \limsup_n \mathbf{P}^n\left[X_{n,\Gamma}(t) \notin \Gamma_k,\ \exists 0 \le t \le N\right] = 0 \qquad \forall N > 0. \tag{4.5}$$

Note that

$$\begin{aligned}&\limsup_n \mathbf{E}^n\left[\sup_{t\in[0,N]}\left|g^m\big(X_{n,\Gamma}(t)\big)-f^2\big(X_{n,\Gamma}(t)\big)\right|\right]\\&\quad\le \sup_{x\in\Gamma_k}\left|g^m(x)-f^2(x)\right|\\&\qquad+\big(\|f^2\|+\vee_m\|g^m\|\big)\limsup_n \mathbf{P}^n\big(X_{n,\Gamma}(t)\notin\Gamma_k,\ \exists 0\le t\le N\big).\end{aligned}$$

By (4.5), $\forall N>0$, $\varepsilon>0$, we can choose k_0, such that

$$\left(\|f^2\|+\bigvee_m\|g^m\|\right)\limsup_n \mathbf{P}^n\big(X_{n,\Gamma}(t)\notin\Gamma_{k_0},\ \exists 0\le t\le N\big)\le\frac{\varepsilon}{2}.$$

Then choose m_0, such that $\sup_{x\in\Gamma_{k_0}}|g^{m_0}(x)-f^2(x)|\le\varepsilon/2$. Hence,

$$\limsup_n \mathbf{E}^n\left[\sup_{t\in[0,N]}\left|g^{m_0}\big(X_{n,\Gamma}(t)\big)-f^2\big(X_{n,\Gamma}(t)\big)\right|\right]\le\varepsilon. \tag{4.6}$$

By Lemma 3.4, we have

$$f_n\big(X_{n,\Gamma}(t)\big)-\int_0^t A_nf_n\big(X_n(s)\big)1_{[s<S_\Gamma(X_n)]}\,ds$$

and

$$g_n^{m_0}\big(X_{n,\Gamma}(t)\big)-\int_0^t A_ng_n^{m_0}\big(X_n(s)\big)1_{[s<S_\Gamma(X_n)]}\,ds$$

are $(\mathcal{F}_t^{X_n})_{t\ge 0}$ martingales for sufficiently large n. With reference to Lemma 2.8, let

$$Y_n(t)=f_n\big(X_{n,\Gamma}(t)\big),\ \widetilde{Y}_n(t)=g_n^{m_0}\big(X_{n,\Gamma}(t)\big)$$

and

$$Z_n(s)=A_nf_n\big(X_n(s)\big)1_{[s<S_\Gamma(X_n)]},\ \widetilde{Z}_n(s)=A_ng_n^{m_0}\big(X_n(s)\big)1_{[s<S_\Gamma(X_n)]}.$$

So it remains to show that the conditions of Lemma 2.8 hold. By (4.6), we have

$$\begin{aligned}&\limsup_n \mathbf{E}^n\left[\sup_{t\in[0,N]}\left|\widetilde{Y}_n(t)-f^2\big(X_{n,\Gamma}(t)\big)\right|\right]\\&\quad\le\varepsilon+\limsup_n\sup_{x\in\Gamma_i}\left|g_n^{m_0}(x)-g^{m_0}(x)\right|\\&\qquad+\left(\|g^{m_0}\|+\bigvee_n\|g_n^{m_0}\|\right)\limsup_n \mathbf{P}^n\big(X_{n,\Gamma}(t)\notin\Gamma_i,\ \exists 0\le t\le N\big)\\&\quad=\varepsilon+\left(\|g^{m_0}\|+\bigvee_n\|g_n^{m_0}\|\right)\limsup_n \mathbf{P}^n\big(X_{n,\Gamma}(t)\notin\Gamma_i,\ \exists 0\le t\le N\big).\end{aligned}\tag{4.7}$$

Let $i \uparrow \infty$ in (4.7), we get (ii) of Lemma 2.8. On the other hand,

$$\limsup_n \mathbf{E}^n \left[\sqrt{\int_0^N \left[\widetilde{Z}_n(s)\right]^2 ds} \right] \le \sqrt{N} \limsup_n \sup_{x \in \Gamma} \left| A_n g_n^{m_0}(x) \right| < \infty,$$

which implies Lemma 2.8(iv). Similar arguments give Lemma 2.8(i) and (iii). Therefore, $\{f \circ X_{n,\Gamma}\}_{n \ge 1}$ is tight and we complete the proof by Lemma 2.7. □

4.4. THEOREM. *Let A, μ, X, A_n, μ_n and X_n be the same as in Theorem* 4.1. *Suppose that the local martingale problem for A is well posed. If* (4.1), (4.3) *and* (4.4) *hold*, $(X_{n,\Gamma})_{n \ge 1}$ *satisfies the* CCC (2.4), $\forall \Gamma \in \mathcal{U}(\mathcal{E})$ [*or equivalently* (X_n) *satisfies the* CCC (2.4)] *and*

$$\mathcal{D}(A) \text{ is dense in } \mathbf{C}^b(\mathcal{E}) \text{ with respect to LIM}^*, \tag{4.8}$$

then $X_n \Rightarrow X$.

PROOF. By the local compactness of $\mathcal{E}$, every compact set has a compact neighborhood, so we can choose $\{\Gamma_k\} \subset \mathcal{U}(\mathcal{E})$ such that $\Gamma_k \subset (\Gamma_{k+1})^o$, $\forall k$, and $\lim_{k \uparrow \infty} \mathbf{P}(X_t \in \Gamma_k,\ \forall t \le N) = 1$, $\forall N > 0$. Thus this theorem follows from Theorems 3.7, 4.3 and 4.1. □

4.5. REMARK. It can be seen that the CCC (2.4) of $(X_{n,\Gamma})_{n \ge 1}$ in Theorems 4.3 and 4.4 [or equivalently (4.5)] is equivalent to

$$\lim_{k \uparrow \infty} \limsup_n \mathbf{P}^n \left[X_n\big(S_\Gamma(X_n)\big) \notin \Gamma_k \text{ and } S_\Gamma(X_n) \le N \right] = 0 \qquad \forall N > 0, \tag{4.9}$$

for a sequence $\{\Gamma_k\} \subset \mathcal{U}(\mathcal{E})$.

4.6. REMARK. If $\mathcal{E} = \mathbf{R}^m$, then (4.9) is equivalent to that (μ_n) is tight and

$$\lim_{k \uparrow \infty} \limsup_n \mathbf{P}^n \left[\left| \Delta X_n\big(S_\Gamma(X_n)\big) \right| \ge k,\ S_\Gamma(X_n) \le N \right] = 0 \qquad \forall N > 0.$$

4.7. REMARK. If $\{X_n\}$ are continuous processes and $\{\mu_n\}$ is tight, then the CCC (2.4) of $\{X_{n,\Gamma}\}$ always holds, $\forall \Gamma \in \mathcal{U}(\mathcal{E})$. In fact, we can choose a sequence of compact sets $\Gamma \subset \Gamma_1 \subset \Gamma_2 \subset \cdots$ such that $\lim_{k \uparrow \infty} \limsup_n \mu_n((\Gamma_k)^c) = 0$. Note that $X_n(t) \in \Gamma^o$ if $0 \le t < S_\Gamma(X_n)$ and $X_n(S_\Gamma(X_n)) \in \partial\Gamma \subset \Gamma_k$ when $0 < S_\Gamma(X_n) < \infty$, the claim is true because

$$\mathbf{P}^n\left[X_{n,\Gamma}(t) \notin \Gamma_k,\ \exists t \ge 0\right] = \mathbf{P}^n\left[S_\Gamma(X_n) = 0,\ X_n(0) \notin \Gamma_k\right] \le \mu_n\big((\Gamma_k)^c\big).$$

4.8. REMARK. For $H \subset \mathbf{C}^b(\mathcal{E})$, either of the following conditions implies that H is dense in $\mathbf{C}^b(\mathcal{E})$ with respect to LIM*:

$$H \text{ is dense in } \widehat{\mathbf{C}}(\mathcal{E}) \text{ with respect to } \|\cdot\|. \tag{4.10}$$

$$H \text{ is dense in } \overline{\mathbf{C}}(\mathcal{E}) \text{ with respect to } \|\cdot\|. \tag{4.11}$$

As a matter of fact, it suffices to show that $\widehat{\mathbf{C}}(\mathcal{E})$ and $\overline{\mathbf{C}}(\mathcal{E})$ are dense in $\mathbf{C}^b(\mathcal{E})$ with respect to LIM*. Clearly, if $\mathcal{E}$ is compact, then $\widehat{\mathbf{C}}(\mathcal{E}) = \overline{\mathbf{C}}(\mathcal{E}) = \mathbf{C}^b(\mathcal{E})$. Thus, the claim follows from the σ-compactness of $\mathcal{E}$ and the definition of LIM*.

4.9. REMARK. It is also worthwhile to note that the CCC (2.4) of (X_n) in Theorem 4.4. is very important. The counterexample in Remark 4.2 also shows that except (2.4), all of the other conditions of Theorem 4.4 hold, but X_n does not converge weakly to X.

We now prove the result in Example 1.1 to illustrate how to apply the theory in this paper.

4.10. PROOF OF EXAMPLE 1.1. (a) $\Rightarrow$ (b). The conditions (4.1)–(4.4) hold clearly. Take $\Gamma_k = \{0, 1, \ldots, k\}$. Then $\Gamma_k \subset (\Gamma_{k+1})^o$, $\forall k$ and

$$\begin{aligned}
&\lim_{m\uparrow\infty} \limsup_n \mathbf{P}^n\Big[X_n\big(S_{\Gamma_k}(X_n)\big) \notin \Gamma_m\Big] \\
&\quad \le \lim_{m\uparrow\infty} \limsup_n \left[\tau_n\big((\Gamma_m)^c\big) + \sup_{i\le k} \frac{\sum_{j=m+1}^\infty q_{ij}^n}{-q_{ii}^n}\right] = 0,
\end{aligned}$$

since $\sum_{j=m+1}^\infty q_{ij}^n = -\sum_{j=0}^m q_{ij}^n \to -\sum_{j=0}^m q_{ij} = \sum_{j=m+1}^\infty q_{ij}$. Hence (b) follows from Remark 3.3(i), Theorem 4.3 and Theorem 4.1.

(c) $\Rightarrow$ (a). Take $G_i = \{i\}$. Then we have $T := S_{G_i}(X) = S_{G_i}+(X)$. Thus, by Lemma 2.3(i), $T_n := S_{G_i}(X_n)$ converges weakly to T, which yields $q_{ii}^n \to q_{ii}$ as $n \to \infty$ because T_n and T are exponential random variables with parameters $-q_{ii}^n$ and $-q_{ii}$, respectively. On the other hand, it is clear that $X(T^-) = i$ and $(G_i)^o = G_i$, so $X_n(T_n)$ converges in distribution to $X(T)$ by Lemma 2.3(ii), which gives $\forall j \neq i$,

$$\begin{aligned}
q_{ij}^n &= -q_{ii}^n \mathbf{P}^n\big[X_n(T_n) = j\big] \\
&\to -q_{ii}\mathbf{P}\big[X(T) = j\big] = q_{ij}.
\end{aligned}$$

□

5. Weak convergence of Markov processes with generators. In this section, we consider weak convergence of Markov processes with generators defined in Dynkin (1965) or Ethier and Kurtz (1986). For each $n = 1, 2, \ldots$, let $\{T_n(t)\}$ and $\{T(t)\}$ be strongly continuous contraction semigroups on $\widehat{\mathbf{C}}(\mathcal{E})$ with generators A_n and A, and resolvents $\{G_n(\lambda)\}_{\lambda>0}$ and $\{G(\lambda)\}_{\lambda>0}$. In order to investigate necessary and sufficient conditions, we need the following technical lemmas.

5.1. LEMMA. *For $(g_n)_{n\ge 1}, g \in \mathbf{B}(\mathcal{E})$, the following are equivalent*:

(i) $\forall x_n \to x$, $g_n(x_n) \to g(x)$.

(ii) *g is continuous and* $\mathrm{LIM}_n g_n = g$.

5.2. LEMMA. *For the following four statements, we have* (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iv) $\Rightarrow$ (iii).

(i) $\forall f \in \widehat{\mathbf{C}}(\mathcal{E})$, $t \geq 0$, $\mathrm{LIM}_n\, T_n(t)f = \mathrm{LIM}_n^*\, T_n(t)f = T(t)f$.
(ii) $\forall f \in \widehat{\mathbf{C}}(\mathcal{E})$, $\lambda > 0$, $\mathrm{LIM}_n\, G_n(\lambda)f = \mathrm{LIM}_n^*\, G_n(\lambda)f = G(\lambda)f$.
(iii) $\forall f \in D(\mathcal{A})$, $\exists f_n \in \mathcal{D}(A_n)$, *such that* $\mathrm{LIM}_n^*\, f_n = f$, $\mathrm{LIM}_n A_n\, f_n = Af$.
(iv) $\forall f \in D(\mathcal{A})$, $\exists f_n \in \mathcal{D}(A_n)$, *such that* $\mathrm{LIM}_n^*\, f_n = f$, $\mathrm{LIM}_n^* A_n\, f_n = Af$.

PROOF [cf. Ethier and Kurtz (1986), pages 28 and 29]. Note that $\|T_n(t)f\| \leq \|f\|$. We have

$$\mathrm{LIM}_n\, T_n(t)f = \mathrm{LIM}_n^*\, T_n(t)f \quad \text{and} \quad \mathrm{LIM}_n\, G_n(\lambda)f = \mathrm{LIM}_n^*\, G_n(\lambda)f.$$

It is well known that $\forall g \in \widehat{\mathbf{C}}(\mathcal{E})$,

$$G_n(\lambda)g = \int_0^\infty \exp(-\lambda t)T_n(t)g\, dt,$$
$$G(\lambda)g = \int_0^\infty \exp(-\lambda t)T(t)g\, dt$$

[see Dynkin (1965) or Ethier and Kurtz (1986)], so we have (i) $\Rightarrow$ (ii). On the other hand, the Laplace transformation property gives (ii) $\Rightarrow$ (i). Thus, it remains to show (ii) $\Rightarrow$ (iv). For all $f \in \mathcal{D}(A)$, take $g = (\lambda_0 - A)f \in \widehat{\mathbf{C}}(\mathcal{E})$ for a $\lambda_0 > 0$; then

$$f_n = G_n(\lambda_0)g \in \mathcal{D}(A_n).$$

Because

$$A_n f_n = -(\lambda_0 - A_n)f_n + \lambda_0 f_n = -g + \lambda_0 f_n,$$

we have $\mathrm{LIM}_n^* f_n = \mathrm{LIM}_n^*\, G_n(\lambda_0)g = G(\lambda_0)g = f$, $\mathrm{LIM}_n^* A_n f_n = -g + \lambda_0 f = Af$. □

Let X denote the solution of the local martingale problem for (A, μ), and let X_n denote the solution of the local martingale problem for (A_n, μ_n).

5.3. THEOREM. *The following are equivalent*:

(i) *Condition* (i) *of Lemma* 5.2 *holds and* $\forall \mu_n \Rightarrow \mu$, $(X_{n,\Gamma})$ *satisfies the* CCC (2.4), $\forall \Gamma \in \mathcal{U}(\mathcal{E})$.
(ii) *Condition* (ii) *of Lemma* 5.2 *holds and* $\forall \mu_n \Rightarrow \mu$, $(X_{n,\Gamma})$ *satisfies the* CCC (2.4), $\forall \Gamma \in \mathcal{U}(\mathcal{E})$.
(iii) *Condition* (iii) *of Lemma* 5.2 *holds and* $\forall \mu_n \Rightarrow \mu$, $(X_{n,\Gamma})$ *satisfies the* CCC (2.4), $\forall \Gamma \in \mathcal{U}(\mathcal{E})$.
(iv) *Condition* (iv) *of Lemma* 5.2 *holds and* $\forall \mu_n \Rightarrow \mu$, $(X_{n,\Gamma})$ *satisfies the* CCC (2.4), $\forall \Gamma \in \mathcal{U}(\mathcal{E})$.
(v) $\forall \mu_n \Rightarrow \mu$, $X_n \Rightarrow X$.
(vi) $\forall x_n \to x$, *take* $\mu_n = \delta_{\{x_n\}}$, $\mu = \delta_{\{x\}}$, $X_n \Rightarrow X$.

PROOF. Note that $\mathcal{D}(A)$ is dense in $\widehat{\mathbf{C}}(\mathcal{E})$ with respect to $\|\cdot\|$ in this case [see Dynkin (1965), page 30, or Ethier and Kurtz (1986), pages 9–14], we have that $\mathcal{D}(A)$ satisifes (4.8) by Remark 4.8. Note also that the local martingale problem is the same as the martingale problem for this case, and it is known that the martingale problem for A is well posed. Thus, (iii) $\Rightarrow$ (v) is a corollary of Theorem 4.4. If (vi) holds, by Lemma 5.1, we have for every bounded continuous function h on $(D_{[0,\infty)}, \rho)$ that

$$\mathbf{E}h(X_n) = \int \mathbf{E}h\big(X_n^{\delta_{\{x\}}}\big)\mu_n(dx) \to \int \mathbf{E}h\big(X^{\delta_{\{x\}}}\big)\mu(dx) = \mathbf{E}h(X).$$

Here we use $X_n^{\delta_{\{x\}}}$ and $X^{\delta_{\{x\}}}$ to identify the solutions of the local martingale problems for $(A_n, \delta_{\{x\}})$ and $(A, \delta_{\{x\}})$, respectively. So (vi) implies (v). Now, we prove (vi) $\Rightarrow$ (i). For all $x_n \to x$, note that $X^{\delta_{\{x\}}}$ is stochastically continuous [see Ethier and Kurtz (1986), page 181], and we have

$$T_n(t)f(x_n) = \mathbf{E}f\big(X_n^{\delta_{\{x_n\}}}(t)\big) \to \mathbf{E}f\big(X^{\delta_{\{x\}}}(t)\big) = T(t)f(x).$$

Thus (i) follows from Lemma 5.1, (v) and Lemma 2.5. □

By Remark 4.7, we have the following corollary.

5.4. COROLLARY. *Suppose that the solution of the (local) martingale problem for (A_n, μ_n) is continuous, $\forall n$, $\forall \mu_n \in \mathcal{P}(\mathcal{E})$. Then the conditions* (i)–(iv) *of Lemma* 5.2 *and* (v) *and* (vi) *of Theorem* 5.3 *are equivalent.*

5.5. REMARK. All of the results in this section are valid if we replace $\widehat{\mathbf{C}}(\mathcal{E})$ with $\overline{\mathbf{C}}(\mathcal{E})$.

6. Convergence of diffusion processes with jumps. As an application of this article, we give some sufficient conditions for weak convergence of diffusion processes with jumps. The conditions are slightly weaker than those in Jacod and Shiryaev (1987).

In this section, we take $\mathcal{E} = \mathbf{R}^m$. Let $\widehat{\mathbf{C}}^2$ be the set of bounded uniformly continuous functions defined on $\mathbf{R}^m$ that are of class C^2. If $f \in \widehat{\mathbf{C}}^2$, set

$$A_n f(x) = \sum_{i \le m} b_n^i(x)\frac{\partial f(x)}{\partial x^i} + \frac{1}{2}\sum_{i,j \le m} c_n^{ij}(x)\frac{\partial^2 f(x)}{\partial x^i \partial x^j} + \int_{\mathbf{R}^m} (f(x+y) - f(x))N_n(x, dy), \qquad n = 1, 2, \dots,$$

and

$$Af(x) = \sum_{i \le m} b^i(x)\frac{\partial f(x)}{\partial x^i} + \frac{1}{2}\sum_{i,j \le m} c^{ij}(x)\frac{\partial^2 f(x)}{\partial x^i \partial x^j} + \int_{\mathbf{R}^m} (f(x+y) - f(x))N(x, dy),$$

where b_n, b Borel functions: $\mathbf{R}^m \mapsto \mathbf{R}^m$ and c_n, c Borel functions: $\mathbf{R}^m \mapsto \mathbf{R}^m \otimes \mathbf{R}^m$ with values in the set of symmetric nonnegative matrices, and $N_n(x, dy)$, $N(x, dy)$ are Borel transition kernels from $\mathbf{R}^m$ into itself with $\int(|y|^2 \wedge 1) N_n(x, dy) < \infty$ and $\int(|y|^2 \wedge 1) N(x, dy) < \infty$.

Let $h(y)$: $\mathbf{R}^m \mapsto \mathbf{R}^m$ be a continuous truncation function, that is, $\exists 0 < a < \infty$, such that $|h(y)| \leq |y| \wedge a$ and

$$h(x) = \begin{cases} x, & \text{if } |x| \leq a/2, \\ 0, & \text{if } |x| \geq a. \end{cases}$$

Set

$$\widetilde{b}_n^i(x) = b_n^i(x) + \int h^i(y) N_n(x, dy), \qquad \widetilde{c}_n^{ij}(x) = c_n^{ij}(x) + \int h^i(y) h^j(y) N_n(x, dy),$$

$$\widetilde{b}^i(x) = b^i(x) + \int h^i(y) N_n(x, dy), \qquad \widetilde{c}^{ij}(x) = c^{ij}(x) + \int h^i(y) h^j(y) N(x, dy),$$

$$N_n(x, f) = \int f(y) N_n(x, dy), \quad N(x, f) = \int f(y) N(x, dy).$$

By Lemma 5.1 and Remark 4.13 of Jacod and Shiryaev [(1987), page 516], we have the following proposition.

6.1. PROPOSITION. *The following are equivalent*:

(i) *For all* $f \in \widehat{\mathbf{C}}^2$, $\forall x_n \to x$, $A_n f(x_n) \to A f(x)$.

(ii) $\forall x_n \to x$: (a) $\widetilde{b}_n(x_n) \to \widetilde{b}(x)$; (b) $\widetilde{c}_n(x_n) \to \widetilde{c}(x)$; (c) $\forall f \in \overline{\mathbf{C}}(\mathbf{R}^m)$ *zero on a neighborhood of* 0, $N_n(x_n, f) \to N(x, f)$.

Let $A_n \equiv A$, $\forall n \geq 1$ in Proposition 6.1. We then have:

6.2. COROLLARY. *For all* $f \in \widehat{\mathbf{C}}^2$, Af *is continuous if and only if* $\widetilde{b}(x)$, $\widetilde{c}(x)$ *and* $N(x, g)$ *are continuous for all* $g \in \overline{\mathbf{C}}(\mathbf{R}^m)$ *zero on a neighbourhood of* 0.

Applying Remark 4.6 and Theorem 4.4 gives the following theorem. Note that conditions (6.3) and (6.4) of the theorem are slightly weaker than conditions 4.9 and 4.11 in Jacod and Shiryaev [(1987), page 515 and 516].

6.3. THEOREM. *Let* $\{\mu, \mu_n, n \geq 1\} \subset \mathcal{P}(\mathbf{R}^m)$. *Assume*, $\forall n$, *that the local martingale problem for* (A_n, μ_n) *has a solution* X_n, *and* X *is a solution of the local martingale problem for* (A, μ). *If*

(6.1) *for all* $f \in \widehat{\mathbf{C}}^2$, Af *is continuous*,

(6.2) *the local martingale problem for A is well-posed*

[*see Jacod and Shiryaev* (1987), *pages* 145 *and* 146],

(6.3) *for each* $f \in \widehat{\mathbf{C}}^2$, *there exists* $f_n \in \widehat{\mathbf{C}}^2$, *such that*

$$\mathrm{LIM}^*_n f_n = f,\ \mathrm{LIM}_n A_n f_n = Af,$$

(6.4) $$\lim_{p\uparrow\infty} \limsup_{n} \sup_{|x|\le r} N_n\big(x, 1_{|y|\ge p}\big) = 0 \qquad \forall r > 0,$$

(6.5) $$\mu_n \Rightarrow \mu,$$

then $X_n \Rightarrow X$.

6.4. COROLLARY. *Suppose conditions* (6.1), (6.2) *and* (6.5) *hold, and*

(6.6) $$\lim_{p\uparrow\infty} \sup_{|x|\le r} N\big(x, 1_{|y|\ge p}\big) = 0 \qquad \forall r > 0.$$

If

(6.7) $$\textit{for all } f \in \widehat{\mathbf{C}}^2, \qquad \mathrm{LIM}_n A_n f = Af,$$

or equivalently,

(6.8) $$\mathrm{LIM}_n \widetilde{b}_n = \widetilde{b},\quad \mathrm{LIM}_n \widetilde{c}_n = \widetilde{c}, \mathrm{LIM}_n N_n(\cdot, f) = N(\cdot, f) \quad \forall f \in \overline{\mathbf{C}}(\mathbf{R}^m) \textit{ zero on a neighbourhood of } 0,$$

then $X_n \Rightarrow X$.

PROOF. It is obvious that $\widetilde{b}, \widetilde{c}$, and $N(\cdot, f)$ are continuous by (6.1) and Corollary 6.2. Hence, applying Lemma 5.1 and Proposition 6.1 gives that (6.7) is equivalent to (6.8). On the other hand, (6.4) follows from (6.8) and (6.6), so the proof is complete by Theorem 6.3. □

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REFERENCES

BILLINGSLEY, P. (1968). *Convergence of Probability Measures.* Wiley, New York.

DYNKIN, E. B. (1965). *Markov Processes* **1**. Springer, New York.

ETHIER, S. N. and KURTZ, T. G. (1986). *Markov Processes, Characterization and Convergence.* Wiley, New York.

JACOD, J. (1979). *Calcul Stochastique et Problèmes de Martingales. Lecture Notes in Math.* **714**. Springer, Berlin.

JACOD, J. and SHIRYAEV, A. N. (1987). *Limit Theorems for Stochastic Processes*. Springer, Berlin.

PARTHASARATHY, K. R. (1967). *Probability Measures on Metric Spaces.* Academic Press, New York.

REVUZ, D. and YOR, M. (1991). *Continuous Martingales and Brownian Motion.* Springer, Berlin.

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