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ASYMPTOTIC EXPANSIONS FOR THE DISTRIBUTIONS OF STOPPED RANDOM WALKS AND FIRST PASSAGE TIMES[1]

By Tze Leung Lai and Julia Qizhi Wang

Stanford University

Let $S_n = X_1 + \cdots + X_n, n \geq 1$, be a d-dimensional random walk and let $T_a = \inf\{n \geq n_a: ng(S_n/n) \geq a\}$, where $n_a = o(a)$. Let $\theta = g(EX_1), \hat{\theta}_n = g(S_n/n)$ and $\Delta_a = T_a\hat{\theta}_{T_a} - a$. Edgeworth-type expansions are developed for $P\{T_a = n,\ y_1 \leq \Delta_a \leq y_2\}$ and for the distribution functions of T_a and of $\sqrt{T_a}(h(\hat{\theta}_{T_a}) - h(\theta))$, where h is a real-valued function such that $h'(\theta) \neq 0$.

1. Introduction. Let $X, X_1, X_2, \ldots$ be i.i.d. d-dimensional random vectors such that

$$\limsup_{\|t\| \to \infty} \left|E\exp\bigl(i\langle t, X\rangle\bigr)\right| < 1 \quad \text{and} \quad E\|X\|^4 < \infty. \tag{1.1}$$

Let $\mu = EX$ and $S_n = X_1 + \cdots + X_n$. Let $g: \mathbf{R}^d \to \mathbf{R}$ be a smooth function such that $g(\mu) > 0$. First passage times of the form

$$T_a = \inf\Bigl\{n \geq n_a: ng\bigl(S_n/n\bigr) \geq a\Bigr\}, \tag{1.2}$$

in which $n_a = o(a)$ is nonrandom (representing a required minimal sample size), play an important role in sequential statistical methodology. Motivated by these applications, Woodroofe and Keener (1987) developed the following asymptotic expansion of the distribution function of T_a in the case $d = 1 = n_a$ and under the assumption that g is twice continuously differentiable in some neighborhood of μ with $g'(\mu) > 0$. Let $v = \operatorname{Var}(X), \sigma = g'(\mu)v^{1/2}$ and $\theta = g(\mu)$. For positive integers n such that $n = a/\theta + O(\sqrt{a})$, letting $t_{n,a} = (a - \theta n)/(\sigma\sqrt{n})$, they showed that

$$P\{T_a \leq n\} = 1 - \Phi(t_{n,a}) + n^{-1/2}Q(t_{n,a})\phi(t_{n,a}) + o\bigl(n^{-1/2}\bigr), \tag{1.3}$$

where Φ and ϕ denote the standard normal distribution and density functions and

$$\begin{aligned} Q_1(z) &= -\frac{g''(\mu)vz^2}{2\sigma} + \frac{E(X-\mu)^3}{6v^{3/2}}\bigl(1 - z^2\bigr), \\ Q(z) &= \sigma^{-1}\int_{-\infty}^{0} P\{M < x\}\,dx - Q_1(z), \\ M &= \inf_{n \geq 1} \sum_{j=1}^{n} \{g'(\mu)(X_j - \mu) + g(\mu)\}. \end{aligned} \tag{1.4}$$

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Assuming furthermore that $P\{T_a \leq a/2\theta\} = o(a^{-1/2})$, they also obtained the following asymptotic expansion for the distribution function of the normalized sum $Z_n = (nv)^{-1/2}(S_n - n\mu)$ at $n = T_a$:

$$P\{Z_{T_a} < z\} = \Phi(z) + N_{a,z}^{-1/2} q(a,z)\phi(z) + o(a^{-1/2}), \tag{1.5}$$

where

$$\begin{aligned} \gamma(a,z) &= \inf\left\{x > a/2\theta\colon xg(\mu + zv^{1/2}x^{-1/2}) \geq a\right\}, \\ N_{a,z} &= [\gamma(a,z)]\ ([\cdot] \text{ being the greatest integer function}), \\ \delta_{a,z} &= \gamma(a,z) - N_{a,z}, \\ q(a,z) &= \sigma^{-1}\{\theta\delta_{a,z} + g''(\mu)vz^2/2\} - Q(z) \\ &\quad - \sigma^{-1}\sum_{r=1}^{\infty}\int_{(r-\delta_{a,z})\theta}^{\infty} P\{M \geq x\}\,dx. \end{aligned} \tag{1.6}$$

For $d = 1$, the special case $g(x) = x$ in (1.2) gives the first time that the random walk S_n crosses the level a, whereas the special case $g(x) = x^2/2$ in (1.2) gives the first time that S_n crosses the square-root boundary $(2an)^{1/2}$. Takahashi (1987) analyzed the latter case directly for normally distributed X_j, without reexpressing the square-root boundary crossing time of S_n as a first passage time (1.2) of $\frac{1}{2}n(S_n/n)^2$, and was able to improve (1.5) to a higher-order asymptotic expansion, with $o(a^{-1})$ remainder, when $\delta_{a,z} = 0$. He also gave some numerical results in this case showing that the higher-order approximation is more accurate than the Woodroofe–Keener approximation (1.5), and that both these approximations are substantial improvements over the simple normal approximation $\Phi(z)$ to the left-hand side of (1.5). However, his arguments are applicable only under the very restrictive assumption of Gaussian X_j and linear or quadratic g. Without this assumption, we shall develop a new approach to show that (1.3) and (1.5) can indeed be refined to higher-order expansions with $o(a^{-1})$ remainders and also generalize the results to the case of d-dimensional X_j. These higher-order expansions for the general d-dimensional case are stated and discussed in Section 2, and their proofs are given in Section 4. Section 3 gives some preliminary lemmas that are related to nonlinear renewal theory, fluctuation theory of random walks and multivariate Edgeworth expansions.

2. Asymptotic expansions of the distributions of T_a and $h \circ g(S_{T_a}/T_a)$. In this section we shall consider the general d-dimensional case. Define

$$\mu = EX, \qquad V = \operatorname{Cov}(X), \qquad \theta = g(\mu), \qquad \widehat{\theta}_n = g(S_n/n). \tag{2.1}$$

The function $g\colon \mathbf{R}^d \to \mathbf{R}$ will be assumed to be such that

$$g(\mu) > 0,\ \nabla g(\mu) \neq 0 \text{ and } g \in C^4(U) \tag{2.2}$$

(i.e., g is four times continuously differentiable in U) for some neighborhood U of μ. Denote the partial derivatives of g evaluated at μ by $g_i = \partial g/\partial x_i|_{x=\mu}$, $g_{ii} = \partial^2 g/\partial x_i^2|_{x=\mu}$ and, in general, $g_{i_1,\dots,i_r} = \partial^r g/\partial x_{i_1}\cdots\partial x_{i_r}|_{x=\mu}$, and denote the components of X and μ by $X^{(j)}$ and $\mu^{(j)}$, $j = 1,\dots,d$. Let $v_{i_1,\dots,i_r} = E\{\prod_{j=1}^r (X^{(i_j)} - \mu^{(i_j)})\}$,

$$\sigma = \left(\sum_{1\le i,j\le d} g_i g_j v_{ij}\right)^{1/2} = \|\nabla g(\mu) V^{1/2}\|, \qquad \alpha_1 = \sum_{1\le i,j\le d} g_{ij} v_{ij}/2,$$

$$\alpha_2 = \sum_{1\le i,j,k\le d} g_i g_j g_k v_{ijk} + 3 \sum_{1\le i,j,k,m\le d} g_i g_j g_{km} v_{ik} v_{jm},$$

$$\alpha_3 = \sum_{1\le i,j,k\le d} g_i g_{jk} v_{ijk} + \sum_{1\le i,j,k,m\le d} \left(g_i g_{jkm} v_{im} v_{jk} + g_{ij} g_{km} v_{ik} v_{jm}/2\right),$$

$$\begin{aligned}\alpha_4 = &\sum_{1\le i,j,k,l\le d} g_i g_j g_k g_l (v_{ijkl} - 3 v_{ij} v_{kl}) \\ &+ 12 \sum_{1\le i,j,k,l,m\le d} g_{ij} g_k g_l g_m v_{ik} v_{jlm} \\ &- \sum_{1\le i,j,k,l\le d} \left\{3 g_{ij} g_{kl} \left(v_{ij} v_{kl}/2 + v_{ik} v_{jl}\right) + 6 g_{ijk} g_l v_{ij} v_{kl}\right\} \\ &\times \left(\sum_{1\le m,r\le d} g_m g_r v_{mr}\right) \\ &+ \sum_{1\le i,j,k,l,m,r\le d} \\ &\times \left\{\left(\tfrac{3}{2} g_{ij} g_{kl} + \tfrac{2}{3} g_{ijk} g_l\right) g_m g_r v_{(i,j,k,l,m,r)} - 6 g_{ij} g_{kl} g_m g_r v_{im} v_{jr} v_{kl}\right\},\end{aligned} \tag{2.3}$$

where $v_{(i,j,k,l,m,r)} = v_{ij} v_{kl} v_{mr} + v_{ik} v_{jl} v_{mr} + \cdots$ is a sum of 15 terms. The assumption (1.1) implies that V is nonsingular and, therefore, $\sigma > 0$ if (2.2) also holds. Define

$$\begin{aligned} Q_1(z) &= -\sigma^{-1}\alpha_1 + \left(6\sigma^3\right)^{-1}\alpha_2\left(1 - z^2\right), \\ Q_2(z) &= -\frac{\alpha_3 + \alpha_1^2}{2\sigma^2} z - \frac{\alpha_4 + 4\alpha_1\alpha_2}{24\sigma^4}\left(z^3 - 3z\right) \\ &\quad - \frac{\alpha_2^2}{72\sigma^6}\left(z^5 - 10z^3 + 15z\right). \end{aligned} \tag{2.4}$$

In the case $d = 1$ and $g'(\mu) > 0$, $Q_1(z)$ agrees with that in (1.4), in which $\sigma = g'(\mu)\sqrt{v}$. Moreover, it follows from Theorem 2 of Bhattacharya and Ghosh (1978) that

$$P\{n^{1/2}(\hat{\theta}_n - \theta)/\sigma < z\} = \Phi(z) + n^{-1/2} Q_1(z)\phi(z) + n^{-1} Q_2(z)\phi(z) + o\left(n^{-1}\right). \tag{2.5}$$

The polynomials Q_1 and Q_2 also play an important role in the asymptotic expansion of the distribution of T_a given below in Theorem 1, which is a generalization and refinement of (1.3). Another important ingredient in the expansion is the random walk $S_k^{(Y)}$ defined by

$$Y_i = \langle X_i - \mu, \nabla g(\mu)\rangle + \theta, \qquad S_k^{(Y)} = \sum_{i=1}^{k} Y_i, \tag{2.6}$$

which is used to approximate $T_a g(S_{T_a}/T_a) - (T_a - k)g(S_{T_a-k}/(T_a - k))$. In the case $d = 1$, this reduces to the random walk whose minimum is the quantity M in the Woodroofe–Keener expansion. Our higher-order expansion also involves

$$M = \inf_{n\geq 1} S_n^{(Y)}, \qquad \tau_-(\mu) = \inf\{n \geq 1\colon\ S_n^{(Y)} < u\} \qquad (\inf \emptyset = \infty). \tag{2.7}$$

Furthermore, it involves besides $S_n^{(Y)}$ the following "second-order" random walk approximation. Let $G = (g_{ij})_{1\leq i,\,j\leq d}$ be the Hessian matrix and let $\eta_1, \dots, \eta_{d-1}$ be independent standard normal random variables that are independent of $X_1, X_2, \dots$. Let A be an orthogonal $d \times d$ matrix whose first row is $\sigma^{-1}\nabla g(\mu)V^{1/2}$ and let A^T denote its transpose. For $a > 0$ and $z \in \mathbf{R}$ define

$$\begin{aligned} Y_i(a,z) &= Y_i + (\theta/a)^{1/2}\langle (X_i - \mu)GV^{1/2}A^T, (z, \eta_1, \dots, \eta_{d-1})\rangle, \\ S_k(a,z) &= \sum_{i=1}^{k} Y_i(a,z), \qquad M_a(z) = \inf_{k\geq 1} S_k(a,z). \end{aligned} \tag{2.8}$$

In the case $d = 1$, $Y_i(a,z) = Y_i + z(v\theta/a)^{1/2}(X_i - \mu)g''(\mu)\operatorname{sgn} g'(\mu)$, where $v = \operatorname{Var}(X)$. Note that $Y_i(a,z) = Y_i$ if $G = 0$. In fact, $Y_i(a,z)$ arises from a second-order Taylor expansion of $g(x + \mu) - g(x)$ and from an Edgeworth expansion of a certain nonlinear transformation of S_n/n that yields the normal random variables $\eta_1, \dots, \eta_{d-1}$ from the last $d - 1$ coordinates of this nonlinear transformation (from $\mathbf{R}^d$ into $\mathbf{R}^d$), the first coordinate of which involves g, as will be shown in Section 4 (see, in particular, the proof of Lemma 7). Under conditions (1.1) and (2.2), it will be shown in Lemma 4 of Section 3 that the quantities m_0, m_1 and λ associated with (2.7) in our asymptotic expansion are finite, where

$$\begin{aligned} m_j &= \int_{-\infty}^{0} u^j P\{M < u\}\,du \qquad (j = 0, 1), \\ \lambda &= \int_{-\infty}^{0} E\{S_{\tau_-(u)}^{(Y)} - \theta\tau_-(u)\} I_{\{\tau_-(u)<\infty\}}\,du. \end{aligned} \tag{2.9}$$

With σ and $Q_1(z)$, $Q_2(z)$ defined by (2.3) and (2.4) and $M_a(z)$ defined by (2.8), let

$$\begin{aligned} p_{1,a}(z) &= -Q_1(z) + \sigma^{-1}\int_{-\infty}^{0} P\{M_a(z) < u\}\,du, \\ p_2(z) &= -Q_2(z) + \sigma^{-1}m_0\{Q_1'(z) - zQ_1(z)\} + \sigma^{-2}(\lambda - m_1)z. \end{aligned} \tag{2.10}$$

The functions $p_{1,a}$ and p_2 are central to the following asymptotic expansion for T_a.

THEOREM 1. *Under the assumptions* (1.1) *on* X *and* (2.2) *on* g, *suppose that the* n_a *appearing in the definition* (1.2) *of* T_a *satisfies*

$$\lim_{a\to\infty} a^{-1}n_a = 0, \qquad \liminf_{a\to\infty} a^{-1/3}n_a > 0. \tag{2.11}$$

Then as $a\,(\in \mathbf{R}) \to \infty$ *and* $n(\in \mathbf{Z} = \text{set of integers}) \to \infty$ *such that* $n = a/\theta + O(\sqrt{a})$,

$$\begin{aligned} P\{T_a \le n\} &= 1 - \Phi(t_{n,a}) + n^{-1/2}p_{1,a}(t_{n,a})\phi(t_{n,a}) \\ &\quad + n^{-1}p_2(t_{n,a})\phi(t_{n,a}) + o(n^{-1}), \end{aligned} \tag{2.12}$$

where $t_{n,a} = (a - n\theta)/(\sigma\sqrt{n})$.

Let $\Delta_a = T_a g(S_{T_a}/T_a) - a$ denote the overshoot over the boundary at the stopping time (1.2). Lalley (1984) has shown that for $n = a/\theta + O(\sqrt{a})$,

$$P\{T = n, \Delta_a \le y\} \sim \sigma^{-1}n^{-1/2}\phi(t_{n,a}) \int_0^y P\{M \ge u\}\,du \quad \text{as } a \to \infty, \tag{2.13}$$

for every $y \ge 0$. A refinement of this result in the form of a higher-order asymptotic expansion is given by Theorem 2. When $d = 1$, a similar result under stronger assumptions has been established by Keener (1987), who used different methods and considered instead of (1.2) stopping times of the form $\tau_a = \inf\{n\colon S_n > af(n/a)\}$.

THEOREM 2. *With the same notation and assumptions as in Theorem* 1, *let*

$$\begin{aligned} \Lambda(u) = E(Y_1 - \theta)I_{\{M \ge u\}} - \int_u^\infty & E\{S^{(Y)}_{\tau_-(u-y)} - \theta\tau_-(u-y)\} \\ & \times I_{\{\tau_-(u-y) < \infty\}}P(Y \in dy). \end{aligned} \tag{2.14}$$

Then as a $(\in \mathbf{R}) \to \infty$ *and* $n(\in \mathbf{Z}) \to \infty$ *such that* $n = a/\theta + O(\sqrt{a})$,

$$\begin{aligned} P\{T_a = n, y_1 \le \Delta_a \le y_2\} &= \frac{\phi(t_{n,a})}{\sigma\sqrt{n}} \int_{y_1}^{y_2} P\{M_a(t_{n,a}) \ge u\}\,du \\ &\quad + \frac{\phi(t_{n,a})}{\sigma n} \int_{y_1}^{y_2} \bigg\{ \Big[Q_1'(t_{n,a}) - t_{n,a}Q_1(t_{n,a}) - \frac{u}{\sigma}t_{n,a} \Big] \\ &\qquad \times P(M \ge u) + \frac{t_{n,a}}{\sigma}\Lambda(u) \bigg\}\,du + o(n^{-1}), \end{aligned} \tag{2.15}$$

uniformly in $y_2 > y_1 \ge 0$.

We next provide a higher-order refinement of (1.5) and generalize it from the case $d = 1$ to $d \ge 1$. First note that the normalized sum $Z_n = (nv)^{-1/2}(S_n - n\mu)$

in (1.5) can be expressed in terms of the basic entity $\widehat{\theta}_n = g(S_n/n)$ in the stopping time (1.2) via

$$Z_n = (n/v)^{1/2}\big(h(\widehat{\theta}_n) - h(\theta)\big) \quad \text{if } |n^{-1}S_n - \mu| < \varepsilon, \tag{2.16}$$

where $\varepsilon > 0$ is sufficiently small such that g is strictly increasing on $I = [\theta - \varepsilon, \theta + \varepsilon]$ and $h(x) = g^{-1}(x)$ for $x \in I$, recalling that $g'(\mu)$ is assumed to be positive by Woodroofe and Keener (1987). More generally, we shall consider h: $\mathbf{R} \to \mathbf{R}$ such that h is four times continuously differentiable in some neighborhood of θ and $h'(\theta) \neq 0$. Let

$$Z_n = n^{1/2}\big(h(\widehat{\theta}_n) - h(\theta)\big)/\widetilde{\sigma} = n^{1/2}\Big\{\widetilde{g}\big(S_n/n\big) - \widetilde{g}(\mu)\Big\}\Big/\widetilde{\sigma}, \tag{2.17}$$

where $\widetilde{g} = h \circ g$ and $\widetilde{\sigma} = (\sum_{i=1}^d \sum_{j=1}^d \widetilde{g}_i \widetilde{g}_j v_{ij})^{1/2} = \sigma|h'(\theta)|$. Define $\widetilde{\alpha}_1, \widetilde{\alpha}_2, \widetilde{\alpha}_3, \widetilde{\alpha}_4$ as in (2.3), but with $\widetilde{g}_i, \widetilde{g}_{ij}, \widetilde{g}_{ijk}$ replacing g_i, g_{ij}, g_{ijk} and define $\widetilde{Q}_1(z)$, $\widetilde{Q}_2(z)$ as in (2.4), but with $\widetilde{\sigma}$ replacing σ and $\widetilde{\alpha}_i$ replacing α_i. Here $\widetilde{g}_{i_1,\dots,i_r}$ denotes a partial derivative of $\widetilde{g}$ evaluated at μ, as before. By Theorem 2 of Bhattacharya and Ghosh (1978),

$$P\{Z_n < z\} = \Phi(z) + n^{-1/2}\widetilde{Q}_1(z)\phi(z) + n^{-1}\widetilde{Q}_2(z)\phi(z) + o\big(n^{-1}\big). \tag{2.18}$$

We shall give an asymptotic expansion of the distribution of Z_{T_a} and compare it with (2.18). To begin with, note that h is strictly monotone in $[\theta-\varepsilon, \theta+\varepsilon]$ for some $\varepsilon > 0$ because $h'(\theta) \neq 0$. Let ψ be a monotone function such that $\psi(u) = h^{-1}(u)$ for $u \in [\theta - \varepsilon, \theta + \varepsilon]$. Define

$$\gamma(a,z) = \inf\Big\{u \geq n_a \colon u\psi\big(h(\theta) + z\widetilde{\sigma}u^{-1/2}\big) \geq a\Big\}, \qquad N_{a,z} = \big[\gamma(a,z)\big]. \tag{2.19}$$

Let $\delta_{a,z} = \gamma(a,z) - N_{a,z}$,

$$\begin{aligned} l_0(a,z) &= \sigma^{-1}\sum_{j=1}^{\infty}\int_{(j-\delta_{a,z})(\theta + N_{a,z}^{-1/2}z\sigma/2)}^{\infty} P\{M_a(z) \geq u\}\,du, \\ l_1(a,z) &= \frac{1}{\sigma}\sum_{j=1}^{\infty}\int_{\theta(j-\delta_{a,z})}^{\infty}\bigg\{\bigg(Q_1'(z) - zQ_1(z) - \frac{zu}{\sigma}\bigg) \\ &\qquad \times P(M \geq u) + \frac{z}{\sigma}\Lambda(u)\bigg\}\,du, \end{aligned} \tag{2.20}$$

where $Q_1(z)$ and $Q_2(z)$ are defined in (2.4) and $\Lambda(u)$ is defined in (2.14). Let $\rho(a,z) = \sigma^{-1}\{\theta\delta_{a,z} + \psi''(h(\theta))\widetilde{\sigma}^2 z^2/2\}$,

$$q_1(a,z) = -p_{1,a}(z) - l_0(a,z) + \rho(a,z), \tag{2.21}$$

$$\begin{aligned} q_2(a,z) &= -p_2(z) - l_1(a,z) \\ &\quad + \sigma^{-1}z\sum_{j=1}^{\infty}\{\rho(a,z) - \sigma^{-1}\theta j\}\int_{\theta(j-\delta_{a,z})}^{\infty} P\{M \geq u\}\,du \\ &\quad + \sigma^{-1}\psi'''\big(h(\theta)\big)\widetilde{\sigma}^3 z^3/6 - z\rho^2(a,z)/2 \\ &\quad + \{\sigma^{-1}zm_0 - zQ_1(z) + Q_1'(z)\}\rho(a,z), \end{aligned} \tag{2.22}$$

where $p_{1,a}(z)$ and $p_2(z)$ are given in (2.10). The functions q_1 and q_2 and the nonrandom approximation $N_{a,z}$ in (2.19) to the stopping time T_a are central to the following asymptotic expansion of the distribution of Z_{T_a}.

THEOREM 3. *With the same notation and assumptions as in Theorem* 1, *in the case* $h'(\theta) > 0$, *we have uniformly in z belonging to compact subsets of* $\mathbf{R}$,

$$(2.23)\quad P\{Z_{T_a} < z\} = \Phi(z) + N_{a,z}^{-1/2} q_1(a,z)\phi(z) + N_{a,z}^{-1} q_2(a,z)\phi(z) + o(a^{-1})$$

as $a \to \infty$. *In the case* $h'(\theta) < 0$, *replacing h by* $-h$, *the asymptotic expansion* (2.23) *still holds for* $P\{-Z_{T_a} < z\}$.

It is interesting to compare (2.23) with the Edgeworth expansion (2.18) for the normalized statistic Z_n based on a nonrandom number n of sample observations. The polynomials $\widetilde{Q}_1(z)$ and $\widetilde{Q}_2(z)$ depend on the moments of X and the partial derivatives of $\widetilde{g} = g \circ h$, and (2.18) is an asymptotic expansion in powers of $n^{-1/2}$. Because the random variable Z_{T_a} in (2.23) involves not only the sequence of normalized statistics $\{Z_n\}$, but also the first passage time T_a for the sequence $\{ng(S_n/n)\}$, the asymptotic expansion (2.23) resolves this complexity by using the nonrandom quantity $N_{a,z}$ as a first approximation to T_a and is an asymptotic expansion in powers $N_{a,z}^{-1/2}$. The functions $q_1(a,z)$ and $q_2(a,z)$ in (2.23) are analogous to the polynomials $\widetilde{Q}_1(z)$ and $\widetilde{Q}_2(z)$ in (2.18), but are considerably more complicated. Not only do they have to make Edgeworth-type corrections in the normal approximations of Z_n and of $ng(S_n/n)$, but they also have to account for the oscillations of T_a around $N_{a,z}$. This leads to the fluctuation-theoretic quantities m_0, m_1 and λ defined in (2.9) and $l_0(a,z)$ and $l_1(a,z)$ defined in (2.20), associated with the random walks $\{S_k^{(Y)}\}$ and $\{S_k(a, z)\}$ defined in (2.6) and (2.8). Because T_a is integer-valued, approximating its distribution by a (continuous) normal distribution also entails certain adjustments that lead to the sawtooth function $\delta_{a,z}$. Except for these continuity adjustments and fluctuation-theoretic quantities, the asymptotic expansion (2.23) is markedly similar to the Edgeworth expansion (2.18) with n replaced by $N_{a,z}$.

For nonrandom sample sizes n, the Edgeworth expansion (2.18) has recently been used to show that $P\{Z_N < z\}$ can be alternatively approximated by Efron's (1979) bootstrap method with an error $O_p(n^{-1})$. This plays an important role in the theory of bootstrap confidence intervals for $h(\theta)$ [cf. Hall (1988)]. The bootstrap is a resampling method based on the empirical distribution $\widehat{F}_n = n^{-1}\sum_{i=1}^n \delta_{X_i}$. Let $X_1^*, \dots, X_n^*$ be i.i.d with common distribution $\widehat{F}_n$ and let $S_n^* = \sum_{i=1}^n X_i^*$ and $Z_n^* = \sqrt{n}\{\widetilde{g}(S_n^*/n) - \widetilde{g}(\widehat{\mu}_n)\}/\|\nabla\widetilde{g}(\widehat{\mu}_n)\widehat{V}_n^{1/2}\|$, where $\widehat{\mu}_n$ and $\widehat{V}_n$ are the mean and covariance matrix of the empirical distribution $\widehat{F}_n$. Under (1.1) and (2.2), it has been shown that $P\{Z_n^* < z \mid \widehat{F}_n\}$ also satisfies an Edgeworth expansion of the form

$$(2.24)\qquad P\{Z_n^* < z \mid \widehat{F}_n\} = \Phi(z) + n^{-1/2}\widetilde{Q}_{1,n}(z)\phi(z) + O_p(n^{-1}),$$

where $\sup_z |\widetilde{Q}_{1,n}(z) - \widetilde{Q}_1(z)|\phi(z) = O_p(n^{-1/2})$ [cf. Hall (1988)]. Combining (2.18) with (2.24) yields $P\{Z_n < z\} = P\{Z_n^* < z \,|\, \widehat{F}_n\} + O_p(n^{-1})$. Therefore, a second-order approximation, with an $O_p(n^{-1})$ error term, to the probability $P\{Z_n < z\}$ is provided by $P\{Z_n^* < z \,|\, \widehat{F}_n\}$, which can be evaluated to arbitrary accuracy by using simulation, without assuming knowledge of μ, V and the third-order moments v_{ijk}.

In the sequential setting where the fixed sample size n is replaced by the random sample size T_a, it is straightforward to extend the bootstrap method to provide an approximation to $P\{Z_{T_a} < z\}$. The quantity $P\{Z_n^* < z \,|\, \widehat{F}_n\}$ is now replaced by $P\{Z_{T_a^*}^* < z \,|\, \widehat{F}_{T_a}\}$, where $X_1^*, X_2^*, \dots$ are i.i.d. with common distribution $\widehat{F}_{T_a}$ and $T_a^* = \inf\{n \geq n_a \colon ng(S_n^*/n) \geq a\}$. Making use of certain properties of empirical characteristic functions, Lai (1994) has modified the proof of Theorem 3 to develop an analogous Edgeworth-type expansion for $P\{Z_{T_a^*}^* < z \,|\, \widehat{F}_{T_a}\}$, which can be evaluated directly by simulation without assuming knowledge of the underlying distribution F of X. An important application of the Edgeworth-type expansions of $P\{Z_{T_a} < z + a^{-1/2}U\}$ and of $P\{Z_{T_a^*}^* < z + a^{-1/2}U \,|\, \widehat{F}_{T_a}\}$, with $O(a^{-1})$ and $O_p(a^{-1})$ remainders, where U is a bounded random variable with a continuous density function and independent of $\{X_1^*, X_2^*, \dots, X_1, X_2, \dots\}$, is the development of a theory of bootstrap confidence intervals based on sequential samples, analogous to that for samples of fixes size [cf. Lai (1994)]. In this connection, note that the Woodroofe–Keener asymptotic formula (1.5) has a remainder $o(a^{-1/2})$ instead of our desired $O(a^{-1})$.

3. Lemmas on Edgeworth expansions and fluctuation theory of random walks.

LEMMA 1. *Let $X_1, X_2, \dots$ be i.i.d d-dimensional random vectors satisfying* (1.1) *and let ξ be a standard normal d-dimensional random vector independent of $\{X_i\}$. Let $\mu = EX$, $V = \mathrm{Cov}(X)$ and A be an orthogonal $d \times d$ matrix. Then $m^{-1/2}(S_m - m\mu + m^{-2}\xi)V^{-1/2}A^T$ has a density function f_m satisfying the Edgeworth expansion*

$$(3.1)\qquad f_m(w) = (2\pi)^{-d/2} e^{-\|w\|^2/2}\{1 + m^{-1/2}P_1(w) + m^{-1}P_2(w)\} + o(m^{-1})$$

uniformly in $w \in \mathbf{R}^d$, where $P_j(w)$ is a polynomial in w of degree j whose coefficients involve A, V and the cumulants of X up to order $j+2$ ($j = 1, 2$).

PROOF. Let $U_m = m^{-1/2}(S_m - m\mu + m^{-2}\xi), i = \sqrt{-1}$, and note that

$$(3.2)\qquad E\exp(i\langle t, U_m\rangle) = \left\{E\exp\left(i\langle t/\sqrt{m}, X - \mu\rangle\right)\right\}^m \times \exp\{-m^{-5}\|t\|^2/2\}, \qquad t \in \mathbf{R}^d.$$

Hence U_m has an integrable characteristic function and, therefore, also a density function ψ_m. Take any $\delta > 0$. By (1.1) there exists $0 < q < 1$ such that $|Ee^{i\langle t,X\rangle}| \le q$ for $\|t\| \ge \delta$. Because

$$\begin{aligned}\int_{\|t\|\ge m^3} \exp\Big(-\big\|m^{-5/2}t\big\|^2/2\Big)\,dt \\ = m^{5d/2}\int_{\|x\|\ge\sqrt{m}} e^{-\|x\|^2/2}\,dx = o\big(m^{5d/2}e^{-m}\big),\end{aligned} \tag{3.3}$$

$$\int_{m^3\ge\|t\|\ge\delta\sqrt{m}} \Big|E\exp\Big(i\big\langle t/\sqrt{m},X\big\rangle\Big)\Big|^m\,dt = O\big(m^{3d}q^m\big), \tag{3.4}$$

we can apply standard Taylor expansions and Fourier inversion arguments to show from (3.2) that ψ_m has the Edgeworth expansion

$$\sup_{x\in\mathbf{R}^d}\left|\psi_m(x) - \left\{1+\sum_{j=1}^{2} m^{-j/2}P_j^*(x)\right\}\phi^*(x)\right| = o\big(m^{-1}\big), \tag{3.5}$$

where ϕ^* denotes the multivariate normal density with mean 0 and covariance matrix V, and P_j^* is a polynomial of degree j whose coefficients involve the cumulants of X up to order $j+2$ [cf. Section 19 of Bhattacharya and Rao (1976)]. Noting that

$$\operatorname{Cov}\big(U_mV^{-1/2}A^T\big) = AV^{-1/2}\big(V+m^{-5}I\big)V^{-1/2}A^T = I + O\big(m^{-5}\big),$$

it follows from (3.5) that the density function f_m of $U_mV^{-1/2}A^T$ has the expansion (3.1). □

A basic idea in the proof of Theorems 1–3 is to approximate $m^{-1/2}(S_m - m\mu)$ by $m^{-1/2}(S_m - m\mu + m^{-2}\xi)$, whose density function has the Edgeworth expansion (3.1), which will be used to evaluate certain probabilities. Another basic idea is to approximate $\{tg(S_t/t) - mg((S_m+m^{-2}\xi)/m),\ m < t \le m + Cm^{1/3}\}$ by a quadratic function of certain random walks given in Lemma 3(ii), whose proof uses the following lemma.

LEMMA 2. *Let $Z, Z_1, \ldots$ be i.i.d. random variables such that $EZ = 0$. For every $\alpha > \frac{1}{2}$, there exists an absolute constant A_α (depending only on α) such that*

$$\begin{aligned}P\left\{\max_{t\le k}\left|\sum_{i=1}^{t} Z_i\right| \ge \varepsilon k^\alpha\right\} + P\left\{\sup_{t\ge k} t^{-\alpha}\left|\sum_{i=1}^{t} Z_i\right| \ge \varepsilon\right\} \\ \le A_\alpha k^{-(4\alpha-1)}\left\{\varepsilon^{-4}EZ^4 I_{\{|Z|\ge(2-\alpha^{-1})\varepsilon k^\alpha/14\}} + k^{-\alpha}\big(\varepsilon^{-2}EZ^2\big)^{(5\alpha-1)/(2\alpha-1)}\right\}.\end{aligned}$$

PROOF. Let ν be the smallest integer greater than or equal to $(5\alpha-1)/(2\alpha-1)$. As shown in Chow and Lai [(1975), page 55],

$$P\left\{\max_{t\le k}\sum_{i=1}^{t} Z_i \ge \varepsilon k^\alpha\right\} \le kP\{Z\ge \varepsilon k^\alpha/(2\nu)\} + P^\nu\left\{\max_{t\le k}\sum_{i=1}^{t} Z_i \ge \varepsilon k^\alpha/(2\nu)\right\}$$
$$\le k^{1-4\alpha}(2\nu/\varepsilon)^4 EZ^4 I_{\{Z\ge \varepsilon k^\alpha/2\nu\}}$$
$$+\left\{C_\alpha(2\nu/\varepsilon)^2 k^{-(2\alpha-1)}EZ^2\right\}^{(5\alpha-1)/(2\alpha-1)}$$

for some absolute constant C_α. Note that $\nu < 7\alpha/(2\alpha-1) = 7/(2-\alpha^{-1})$. Moreover,

$$P\left\{\sup_{t\ge k} t^{-\alpha}\left|\sum_{i=1}^{t} Z_i\right| \ge \varepsilon\right\} \le \sum_{j=0}^{\infty} P\left\{\max_{2^j k\le t\le 2^{j+1}k}\left|\sum_{i=1}^{t} Z_i\right| \ge 2^{-\alpha}\varepsilon\left(2^{j+1}k\right)^\alpha\right\}.$$

Hence the desired conclusion follows. □

LEMMA 3. *Let $X_1, X_2, \dots$ be i.i.d. d-dimensional random vectors such that $E\|X_1\|^4 < \infty$ and $EX_1 = \mu$ and let ξ be a d-dimensional random vector independent of $\{X_i\}$ such that $E\|\xi\| < \infty$. Let $S_t = \sum_{i=1}^t X_i$. Suppose that g: $\mathbf{R}^d \to \mathbf{R}$ satisfies* (2.2). *Let $g_{ij} = \partial^2 g/\partial u_i\,\partial u_j|_{u=\mu}$ and $G = (g_{ij})_{1\le i,j\le d}$. Let $C>0$ and $\varepsilon>0$.*

(i) $P\{\max_{n/2\le t\le n-cn^{1/3}} |ng(S_n/n) - tg(S_t/t) - (n-t)g(\mu)|/(n-t) \ge \varepsilon\} = o(n^{-1})$.

(ii) *Let $b_m = m + O(\sqrt{m})$. Then for every $\delta > 0$,*

$$P\left\{\max_{m<t\le m+Cm^{1/3}}\left| tg\left(\frac{S_t}{t}\right) - mg\left(\frac{S_m+m^{-2}\xi}{m}\right) - (t-m)g(\mu)\right.\right.$$
$$-\sum_{i=m+1}^{t}\left\langle X_i-\mu, \nabla g(\mu) + \frac{1}{\sqrt{b_m}}\left(\frac{S_m - m\mu + m^{-2}\xi}{\sqrt{m}}\right)\frac{G}{\sqrt{b_m}}\right\rangle$$
$$\left.\left. -\frac{1}{2m}\left(\sum_{i=m+1}^{t}(X_i-\mu)\right)G\left(\sum_{i=m+1}^{t}(X_i-\mu)\right)^T\right| \ge m^{-2/3+\delta}\right\} = o\left(m^{-1}\right).$$

PROOF. We shall only prove (ii), because the proof of (i) is similar and simpler. First note that $P\{\|\xi\|>m\} = o(m^{-1})$ because $E\|\xi\|<\infty$. Take (arbitrarily small) $\delta > 0$ and define $\Omega = \{\max_{m\le t\le m+Cm}\|S_t - t\mu\| \le m^{1/2+\delta/3}, \|\xi\|\le m\}$. By Lemma 2 (with $\alpha = 1/2+\delta/3$), $P(\Omega) = 1-o(m^{-1})$. Let $\widetilde{S}_t = S_t - t\mu$. Defining $r(x)$ as the remainder in the Taylor expansion $g(x+\mu) = g(\mu) + \langle x, \nabla g(\mu)\rangle + \frac{1}{2}xGx^T + r(x)$, note that $r(x) = O(\|x\|^3)$ and $\nabla r(x) = O(\|x\|^2)$ as $x \to 0$. Writing $tr(x) - mr(y) = (t-m)r(x) + \langle m\nabla r(\hat{x}), x-y\rangle$ with $\hat{x}$ lying between x and y, it follows that on Ω, for $m \le t \le m + Cm^{1/3}$,

$$
\begin{aligned}
&tg\big(\mu + t^{-1}\widetilde{S}_t\big) - mg\big(\mu + m^{-1}\widetilde{S}_m\big)\\
&\quad = (t-m)g(\mu) + \big\langle \widetilde{S}_t - \widetilde{S}_m, \nabla g(\mu)\big\rangle\\
&\qquad + \big(t^{-1}\widetilde{S}_t G \widetilde{S}_t^T - m^{-1}\widetilde{S}_m G\widetilde{S}_m^T\big)/2 + O\big(m^{1/3-3/2+\delta}\big)\\
&\qquad + O\Big(m^{2\delta/3}\big\| t^{-1}\widetilde{S}_t - m^{-1}\widetilde{S}_m\big\|\Big),\\
&\big\| t^{-1}\widetilde{S}_t - m^{-1}\widetilde{S}_m\big\|\\
&\quad = O\Big(m^{-1}\big\|\widetilde{S}_t - \widetilde{S}_m\big\|\Big) + O\big(m^{-7/6+\delta/3}\big),\\
&t^{-1}\widetilde{S}_t G\widetilde{S}_t^T - m^{-1}\widetilde{S}_m G\widetilde{S}_m^T\\
&\quad = 2m^{-1}\widetilde{S}_m G(\widetilde{S}_t - \widetilde{S}_m)^T + \big(t^{-1} - m^{-1}\big)\widetilde{S}_t G\widetilde{S}_t^T\\
&\qquad + m^{-1}(\widetilde{S}_t - \widetilde{S}_m)G(\widetilde{S}_t - \widetilde{S}_m)^T\\
&\quad = 2b_m^{-1/2}m^{-1/2}\big(\widetilde{S}_m + m^{-2}\xi\big)G(\widetilde{S}_t - \widetilde{S}_m)^T\\
&\qquad + m^{-1}(\widetilde{S}_t - \widetilde{S}_m)G(\widetilde{S}_t - \widetilde{S}_m)^T + O\Big(m^{-1+\delta/3}\big\|\widetilde{S}_t - \widetilde{S}_m\big\|\Big)\\
&\qquad + o(m^{-2/3+\delta}),\\
&m\Big\{g\big(\mu + m^{-1}\widetilde{S}_m\big) - g\big(\mu + m^{-1}\widetilde{S}_m + m^{-3}\xi\big)\Big\}\\
&\quad = O\big(m^{-2}\|\xi\|\big) = O\big(m^{-1}\big).
\end{aligned}
$$

By Lemma 2, $P\{\max_{m\le t\le m+Cm^{1/3}} \|\widetilde{S}_t - \widetilde{S}_m\| \ge m^{1/3}\} = o(m^{-1})$. Hence (ii) follows. □

Note the resemblance between Lemma 3 and similar ideas in nonlinear renewal theory [cf. Woodroofe (1982)]. Throughout the rest of this section we shall let $Y, Y_1, Y_2, \dots$ be i.i.d. random variables such that $EY = \theta > 0$ and let $S_n^{(Y)} = \sum_{i=1}^n Y_i$. Define M and $\tau_-(u)$ by (2.7). From the fluctuation theory of random walks, there exists for $k = 1, 2, \dots$ an absolute constant C_k (depending only on k) such that

$$
\begin{aligned}
\left|\int_{-\infty}^0 u^{k-1}P\{M < u\}\,du\right| &= k^{-1}E(M^-)^k\\
&\le C_k\Big\{\theta^{-1}E\big((Y-\theta)^-\big)^{k+1} + \big(\theta^{-1}\operatorname{Var} Y\big)^k\Big\}
\end{aligned}
\tag{3.6}
$$

[cf. Chow and Lai (1975), page 63]. This gives a bound for the m_j defined in

(2.9). The following lemma provides a bound for the other fluctuation-theoretic quantity λ in (2.9).

LEMMA 4. (i) $\sup_{u \le 0} | S^{(Y)}_{\tau_-(u)} | I_{\{\tau_-(u) < \infty\}} = M^-$.

(ii) $(M^-)^2/2 \le \left| \int_{-\infty}^0 S^{(Y)}_{\tau_-(u)} I_{\{\tau_-(u) < \infty\}}\, du \right| \le (M^-)^2$.

(iii) *Letting* $\tau = \tau_-(0)$ *and* $Z = S^{(Y)}_\tau I_{\{\tau < \infty\}}$, *we have*

$$E\left(\int_{-\infty}^0 \tau_-(u) I_{\{\tau_-(u) < \infty\}}\, du \right) = \frac{E\big(|Z| \tau I_{\{\tau < \infty\}}\big)}{P\{\tau = \infty\}} + \frac{(E|Z|) E\big(\tau I_{\{\tau < \infty\}}\big)}{P^2\{\tau = \infty\}},$$

$$E\left(\int_{-\infty}^0 S^{(Y)}_{\tau_-(u)} I_{\{\tau_-(u) < \infty\}}\, du \right) = -\frac{EZ^2}{P\{\tau = \infty\}} - \left\{ \frac{EZ}{P\{\tau = \infty\}} \right\}^2.$$

(iv) *For every* $p > 0$, *there exists an absolute constant* A_p *such that*

$$\sup_{u \le 0} E\tau_-^p(u) I_{\{\tau_-(u) < \infty\}} \le A_p \left\{ E\big((Y - \theta)^- / \theta\big)^{p+1} + \big(\theta^{-2} \operatorname{Var} Y\big)^p \right\}.$$

PROOF. To prove (iv), first note that

$$\begin{aligned} E\tau_-^p(u) I_{\{\tau_-(u) < \infty\}} &= p \int_0^\infty t^{p-1} P\{\infty > \tau_-(u) \ge t\}\, dt \\ &\le p \int_0^\infty t^{p-1} P\left\{ \sup_{k \ge t} k^{-1} \sum_{i=1}^k (\theta - Y_i) \ge \theta \right\} dt \quad \text{for } u \le 0. \end{aligned}$$

Hence (iv) follows from Theorem 1 and Lemma 2 of Chow and Lai (1975).

Let $(\sum_1^k \tau_i, S^{(Y)}_{\tau_1 + \cdots + \tau_k})$, $1 \le k \le K := \max\{i: \tau_i < \infty\}$ ($K = 0$ if $\tau_1 = \infty$), be the descending ladder points of the random walk $\{S^{(Y)}_n\}$, that is, $\tau_1 = \tau_-(0)$, $\tau_1 + \tau_2 = \inf\{n > \tau_1: S^{(Y)}_n < S^{(Y)}_{\tau_1}\}$ and so forth [cf. Feller (1971), page 390]. Letting $S^{(Y)}_0 = 0$, note that

$$(3.7)\qquad \begin{aligned} &\tau_-(u) = \infty \quad \text{if } u < \sum_{j=1}^K Z_j, \text{ where } Z_j = S^{(Y)}_{\tau_1 + \cdots + \tau_j} - S^{(Y)}_{\tau_1 + \cdots + \tau_{j-1}}, \\ &\big(\tau_-(u), S^{(Y)}_{\tau_-(u)}\big) = \left(\sum_{i=1}^j \tau_i, \sum_{i=1}^j Z_i \right) \quad \text{if } \sum_{i=1}^j Z_i \le u < \sum_{i=1}^{j-1} Z_i \text{ with } j \le K. \end{aligned}$$

We use the convention $\sum_{i=1}^k = 0$ if $k = 0$. Hence

$$\int_{-\infty}^0 S^{(Y)}_{\tau_-(u)} I_{\{\tau_-(u) < \infty\}}\, du = -\sum_{j=1}^K \left(\sum_{i=1}^j Z_i \right) Z_j = -\tfrac{1}{2} \left\{ \sum_{j=1}^K Z_j^2 + \left(\sum_{j=1}^K Z_j \right)^2 \right\}.$$

Because $M^- = -\sum_{j=1}^K Z_j$, (ii) follows. Moreover, (i) also follows from (3.7), which also yields

$$\begin{aligned}\int_{-\infty}^0 \tau_-(u) I_{\{\tau_-(u)<\infty\}}\,du &= \sum_{j=1}^K \left(\sum_{i=1}^j \tau_i\right)|Z_j| \\ &= \sum_{j=1}^\infty |Z_j|\tau_j I_{\{\tau_1<\infty,\dots,\tau_j<\infty\}} \\ &\quad+ \sum_{j=2}^\infty\sum_{i=1}^{j-1} |Z_j|\tau_i I_{\{\tau_1<\infty,\dots,\tau_j<\infty\}}.\end{aligned} \tag{3.8}$$

In the preceding second equality, we extend the definition of τ_j as $\tau_j = \infty$ if $\tau_{j-1} = \infty$, letting $\tau_j = \inf\{n > \tau_{j-1}: S_n^{(Y)} < S_{\tau_{j-1}}^{(Y)}\}$ $(\inf \emptyset = \infty)$ if $\tau_{j-1} < \infty$ as before. Moreover, define Z_j as before if $\tau_j < \infty$ and set $Z_j = 0$ if $\tau_j = \infty$. From (3.8) it follows that

$$\begin{aligned}&E\left(\int_{-\infty}^0 \tau_-(u) I_{\{\tau_-(u)<\infty\}}\,du\right)\\ &= \left(E|Z|\,\tau I_{\{\tau<\infty\}}\right)\sum_{j=1}^\infty P^{j-1}\{\tau<\infty\}\\ &\quad+ \left(E|Z|\,I_{\{\tau<\infty\}}\right)\left(E\tau I_{\{\tau<\infty\}}\right)\sum_{j=2}^\infty (j-1)P^{j-2}\{\tau<\infty\}.\end{aligned}$$

Likewise $-E(\int_{-\infty}^0 S_{\tau_-(u)}^{(Y)} I_{\{\tau_-(u)<\infty\}}\,du)$ can be written as

$$\begin{aligned}&E\left\{\sum_{j=1}^\infty Z_j\left(\sum_{i=1}^j Z_i\right) I_{\{\tau_1<\infty,\dots,\tau_j<\infty\}}\right\}\\ &= E\sum_{j=1}^\infty Z_j^2 I_{\{\tau_1<\infty,\dots,\tau_j<\infty\}} + E\sum_{j=2}^\infty\sum_{i=1}^{j-1} Z_j Z_i I_{\{\tau_1<\infty,\dots,\tau_j<\infty\}}\\ &= \left(EZ^2 I_{\{\tau<\infty\}}\right)\sum_{j=1}^\infty P^{j-1}\{\tau<\infty\}\\ &\quad+ \left(EZI_{\{\tau<\infty\}}\right)^2\sum_{j=2}^\infty (j-1)P^{j-2}\{\tau<\infty\},\end{aligned}$$

implying (iii). □

In Section 4 the integrals that appear in (2.9), (2.14) and (2.20) will be obtained as limits of certain Riemann sums. Let ε_n be positive constants such that

$\lim_{n\to\infty}\varepsilon_n = 0$ and let $I = I(n)$ and $J = J(n,c)$ be positive integers such that $I(n) \to \infty$ and $J(n,c) \to \infty$ as $n \to \infty$, uniformly in $c \geq 0$. Let

$$\begin{gathered}\delta_{-1}(n) > 0 = \delta_0(n) > \cdots > \delta_I(n),\\ c_{-1}(n,c) < c = c_0(n,c) < \cdots < c_J(n,c)\end{gathered} \tag{3.9}$$

be such that $\delta_I(n) \to -\infty$ and $c_J(n,c) \to \infty$, uniformly in $c \geq 0$, and for all $j \geq -1$

$$\begin{aligned}&\delta_j(n) - \delta_{j+1}(n) \leq \varepsilon_n, \qquad c_{j+1}(n,c) - c_j(n,c) \leq (c+1)\varepsilon_n,\\ &\left|1 - \big(\delta_j(n) - \delta_{j+1}(n)\big)/\big(\delta_{j+1}(n) - \delta_{j+2}(n)\big)\right|\\ &\quad + \left|1 - \big(c_{j+1}(n,c) - c_j(n,c)\big)/\big(c_{j+2}(n,c) - c_{j+1}(n,c)\big)\right| \leq \varepsilon_n.\end{aligned} \tag{3.10}$$

The proof of Theorems 1–3 uses (3.9) as partitions to form the Riemann sums that approximate the integrals $\int_{-\infty}^0$ and $\int_c^\infty$ in (2.9), (2.14) and (2.20). Moreover, we shall replace the $S_k^{(Y)}$ and M that appear in the integrals by their perturbed versions $S_k(a,z)$ and $M_a(z)$ in the approximating Riemann sums. The following lemma, which considers somewhat more general perturbed sums than $S_k(a,z)$, uses Lemma 4 to establish the desired convergence of such Riemann-type sums.

LEMMA 5. *Assume that $EY^4 < \infty$ and let X_n be random vectors such that $(Y_1,X_1), (Y_2,X_2),\dots$ are i.i.d. Let U be a random vector independent of $\{X_1,Y_1, X_2, Y_2,\dots\}$ and let $\mathcal{F}$ be a class of real-valued functions ψ such that $\sup_{\psi\in\mathcal{F}} E(\psi(X_1,U))^4 < \infty$ and $E[\psi(X_1,U)|U] = 0$ a.s. for every $\psi \in \mathcal{F}$. Let $S_{k,\varepsilon}(\psi) = \sum_{i=1}^k (Y_i + \epsilon\psi(X_i,U))$. Then for every $h \geq -1$, as $\varepsilon \to 0$, $n \to \infty$ and $k \to \infty$,*

$$\begin{aligned}&\sum_{j=h}^{I-1}\{\delta_j(n) - \delta_{j+1}(n)\}E(S_{k,\varepsilon}(\psi) - k\theta)I_{\{\min_{i\leq k} S_{i,\varepsilon}(\psi) < \delta_j(n)\}}\\ &\qquad \to \int_{-\infty}^0 E\big(S^{(Y)}_{\tau_-(u)} - \theta\tau_-(u)\big)I_{\{\tau_-(u)<\infty\}}\,du,\end{aligned} \tag{3.11}$$

$$\begin{aligned}&\sum_{j=h}^{J-1}\{c_{j+1}(n,c) - c_j(n,c)\}E\big(S_{k,\varepsilon}(\psi) - k\theta\big)I_{\{\min_{i\leq k} S_{i,\varepsilon}(\psi) \geq c_j(n,c)\}}\\ &\qquad = \int_c^\infty E(Y_1 - \theta)I_{\{M\geq u\}}\,du\\ &\qquad\quad - \int_c^\infty\int_u^\infty E\big\{S^{(Y)}_{\tau_-(u-y)} - \theta\tau_-(u-y)\big\}\\ &\qquad\quad \times I_{\{\tau_-(u-y)<\infty\}}\,dH(y)\,du + o\big((1+c)^{-2}\big)\end{aligned} \tag{3.12}$$

uniformly in $c \geq 0$ and $\psi \in \mathcal{F}$, where H is the distribution function of Y.

PROOF. Let $\widetilde{S}_t = S_{t,\varepsilon}(\psi)$ and $\widetilde{\tau}(u) = \inf\{t: \widetilde{S}_t < u\}$. For notational simplicity we shall write δ_j and c_j instead of $\delta_j(n)$ and $c_j(n,c)$. To prove (3.11), note that

$$
\begin{aligned}
E(\widetilde{S}_k - k\theta)I_{\{\tilde{\tau}(\delta_j)\le k\}} &= E\left\{\sum_{i=1}^{k} E\left[(\widetilde{S}_k - k\theta)I_{\{\tilde{\tau}(\delta_j)=i\}} \mid U, Y_1, X_1, \dots, Y_i, X_i\right]\right\} \\
&= E\left\{\sum_{i=1}^{k} (\widetilde{S}_i - i\theta)I_{\{\tilde{\tau}(\delta_j)=i\}}\right\} \\
&= E\big(\widetilde{S}_{\tilde{\tau}(\delta_j)} - \theta\tilde{\tau}(\delta_j)\big)I_{\{\tilde{\tau}(\delta_j)\le k\}},
\end{aligned}
\tag{3.13}
$$

$$
\begin{aligned}
(1-\varepsilon_n)\int_{\delta_I}^{\delta_{h+1}} f(u)I_{\{\tilde{\tau}(u)\le k\}}\,du &\le \sum_{j=h}^{I-1}(\delta_j - \delta_{j+1})f(\delta_j)I_{\{\tilde{\tau}(\delta_j)\le k\}} \\
&\le \int_{-\infty}^{\delta_{-1},\,-1} f(u)I_{\{\tilde{\tau}(u)<\infty\}}\,du,
\end{aligned}
\tag{3.14}
$$

for either $f(u) = \tilde{\tau}(u)$ or $f(u) = |\widetilde{S}_{\tilde{\tau}(u)}|$. Because $E|Y|^4 + \sup_{\psi\in\mathcal{F}}\int E|\psi(X_1,z)|^4\,dG(z) < \infty$, where G is the distribution function of U, we can apply Lemma 4 to the sequences $\{Y_i + \epsilon\psi(X_i,z)\}$ and $\{Y_i\}$. Combining the representations in Lemma 4(iii) with related uniform integrability results implied by Lemma 4(i) and (3.6) and by Lemma 4(iv), it can be shown that as $\varepsilon \to 0$,

$$
\begin{aligned}
E\left(\int_{-\infty}^{0} \tilde{\tau}(u)I_{\{\tilde{\tau}(u)<\infty\}}\,du\right) &\to E\left(\int_{-\infty}^{0} \tau_-(u)I_{\{\tau_-(u)<\infty\}}\,du\right), \\
E\left(\int_{-\infty}^{0} \widetilde{S}_{\tilde{\tau}(u)}I_{\{\tilde{\tau}(u)<\infty\}}\,du\right) &\to E\left(\int_{-\infty}^{0} S^{(Y)}_{\tau_-(u)}I_{\{\tau_-(u)<\infty\}}\,du\right)
\end{aligned}
\tag{3.15}
$$

uniformly in $\psi \in \mathcal{F}$. From (3.13)–(3.15), we obtain (3.11) as $\varepsilon \to 0$, $k \to \infty$ and $n \to \infty$.

Let $\widetilde{Y}_i = Y_i + \varepsilon\psi(X_i, U)$ and write $\widetilde{S}_k - k\theta = (Y_1 - \theta) + \sum_{i=1}^{k-1}(\widetilde{Y}_{i+1} - \theta)$ for $k \ge 2$. Let $\widetilde{H}$ be the distribution function of $\widetilde{Y}_1$. For $u \ge 0$ and $k \ge 2$,

$$
\begin{aligned}
&E(\widetilde{S}_k - k\theta)I_{\{\min_{r\le k}\Sigma_{i=1}^r \tilde{Y}_i \ge u\}} \\
&\quad = E(\widetilde{Y}_1 - \theta)I_{\{\min_{r\le k}\Sigma_{i=1}^r \tilde{Y}_i \ge u\}} \\
&\qquad + \int_u^{\infty} E\left[\sum_{i=1}^{k-1}(\widetilde{Y}_{i+1} - \theta)I_{\{\min_{r\le k-1}(\Sigma_{j=1}^r \tilde{Y}_{j+1} + y)\ge u\}}\right] d\widetilde{H}(y), \\
&E\{\widetilde{S}_{k-1} - (k-1)\theta\}I_{\{\min_{r\le k-1}\Sigma_{i=1}^r \tilde{Y}_i \ge u-y\}} \\
&\quad = -E\{\widetilde{S}_{k-1} - (k-1)\theta\}I_{\{\min_{r\le k-1}\Sigma_{i=1}^r \widetilde{Y}_i < u-y\}} \\
&\quad = -E\{\widetilde{S}_{k-1} - (k-1)\theta\}I_{\{\tilde{\tau}(u-y)\le k-1\}} \\
&\quad = -E\{\widetilde{S}_{\tilde{\tau}(u-y)} - \theta\tilde{\tau}(u-y)\}I_{\{\tilde{\tau}(u-y)\le k-1\}}
\end{aligned}
$$

as in (3.13). Let $B(\varepsilon,\psi) = E\widetilde{M}^- + \sup_{u\le 0}\theta E\tilde{\tau}(u)I_{\{\tilde{\tau}(u)<\infty\}}$, where $\widetilde{M} = \inf_{t\ge 1}\widetilde{S}_t$. By (3.6) and Lemma 4(iv), $\sup_{0\le\varepsilon\le 1,\psi\in\mathcal{F}} B(\varepsilon,\psi) < \infty$. Moreover, $\sup_{u\le 0} E|\widetilde{S}_{\tilde{\tau}(u)} - \theta\tilde{\tau}(u)|I_{\{\tilde{\tau}(u)<\infty\}} \le B(\varepsilon,\psi)$ by Lemma 4(i), and it follows from (3.10) that

$$\sum_{j=h}^{J-1}(c_{j+1}-c_j)\Big\{E\big(|\widetilde{Y}_1|+\theta\big)I_{\{\widetilde{Y}_1\ge c_j\}} + B(\varepsilon,\psi)P(\widetilde{Y}_1\ge c_j)\Big\}$$

$$\le (1+\varepsilon_n)\int_{(1-\varepsilon_n)c-\varepsilon_n}^{\infty} E\{|\widetilde{Y}_1|+\theta+B(\varepsilon,\psi)\}I_{\{\widetilde{Y}_1\ge u\}}\,du$$

$$= o(c^{-2}) \quad \text{as } n\to\infty \text{ and } c\to\infty,$$

noting that $\sup_{0\le\varepsilon\le 1,\psi\in\mathcal{F}} E(\widetilde{Y}_1^4) < \infty$. Hence an argument similar to that used above to prove (3.11) can be used to prove (3.12). □

4. Proof of Theorems 1–3. In this section we first state a key lemma (Lemma 6), of which Theorems 1 and 2 are then shown to be simple corollaries. We next preface the proof of this key lemma by two additional lemmas, which we use in conjunction with the basic lemmas in Section 3 to prove Lemma 6. We then prove Theorem 3 by a modification of the arguments used to prove Lemma 6 and by applying Theorem 1 and some of the lemmas in Section 3.

LEMMA 6. *With the same notation and assumptions as in Theorem* 1, *let* $t = t_{n,a} = (a-n\theta)/\sqrt{n}\sigma$. *For any* $\beta > 0$, *as* $a\to\infty$ *and* $n\to\infty$ *such that* $|t|\le\beta$,

$$\begin{aligned} P\{T_a\le n, n\hat{\theta}_n < a\} &= \frac{\phi(t)}{\sigma\sqrt{n}}\int_{-\infty}^{0} P\{M_a(t)<u\}\,du \\ &\quad + \frac{\phi(t)}{\sigma n}\int_{-\infty}^{0}\bigg\{\frac{t}{\sigma}E\big(S^{(Y)}_{\tau_-(u)} - \theta\tau_-(u)\big)I_{\{\tau_-(u)<\infty\}} \\ &\quad + \Big(Q_1'(t) - tQ_1(t) - \frac{tu}{\sigma}\Big)P(M<u)\bigg\}\,du + o(n^{-1}), \end{aligned} \tag{4.1}$$

$$\begin{aligned} P\{T_a = n, n\hat{\theta}_n \ge a+c\} &= \frac{\phi(t)}{\sigma\sqrt{n}}\int_{c}^{\infty} P\{M_a(t)\ge u\}\,du \\ &\quad + \frac{\phi(t)}{\sigma n}\int_{c}^{\infty}\bigg\{\frac{t}{\sigma}\Lambda(u) + \Big(Q_1'(t) - tQ(t) - \frac{tu}{\sigma}\Big)P(M\ge u)\bigg\}\,du + o(n^{-1}) \end{aligned} \tag{4.2}$$

uniformly in $c \geq 0$ *and* $|t| \leq \beta$. *Moreover, for every* $\varepsilon > 0$,

$$P\left\{|a^{-1}T_a - \theta^{-1}| > \varepsilon\right\} = o(a^{-1}) \quad as \quad a \to \infty. \tag{4.3}$$

PROOF OF THEOREM 1. Because $P\{T_a \leq n\} = P\{n\widehat{\theta}_n \geq a\} + P\{T_a \leq n,\ n\widehat{\theta}_n < a\}$ and because $P\{n\widehat{\theta}_n \geq a\} = P\{\sqrt{n}(\widehat{\theta}_n - \theta)/\sigma \geq t_{n,a}\}$, the asymptotic expansion (2.12) follows from (4.1) and (2.5). □

PROOF OF THEOREM 2. The desired conclusion (2.15) follows from (4.2) because $P\{T_a = n,\ \Delta_a \geq y_1\} - P\{T_a = n,\ \Delta_a \geq y_2\} \leq P\{T_a = n,\ y_1 \leq \Delta_a \leq y_2\} \leq P\{T_a = n,\ \Delta_a \geq y_1\} - P\{T_a = n,\ \Delta_a \geq y_2 + n^{-1}\}$. □

To prove Lemma 6, the first step is to approximate the left-hand side of (4.1) as

$$\begin{aligned} &P\{T_a \leq n, n\widehat{\theta}_n < a\} \\ &\quad = \sum_{j \geq 0:\, |\delta_j| \leq n^{1/3}} P\left\{\max_{n-k \leq r < n} r\widehat{\theta}_r \geq a, a + \delta_{j+1} \leq n\widehat{\theta}_n < a + \delta_j\right\} + o(n^{-1}) \end{aligned} \tag{4.4}$$

and to use a similar discretization of the interval $[c, \infty)$ in which the overshoot $T_a\widehat{\theta}_{T_a} - a$ is assumed to lie for the event on the left-hand side of (4.2). Specifically, the $\delta_j = \delta_j(n)$ in (4.4) and the partition $c_j = c_j(n, c)$ used to prove (4.2) are defined for $n \geq 3$ and $g \geq 0$ by

$$\begin{aligned} &c_j = c + j(c+1)(n\log n)^{-1/2} \quad \text{and} \quad \delta_j = -j(n\log n)^{-1/2} \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{for } -1 \leq j \leq (n\log n)^{1/2}, \\ &c_{j+1} = c_j + c_j(n\log n)^{-1/2} \quad \text{and} \quad \delta_{j+1} = \delta_j - |\delta_j|(n\log n)^{-1/2} \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{for } j \geq (n\log n)^{1/2}. \end{aligned} \tag{4.5}$$

The $c_j(n, c)$ and $\delta_j(n)$ defined in (4.5) satisfy the assumptions on the partitions (3.9) in Lemma 5, for which we let $J = J(n, c)$ and $I = I(n)$ be such that $c_J \leq n^{1/3} < c_{J+1}$ and $|\delta_I| \leq n^{1/3} < |\delta_{I+1}|$. We shall apply Lemma 5 in conjunction with Lemma 1 on Edgeworth expansions of multivariate densities to derive the fluctuation-theoretic integral that appear on the right-hand sides of (4.1) and (4.2). This is the content of the following lemma whose proof, and those of Lemmas 8 and 9, are given at the end of this section.

LEMMA 7. *Let* ξ *be a standard normal random vector independent of* $\{X_i\}$. *Denote* $m^{-1/2}(S_m - m\mu + m^{-2}\xi)V^{-1/2}A^T$ *by* $W_m = (W_m^{(1)}, \ldots, W_m^{(d)})$ *and define*

$$\begin{aligned} \widehat{Y}_{i,m}(a, z) &= \langle X_i - \mu, \nabla g(\mu)\rangle \\ &\quad + \theta + (\theta/a)^{1/2}\left\langle (X_i - \mu)GV^{1/2}A^T, \left(z, W_m^{(2)}, \ldots, W_m^{(d)}\right)\right\rangle. \end{aligned}$$

Let $a_n \sim n\theta$ and let $K_2 > K_1 \geq 2\theta^{-1}$. Then for fixed $\beta > 0$ and integer $h \geq -1$, as $n \to \infty$,

$$\begin{aligned}\sup_{\substack{|z|\leq\beta\\ K_1 n^{1/3}\leq k\leq K_2 n^{1/3}}} \Bigg| \sum_{j\geq 0: c_j \leq n^{1/3}} & P\Bigg\{ \Bigg\| \sum_{i=n-k+1}^{n} (X_i-\mu)\Bigg\| \leq n^{1/3}, \\ & \min_{n-k<r\leq n} \sum_{i=r}^{n} \widehat{Y}_{i,n-k}(a_n,z) \geq c_{j+h}, \\ & z+\frac{c_j}{\sigma\sqrt{n}} \leq \frac{\sqrt{n}}{\sigma}\left(\widehat{\theta}_n-\theta\right) < z+\frac{c_{j+1}}{\sigma\sqrt{n}}\Bigg\} \\ & -\frac{\phi(z)}{\sigma\sqrt{n}}\int_c^{\infty} P\{M_{a_n}(z)\geq u\}\,du \\ & -\frac{\phi(z)}{\sigma n}\int_c^{\infty}\left\{\frac{z}{\sigma}\Lambda(u)+\left(Q_1'(z)-zQ_1(z)-\frac{zu}{\sigma}\right)P(M\geq u)\right\}du\Bigg| \\ = o\left((1+c)^{-2}n^{-1}\right) & + O\left(n^{-3/2}\right)\end{aligned} \tag{4.6}$$

uniformly in $0 \leq c \leq n^{1/3}$ and

$$\begin{aligned}\sup_{\substack{|z|\leq\beta\\ K_1 n^{1/3}\leq k\leq K_2 n^{1/3}}} \Bigg| \sum_{j\geq 0: |\delta_j| \leq n^{1/3}} & P\Bigg\{ \Bigg\| \sum_{i=n-k+1}^{n} (X_i-\mu)\Bigg\| \leq n^{1/3}, \\ & \min_{n-k<r\leq n} \sum_{i=r}^{n} \widehat{Y}_{i,n-k}(a_n,z) < \delta_{j+h}, \\ & z+\frac{\delta_{j+1}}{\sigma\sqrt{n}} \leq \frac{\sqrt{n}}{\sigma}\left(\widehat{\theta}_n-\theta\right) < z+\frac{\delta_j}{\sigma\sqrt{n}}\Bigg\} \\ & -\frac{\phi(z)}{\sigma\sqrt{n}}\int_{-\infty}^{0} P\{M_{a_n}(z) < u\}\,du \\ & -\frac{\phi(z)}{\sigma n}\int_{-\infty}^{0}\Bigg\{\frac{z}{\sigma}E\left(S^{(Y)}_{\tau_-(u)}-\theta\tau_-(u)\right)I_{\{\tau_-(u)<\infty\}} \\ & +\left(Q_1'(z)-zQ_1(z)-\frac{zu}{\sigma}\right)P(M<u)\Bigg\}du\Bigg| \\ = o\left(n^{-1}\right). & \end{aligned} \tag{4.7}$$

In Section 3, we introduced Lemma 3 (ii) to approximate $\{tg(S_t/t)-mg((S_m+m^{-2}\xi)/m),\ m<t\leq m+C^{1/3}m\}$ by a quadratic function of certain random walks so that the remainder does not exceed $m^{-2/3+\delta}$ with probability $1-o(m^{-1})$ for every $\delta > 0$. Setting $m = n-[3\theta^{-1}n^{1/3}]$, an important step in the proof of Lemma 6 is to remove the second-degree term in Lemma 3(ii) so that $\{n\widehat{\theta}_n - r\widehat{\theta}_r,\ m \leq r < n\}$ can be approximated simply by the random walk $\{\sum_{i=r+1}^{n}\widehat{Y}_{i,m}(a_n,z),\ m \leq r < n\}$ used in Lemma 7. This is the content of the following lemma, in which we use the notation $a \vee b$ to denote $\max(a,b)$.

LEMMA 8. *Let $k_n = [3\theta^{-1}n^{1/3}]$. Then for every $\beta > 0$, as $n \to \infty$,*

$$\sup_{\substack{|z| \le \beta \\ 0 \le c \le n^{1/3}}} \sum_{j \ge 0: c_j \le n^{1/3}} P\Bigg\{ \Bigg\| \sum_{i=n-k_n+1}^{n} (X_i - \mu) \Bigg\| \le n^{1/3},$$
$$z + \frac{c_j}{\sigma\sqrt{n}} \le \frac{\sqrt{n}}{\sigma}(\widehat{\theta}_n - \theta) < z + \frac{c_{j+1}}{\sigma\sqrt{n}}, \tag{4.8}$$
$$\max_{n-k_n < r \le n} n^{-1} \Bigg\| \sum_{i=n-k_n+1}^{n} (X_i - \mu) \Bigg\|^2 \ge (c_j \vee 1) n^{-13/24} \Bigg\} = o(n^{-1}),$$

$$\sup_{|z| \le \beta} \sum_{j \ge 0: |\delta_j| \le n^{1/3}} P\Bigg\{ \Bigg\| \sum_{i=n-k_n+1}^{n} (X_i - \mu) \Bigg\| \le n^{1/3},$$
$$z + \frac{\delta_{j+1}}{\sigma\sqrt{n}} \le \frac{\sqrt{n}}{\sigma}(\widehat{\theta}_n - \theta) < z + \frac{\delta_j}{\sigma\sqrt{n}}, \tag{4.9}$$
$$\max_{n-k_n < r \le n} n^{-1} \Bigg\| \sum_{i=n-k_n+1}^{r} (X_i - \mu) \Bigg\|^2 \ge \big(|\delta_j| \vee 1\big) n^{-13/24} \Bigg\}$$
$$= o(n^{-1}).$$

Let $m = n - k_n$ and $a_n = n\theta + O(\sqrt{n})$. Then for every fixed $\beta > 0$, as $n \to \infty$,

$$\sup_{\substack{|z| \le \beta \\ 0 \le c \le n^{1/3}}} \sum_{j \ge 0: c_j \le n^{1/3}} P\Bigg\{ \Bigg\| \sum_{i=m+1}^{n} (X_i - \mu) \Bigg\| \le n^{1/3},$$
$$z + \frac{c_j}{\sigma\sqrt{n}} \le \frac{\sqrt{n}}{\sigma}(\widehat{\theta}_n - \theta) < z + \frac{c_{j+1}}{\sigma\sqrt{n}}, \tag{4.10}$$
$$\max_{m \le r < n} \Bigg| n\widehat{\theta}_n - r\widehat{\theta}_r - \sum_{i=r+1}^{n} \widehat{Y}_{i,m}(a_n, z) \Bigg|$$
$$\ge n^{-1/25} \min(c_j - c_{j-1}, c_{j+2} - c_{j+1}) \Bigg\} = o(n^{-1}),$$

$$\sup_{|z| \le \beta} \sum_{j \ge 0: |\delta_j| \le n^{1/3}} P\Bigg\{ \Bigg\| \sum_{i=m+1}^{n} (X_i - \mu) \Bigg\| \le n^{1/3},$$
$$z + \frac{\delta_{j+1}}{\sigma\sqrt{n}} \le \frac{\sqrt{n}}{\sigma}(\widehat{\theta}_n - \theta) < z + \frac{\delta_j}{\sigma\sqrt{n}}, \tag{4.11}$$
$$\max_{m \le r < n} \Bigg| n\widehat{\theta}_n - r\widehat{\theta}_r - \sum_{i=r+1}^{n} \widehat{Y}_{i,m}(a_n, z) \Bigg|$$
$$\ge n^{-1/25} \min(\delta_{j-1} - \delta_j, \delta_{j+1} - \delta_{j+2}) \Bigg\} = o(n^{-1}).$$

PROOF OF LEMMA 6. Because $T_a \geq n_a$, $P\{T_a < a/(\theta+\varepsilon) \text{ or } T_a > a/(\theta-\varepsilon)\}$ is majorized by

$$P\left\{\sup_{i\geq n_a}\left|g\left(\frac{S_i}{i}\right)-g(\mu)\right|\geq\varepsilon\right\}$$

$$\leq P\left\{\sup_{i\geq n_a}\left\|\frac{S_i}{i}-\mu\right\|\geq\max\left(\delta,\frac{\varepsilon}{2\|\nabla g(\mu)\|}\right)\right\}=o\left(a^{-1}\right), \tag{4.12}$$

by (2.11) and Lemma 2 (with $\alpha = 1$), where δ is so chosen that $\|\nabla g(x)\| \leq 2\|\nabla g(\mu)\|$ if $\|x-\mu\| \leq \delta$. Hence (4.3) follows.

To prove (4.1), we shall first establish the representation (4.4) and then apply Lemmas 8 and 7, in which we let $a_n = a$ and $k_n = [3\theta^{-1}n^{1/3}]$. From (4.3) and Lemma 3 (i), it follows that

$$P\{T_a\leq n-k, n\widehat{\theta}_n<a\}=P\left\{n/2\leq T_a\leq n-k,\ n\widehat{\theta}_n<a\right\}+o\left(a^{-1}\right)$$

$$\leq P\left\{\max_{n/2\leq r\leq n-k}(n\widehat{\theta}_n-r\widehat{\theta}_r)<0\right\}+o\left(a^{-1}\right)=o\left(a^{-1}\right) \tag{4.13}$$

as $a\to\infty$ and $n\to\infty$ such that $|t|(=|t_{n,a}|)\leq\beta$.

Let $\widetilde{S}_i = S_i - i\mu$. By Lemma 2, $P\{\max_{n/2\leq i\leq n}\|\widetilde{S}_i\|^2/i \geq n^{1/4}\} = o(n^{-1})$. In view of the Taylor expansion $g(x+\mu) = g(\mu) + \langle x, \nabla g(\mu)\rangle + O(\|x\|^2)$ as $x\to 0$, it then follows that for any $\varepsilon > 0$,

$$P\left\{n\widehat{\theta}_n-r\widehat{\theta}_r\leq-\varepsilon n^{1/3}\ \text{for some } n-k\leq r\leq n\right\}$$

$$\leq P\left\{\max_{n-k\leq r\leq n}\|\widetilde{S}_n-\widetilde{S}_r\|\geq\frac{\varepsilon n^{1/3}}{2\|\nabla g(\mu)\|}\right\} \tag{4.14}$$

$$+o\left(n^{-1}\right)=o\left(n^{-1}\right)\quad\text{by Lemma 2.}$$

From (4.13) and (4.14), (4.4) follows. Because $a = n\theta + \sqrt{n}\sigma t$,

$$P\left\{\min_{n-k\leq r<n}(n\widehat{\theta}_n-r\widehat{\theta}_r)\leq\delta_{j+1}, t+\frac{\delta_{j+1}}{\sigma\sqrt{n}}\leq\frac{\sqrt{n}}{\sigma}(\widehat{\theta}_n-\theta)<t+\frac{\delta_j}{\sigma\sqrt{n}}\right\}$$

$$\leq P\left\{\max_{n-k\leq r<n}r\widehat{\theta}_r\geq a,\ a+\delta_{j+1}\leq n\widehat{\theta}_n<a+\delta_j\right\} \tag{4.15}$$

$$\leq P\left\{\min_{n-k\leq r<n}(n\widehat{\theta}_n-r\widehat{\theta}_r)\leq\delta_j,\ t+\frac{\delta_{j+1}}{\sigma\sqrt{n}}\leq\frac{\sqrt{n}}{\sigma}(\widehat{\theta}_n-\theta)<t+\frac{\delta_j}{\sigma\sqrt{n}}\right\}.$$

Combining (4.15) with (4.4) and (4.11) yields

$$\sum_{j \geq 0: |\delta_j| \leq n^{1/3}} P\left\{ \left\| \sum_{i=m+1}^{n} (X_i - \mu) \right\| \leq n^{1/3}, \min_{m < r \leq n} \sum_{i=r}^{n} \widehat{Y}_{i,m}(a,t) < \delta_{j+2}, \right.$$

$$\left. t + \frac{\delta_{j+1}}{\sigma\sqrt{n}} \leq \frac{\sqrt{n}}{\sigma}\left(\widehat{\theta}_n - \theta\right) < t + \frac{\delta_j}{\sigma\sqrt{n}} \right\} + o\left(n^{-1}\right)$$

$$\begin{aligned} (4.16) \quad &\leq \sum_{j \geq 0: |\delta_j| \leq n^{1/3}} P\left\{ \max_{m \leq r < n} r\widehat{\theta}_r \geq a, a + \delta_{j+1} \leq n\widehat{\theta}_n < a + \delta_j \right\} \\ &= P\left\{T_a \leq n, n\widehat{\theta}_n < a\right\} + o\left(n^{-1}\right) \\ &\leq \sum_{j \geq 0: |\delta_j| \leq n^{1/3}} P\left\{ \left\| \sum_{i=m+1}^{n} (X_i - \mu) \right\| \leq n^{1/3}, \min_{m < r \leq n} \sum_{i=r}^{n} \widehat{Y}_{i,m}(a,t) < \delta_{j-1}, \right. \\ &\qquad \left. t + \frac{\delta_{j+1}}{\sigma\sqrt{n}} \leq \frac{\sqrt{n}}{\sigma}\left(\widehat{\theta}_n - \theta\right) < t + \frac{\delta_j}{\sigma\sqrt{n}} \right\} + o\left(n^{-1}\right) \end{aligned}$$

uniformly in $|t| \leq \beta$. Applying Lemma 7 to the upper and lower bounds of $P\{T_a \leq n, n\widehat{\theta}_n < a\} + o(n^{-1})$ in (4.16) then gives the desired conclusion (4.1). The proof of (4.2) is similar. Note that the left-hand side of (4.2) is $o(n^{-1})$ for $c \geq n^{1/3}$ by an argument similar to (4.14) and that the right-hand side of (4.2) is also $o(n^{-1})$ for $c \geq n^{1/3}$, so we can restrict to the range $0 \leq c \leq n^{1/3}$ for which Lemma 7 is applicable. □

The preceding argument used to prove Lemma 6 can be modified to give the proof of Theorem 3, which we present next before giving the proof of Lemmas 7 and 8.

PROOF OF THEOREM 3. For notational simplicity write N for $N_{a,z}$ and δ for $\delta_{a,z}$. Fix any $\beta > 0$ and suppose that $h'(\theta) > 0$. Noting that $\psi(h(\theta)) = \theta$, $\psi'(h(\theta)) = 1/h(\theta)$ and $\sigma = \widetilde{\sigma}/h'(\theta)$, Taylor expansion yields

$$\begin{aligned} \psi\left(h(\theta) + z\widetilde{\sigma}u^{-1/2}\right) &= \theta + z\sigma u^{-1/2} + \tfrac{1}{2}z^2\widetilde{\sigma}^2 u^{-1}\psi''\left(h(\theta)\right) \\ &\quad + \tfrac{1}{6}z^3\widetilde{\sigma}^3 u^{-3/2}\psi'''\left(h(\theta)\right) + O\left(u^{-2}\right) \end{aligned}$$

as $u \to \infty$ uniformly in $|z| \leq \beta$. Hence (2.19) implies that as $a \to \infty$,

$$\begin{aligned} a &= \theta\gamma(a,z) + z\sigma\gamma^{1/2}(a,z) + z^2\widetilde{\sigma}^2\psi''\left(h(\theta)\right)/2 \\ (4.17) \qquad &\quad + z^3\widetilde{\sigma}^3\psi'''\left(h(\theta)\right)\gamma^{-1/2}(a,z)/6 + O\left(a^{-1}\right) \end{aligned}$$

uniformly in $|z| \leq \beta$. Because $N = \gamma(a,z) - \delta$ with $0 \leq \delta < 1$, it follows that $(N + \nu)^{-1/2} = \gamma^{-1/2}(a,z) + O((\nu + 1)N^{-3/2})$ uniformly in $0 \leq \nu \leq N^{1/3}$ and, therefore, by (4.17),

$$\frac{a-\theta(N+\nu)}{\sigma\sqrt{N+\nu}} = -\frac{\theta(\nu-\delta)}{\sigma\sqrt{N}} + z + \frac{z^2\tilde{\sigma}^2\psi''(h(\theta))}{2\sigma\sqrt{N}} + \frac{z^3\tilde{\sigma}^3\psi'''(h(\theta))}{6\sigma N} + O\left(\frac{(\nu+1)^2}{N^{3/2}}\right) \tag{4.18}$$

uniformly in $0 \le \nu \le N^{1/3}$. If $n \ge n_a$ and $|\widehat{\theta}_n - \theta| < \varepsilon$, then by (2.17) and (2.19),

$$Z_n < z \text{ and } n\widehat{\theta}_n \ge a \Leftrightarrow h(\widehat{\theta}_n) < h(\theta) + z\tilde{\sigma}n^{-1/2} \text{ and } n\widehat{\theta}_n \ge a$$
$$\Leftrightarrow a \le n\widehat{\theta}_n < n\psi\big(h(\theta) + z\tilde{\sigma}n^{-1/2}\big) \Rightarrow n > \gamma(a, z).$$

Because $P\{|\widehat{\theta}_{T_a} - \theta| \ge \varepsilon\} = o(a^{-1})$ by (4.12), it then follows that

$$\begin{aligned} P\{Z_{T_a} < z\} &= P\{T_a > N,\ Z_{T_a} < z\} + o(a^{-1}) \\ &= P\{T_a > N\} - P\{T_a > N,\ Z_{T_a} \ge z\} + o(a^{-1}). \end{aligned} \tag{4.19}$$

Let $t_{n,a} = (a - \theta n)/(\sigma\sqrt{n})$, $k = [3\theta^{-1}N^{1/3}]$ and $\rho = \sigma^{-1}\{\theta\delta + z^2\tilde{\sigma}^2\psi''(h(\theta))/2\}$ $[= \rho(a,z)]$. Setting $\nu = 0$ in (4.18) and noting that $Q_1(z+s) = Q_1(z) + sQ_1'(z) + O(s^2)$ and $\Phi(z+s) = \Phi(z) + s\phi(z) - \frac{1}{2}s^2 z\phi(z) + O(s^3)$ while $\phi(z+s) = \phi(z) - sz\phi(z) + O(s^2)$ as $s \to 0$, we obtain from Theorem 1 that uniformly in $|z| \le \beta$,

$$\begin{aligned} P\{T_a > N\} = {} & \Phi(z) + \phi(z)\Big\{\rho N^{-1/2} + z^3\tilde{\sigma}^3\psi'''(h(\theta))N^{-1}/6\sigma \\ & -z\rho^2 N^{-1}/2 + O(N^{-3/2})\Big\} \\ & -N^{-1/2}\phi(z)\Big(1 - N^{-1/2}\rho z + O(N^{-1})\Big) \\ & \times \left\{\sigma^{-1}\int_{-\infty}^{0} P\big(M_a(z) < u\big)\,du - Q_1(z) - N^{-1/2}\rho Q_1'(z) + O(N^{-2/3})\right\} \\ & -N^{-1}p_2(z)\phi(z) + o(N^{-1}), \end{aligned} \tag{4.20}$$

because $p_2(z+s) = p_2(z) + O(s)$ as $s \to 0$ by (2.10) and because for any nonrandom sequence $\varepsilon_N = O(N^{-1/2})$,

$$\int_{-\infty}^{0} P\{M_a(z+\varepsilon_N) < u\}\,du = \int_{-\infty}^{0} P\{M_a(z) < u\}\,du + O(N^{-2/3}). \tag{4.21}$$

To see (4.21), note that $k/N \sim 3\theta^{-1}N^{-2/3}$, $P\{\max_{1\le j\le k}|\Sigma_{i=1}^{j}\langle a^{-1/2}(X_i-\mu)G V^{1/2}A^T, (\varepsilon_N, \eta_1, \dots, \eta_{d-1})\rangle| \ge k/N\} = o(N^{-1})$ by Lemma 2 and that by (3.6) and Lemma 2,

$$\int_{-\infty}^{-N^{1/3}} \Big[P\{M_a(z+\varepsilon_N) < u\} + P\{M_a(z) < u\}\Big]\,du + N^{1/3}P\Big\{\inf_{i\ge k} S_i(a,z) < 0\Big\}$$
$$+ N^{1/3}P\Big\{\inf_{i\ge k} S_i(a, z+\varepsilon_N) < 0\Big\} = O(N^{-2/3}).$$

Noting that $(N+\nu)^{-1/2} - \gamma^{-1/2}(a,z) = -\frac{1}{2}N^{-3/2}(\nu-\delta) + O(N^{-5/2}\nu^2)$, we have

$$\psi\left(h(\theta) + \frac{z\widetilde{\sigma}}{\sqrt{N+\nu}}\right) = \psi\left(h(\theta) + \frac{z\widetilde{\sigma}}{\sqrt{\gamma(a,z)}}\right) - \frac{z\widetilde{\sigma}(\nu-\delta)}{2N^{3/2}h'(\theta)} + O\left(N^{-2}\nu + N^{-5/2}\nu^2\right),$$

uniformly in $1 \le \nu \le N^{5/8}$. Combining this with (2.17) and (2.19) yields on $\{|\widehat{\theta}_{N+\nu} - \theta| < \varepsilon\}$,

$$Z_{N+\nu} \ge z \Leftrightarrow \widehat{\theta}_{N+\nu} \ge \psi\left(h(\theta) + z\widetilde{\sigma}/\sqrt{N+\nu}\right) \Leftrightarrow (N+\nu)\widehat{\theta}_{N+\nu} \ge a + c(\nu,z),$$

where uniformly in $|z| \le \beta$ and $1 \le \nu \le N^{5/8}$,

$$\begin{aligned} c(\nu,z) &= (\nu-\delta)\psi\left(h(\theta) + \frac{z\widetilde{\sigma}}{\sqrt{\gamma(a,z)}}\right) \\ &\quad - \frac{(N+\nu)z\widetilde{\sigma}(\nu-\delta)}{2N^{3/2}h'(\theta)} + O\left(N^{-1}\nu + N^{-3/2}\nu^2\right) \\ &= (\nu-\delta)\left(\theta + \frac{z\sigma}{2\sqrt{N}}\right) + O\left(\nu N^{-7/8}\right) \text{ because } \sigma = \frac{\widetilde{\sigma}}{h'(\theta)} = \widetilde{\sigma}\psi'\left(h(\theta)\right). \end{aligned} \tag{4.22}$$

By (4.17) and an argument similar to (4.3), $P\{T_z > N + N^{5/8}\} = o(N^{-1})$ and therefore

$$\begin{aligned} &P\{T_a > N,\ Z_{T_a} \ge z\} \\ &\quad = \sum_{1 \le \nu \le N^{5/8}} P\{T_a = N+\nu,\ (N+\nu)\widehat{\theta}_{N+\nu} \ge a + c(\nu,z)\} + o\left(N^{-1}\right). \end{aligned} \tag{4.23}$$

Take any $1 < b < 3/2$. In view of (4.22), we have for all large a,

$$\begin{aligned} &\sum_{N^{1/3} < \nu \le N^{5/8}} P\{T_a = N+\nu, (N+\nu)\widehat{\theta}_{N+\nu} \ge a + c(\nu,z)\} \\ &\quad \le \sum_{i=0}^{\infty} P\Big\{(N+\nu)\widehat{\theta}_{N+\nu} - [N + b^i N^{1/3}]\widehat{\theta}_{N+[b^i N^{1/3}]} > b^i N^{1/3}\theta/2 \\ &\qquad\qquad \text{for some } b^i N^{1/3} < \nu \le b^{i+1}N^{1/3}\Big\} \\ &\quad \le \sum_{i=0}^{\infty} P\Bigg\{\max_{b^i N^{1/3} < \nu \le b^{i+1}N^{1/3}} \sum_{j=[b^i N^{1/3}]+1}^{\nu} \langle X_{N+j} - \mu, \nabla g(\mu)\rangle \\ &\qquad\qquad \ge \frac{1}{2}\theta N^{1/3}\left(\frac{b^i}{2} - \left(b^{i+1} - b^i\right)\right)\Bigg\} + o\left(N^{-1}\right) \\ &\quad = o\left(N^{-1}\right)\sum_{i=0}^{\infty} b^{-3i} + o\left(N^{-1}\right) = o\left(N^{-1}\right) \end{aligned} \tag{4.24}$$

by Lemma 2 and an argument similar to that of Lemma 3(i), noting that $b - 1 < \frac{1}{2}$. Let $m = N - k$. An argument similar to (4.4), (4.14) and (4.24) can be used to show that

$$\begin{aligned}(4.25)\qquad &\sum_{\nu=1}^{[N^{1/3}]} P\{T_a = N+\nu, (N+\nu)\widehat{\theta}_{N+\nu} \geq a + c(\nu,z)\} \\ &= \sum_{\nu=1}^{[N^{1/3}]} P\Big\{\max_{m \leq r < N+\nu} r\widehat{\theta}_r < a,\, a + (N+\nu)^{1/3} \\ &\qquad > (N+\nu)\widehat{\theta}_{N+\nu} \geq a + c(\nu,z)\Big\} + o(N^{-1}).\end{aligned}$$

The desired asymptotic expansion (2.23) follows from (4.19), (4.20), (4.23)–(4.25) and Lemma 9. □

LEMMA 9. *With the same notation and assumptions as in Theorem* 3, *let* $N = N_{a,z}, \delta = \delta_{a,z}, \rho = \rho(a,z)$, $m = N - [3\theta^{-1}N^{1/3}]$ *and define* $c(\nu,z)$ *as in* (4.22). *If* $h'(\theta) > 0$, *then as* $a \to \infty$,

$$\begin{aligned}&\sum_{\nu=1}^{[N^{1/3}]} P\Big\{\max_{m \leq r < N+\nu} r\widehat{\theta}_r < a,\, a + (N+\nu)^{1/3} > (N+\nu)\widehat{\theta}_{N+\nu} \geq a + c(\nu,z)\Big\} \\ &= \frac{\phi(z)}{\sigma\sqrt{N}} \sum_{\nu=1}^{\infty} \int_{(\nu-\delta)(\theta + z\sigma/2\sqrt{N})}^{\infty} P\{M_a(z) \geq u\}\, du \\ (4.26)\qquad &- \frac{z\phi(z)}{\sigma N} \sum_{\nu=1}^{\infty} \left(\rho - \frac{\theta\nu}{\sigma}\right) \int_{(\nu-\delta)\theta}^{\infty} P\{M \geq u\}\, du \\ &+ \frac{\phi(z)}{\sigma N} \sum_{\nu=1}^{\infty} \int_{(\nu-\delta)\theta}^{\infty} \left\{\frac{z}{\sigma}\Lambda(u) + \left(Q_1'(z) - zQ(z) - \frac{zu}{\sigma}\right) P(M \geq u)\right\} du \\ &+ o(N^{-1})\end{aligned}$$

uniformly in $|z| \leq \beta$, *for every fixed* $\beta > 0$.

We now give the proofs of Lemmas 7–9, from which the reason for the particular choice (4.5) of the partitions $\delta_j(n)$ and $c_j(n,c)$ will become clear. Throughout the sequel we shall let f_m denote the density function of $m^{-1/2}(S_m - m\mu + m^{-2}\xi)V^{-1/2}A^T$. By Lemma 1, f_m has the Edgeworth expansion (3.1), which we shall use to integrate over certain sets in the proof of Lemma 7. To perform this integration, we shall use a change of variables and Taylor's expansion of its Jacobian, similar to Lemma 2.1 of Bhattacharya and Ghosh (1978). Moreover, in view of the restriction of $\sqrt{n}(\widehat{\theta}_n - \theta)/\sigma$ to certain narrow intervals in the events in Lemma 7, we shall also use ideas similar to the conditioned random walk approach in nonlinear renewal theory, involving time reversal arguments and local limit theorems [cf. Chapter 5 of Woodroofe (1982)].

PROOF OF LEMMA 7. Let $S_{n,k} = X_n + \cdots + X_{n-k+1}$ and decompose $\sqrt{n}(\widehat{\theta}_n - \theta)/\sigma$ as a sum of three terms:

$$
\begin{aligned}
\frac{\sqrt{n}}{\sigma}(\widehat{\theta}_n - \theta) = {} & \frac{\sqrt{n}}{\sigma}\left\{g\left(\frac{S_n}{n}\right) - g\left(\frac{S_n + (n-k)^{-2}\xi}{n}\right)\right\} \\
& + \frac{\sqrt{n}}{\sigma}\left\{g\left(\mu + \frac{S_{n,k} - k\mu}{n}\right) - g(\mu)\right\} \\
& + \frac{\sqrt{n}}{\sigma}\left\{g\left(\mu + \frac{\sqrt{n-k}}{n}W_{n-k}AV^{1/2} + \frac{S_{n,k} - k\mu}{n}\right)\right. \\
& \qquad \left. - g\left(\mu + \frac{S_{n,k} - k\mu}{n}\right)\right\}.
\end{aligned}
\tag{4.27}
$$

Here A is an orthogonal $d \times d$ matrix whose first row is $\sigma^{-1}\nabla g(\mu)V^{1/2}$, as in (2.8). Let

$$
\rho_{n,k,x}(w) = \frac{\sqrt{n}}{\sigma}\left\{g\left(\left(\frac{n-k}{n}\right)\frac{wAV^{1/2}}{\sqrt{n-k}} + \frac{x}{n} + \mu\right) - g\left(\frac{x}{n} + \mu\right)\right\}, \qquad w \in \mathbf{R}^d,
\tag{4.28}
$$

so the last summand in (4.27) is $\rho_{n,k,S_{n,k}-k\mu}(W_{n-k})$. Because $(\sigma^{-1}\nabla g(\mu)V^{1/2})^T$ is the first column vector of A^T and is orthogonal to the other column vectors of A^T, it follows that $\sigma^{-1}\nabla g(\mu)V^{1/2}A^T = (1, 0, \dots, 0)$ and, therefore,

$$
\begin{aligned}
\sigma^{-1}\langle \nabla g(\mu), wAV^{1/2}\rangle &= \sigma^{-1}\nabla g(\mu)V^{1/2}A^T w^T \\
&= w^{(1)} \quad \text{for } w = \left(w^{(1)}, \dots, w^{(d)}\right).
\end{aligned}
\tag{4.29}
$$

In view of (4.28) and (4.29) and recalling that $k \le K_2 n^{1/3}$, Taylor's expansion yields

$$
\rho_{n,k,x}(w) = w^{(1)} + \sigma^{-1}n^{-1/2}wAV^{1/2}GV^{1/2}A^T w^T/2 + O\left(n^{-2/3}\log n\right)
\tag{4.30}
$$

uniformly in $\|x\| \le n^{1/3}$ and $\|w\| \le \log n$.

Let $\|x\| \le n^{1/3}$ and consider the transformation $u = T(w) = (\rho_{n,k,x}(w), w^{(2)}, \dots, w^{(d)})$ for $\|w\| \le \log n$. In view of (4.30), it has an inverse

$$
w = T^{-1}(u) = \left(u^{(1)} + n^{-1/2}p(u) + O\left(n^{-2/3}\log n\right), u^{(2)}, \dots, u^{(d)}\right)
\tag{4.31}
$$

uniformly in $\|x\| \le n^{1/3}$ and $\|w\| \le \log n$, and in this region the Jacobian satisfies

$$
\det\left(\frac{\partial w^{(i)}}{\partial u^{(j)}}\right)_{1 \le i,j \le d} = 1 + n^{-1/2}q(u) + O\left(n^{-2/3}\log n\right),
\tag{4.32}
$$

where $p(u)$ and $q(u)$ are polynomials whose coefficients do not depend on n and x. Letting $\Delta_{n,x} = \sqrt{n}\{g(\mu + n^{-1}x) - g(\mu)\}/\sigma$, note that

$$\begin{aligned}\Delta_{n,x} &= \sigma^{-1}n^{-1/2}\langle x, \nabla g(\mu)\rangle + O\big(n^{-3/2}\|x\|^2\big)\\ &= \sigma^{-1}n^{-1/2}\Big\{\langle x, \nabla g(\mu)\rangle + \big(\theta/a_n\big)^{1/2}\big\langle xGV^{1/2}A^T, w\big\rangle\Big\} + O\big(n^{-2/3}\log n\big)\end{aligned} \tag{4.33}$$

uniformly in $\|x\| \le n^{1/3}$ and $\|w\| \le 2\log n$. Because the η_i in (2.8) are independent standard normal random variables that are independent of $X_1, X_2, \dots,$ we have

$$\begin{aligned}&\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} P\Bigg\{\min_{r\le k}\sum_{i=1}^{r}\bigg(Y_i + \Big\langle\big(\theta/a_n\big)^{1/2}(X_i-\mu)GV^{1/2}A^T,\\ &\qquad\qquad \big(z, u^{(2)}, \dots, u^{(d)}\big)\Big\rangle\bigg) \ge c_{j+h}\Bigg\}\prod_{i=2}^{d}\phi\big(u^{(i)}\big)du^{(i)}\\ &= P\Bigg\{\min_{r\le k}\sum_{i=1}^{r}\bigg(Y_i + \Big\langle\big(\theta/a_n\big)^{1/2}(X_i-\mu)GV^{1/2}A^T,\\ &\qquad\qquad (z, \eta_1, \dots, \eta_{d-1})\Big\rangle\bigg) \ge c_{j+h}\Bigg\}\\ &= P\Big\{\min_{r\le k} S_r(a_n, z) \ge c_{j+h}\Big\}.\end{aligned} \tag{4.34}$$

Let F_k denote the joint distribution function of $(X_1 - \mu, X_1 + X_2 - 2\mu, \dots, X_1 + \dots + X_k - k\mu)$. For $\alpha = \pm 1$ and $z \in \mathbf{R}$, let

$$\begin{aligned}\Omega_{\alpha,j,z} = \Bigg\{&(w, s_1, \dots, s_k) \in \mathbf{R}^{d(k+1)}: \|s_k\| \le n^{1/3},\\ &\min_{1\le i\le k}\bigg(\big\langle s_i, \nabla g(\mu)\big\rangle + i\theta\\ &\qquad + \Big\langle\big(\theta/a_n\big)^{1/2}s_iGV^{1/2}A^T, \big(z, w^{(2)}, \dots, w^{(d)}\big)\Big\rangle\bigg)\\ &\ge c_{j+h}, z + c_j/(\sigma\sqrt{n}) - \Delta_{n,s_k} - \alpha n^{-2} \le \rho_{n,k,s_k}(w)\\ &\qquad\qquad \le z + c_{j+1}/(\sigma\sqrt{n}) - \Delta_{n,s_k} + \alpha n^{-2}\Bigg\}.\end{aligned}$$

Noting that $\phi(z+t) = \phi(z) - tz\phi(z) + O(t^2)$ as $t \to 0$, we have, in view of (4.5), that

$$\begin{aligned}&\int_{z+c_j/(\sigma\sqrt{n})-\Delta_{n,x}-\alpha n^{-2}}^{z+c_{j+1}/(\sigma\sqrt{n})-\Delta_{n,x}+\alpha n^{-2}} \phi\Big(u^{(1)} + n^{-1/2}p\big(u^{(1)}, \dots, u^{(d)}\big) + O\big(n^{-2/3}\log n\big)\Big)\,du^{(1)}\\ &= \Big\{\phi(z) + \Big[\Delta_{n,x} - \sigma^{-1}n^{-1/2}c_j - n^{-1/2}p\big(z, u^{(2)}, \dots, u^{(d)}\big)\\ &\qquad\qquad + O\big(n^{-2/3}\log n\big)\Big]\big(z\phi(z) + o(1)\big)\Big\}\\ &\quad \times\big\{\sigma^{-1}n^{-1/2}(c_{j+1} - c_j) + 2\alpha n^{-2}\big\}\end{aligned} \tag{4.35}$$

uniformly in $j \ge 0$ with $c_j \le n^{1/3}$, $\|x\| \le n^{1/3}$, $|u^{(2)}| + \dots + |u^{(d)}| \le \log n$ and $|z| \le \beta$. Because $k = O(n^{1/3})$ and $\|s_k\| \le n^{1/3}$ on $\Omega_{\alpha,j,z}$, it then follows from (3.1), (4.31)–(4.35) and the change of variables $w = T^{-1}(u)$ that

$$\begin{aligned}
&\int_{\Omega\alpha, j, z} f_{n-k}(w)\,dw\,dF_k(s_1,\dots,s_k)\\
&\quad= \left(\frac{c_{j+1}-c_j}{\sigma\sqrt{n}} + \frac{2\alpha}{n^2}\right)\\
&\qquad\times\Bigg\{\phi(z)P\Big[\min_{r\le k} S_r(a_n,z)\ge c_{j+h}\Big] - \frac{z\phi(z)+o(1)}{\sigma\sqrt{n}}c_jP\Big[\min_{r\le k} S_r(a_n,z)\ge c_{j+h}\Big]\\
&\qquad\quad+\frac{z\phi(z)+o(1)}{\sigma\sqrt{n}}E\big(S_k(a_n,z)-\theta k\big)I_{\{\min_{r\le k} S_r(a_n,z)\ge c_{j+h}\}}\\
&\qquad\quad+O\left(P\left[\left\|\sum_{i=1}^k(X_i-\mu)\right\|>n^{1/3},\ \min_{r\le k}S_r(a_n,z)\ge c_{j+h}\right]\right)\\
&\qquad\quad+O\left(n^{-1/2}E\left\|\sum_{i=1}^k(X_i-\mu)\right\|I_{\{\|\Sigma_1^k(X_i-\mu)\|>n^{1/3},\ \min_{r\le k}S_r(a_n,z)\ge c_{j+h}\}}\right)\\
&\qquad\quad+\frac{\phi(z)+o(1)}{\sqrt{n}}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}[P_1-zp+q]\big(z,u^{(2)},\dots,u^{(d)}\big)\\
&\qquad\qquad\times P\left[\min_{r\le k}\sum_{i=1}^r\left(Y_i+\left\langle\big(\theta/a_n\big)^{1/2}(X_i-\mu)GV^{1/2}A^T,\right.\right.\right.\\
&\qquad\qquad\qquad\left.\left.\left.\big(z,u^{(2)},\dots,u^{(d)}\big)\right\rangle\right)\ge c_{j+h}\right]\prod_{i=2}^d\phi\big(u^{(i)}\big)\,du^{(i)}\\
&\qquad\quad+O\big(n^{-2/3}\log n\big)P\Big[\min_{r\le k}S_r(a_n,z)\ge c_{j+h}\Big]\Bigg\}
\end{aligned}\tag{4.36}$$

uniformly in $j\ge 0$ with $c_j\le n^{1/3}$ and $|z|\le\beta$, where $P_1(u)$, $p(u)$ and $q(u)$ are polynomials given in (3.1), (4.31) and (4.32).

Because $\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}[P_1-zp+q](z,u^{(2)},\dots,u^{(d)})\prod_{i=2}^d\phi(u^{(i)})\,du^{(i)} = Q_1'(z) - z\,Q_1(z)$ by (4.31), (4.32) and an argument similar to the proof of Lemma 2.1 of Bhattacharya and Ghosh (1978), and because $P\{\max_{r\le k}|\Sigma_{i=1}^r\langle X_i-\mu,u\rangle|/a_n^{1/2}\ge(1+c)n^{-1/6}\} = o(\|u\|^4(1+c)^{-4}n^{-1})$ uniformly in $\|u\|\le\log n$ and $0\le c\le n^{1/3}$ by Lemma 2, it follows from (4.5) that

$$\begin{aligned}
&\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\Bigg\{\sum_{j\ge 0:\,c_j\le n^{1/3}}(c_{j+1}-c_j)\\
&\qquad\times P\left[\min_{r\le k}\sum_{i=1}^r\left(Y_i+\left\langle\big(\theta/a_n\big)^{1/2}(X_i-\mu)GV^{1/2}A^T,\right.\right.\right.\\
&\qquad\qquad\left.\left.\left.\big(z,u^{(2)},\dots,u^{(d)}\big)\right\rangle\right)\ge c_{j+h}\right]\Bigg\}\\
&\quad\times[P_1-zp+q]\big(z,u^{(2)},\dots,u^{(d)}\big)\prod_{i=2}^d\phi\big(u^{(i)}\big)\,du^{(i)}\\
&\quad=\left\{\int_c^{\infty}P(M\ge t)\,dt\right\}\{Q_1'(z)-zQ_1(z)\}+O\big((1+c)^{-3}n^{-1/6}\big)
\end{aligned}\tag{4.37}$$

uniformly in $|z| \le \beta$ and $0 \le c \le n^{1/3}$, noting that for $t \ge 1$, $P\{\inf_{r \ge 1} \sum_{i=1}^r Y_i \ge t\} \le P\{Y_1 \ge t\} = O(t^{-4})$. Moreover,

$$\sum_{j \ge 0: c_j \le n^{1/3}} (c_{j+1} - c_j) c_j P\Big\{ \min_{r \le k} S_r(a_n, z) \ge c_{j+h} \Big\} = \int_c^\infty tP(M \ge t)\, dt + o\big((1+c)^{-2}\big) \tag{4.38}$$

and, by Lemma 5 and (2.14),

$$\sum_{j \ge 0: c_j \le n^{1/3}} (c_{j+1} - c_j) E\big(S_k(a_n, z) - \theta k\big) I_{\{\min_{r \le k} S_r(a_n,z) \ge c_{j+h}\}} = \int_c^\infty \Lambda(t)\, dt + o\big((1+c)^{-2}\big) \tag{4.39}$$

uniformly in $|z| \le \beta$ and $0 \le c \le n^{1/3}$. Because $E\|X\|^4 < \infty$ and $k \le K_2 n^{1/3}$,

$$\begin{aligned} &n^{-1/2} E \left\| \sum_{i=1}^k (X_i - \mu) \right\| I_{\{\|\sum_1^k (X_i - \mu)\| > n^{1/3},\, \min_{r \le k} S_r(a_n,z) \ge c_{j+h}\}} \\ &\quad \le 8n^{-1/2-1} \left\{ E\|X_1 - \mu\|^4 + E \left\| \sum_{i=2}^k (X_i - \mu) \right\|^4 I_{\{Y_1(a_n,z) \ge c_{j+h}\}} \right\} \\ &\quad = O\Big(n^{-3/2} + n^{-5/6} P\big\{Y_1(a_n,z) \ge c_{j+h}\big\}\Big), \end{aligned} \tag{4.40}$$

$$\begin{aligned} &P\left\{ \left\| \sum_{i=1}^k (X_i - \mu) \right\| > n^{1/3},\ \min_{r \le k} S_r(a_n,z) \ge c_{j+h} \right\} \\ &\quad \le P\{\|X_1 - \mu\| > n^{1/3}/2\} + P\left\{ \left\| \sum_{i=2}^k (X_i - \mu) \right\| > n^{1/3}/2 \right\} \\ &\qquad \times P\big\{Y_1(a_n,z) \ge c_{j+h}\big\} \\ &\quad = o\big(n^{-4/3}\big) + o\Big(n^{-1} P\big\{Y_1(a_n,z) \ge c_{j+h}\big\}\Big) \quad \text{by Lemma 2.} \end{aligned} \tag{4.41}$$

By (4.5), $\sum_{j \ge 0: c_j \le n^{1/3}} (c_{j+1} - c_j) = n^{1/3} + o(1)$. Moreover, for $h \ge -1$ and all large n,

$$\begin{aligned} &\int_{c - (c+1)(n \log n)^{-1/2}}^{n^{1/3}+1} P\Big\{ \min_{r \le k} S_r(a_n,z) \ge t \Big\}\, dt \\ &\quad \ge \sum_{j \ge 0: c_j \le n^{1/3}} (c_{j+1} - c_j) P\Big\{ \min_{r \le k} S_r(a_n,z) \ge c_{j+h} \Big\} \\ &\quad \ge \int_{c + (c+1)(h+1)(n \log n)^{-1/2}}^{n^{1/3}-1} P\Big\{ \min_{r \le k} S_r(a_n,z) \ge t \Big\}\, dt. \end{aligned} \tag{4.42}$$

Because $k \geq K_1 n^{1/3}$ with $K_1 \geq 2\theta^{-1}$ and $E\|X\|^4 < \infty$, we obtain by Lemma 2 that

$$\begin{aligned}(4.43)\quad &\int_{-1}^{n^{1/3}+1} P\Big\{\inf_{r>k} S_r(a_n,z) < t\Big\}\,dt \\ &\leq \big(n^{1/3}+2\big)P\left\{\sup_{r>k} r^{-1}\sum_{i=1}^{r}\big(\theta - Y_i(a_n,z)\big) \geq \frac{\theta}{4}\right\} = O\big(n^{-1}\big)\end{aligned}$$

uniformly in $|z| \leq \beta$. Moreover, $|Y_1(a_n,z) - \theta| \leq \|X_1 - \mu\|\{C_1 + C_2 n^{-1/2}(|z| + \Sigma_{i=1}^{d-1}|\eta_i|)\}$ for some positive constants C_1 and C_2 and $P\{Y_1(a_n,z) \geq t\} \leq t^{-4}EY_1^4(a_n,z)$, so

$$\begin{aligned}(4.44)\quad &\int_{n^{1/3}-1}^{\infty} P\Big\{\inf_{r\geq 1} S_r(a_n,z) \geq t\Big\}\,dt \\ &\leq \int_{n^{1/3}-1}^{\infty} P\big\{Y_1(a_n,z) \geq t\big\}\,dt = O\big(n^{-1}\big).\end{aligned}$$

From (4.42)–(4.44), it follows that uniformly in $0 \leq c \leq n^{1/3}$ and $|z| \leq \beta$,

$$\begin{aligned}(4.45)\quad &\sum_{j\geq 0:\, c_j \leq n^{1/3}} (c_{j+1} - c_j)P\Big\{\min_{r\leq k} S_r(a_n,z) \geq c_{j+h}\Big\} \\ &= \int_c^{\infty} P\Big\{\inf_{r\geq 1} S_r(a_n,z) \geq t\Big\}\,dt \\ &\quad + O\Big(\big\{(c+1)(n\log n)^{-1/2}\big\}(c+1)^{-4}\Big) + O\big(n^{-1}\big).\end{aligned}$$

Combining (4.36) with (4.45) and (4.37)–(4.41) yields

$$\begin{aligned}(4.46)\quad &\sum_{j\geq 0:\, c_j \leq n^{1/3}} \int_{\Omega_{\alpha,j,z}} f_{n-k}(w)\,dw\,dF_k(s_1,\dots,s_k) \\ &= \sigma^{-1}n^{-1/2}\phi(x)\int_c^{\infty} P\big\{M_{a_n}(z) \geq t\big\}\,dt \\ &\quad + \sigma^{-2}n^{-1}z\phi(z)\left\{\int_c^{\infty}\Lambda(t)\,dt - \int_c^{\infty} tP(M \geq t)\,dt\right\} \\ &\quad + \sigma^{-1}n^{-1}\phi(z)\big\{Q_1'(z) - zQ_1(z)\big\}\int_c^{\infty} P(M \geq t)\,dt \\ &\quad + o\big((1+c)^{-2}n^{-1}\big) + O\big(n^{-3/2}\big)\end{aligned}$$

uniformly in $|z| \leq \beta$, $K_1 n^{1/3} \leq k \leq K_2 n^{1/3}$ and $0 \leq c \leq n^{1/3}$, noting that $\Sigma_{j\leq 0:\, c_j \leq n^{1/3}}(c_{j+1} - c_j + 1) \sim n^{1/3}$ by (4.5). Because ξ is normal, $P\{\|\xi\| > \log n\} =$

$o(n^{-2})$. Because $E\|X\|^4 < \infty$, $P\{\|n^{-1}S_n - \mu\| > \varepsilon\} = O(n^{-2})$ for every $\varepsilon > 0$. Taking ε sufficiently small then gives

$$\sup_{K_1 n^{1/3} \le k \le K_2 n^{1/3}} P\left\{ \frac{\sqrt{n}}{\sigma} \left| g\left(\frac{S_n}{n}\right) - g\left(\frac{S_n + (n-k)^{-2}\xi}{n}\right) \right| \ge n^{-2} \right\} = O(n^{-2}). \tag{4.47}$$

Because f_{n-k} is the density function of $(n-k)^{-1/2}\{S_{n-k} - (n-k)\mu + (n-k)^{-2}\xi\}V^{-1/2}A^T$, which is independent of $(X_n, X_{n-1}, \dots, X_{n-k+1})$, it follows from (4.27) and the definitions of $\widehat{Y}_{t,n-k}(a_n, z)$ and of $\Omega_{\alpha,j,z}$ for $\alpha = \pm 1$ that

$$\begin{aligned}
&\sum_{j \ge 0: c_j \le n^{1/3}} \int_{\Omega_{-1,j,z}} f_{n-k}(w)\, dw\, dF_k(s_1, \dots, s_k) - (4.47) \\
&\quad \le \sum_{j \ge 0: c_j \le n^{1/3}} P\Bigg\{ \left\| \sum_{t=n-k+1}^{n} (X_t - \mu) \right\| \le n^{1/3}, \\
&\qquad\qquad \min_{0 \le i < k} \sum_{t=n-i}^{n} \widehat{Y}_{t,n-k}(a_n, z) \ge c_{j+h}, \\
&\qquad\qquad z + \frac{c_j}{\sigma\sqrt{n}} \le \frac{\sqrt{n}(\widehat{\theta}_n - \theta)}{\sigma} < z + \frac{c_{j+1}}{\sigma\sqrt{n}} \Bigg\} \\
&\quad \le \sum_{j \ge 0: c_j \le n^{1/3}} \int_{\Omega_{1,j,z}} f_{n-k}(w)\, dw\, dF_k(s_1, \dots, s_k) + (4.47).
\end{aligned}$$

Hence the desired conclusion (4.6) follows from (4.46). The proof of (4.7) is similar, noting that in analogy with (4.44) we now have

$$\int_{-\infty}^{-n^{1/3}+1} P\left\{ \inf_{i \ge 1} S_i(a_n, z) < u \right\} du \le \int_{-\infty}^{-n^{1/3}+1} |u|^{-3} E\big(M_{a_n}^-(z)\big)^3\, du = O\big(n^{-2/3}\big),$$

because $\sup_n E(M_{a_n}^-(z))^3 < \infty$ by (3.6). □

PROOF OF LEMMA 8. We shall only prove (4.9) and (4.11), because (4.8) and (4.10) can be proved by similar arguments. Using the same notation as that in the proof of Lemma 7, let $k = k_n$ and define

$$\begin{aligned}
E_{j,z} = \Bigg\{ &(w, s_1, \dots, s_k) \in \mathbf{R}^{d(k+1)} \colon \|s_k\| \le n^{1/3}, \\
&n^{-1} \max\Big(\|s_k\|^2, \max_{1 \le i < k} \|s_k - s_i\|^2 \Big) \ge \big(|\delta_j| \vee 1\big) n^{-13/24}, \\
&z + \frac{\delta_{j+1}}{\sigma\sqrt{n}} - \Delta_{n,s_k} - n^{-2} \le \rho_{n,k,s_k}(w) < z + \frac{\delta_j}{\sigma\sqrt{n}} - \Delta_{n,s_k} + n^{-2} \Bigg\}.
\end{aligned}$$

From (4.27) and the definition of $E_{j,z}$, it follows that the sum of probabilities in (4.9) is majorized by

$$\sum_{j\geq 0:\,|\delta_j|\leq n^{1/3}} \int_{E_{j,z}} f_{n-k}(w)\,dw\,dF_k(s_1,\dots,s_k)$$

$$+P\left\{\frac{\sqrt{n}}{\sigma}\left|g\left(\frac{S_n}{n}\right)-g\left(\frac{S_n+(n-k)^{-2}\xi}{n}\right)\right|\geq n^{-2}\right\}$$

$$\leq \sum_{0\leq j\leq (n\log n)^{1/2}} \left(\frac{\delta_j-\delta_{j+1}}{\sigma\sqrt{n}}+\frac{2}{n^2}\right)(\phi(z)+o(1))$$

$$\times P\left\{\max_{1\leq t\leq k} n^{-1}\left\|\sum_{i=1}^{t}(X_i-\mu)\right\|^2\geq n^{-13/24}\right\} \tag{4.48}$$

$$+\sum_{\substack{j\geq (n\log n)^{1/2}\\ |\delta_j|\leq n^{1/3}}} \left(\frac{\delta_j-\delta_{j+1}}{\sigma\sqrt{n}}+\frac{2}{n^2}\right)(\phi(z)+o(1))$$

$$\times P\left\{\max_{1\leq t\leq k}\left\|\sum_{i=1}^{t}(X_i-\mu)\right\|\geq |\delta_j|^{1/2}n^{11/48}\right\}+O(n^{-2}),$$

in view of (4.47), (4.31), (4.32) and (4.35) together with the change of variables $w=T^{-1}(u)$. By Lemma 2 (with $\alpha=11/16$), $P\{\max_{t\leq k}\|\sum_{i=1}^{t}(X_i-\mu)\|\geq n^{11/48}\}=o(n^{-7/12})$ and $P\{\max_{t\leq k}\|\sum_{i=1}^{t}(X_i-\mu)\|\geq|\delta_j|^{1/2}n^{11/48}\}=o(|\delta_j|^{-2}n^{-7/12})$ uniformly in $1\leq|\delta_j|\leq n^{1/3}$. From (4.5), it follows that $\sum_{0\leq j\leq(n\log n)^{1/2}}(\delta_j-\delta_{j+1})=1+o(1)$. Moreover, $\sum_{j\geq(n\log n)^{1/2},\,|\delta_j|\leq n^{1/3}}(\delta_j-\delta_{j+1})/\delta_j^2=O(1)$. Hence (4.48) is $o(n^{-1})$, implying (4.9).

We next make use of (4.9) to prove (4.11). Noting that $m^{-1/2}(S_m-m\mu+m^{-2}\xi)=W_mAV^{1/2}$, it follows from Lemma 3(ii) that on an event Ω_n with $P(\Omega_n)=1-o(n^{-1})$,

$$r\widehat{\theta}_r=mg\big((S_m+m^{-2}\xi)/m\big)+(r-m)\theta+\sum_{i=m+1}^{r}\langle X_i-\mu,\,\nabla g(\mu)\rangle$$

$$+\big(\theta/a_n\big)^{1/2}\sum_{i=m+1}^{r}\langle (X_i-\mu)GV^{1/2}A^T,W_m\rangle$$

$$+O\left(n^{-1}\left\|\sum_{i=m+1}^{r}(X_i-\mu)\right\|^2\right)+O\big(n^{-4/7}\big)$$

uniformly in $m\leq r\leq n$. Hence on Ω_n, we have uniformly in $m\leq r<n$ and

$|z| \le \beta$,

$$
\begin{aligned}
n\widehat{\theta}_n - r\widehat{\theta}_r = \sum_{i=r+1}^{n} \widehat{Y}_{i,m}(a_n, z) + O\left(n^{-1} \max_{m<s\le n} \left\| \sum_{i=m+1}^{s} (X_i - \mu) \right\|^2\right) \\
+ O\left(n^{-1/2} \max_{m<s\le n} \left\| \sum_{i=s}^{n} \left\langle (X_i - \mu) G V^{1/2} A^T, \left(W_m^{(1)} - z, 0, \dots, 0\right) \right\rangle \right\|\right) \\
+ O\left(n^{-4/7}\right),
\end{aligned} \tag{4.49}
$$

because $\widehat{Y}_{i,m}(a_n, z) = \theta + \langle X_i - \mu, \nabla g(\mu) \rangle + (\theta/a_n)^{1/2} \langle (X_i - \mu) G V^{1/2} A^T, (z, W_m^{(2)}, \dots, W_M^{(d)}) \rangle$. Let

$$
E_n = \left\{ \max_{m<s\le n} \left\| \sum_{i=s}^{n} (X_i - \mu) \right\| \le n^{1/3}, \|W_m\| \le \log m, \right.
\left. \frac{\sqrt{n}}{\sigma} \left| g\left(\frac{S_n}{n}\right) - g\left(\frac{S_n + m^{-2}\xi}{n}\right) \right| \le n^{-2} \right\},
$$

$$
E_{n,j,z} = E_n \cap \left\{ z + \frac{\delta_{j+1}}{\sigma\sqrt{n}} \le \frac{\sqrt{n}}{\sigma}(\widehat{\theta}_n - \theta) < z + \frac{\delta_j}{\sigma\sqrt{n}} \right\}.
$$

By (4.47) and Lemmas 1 and 2, $P(E_n) = 1 - o(n^{-1})$. From (4.27), (4.28), (4.30) and (4.33), it follows that on the event $E_{n,j,z}$,

$$
W_m^{(1)} - z = -\sigma^{-1} n^{-1/2} \sum_{i=m+1}^{n} \langle X_i - \mu, \nabla g(\mu) \rangle + O\left(n^{-1/2} (\log n)^2\right) + O\left(n^{-1/2} |\delta_{j+1}|\right).
$$

Combining this with (4.49) yields that on $\Omega_n \cap E_{n,j,z}$ with $|z| \le \beta$,

$$
\begin{aligned}
\sup_{m \le r < n} \left| n\widehat{\theta}_n - r\widehat{\theta}_r - \sum_{i=r+1}^{n} \widehat{Y}_{i,m}(a_n, z) \right| \\
= O\left(n^{-4/7}\right) + O\left(n^{-2/3} (\log n)^2\right) + O\left(n^{-2/3} |\delta_{j+1}|\right) \\
+ O\left(n^{-1} \max_{m<s\le n} \left\| \sum_{i=m+1}^{s} (X_i - \mu) \right\|^2\right),
\end{aligned} \tag{4.50}
$$

noting that $\|\sum_{i=m+1}^{n} (X_i - \mu)\| \, \|\sum_{i=s}^{n} (X_i - \mu)\| \le 2\|\sum_{i=m+1}^{n} (X_i - \mu)\|^2 + \|\sum_{i=m+1}^{s-1} \times (X_i - \mu)\|^2$. By (4.9) there exists $E_{n,j,z}^* \subset E_{n,j,z}$ such that

$$
\sup_{|z| \le \beta} \sum_{j \ge 0: |\delta_j| \le n^{1/3}} P(E_{n,j,z} - E_{n,j,z}^*) = o\left(n^{-1}\right), \tag{4.51a}
$$

$$
\max_{m<s\le n} n^{-1} \left\| \sum_{i=m+1}^{s} (X_i - \mu) \right\|^2 < \left(|\delta_j| \vee 1\right) n^{-13/24} \quad \text{on } E_{n,j,z}^*. \tag{4.51b}
$$

From (4.5), (4.50) and (4.51b), it follows that for all $|z| \leq \beta$ and large n,

$$\text{(4.52)} \qquad \sup_{m \leq r < n} \left| n\widehat{\theta}_n - r\widehat{\theta}_r - \sum_{i=r+1}^{n} Y_{i,m}(a_n, z) \right| \leq n^{-1/25} \min\{\delta_{j-1} - \delta_j, \delta_{j+1} - \delta_{j+2}\} \quad \text{on } \Omega_n \cap E^*_{n,j,z}.$$

From (4.51a) and (4.52), (4.11) follows, recalling that $P(E_n) = 1 - o(n^{-1})$. □

PROOF OF LEMMA 9. We shall use the same notation as that in the proofs of Theorem 3 and Lemma 7 and modify the proofs of Lemmas 6 and 8 to prove (4.26). By Lemma 3(ii), there exists an event Ω_N, with $P(\Omega_N) = 1 - o(N^{-1})$, on which (4.49) with $a_n = a$ holds uniformly in $m(= N - k) \leq r \leq n \leq N + N^{1/3}$ and $|z| \leq 2\beta$. Let

$$D_N = \bigcap_{m \leq n \leq N + N^{1/3}} \left\{ \left\| \sum_{i=m+1}^{n} (X_i - \mu) \right\| \leq n^{1/3},\ \|W_m\| \leq \log m, \right. \left. \frac{\sqrt{n}}{\sigma} \left| g\left(\frac{S_n}{n}\right) - g\left(\frac{S_n + m^{-2}\xi}{n}\right) \right| \leq n^{-2} \right\}.$$

By (4.47) and Lemmas 1 and 2, $P(D_N) = 1 - o(N^{-1})$.

Fix $\nu \in \{1, \dots, N^{1/3}\}$ and $z \in [-\beta, \beta]$, and simply write c instead of $c(\nu, z)$ and t instead of $t_{N+\nu,a} = \{a - \theta(N + \nu)\}/(\sigma\sqrt{N + \nu})$. Note the relationship between t and z via $N = N_{a,z} = [\gamma(a, z)]$. Define c_j by (4.5) [in which $c = c(\nu, z)$] and let

$$D_{\nu,j,t} = D_N \cap \left\{ t + \frac{c_j}{\sigma\sqrt{N + \nu}} \leq \frac{\sqrt{N + \nu}}{\sigma}(\widehat{\theta}_{N+\nu} - \theta) < t + \frac{c_{j+1}}{\sigma\sqrt{N + \nu}}, \right. \left. (N + \nu)\widehat{\theta}_{N+\nu} - (N + \nu - 1)\widehat{\theta}_{N+\nu-1} \geq c \right\}.$$

From (4.49), it follows that on $\Omega_N \cap D_N$,

$$n\widehat{\theta}_n - r\widehat{\theta}_r = \sum_{i=r+1}^{n} Y_i + O(N^{-1/6} \log N) \quad \text{uniformly in } m \leq r < n \leq N + N^{1/3},$$

recalling that $Y_i = \theta + \langle X_i - \mu, \nabla g(\mu) \rangle$. Therefore, for all large N, $Y_{N+\nu} \geq c - 1$ on $\Omega_N \cap D_N \cap \{(N + \nu)\widehat{\theta}_{N+\nu} - (N + \nu - 1)\widehat{\theta}_{N+\nu-1} \geq c\}$. It will be shown later that

$$\text{(4.53)} \qquad \sup_{|t| \leq 2\beta} \sum_{j \geq 0: c_j \leq 2N^{1/3}} P\left\{ \left\| \sum_{i=m+1}^{n} (X_i - \mu) \right\| \leq n^{1/3},\ Y_n \geq c - 1, \right. t + \frac{c_j}{\sigma\sqrt{n}} \leq \frac{\sqrt{n}}{\sigma}(\widehat{\theta}_n - \theta) < t + \frac{c_{j+1}}{\sigma\sqrt{n}}, \left. \max_{m < r \leq n} n^{-1} \left\| \sum_{i=m+1}^{r} (X_i - \mu) \right\|^2 \geq (c_j \vee 1) n^{-13/24} \right\}$$

$$= o\big((1 + c)^{-2} n^{-1}\big) \quad \text{uniformly in } m < n \leq N + N^{1/3} \text{ and } 0 \leq c \leq N^{1/3}.$$

Hence, as in (4.51), there exists $D^*_{\nu,j,t} \subset D_{\nu,j,t}$ such that uniformly in $0 \le c \le N^{1/3}$,

$$\text{(4.54a)} \qquad \sup_{\substack{|t| \le 2\beta \\ 1 \le \nu \le N^{1/3}}} \sum_{j \ge 0: c_j \le 2N^{1/3}} P\big(D_{\nu,j,t} - D^*_{\nu,j,t}\big) = o\big((1+c)^{-2}N^{-1}\big),$$

$$\text{(4.54b)} \qquad \max_{m < r \le N+\nu} (N+\nu)^{-1} \left\| \sum_{i=m+1}^{r} (X_i - \mu) \right\|^2 < (c_j \vee 1)(N+\nu)^{-13/24} \quad \text{on } D^*_{\nu,j,t}.$$

Therefore, we can use an argument similar to that for (4.52) to show that on $\Omega_N \cap D^*_{\nu,j,t}$,

$$\sup_{m \le r < N+\nu} \left| (N+\nu)\widehat{\theta}_{N+\nu} - r\widehat{\theta}_r - \sum_{i=r+1}^{N+\nu} \widehat{Y}_{i,m}(a,t) \right| < N^{-1/25} \min\{c_j - c_{j-1}, c_{j+2} - c_{j+1}\},$$

for $1 \le \nu \le N^{1/3}$ and all large N. Combining this with (4.54a) and an argument analogous to that of (4.15) and (4.16), we obtain that uniformly in $|t| \le 2\beta$,

$$\begin{aligned}
&\sum_{\nu=1}^{[N^{1/3}]} \sum_{j \ge 0: c_j \le (N+\nu)^{1/3}} P\Bigg\{ \left\| \sum_{i=m+1}^{N+\nu} (X_i - \mu) \right\| \le (N+\nu)^{1/3}, \\
&\qquad \min_{m < r \le N+\nu} \sum_{i=r}^{N+\nu} Y_{i,m}(a,t) \ge c_{j+2}, \\
&\qquad t + \frac{c_j}{\sigma\sqrt{N+\nu}} \le \frac{\sqrt{N+\nu}}{\sigma}\big(\widehat{\theta}_{N+\nu} - \theta\big) < t + \frac{c_{j+1}}{\sigma\sqrt{N+\nu}} \Bigg\} \\
&\quad + \sum_{\nu=1}^{[N^{1/3}]} o\big((1+c)^{-2}N^{-1}\big) + o\big(N^{-1}\big) \\
\text{(4.55)} \quad &\le \sum_{\nu=1}^{[N^{1/3}]} P\Big\{ \max_{m \le r < N+\nu} r\widehat{\theta}_r < a,\ a + (N+\nu)^{1/3} > (N+\nu)\widehat{\theta}_{N+\nu} \ge a + c \Big\} \\
&\le \sum_{\nu=1}^{[N^{1/3}]} \sum_{j \ge 0: c_j \le (N+\nu)^{1/3}} P\Bigg\{ \left\| \sum_{i=m+1}^{N+\nu} (X_i - \mu) \right\| \le (N+\nu)^{1/3}, \\
&\qquad \min_{m < r \le N+\nu} \sum_{i=r}^{N+\nu} Y_{i,m}(a,t) \ge c_{j-1}, \\
&\qquad t + \frac{c_j}{\sigma\sqrt{N+\nu}} \le \frac{\sqrt{N+\nu}}{\sigma}(\widehat{\theta}_{N+\nu} - \theta) \\
&\qquad\qquad < t + \frac{c_{j+1}}{\sigma\sqrt{N+\nu}} \Bigg\} \\
&\quad + \sum_{\nu=1}^{[N^{1/3}]} o\big((1+c)^{-2}N^{-1}\big) + o\big(N^{-1}\big),
\end{aligned}$$

where the $o(N^{-1})$ term represents an upper bound of the probability of the complement of $\Omega_N \cap D_N$. From (4.18), it follows that $\phi(t)(=\phi(t_{N+\nu,a}))=\phi(z)-N^{-1/2}z\phi(z)(\rho-\sigma^{-1}\theta\nu)+O(N^{-5/6})$ uniformly in $|z|\le\beta$ and $1\le\nu\le N^{1/3}$. Because $c=c(\nu,z)=(\nu-\delta)(\theta+z\sigma/2\sqrt{N})+O(\nu N^{-7/8})$ by (4.22), the desired conclusion (4.26) follows from (4.55) and (4.6), making use of arguments like those in (4.21), (4.37), (4.44) and (4.45), and noting that $\sup_{|z|\le\beta}\sum_{\nu=1}^{\infty}(1+c(\nu,z))^{-2}=\sum_{\nu=1}^{\infty}O(\nu^{-2})<\infty$.

It remains to prove (4.53), which is a refinement of (4.8). Note that for $m<n\le N+N^{1/3}$,

$$\begin{aligned}
P&\left\{\max_{1\le s\le n-m} n^{-1}\left\|\sum_{i=m+1}^{m+s}(X_i-\mu)\right\|^2\ge(c_j\vee 1)n^{13/24},\ Y_n\ge c-1\right\}\\
&\le P\left\{\|X_n-\mu\|\ge\left(c_j^{1/2}\vee 1\right)n^{11/48}/2\right\}\\
&\quad+P\left\{\max_{1\le s<n-m}\left\|\sum_{i=m+1}^{m+s}(X_i-\mu)\right\|\ge\left(c_j^{1/2}\vee 1\right)n^{11/48}/2\right\}P\{Y_n\ge c-1\}.
\end{aligned}$$

Moreover, $P\{Y\ge c-1\}=O((1+c)^{-4})$ and as $n\to\infty$,

$$\begin{aligned}
&\sum_{j=0}^{\infty}(c_{j+1}-c_j)n^{-1/2}P\left\{\|X-\mu\|\ge\left(c_j^{1/2}\vee 1\right)n^{11/48}/2\right\}\\
&\quad\le n^{-1/2}\int_{c-(c+1)(n\log n)^{1/2}}^{\infty}P\{\|X-\mu\|\ge(\sqrt{u}\vee 1)n^{11/48}/2\}\,du\\
&\quad=O\left(n^{-17/12}(1+c)^{-1}\right)\\
&\quad=o\left((1+c)^{-2}n^{-1}\right)
\end{aligned}$$

uniformly in $0\le c\le 4n^{1/3}$. Hence a straightforward modification of the proof of (4.8) can be used to prove (4.53). □

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DEPARTMENT OF STATISTICS
STANFORD UNIVERSITY
STANFORD, CALIFORNIA 94305