

LÉVY MEASURES OF INFINITELY DIVISIBLE RANDOM VECTORS AND SLEPIAN INEQUALITIES¹

GENNADY SAMORODNITSKY AND MURAD S. TAQQU

Cornell University and Boston University

We study Slepian inequalities for general non-Gaussian infinitely divisible random vectors. Conditions for such inequalities are expressed in terms of the corresponding Lévy measures of these vectors. These conditions are shown to be nearly best possible, and for a large subfamily of infinitely divisible random vectors these conditions are necessary and sufficient for Slepian inequalities. As an application we consider symmetric α -stable Ornstein–Uhlenbeck processes and a family of infinitely divisible random vectors introduced by Brown and Rinott.

1. Introduction. Let \mathbf{X} and \mathbf{Y} be two random vectors in R^d . If for any $\lambda \in R^d$,

$$(1.1) \quad P(\mathbf{X} > \lambda) \geq P(\mathbf{Y} > \lambda),$$

then the random vectors \mathbf{X} and \mathbf{Y} are said to satisfy the *right* Slepian inequality. If for any $\lambda \in R^d$,

$$(1.2) \quad P(\mathbf{X} < \lambda) \geq P(\mathbf{Y} < \lambda),$$

then the random vectors \mathbf{X} and \mathbf{Y} are said to satisfy the *left* Slepian inequality. [Throughout this paper the notation $\mathbf{x} > \mathbf{y}$ for $\mathbf{x} = (x_1, \dots, x_d)$, $\mathbf{y} = (y_1, \dots, y_d)$ means that $x_i > y_i$ for every $i = 1, \dots, d$, whereas the notation $\mathbf{x} \geq \mathbf{y}$ means that $x_i \geq y_i$ for every $i = 1, \dots, d$, and similarly with reverse inequalities.] If \mathbf{X} and \mathbf{Y} satisfy both (1.1) and (1.2), then we say that these two vectors satisfy the two-sided Slepian inequality.

Because probability measures are continuous from above and from below, it is clear that \mathbf{X} and \mathbf{Y} satisfy the right Slepian inequality if and only if for any $\lambda \in R^d$,

$$(1.3) \quad P(\mathbf{X} \geq \lambda) \geq P(\mathbf{Y} \geq \lambda),$$

and they satisfy the left Slepian inequality if and only if for any $\lambda \in R^d$,

$$(1.4) \quad P(\mathbf{X} \leq \lambda) \geq P(\mathbf{Y} \leq \lambda).$$

As a result, we may switch from one version of the Slepian inequalities to the other depending on the situation.

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Our terminology is (only) slightly unorthodox. First, the original Slepian inequality [15] has been proved for centered Gaussian vectors. Second, (1.1) and (1.2) are equivalent if the random vectors \mathbf{X} and \mathbf{Y} are symmetric, but because (1.1) and (1.2) are, in general, not always equivalent, it is necessary to consider both inequalities. The right (left) Slepian inequality has the following interpretation: the probability that the components are \mathbf{X} are all very large (very small) is greater than the corresponding probability for \mathbf{Y} . Hence the components of \mathbf{X} are “more positively dependent” than those of \mathbf{Y} .

The original Slepian result for centered Gaussian vectors can be formulated as follows. If \mathbf{X} and \mathbf{Y} are two zero mean normal random vectors, then (1.1) and (1.2) hold if and only if

$$(1.5) \quad \begin{aligned} EX_i^2 &= EY_i^2 \quad \text{for each } i = 1, \dots, d, \\ E(X_i X_j) &\geq E(Y_i Y_j) \quad \text{for every } i, j = 1, \dots, d. \end{aligned}$$

This result has been used extensively for studying Gaussian processes, especially their sample paths (see, e.g., [7], [1] and [10]). Its importance has generated a lot of interest in extensions to as wide a class of (non-Gaussian) stochastic processes as possible. The task, however, has turned out not to be easy. The main difficulty seems to be that natural extensions of the conditions (1.5) do not, in general, suffice for Slepian inequalities. This is not surprising if one remembers that “a few numbers” like those appearing in (1.5) do characterize a Gaussian law, but this is not usually the case for many other laws of interest.

A weaker version of Slepian inequality, the Sudakov version of Slepian’s comparison principle, for Gaussian random vectors ([7]) was extended to symmetric α -stable random vectors with $\alpha > 1$ by Marcus and Pisier [11] and to general *type G* infinitely divisible random vectors by Samorodnitsky and Taqqu [14]. However, these extensions are not entirely satisfactory, in the sense that the result of Marcus and Pisier involves a dimension-dependent numerical constant that blows up as the dimension increases, and the conditions under which the result of Samorodnitsky and Taqqu holds are often difficult to verify.

The only instance known to the authors of a successful nontrivial extension of the full Slepian inequality to non-Gaussian situations is due to Brown and Rinott [3]. It deals with an especially simple family of infinitely divisible random vectors. Our wish to understand the basic features that make the Slepian inequality work in this case has led us to the present research. We discuss the Brown and Rinott family later in the paper.

Our goal is to extend the Slepian inequality, that is the conclusions (1.2) and (1.1), to the class of all infinitely divisible random vectors. Our conditions involve comparison, not of covariances as in (1.5), but of Lévy measures. This is, of course, quite natural for this type of random vector.

Our overall approach is akin to that used in [12] and [9] under different circumstances. Professor Pitt kindly pointed out to us the potential usefulness of his approach to our problem.

The paper is organized as follows. In Section 2, we consider a particular subfamily of infinitely divisible random vectors for which our main results are

especially transparent (and complete). In Section 3, the main results are given in the general infinitely divisible case, and Sections 4 and 5 contain examples: in Section 4, we specialize our results to the symmetric α -stable case and treat, in particular, symmetric α -stable Ornstein–Uhlenbeck processes, whereas in Section 5, we discuss the Brown and Rinott [3] family of infinitely divisible random vectors.

2. Slepian inequalities for infinitely divisible random vectors. A particular case. We consider in this section a subfamily of d -dimensional infinitely divisible random vectors $\mathbf{X} = (X_1, \dots, X_d)$ without a Gaussian component, whose characteristic function $\phi_{\mathbf{X}}(\boldsymbol{\theta}) = E \exp i(\boldsymbol{\theta}, \mathbf{X})$ has the form

$$(2.1) \quad \phi_{\mathbf{X}}(\boldsymbol{\theta}) = \exp \left\{ \int_{R^d - \{0\}} \left(\exp(i(\boldsymbol{\theta}, \mathbf{x})) - 1 \right) \nu(d\mathbf{x}) + i(\boldsymbol{\theta}, \mathbf{c}) \right\}.$$

The Lévy measure ν of \mathbf{X} in (2.1) satisfies

$$(2.2) \quad \int_{R^d - \{0\}} (1 \wedge \|\mathbf{x}\|) \nu(d\mathbf{x}) < \infty$$

and $\mathbf{c} \in R^d$ is the *shift vector* of \mathbf{X} .

Our main results are more transparent when condition (2.2) holds, and this case will be considered first. The general infinitely divisible case will be treated in the next section.

Infinitely divisible random vectors satisfying (2.2) have the nice property that if K is a closed convex cone in R^d supporting the Lévy measure ν of \mathbf{X} and if the shift vector $\mathbf{c} \in K$, then $P(\mathbf{X} \in K) = 1$. Examples include compound Poisson random vectors and α -stable random vectors with $0 < \alpha < 1$.

For a given infinitely divisible random vector \mathbf{X} with characteristic function $\phi_{\mathbf{X}}(\boldsymbol{\theta})$ given by (2.1) and $t \geq 0$, let P^t denote the distribution of an infinitely divisible random vector with characteristic function $(\phi_{\mathbf{X}}(\boldsymbol{\theta}))^t$. Then $P^t * P^s = P^{s+t}$ for any $t, s \geq 0$, where $*$ denotes convolution of probability measures. Moreover, letting $\{\mathbf{X}^{*t}, t \geq 0\}$ be a process with stationary independent increments satisfying $\mathbf{X}^{*0} = 0$ and $\mathbf{X}^{*1} =_d \mathbf{X}$, we have $\mathbf{X}^{*t} \sim P^t$ for $t > 0$. We refer to $\{P^t, t \geq 0\}$ as the convolution semigroup generated by the infinitely divisible random vector \mathbf{X} , and these semigroups play an important role in our arguments. Recall that the generator G of the convolution semigroup $\{P^t, t \geq 0\}$ generated by \mathbf{X} whose Lévy measure satisfies (2.2) can be written in the form

$$(2.3) \quad Gg(\mathbf{y}) = \int_{R^d - \{0\}} (g(\mathbf{x} + \mathbf{y}) - g(\mathbf{y})) \nu(d\mathbf{x}) + (\mathbf{c}, \Delta g(\mathbf{y}))$$

for $\mathbf{y} \in R^d$, where $g: R^d \rightarrow R$ is in the domain \mathcal{D} of the generator G . Recall further that any $g \in C_b^\infty$ (the space of all infinitely differentiable functions $R^d \rightarrow R$ with bounded derivatives) is in the domain \mathcal{D} (see, e.g., [8]).

To maintain the view of the forest beyond the trees, we describe the basic ideas behind our results. Looking back at the Gaussian conditions (1.5) one

observes that they imply that the components of the random vector \mathbf{X} “cluster together” more than the components of the random vector \mathbf{Y} do. The Slepian inequalities (1.1) and (1.2) may then be regarded as an expression of that “clustering.” In the non-Gaussian infinitely divisible case, criteria of clustering are naturally related to Lévy measures. Specifically, let \mathbf{X} and \mathbf{Y} be two infinitely divisible random vectors with Lévy measures $\nu_{\mathbf{X}}$ and $\nu_{\mathbf{Y}}$ satisfying (2.2) and shift vectors $\mathbf{c}_{\mathbf{X}}$ and $\mathbf{c}_{\mathbf{Y}}$ accordingly. If one is interested, say, in the *right* Slepian inequality (1.1), it is intuitive then that the appropriate clustering requirement on the Lévy measures should be

$$(2.4) \quad \nu_{\mathbf{X}}\{\mathbf{x} \in R^d: \mathbf{x} > \lambda\} \geq \nu_{\mathbf{Y}}\{\mathbf{x} \in R^d: \mathbf{x} > \lambda\}$$

for every $\lambda \in R^d$, or a version of this condition with nonstrict inequalities. The condition (2.4), however, can be awkward when $\lambda \in R^d_- := \{\mathbf{x} \in R^d - \{\mathbf{0}\}: \mathbf{x} \leq \mathbf{0}\}$ because the origin, which plays a special role for Lévy measures, belongs to the set $\{\mathbf{x} \in R^d: \mathbf{x} > \lambda\}$. For example, $\nu_{\mathbf{X}}$ and $\nu_{\mathbf{Y}}$ may have an infinite mass in a neighborhood of the origin. We will therefore suppose

$$(2.5) \quad \nu_{\mathbf{X}}\{\mathbf{x} \in R^d: \mathbf{x} > \lambda\} \geq \nu_{\mathbf{Y}}\{x \in R^d: \mathbf{x} > \lambda\} \quad \text{for every } \lambda \in R^d - R^d_-,$$

to which we add the “complementary” condition

$$(2.6) \quad \nu_{\mathbf{X}}\{\mathbf{x} \in R^d: \mathbf{x} \not\geq \lambda\} \leq \nu_{\mathbf{Y}}\{\mathbf{x} \in R^d: \mathbf{x} \not\geq \lambda\} \quad \text{for every } \lambda \in R^d_-,$$

where $\mathbf{x} \not\geq \mathbf{y}$ for \mathbf{x}, \mathbf{y} in R^d means that $x_i < y_i$ for at least one $i = 1, \dots, d$, and similarly with $\mathbf{x} \not\leq \mathbf{y}$. We will also have to take into account the effect of the shift vectors.

In fact, further reflection on conditions (2.5)–(2.6) tells us that these conditions should properly be regarded as corresponding not to (1.1) *alone*, but rather to the *right* Slepian inequality for the whole families of infinitely divisible random vectors arising from the corresponding convolution semigroups. Specifically, let $P_{\mathbf{X}}^t$ and $P_{\mathbf{Y}}^t$ be the convolution semigroups generated by the infinitely divisible random vectors \mathbf{X} and \mathbf{Y} , respectively, and let $\mathbf{X}^{*t} \sim P_{\mathbf{X}}^t$ and $\mathbf{Y}^{*t} \sim P_{\mathbf{Y}}^t$ for $t > 0$ ($\mathbf{X} =_d \mathbf{X}^{*1}$). Then (2.5) and (2.6) should be regarded as corresponding to the family of *right* Slepian inequalities

$$(2.7) \quad P(\mathbf{X}^{*t} > \lambda) \geq P(\mathbf{Y}^{*t} > \lambda) \quad \text{for every } \lambda \in R^d$$

for all $t > 0$. The distinction between (1.1) and (2.7) is critical and we will have more to say about this point in the sequel.

THEOREM 2.1. *Let \mathbf{X} and \mathbf{Y} be two infinitely divisible random vectors in R^d with Lévy measures $\nu_{\mathbf{X}}$ and $\nu_{\mathbf{Y}}$ satisfying (2.2) with characteristic functions given in the form (2.1) and shift vectors $\mathbf{c}_{\mathbf{X}}$ and $\mathbf{c}_{\mathbf{Y}}$, respectively.*

(i) *The following are equivalent:*

(a) *Conditions (2.5) and (2.6) hold and*

$$(2.8) \quad \mathbf{c}_{\mathbf{X}} \geq \mathbf{c}_{\mathbf{Y}}.$$

(b) For every $\lambda \in R^d$, the right Slepian inequalities (2.7) hold for all $t > 0$.

(ii) The following are equivalent:

(a)

$$(2.9) \quad \nu_{\mathbf{X}}\{\mathbf{x} \in R^d: \mathbf{x} < \lambda\} \geq \nu_{\mathbf{Y}}\{\mathbf{x} \in R^d: \mathbf{x} < \lambda\} \quad \text{for every } \lambda \in R^d - R^d_+,$$

$$(2.10) \quad \nu_{\mathbf{X}}\{\mathbf{x} \in R^d: \mathbf{x} \not\leq \lambda\} \leq \nu_{\mathbf{Y}}\{\mathbf{x} \in R^d: \mathbf{x} \not\leq \lambda\} \quad \text{for every } \lambda \in R^d_+$$

and

$$(2.11) \quad \mathbf{c}_{\mathbf{X}} \leq \mathbf{c}_{\mathbf{Y}}.$$

(b) For every $\lambda \in R^d$,

$$(2.12) \quad P(\mathbf{X}^{*t} < \lambda) \geq P(\mathbf{Y}^{*t} < \lambda) \quad \text{for every } \lambda \in R^d$$

for all $t > 0$.

REMARK. The pair of assumptions (2.5) and (2.6) is equivalent to the following assumption, which is in certain circumstances more tractable than the former: For every random vector $\mathbf{W} \in R^d$ whose coordinates W_1, \dots, W_d have bounded densities,

$$(2.13) \quad \int_{R^d - \{0\}} (P(\mathbf{W} < \mathbf{x}) - P(\mathbf{W} < \mathbf{0})) \nu_{\mathbf{X}}(d\mathbf{x}) \geq \int_{R^d - \{0\}} (P(\mathbf{W} < \mathbf{x}) - P(\mathbf{W} < \mathbf{0})) \nu_{\mathbf{Y}}(d\mathbf{x}).$$

Note that both sides of (2.13) are well defined and finite. To verify the equivalence, observe that

$$(2.14) \quad \begin{aligned} & \int_{\mathbf{x} \in R^d - \{0\}} (P(\mathbf{W} < \mathbf{x}) - P(\mathbf{W} < \mathbf{0})) \nu_{\mathbf{X}}(d\mathbf{x}) \\ &= \int_{\mathbf{x} \in R^d - \{0\}} \left[\int_{R^d} (\mathbf{1}(\mathbf{w} < \mathbf{x}) \mathbf{1}(\mathbf{w} \not\leq \mathbf{0}) - \mathbf{1}(\mathbf{w} \not\leq \mathbf{x}) \mathbf{1}(\mathbf{w} < \mathbf{0})) F_{\mathbf{W}}(d\mathbf{w}) \right] \nu_{\mathbf{X}}(d\mathbf{x}) \\ &= \int_{R^d - R^d_-} \nu_{\mathbf{X}}\{\mathbf{x} \in R^d - \{0\}: \mathbf{x} > \mathbf{w}\} F_{\mathbf{W}}(d\mathbf{w}) \\ & \quad - \int_{R^d_-} \nu_{\mathbf{X}}\{\mathbf{x} \in R^d - \{0\}: \mathbf{x} \not\leq \mathbf{w}\} F_{\mathbf{W}}(d\mathbf{w}), \end{aligned}$$

where $F_{\mathbf{W}}$ is the probability law of \mathbf{W} . The implication (2.5) + (2.6) \Rightarrow (2.13) is now obvious. Let us check the converse implication. To establish (2.6) (say), pick first a λ in R^d_- such that both $\lambda < \mathbf{0}$ and λ is a continuity point of both $\nu_{\mathbf{X}}$ and $\nu_{\mathbf{Y}}$. That is, $\nu_{\mathbf{X}}\{\mathbf{x}: x_i = \lambda_i \text{ for some } i\} = 0$, and similarly for $\nu_{\mathbf{Y}}$. (Let us agree to call such points "nice.") For all $\varepsilon > 0$ small enough, the cube $C(\varepsilon) =$

$\prod_1^d(\lambda_i - \varepsilon, \lambda_i + \varepsilon)$ is entirely in R_-^d . Let \mathbf{W} have the uniform distribution over $C(\varepsilon)$. Then (2.13) implies

$$\begin{aligned} & \int_{[-1, 1]^d} \nu_{\mathbf{X}}\{\mathbf{x} \in R^d - \{\mathbf{0}\}: \mathbf{x} \not\geq \lambda + \varepsilon\mathbf{y}\} d\mathbf{y} \\ & \leq \int_{[-1, 1]^d} \nu_{\mathbf{Y}}\{\mathbf{x} \in R^d - \{\mathbf{0}\}: \mathbf{x} \not\geq \lambda + \varepsilon\mathbf{y}\} d\mathbf{y}. \end{aligned}$$

Because λ is nice, we recover (2.6) for all nice $\lambda \in R_-^d$ by letting ε go to zero. Observe now that nice points are dense in R_-^d . For any $\lambda \in R_-^d$ choose a sequence of nice points $\lambda_i, i = 1, 2, \dots$, converging to λ from below. Then

$$\nu_{\mathbf{X}}\{\mathbf{x} \in R^d - \{\mathbf{0}\}: \mathbf{x} \not\geq \lambda_i\} \uparrow \nu_{\mathbf{X}}\{\mathbf{x} \in R^d - \{\mathbf{0}\}: \mathbf{x} \not\geq \lambda\}$$

and we obtain (2.6) for all $\lambda \in R_-^d$. One can show in the same manner that (2.13) implies (2.5) for all $\lambda \in R^d - R_-^d$.

Similarly, the pair of assumptions (2.9) and (2.10) is equivalent to the following assumption: For every random vector $\mathbf{W} \in R^d$ whose coordinates have bounded densities,

$$(2.15) \quad \begin{aligned} & \int_{R^d - \{\mathbf{0}\}} (P(\mathbf{W} > \mathbf{x}) - P(\mathbf{W} > \mathbf{0})) \nu_{\mathbf{X}}(d\mathbf{x}) \\ & \geq \int_{R^d \setminus \{\mathbf{0}\}} (P(\mathbf{W} > \mathbf{x}) - P(\mathbf{W} > \mathbf{0})) \nu_{\mathbf{Y}}(d\mathbf{x}). \end{aligned}$$

We now prove the sufficiency part of Theorem 2.1. The necessity part will be proven in Theorem 3.1.

PROOF. We may and will assume that $\mathbf{c}_{\mathbf{X}} = \mathbf{c}_{\mathbf{Y}} = \mathbf{0}$.

(i) It is obviously enough to prove (2.7) for $t = 1$ [that is, to prove (1.1)].

Fix a $\lambda \in R^d$ and let $f(\mathbf{x}) = \prod_{i=1}^d \mathbf{1}(x_i > \lambda_i), \mathbf{x} \in R^d$. Fix an $\varepsilon > 0$. For each $i = 1, \dots, d$ there clearly is a nondecreasing function $f_{i, \varepsilon}: R \rightarrow R$ in C_b^∞ (of R) such that $\lim_{x \rightarrow -\infty} f_{i, \varepsilon}(x) = 0, \lim_{x \rightarrow \infty} f_{i, \varepsilon}(x) = 1$ and such that

$$|f_{i, \varepsilon}(x) - \mathbf{1}(x > \lambda_i)| \leq \varepsilon$$

for every $x \notin E_{i, \varepsilon}$, where $E_{i, \varepsilon}$ is a Borel set such that

$$P(X_i \in E_{i, \varepsilon}) \leq \varepsilon \quad \text{and} \quad P(Y_i \in E_{i, \varepsilon}) \leq \varepsilon.$$

Further, let $f_\varepsilon(\mathbf{x}) = \prod_{i=1}^d f_{i, \varepsilon}(x_i)$ and observe that $f_\varepsilon \in C_b^\infty$ of R^d , that $|f_\varepsilon(\mathbf{x})| \leq 1$ for every $\mathbf{x} \in R^d$ and, moreover,

$$(2.16) \quad |f(\mathbf{x}) - f_\varepsilon(\mathbf{x})| \leq \varepsilon d$$

for every $\mathbf{x} \notin E_\varepsilon := \{\mathbf{x} \in R^d: x_i \in E_{i,\varepsilon} \text{ for some } i = 1, \dots, d\}$. We remark further that

$$(2.17) \quad P(\mathbf{X} \in E_\varepsilon) \leq \varepsilon d \quad \text{and} \quad P(\mathbf{Y} \in E_\varepsilon) \leq \varepsilon d.$$

Write $f_{i,\varepsilon}(x) = H_i((-\infty, x])$, where for $i = 1, \dots, d$, H_i is a probability measure on R with a bounded density with respect to Lebesgue measure. Then $H = H_1 \times \dots \times H_d$ is a probability measure on R^d . Let $P_{\mathbf{X}}^t$ and $P_{\mathbf{Y}}^t$ be the convolution semigroups generated by \mathbf{X} and \mathbf{Y} accordingly and let $G_{\mathbf{X}}$ and $G_{\mathbf{Y}}$ be the corresponding generators. Note that for every $t > 0$ and $\mathbf{x} \in R^d$,

$$(2.18) \quad P_{\mathbf{Y}}^t f_\varepsilon(\mathbf{x}) := E f_\varepsilon(\mathbf{Y}^{*t} + \mathbf{x}) = E \int_{R^d} \mathbf{1}(\mathbf{z} \leq \mathbf{Y}^{*t} + \mathbf{x}) H(d\mathbf{z}) = P(\mathbf{W} \leq \mathbf{x}),$$

where $\mathbf{W} = \mathbf{Z} - \mathbf{Y}^{*t}$ and \mathbf{Z} is an R^d -valued random vector with the law H independent of \mathbf{Y}^{*t} . We conclude by (2.3) and (2.18) that for every $\mathbf{y} \in R^d$,

$$(2.19) \quad \begin{aligned} G_{\mathbf{X}} P_{\mathbf{Y}}^t f_\varepsilon(\mathbf{y}) &= \int_{R^d - \{0\}} (P_{\mathbf{Y}}^t f_\varepsilon(\mathbf{x} + \mathbf{y}) - P_{\mathbf{Y}}^t f_\varepsilon(\mathbf{y})) \nu_{\mathbf{X}}(d\mathbf{x}) \\ &= \int_{R^d - \{0\}} (P(\mathbf{W} \leq \mathbf{x} + \mathbf{y}) - P(\mathbf{W} \leq \mathbf{y})) \nu_{\mathbf{X}}(d\mathbf{x}), \end{aligned}$$

and similarly

$$(2.20) \quad G_{\mathbf{Y}} P_{\mathbf{Y}}^t f_\varepsilon(\mathbf{y}) = \int_{R^d - \{0\}} (P(\mathbf{W} \leq \mathbf{x} + \mathbf{y}) - P(\mathbf{W} \leq \mathbf{y})) \nu_{\mathbf{Y}}(d\mathbf{x}).$$

The distribution of \mathbf{W} has bounded marginals because it is obtained by convolving the distribution of \mathbf{Z} , which has bounded marginals. Recall that (2.5) and (2.6) are equivalent to (2.13). It then follows from (2.19) and (2.20) that for every $t > 0$ and $\mathbf{y} \in R^d$,

$$(2.21) \quad G_{\mathbf{X}} P_{\mathbf{Y}}^t f_\varepsilon(\mathbf{y}) \geq G_{\mathbf{Y}} P_{\mathbf{Y}}^t f_\varepsilon(\mathbf{y}).$$

Define $h_\varepsilon(t) = P_{\mathbf{X}}^t P_{\mathbf{Y}}^{1-t} f_\varepsilon(\mathbf{0})$, $0 \leq t \leq 1$. Then for every $0 \leq t \leq 1$,

$$h'_\varepsilon(t) = P_{\mathbf{X}}^t (G_{\mathbf{X}} - G_{\mathbf{Y}}) P_{\mathbf{Y}}^{1-t} f_\varepsilon(\mathbf{0}).$$

An immediate conclusion of (2.21) is that $h'_\varepsilon(t) \geq 0$ for every $0 < t < 1$ and so, in particular,

$$(2.22) \quad E f_\varepsilon(\mathbf{X}^{*1}) = h_\varepsilon(1) \geq h_\varepsilon(0) = E f_\varepsilon(\mathbf{Y}^{*1}).$$

Now (2.16) and (2.17) imply $E f(\mathbf{X}^{*1}) \geq E f(\mathbf{Y}^{*1}) - 4\varepsilon d$. That is, for any $\lambda \in R^d$,

$$P(\mathbf{X} > \lambda) \geq P(\mathbf{Y} > \lambda) - 4\varepsilon d.$$

Because this is true for every $\varepsilon > 0$, (1.1) follows.

The proof of the second part of the theorem is identical. \square

3. Slepian inequalities for infinitely divisible random vectors. The general case. The characteristic function of a general d -dimensional infinitely divisible random vector $\mathbf{X} = (X_1, \dots, X_d)$ without a Gaussian component has the form

$$(3.1) \quad \phi_{\mathbf{X}}(\boldsymbol{\theta}) = \exp \left\{ \int_{R^d - \{0\}} \left(\exp(i(\boldsymbol{\theta}, \mathbf{x})) - 1 - i\mathbf{1}(\|\mathbf{x}\| \leq 1)(\boldsymbol{\theta}, \mathbf{x}) \right) \nu(d\mathbf{x}) + i(\boldsymbol{\theta}, \mathbf{b}) \right\}.$$

The Lévy measure of \mathbf{X} , ν , is a σ -finite measure on the Borel subsets of $R^d - \{0\}$ such that $\int_{R^d - \{0\}} (1 \wedge \|\mathbf{x}\|^2) \nu(d\mathbf{x}) < \infty$, and we will also call (with some abuse of terminology) the constant vector \mathbf{b} the shift vector of \mathbf{X} . Incidentally, this name is more appropriate for the vector \mathbf{c} in (2.1) because the vector \mathbf{b} in (3.1) depends on the somewhat arbitrary compensator $\mathbf{1}(\|\mathbf{x}\| \leq 1)(\boldsymbol{\theta}, \mathbf{x})$.

As in the previous section, for a given general infinitely divisible random vector \mathbf{X} with characteristic function $\phi_{\mathbf{X}}(\boldsymbol{\theta})$ given by (3.1) and $t \geq 0$, let P^t denote the distribution of an infinitely divisible random vector with characteristic function $(\phi_{\mathbf{X}}(\boldsymbol{\theta}))^t$. Then again $P^t * P^s = P^{s+t}$ for any $t, s \geq 0$, where $*$ denotes convolution of probability measures and we still refer to $\{P^t, t \geq 0\}$ as the convolution semigroup generated by the infinitely divisible random vector \mathbf{X} . As before, $\mathbf{X}^{*t} \sim P^t$ for $t > 0$. Recall that the generator G of the convolution semigroup $\{P^t, t \geq 0\}$ generated by a general infinitely divisible random vector \mathbf{X} can be written in the form

$$(3.2) \quad Gg(\mathbf{y}) = \int_{R^d - \{0\}} \left(g(\mathbf{x} + \mathbf{y}) - g(\mathbf{y}) - \mathbf{1}(\|\mathbf{x}\| \leq 1)(\mathbf{x}, \Delta g(\mathbf{y})) \right) \nu(d\mathbf{x}) + (\mathbf{b}, \Delta g(\mathbf{y})),$$

for $\mathbf{y} \in R^d$, where $g: R^d \rightarrow R$ is in the domain \mathcal{D} of the generator G . As before, any $g \in C_b^\infty$ (the space of all infinitely differentiable functions $R^d \rightarrow R$ with bounded derivatives) is in the domain \mathcal{D} .

Because the Lévy measure ν of an infinitely divisible random vector is finite outside of a neighborhood of the origin, one can transform any infinitely divisible random vector into one satisfying (2.2) by restricting its Lévy measure to the complement of such a neighborhood—a procedure used repeatedly in the sequel. Specifically, given a vector \mathbf{X} with characteristic function (3.1) and a Borel set $A \subset R^d$ such that $B_\delta := \{\mathbf{x} \in R^d: \|\mathbf{x}\| < \delta\} \subset A$ for some $\delta > 0$, we define an infinitely divisible random vector \mathbf{X}^A as having Lévy measure $\nu^A = \mathbf{1}_{A^c} \nu$ and shift vector $\mathbf{b}^A = \mathbf{b}$. Then its characteristic function can be written in the form (2.1) with the shift vector \mathbf{c}^A given by

$$(3.3) \quad c_i^A = b_i - \int_{A^c} \mathbf{1}(\|\mathbf{x}\| \leq 1) x_i \nu(d\mathbf{x}), \quad i = 1, \dots, d.$$

The intuition that lead us to (2.5) and (2.6) in the beginning of the previous section remains perfectly valid in the general setting as well. However, in

the general case for technical reasons, conditions (2.5) and (2.6) will have to be somewhat modified (strengthened, in fact). We do not know, at this point, whether this is intrinsic to the problem or stems from our approach only.

DEFINITION 3.1. A sequence $\{A_n, n \geq 1\}$ of Borel sets in R^d is said to be *deflating to the origin* if

- (i) $A_1 \supset A_2 \supset \dots$.
- (ii) $\bigcap_{n \geq 1} A_n = \{\mathbf{0}\}$.
- (iii) For every $n \geq 1$ there is a $\delta > 0$ such that $B_\delta = \{\mathbf{x} \in R^d: \|\mathbf{x}\| < \delta\} \subset A_n$.

REMARK. A natural way to produce sequences of sets deflating to the origin is to choose sequences of balls of positive radius decreasing to zero in some norm, not necessarily the Euclidean norm $\|\cdot\|$. All the applications of our results considered in this paper use only sets of this kind. It is conceivable, however, that the greater generality may turn out to be useful in future applications.

Given an infinitely divisible random vector \mathbf{X} with Lévy measure ν and shift vector \mathbf{b} and a sequence of sets $\{A_n, n \geq 1\}$ deflating to the origin, we define a sequence of infinitely divisible random vectors $\{\mathbf{X}^{A_n}, n \geq 1\}$ as described above, by restricting the Lévy measure to the complement of the corresponding set A_n . We record at this point the obvious observation that $\mathbf{X}^{A_n} \Rightarrow \mathbf{X}$ as $n \rightarrow \infty$. The strengthening of the assumptions (2.5) and (2.6) previously mentioned amounts to assuming the following: that these assumptions hold for \mathbf{X} and \mathbf{Y} restricted as before to the complements of two (not necessarily identical) sequences of sets deflating to the origin.

The following theorem is the counterpart of the sufficiency part of Theorem 2.1 in the general case and it is, in fact, an easy consequence of the latter.

THEOREM 3.1. Let \mathbf{X} and \mathbf{Y} be two infinitely divisible random vectors in R^d with characteristic functions (3.1) with Lévy measures $\nu_{\mathbf{X}}$ and $\nu_{\mathbf{Y}}$ and shift vectors $\mathbf{b}_{\mathbf{X}}$ and $\mathbf{b}_{\mathbf{Y}}$, respectively.

(i) Suppose there are two sequences of sets $\{A_n, n \geq 1\}$ and $\{B_n, n \geq 1\}$ deflating to the origin such that for every $n \geq 1$, the following three conditions hold:

$$(3.4) \quad \nu_{\mathbf{X}}\{\mathbf{x} \in A_n^c: \mathbf{x} > \lambda\} \geq \nu_{\mathbf{Y}}\{\mathbf{x} \in B_n^c: \mathbf{x} > \lambda\} \quad \text{for every } \lambda \in R^d - R^d_-,$$

$$(3.5) \quad \nu_{\mathbf{X}}\{\mathbf{x} \in A_n^c: \mathbf{x} \not\geq \lambda\} \leq \nu_{\mathbf{Y}}\{\mathbf{x} \in B_n^c: \mathbf{x} \not\geq \lambda\} \quad \text{for every } \lambda \in R^d_-$$

and

$$(3.6) \quad \mathbf{c}_{\mathbf{X}}^{A_n} \geq \mathbf{c}_{\mathbf{Y}}^{B_n}$$

[cf. (3.3)]. Then for every $\lambda \in R^d$, the right Slepian inequalities (2.7) hold for all $t > 0$.

(ii) Suppose there are two sequences of sets $\{A_n, n \geq 1\}$ and $\{B_n, n \geq 1\}$ de-

flating to the origin such that for every $n \geq 1$ the following three conditions hold:

$$(3.7) \quad \nu_{\mathbf{X}}\{\mathbf{x} \in A_n^c: \mathbf{x} < \lambda\} \geq \nu_{\mathbf{Y}}\{\mathbf{x} \in B_n^c: \mathbf{x} < \lambda\} \quad \text{for every } \lambda \in R^d - R_+^d,$$

where $R_+^d := \{\mathbf{x} \in R^d - \{0\}: \mathbf{x} \geq 0\}$,

$$(3.8) \quad \nu_{\mathbf{X}}\{\mathbf{x} \in A_n^c: \mathbf{x} \not\leq \lambda\} \leq \nu_{\mathbf{Y}}\{\mathbf{x} \in B_n^c: \mathbf{x} \not\leq \lambda\} \quad \text{for every } \lambda \in R_+^d$$

and

$$(3.9) \quad \mathbf{c}_{\mathbf{X}}^{A_n} \leq \mathbf{c}_{\mathbf{Y}}^{B_n}.$$

Then for every $\lambda \in R^d$, the left Slepian inequalities (2.12) hold for all $t > 0$.

REMARKS.

1. It is, of course, obvious how Theorem 3.1 follows from the sufficiency part of Theorem 2.1. Indeed, under the assumptions of, say, part (i) of Theorem 3.1 we get from part (i) of Theorem 2.1 that for every $n \geq 1$ and $\lambda \in R^d$,

$$P((\mathbf{X}^{A_n})^{*t} > \lambda) \geq P((\mathbf{Y}^{B_n})^{*t} > \lambda)$$

for all $t > 0$. Because $(\mathbf{X}^{A_n})^{*t} \Rightarrow \mathbf{X}^{*t}$ as $n \rightarrow \infty$ and $(\mathbf{Y}^{B_n})^{*t} \Rightarrow \mathbf{Y}^{*t}$ as $n \rightarrow \infty$, it follows that (2.7) holds for every $t > 0$.

2. Assuming (3.4) and (3.5) for every $n \geq 1$ is, clearly, a stronger assumption than just (2.5) and (2.6) in the sense that the former imply the latter.
3. The argument of the remark following Theorem 2.1 shows that for every $n \geq 1$ the pair of assumptions (3.4) and (3.5) is equivalent to the following assumption: For every random vector $\mathbf{W} \in R^d$ whose coordinates W_1, \dots, W_d have bounded densities,

$$(3.10) \quad \int_{A_n^c} (P(\mathbf{W} < \mathbf{x}) - P(\mathbf{W} < 0)) \nu_{\mathbf{X}}(d\mathbf{x}) \geq \int_{B_n^c} (P(\mathbf{W} < \mathbf{x}) - P(\mathbf{W} < 0)) \nu_{\mathbf{Y}}(d\mathbf{x}).$$

Similarly, for every fixed $n \geq 1$ the pair of assumptions (3.7) and (3.8) is equivalent to the following assumption: For every random vector $\mathbf{W} \in R^d$ with coordinates having bounded densities,

$$(3.11) \quad \int_{A_n^c} (P(\mathbf{W} > \mathbf{x}) - P(\mathbf{W} > 0)) \nu_{\mathbf{X}}(d\mathbf{x}) \geq \int_{B_n^c} (P(\mathbf{W} > \mathbf{x}) - P(\mathbf{W} > 0)) \nu_{\mathbf{Y}}(d\mathbf{x}).$$

4. In the important particular case when the infinitely divisible random vectors \mathbf{X} and \mathbf{Y} are symmetric, it is trivial (but useful) to note in the context of Theorem 3.1 that any choice of the sequences $\{A_n, n \geq 1\}$ and $\{B_n, n \geq 1\}$ such that the sets A_n and B_n are symmetric around the origin, makes the

conditions (3.6) and (3.9) unnecessary. Moreover, in the symmetric case the two parts in each of Theorems 3.1 and 2.1 coalesce, in the sense that the assumptions of either part of the former imply the conclusions of both parts, and the same is true for the latter theorem with the difference that, in this case, even the assumptions of the two parts become identical.

We now turn to *necessary* conditions for Slepian inequalities.

THEOREM 3.2. *Let \mathbf{X} and \mathbf{Y} be two infinitely divisible random vectors with Lévy measures $\nu_{\mathbf{X}}$ and $\nu_{\mathbf{Y}}$ correspondingly.*

(i) *Suppose that for every $\lambda \in R^d$, the right Slepian inequalities (2.7) hold for every $t > 0$. Then (2.5) and (2.6) hold. Moreover, if $\nu_{\mathbf{X}}$ and $\nu_{\mathbf{Y}}$ satisfy (2.2) and $\mathbf{c}_{\mathbf{X}}$ and $\mathbf{c}_{\mathbf{Y}}$ are corresponding shift vectors in the representation (2.1) of the characteristic functions, then (2.8) holds as well.*

(ii) *Suppose that for every $\lambda \in R^d$, the left Slepian inequalities (2.12) hold for every $t > 0$. Then (2.9) and (2.10) hold. Further, if $\nu_{\mathbf{X}}$ and $\nu_{\mathbf{Y}}$ satisfy (2.2) and $\mathbf{c}_{\mathbf{X}}$ and $\mathbf{c}_{\mathbf{Y}}$ are the corresponding shift vectors in (2.1), then (2.11) holds as well.*

PROOF. (i) Let \mathbf{W} be any random vector in R^d . Define $f(\mathbf{x}) = P(\mathbf{W} < \mathbf{x})$, $\mathbf{x} \in R^d$. Because for every $t > 0$,

$$P_{\mathbf{X}}^t f(\mathbf{0}) = E f(\mathbf{X}^{*t}) = EE\mathbf{1}(\mathbf{W} < \mathbf{X}^{*t}) = \int_{R^d} P(\mathbf{X}^{*t} > \mathbf{w})F_{\mathbf{W}}(d\mathbf{w})$$

and

$$P_{\mathbf{Y}}^t f(\mathbf{0}) = \int_{R^d} P(\mathbf{Y}^{*t} > \mathbf{w})F_{\mathbf{W}}(d\mathbf{w}),$$

we conclude by (2.7) that

$$(3.12) \quad P_{\mathbf{X}}^t f(\mathbf{0}) \geq P_{\mathbf{Y}}^t f(\mathbf{0}) \quad \text{for every } t > 0.$$

Suppose now that \mathbf{W} has a C_b^∞ density with respect to the d -dimensional Lebesgue measure. Then $f \in C_b^\infty$ and so it is in the domains of both generators $G_{\mathbf{X}}$ and $G_{\mathbf{Y}}$. It follows from (3.12) and $P_{\mathbf{X}}^0 f(\mathbf{0}) = f(\mathbf{0}) = P_{\mathbf{Y}}^0 f(\mathbf{0})$ that there is a sequence $t_n \downarrow 0$ such that

$$(P_{\mathbf{X}}^t f(\mathbf{0}))' \Big|_{t=t_n} \geq (P_{\mathbf{Y}}^t f(\mathbf{0}))' \Big|_{t=t_n}, \quad n = 1, 2, \dots$$

That is,

$$P_{\mathbf{X}}^{t_n} G_{\mathbf{X}} f(\mathbf{0}) \geq P_{\mathbf{Y}}^{t_n} G_{\mathbf{Y}} f(\mathbf{0})$$

for every $n = 1, 2, \dots$. Letting $n \rightarrow \infty$, we obtain

$$(3.13) \quad G_{\mathbf{X}} f(\mathbf{0}) \geq G_{\mathbf{Y}} f(\mathbf{0}).$$

Using the representation (3.2) of the generators, we conclude that for every random vector $\mathbf{W} \in R^d$ with a C_b^∞ density,

$$\begin{aligned}
 & \int_{R^d} \left(P(\mathbf{W} < \mathbf{x}) - P(\mathbf{W} < \mathbf{0}) - \mathbf{1}(\|\mathbf{x}\| \leq 1)(\mathbf{x}, \Delta F_{\mathbf{W}}(\mathbf{0})) \right) \nu_{\mathbf{X}}(d\mathbf{x}) \\
 (3.14) \quad & + (b_{\mathbf{X}}, \Delta F_{\mathbf{W}}(\mathbf{0})) \\
 & \geq \int_{R^d} \left(P(\mathbf{W} < \mathbf{x}) - P(\mathbf{W} < \mathbf{0}) - \mathbf{1}(\|\mathbf{x}\| \leq 1)(\mathbf{x}, \Delta F_{\mathbf{W}}(\mathbf{0})) \right) \nu_{\mathbf{Y}}(d\mathbf{x}) \\
 & + (b_{\mathbf{Y}}, \Delta F_{\mathbf{W}}(\mathbf{0})).
 \end{aligned}$$

We first prove (2.5). An earlier argument shows that it is enough to prove it for $\lambda \in R^d - R_-^d$, which is a continuity point of both $\nu_{\mathbf{X}}$ and $\nu_{\mathbf{Y}}$. Let $\lambda_1 = \varepsilon > 0$ (say).

Define $\psi_{\mathbf{X}}(\mathbf{y}) = \nu_{\mathbf{X}}\{\mathbf{x}: \mathbf{x} > \mathbf{y}\}$ and $\psi_{\mathbf{Y}}(\mathbf{y}) = \nu_{\mathbf{Y}}\{\mathbf{x}: \mathbf{x} > \mathbf{y}\}$, $\mathbf{y} \in R^d$. Observe that $\psi_{\mathbf{X}}$ and $\psi_{\mathbf{Y}}$ are bounded on $[\varepsilon/2, \infty) \times R^{d-1}$ and continuous at $\mathbf{y} = \lambda$.

Let $\mathbf{W}^* = \lambda$ a.s. and choose a sequence $\{\mathbf{W}^n, n \geq 1\}$ of random vectors in R^d satisfying

$$(3.15) \quad \mathbf{W}^n \Rightarrow \mathbf{W}^* \quad \text{as } n \rightarrow \infty,$$

$$(3.16) \quad \mathbf{W}^n \text{ has a } C_b^\infty \text{ density,}$$

$$(3.17) \quad W_1^n \geq \varepsilon/2 \quad \text{a.s. } n = 1, 2, \dots$$

Then it follows from (3.17) that $\Delta F_{\mathbf{W}^n}(\mathbf{0}) = \mathbf{0}$, $n = 1, 2, \dots$, and so (3.14) takes, in this case, the form

$$\int_{R^d} P(\mathbf{W}^n < \mathbf{x}) \nu_{\mathbf{X}}(d\mathbf{x}) \geq \int_{R^d} P(\mathbf{W}^n < \mathbf{x}) \nu_{\mathbf{Y}}(d\mathbf{x}), \quad n = 1, 2, \dots$$

Equivalently,

$$E\psi_{\mathbf{X}}(\mathbf{W}^n) \geq E\psi_{\mathbf{Y}}(\mathbf{W}^n), \quad n = 1, 2, \dots$$

Taking the limits as $n \rightarrow \infty$, we conclude that

$$E\psi_{\mathbf{X}}(\mathbf{W}^*) \geq E\psi_{\mathbf{Y}}(\mathbf{W}^*),$$

which is exactly (2.5).

We now turn to verification of (2.6). Of course, it is enough to check it for $\lambda < \mathbf{0}$, which is, in addition, a continuity point of both $\nu_{\mathbf{X}}$ and $\nu_{\mathbf{Y}}$. We pursue a strategy similar to that used in the proof of (2.5). Define $\phi_{\mathbf{X}}(\mathbf{y}) = \nu_{\mathbf{X}}\{\mathbf{x}: \mathbf{x} \not\geq \mathbf{y}\}$ and $\phi_{\mathbf{Y}}(\mathbf{y}) = \nu_{\mathbf{Y}}\{\mathbf{x}: \mathbf{x} \not\geq \mathbf{y}\}$, $\mathbf{y} \in R^d$. Let $\varepsilon = \min_{i=1, \dots, d} (-\lambda_i) > 0$. Observe that $\phi_{\mathbf{X}}$ and $\phi_{\mathbf{Y}}$ are bounded on $(-\infty, -\varepsilon/2]^d$ and are continuous at $\mathbf{y} = \lambda$.

Let again $\mathbf{W}^* = \lambda$ a.s. and choose a sequence $\{\mathbf{W}^n, n \geq 1\}$ satisfying (3.15), (3.16) and

$$(3.18) \quad W_i^n \leq -\varepsilon/2 \quad \text{for every } i = 1, \dots, d \text{ and } n = 1, 2, \dots$$

Then again $\Delta F_{\mathbf{W}^n}(\mathbf{0}) = \mathbf{0}$, $n = 1, 2, \dots$, and so we immediately obtain from (3.14) that

$$\int_{R^d} P(\mathbf{W}^n \not\prec \mathbf{x}) \nu_{\mathbf{X}}(d\mathbf{x}) \leq \int_{R^d} P(\mathbf{W}^n \not\prec \mathbf{x}) \nu_{\mathbf{Y}}(d\mathbf{x}), \quad n = 1, 2, \dots,$$

which is equivalent to

$$E\phi_{\mathbf{X}}(\mathbf{W}^n) \leq E\phi_{\mathbf{Y}}(\mathbf{W}^n), \quad n = 1, 2, \dots$$

Now take the limits as $n \rightarrow \infty$. We obtain

$$E\phi_{\mathbf{X}}(\mathbf{W}^*) \leq E\phi_{\mathbf{Y}}(\mathbf{W}^*),$$

thus proving (2.6).

It remains, therefore, to prove (2.8) under the assumption that the Lévy measures $\nu_{\mathbf{X}}$ and $\nu_{\mathbf{Y}}$ satisfy (2.2).

Because we need to compare the vectors $\mathbf{c}_{\mathbf{X}}$ and $\mathbf{c}_{\mathbf{Y}}$ componentwise, we may as well restrict ourselves to the case $d = 1$. We have by (3.13) and (2.3),

$$\begin{aligned} (3.19) \quad & \int_{-\infty}^{\infty} (P(W < x) - P(W < 0)) \nu_X(dx) + (c_X - c_Y) f_W(0) \\ & \geq \int_{-\infty}^{\infty} (P(W < x) - P(W < 0)) \nu_Y(dx) \end{aligned}$$

for every random variable W with a C_b^∞ density f_W .

Let $W \sim N(0, \sigma^2)$. Observe that

$$\begin{aligned} (3.20) \quad & \int_0^{\infty} (P(W < x) - P(W < 0)) \nu_X(dx) = \int_0^{\infty} P(0 < W < x) \nu_X(dx) \\ & = \int_0^{\infty} \left(\Phi\left(\frac{x}{\sigma}\right) - \frac{1}{2} \right) \nu_X(dx) \\ & \leq \int_0^1 \left(\Phi\left(\frac{x}{\sigma}\right) - \frac{1}{2} \right) \nu_X(dx) + \nu_X([1, \infty)) \\ & = o(\sigma^{-1}) \quad \text{as } \sigma \rightarrow 0 \end{aligned}$$

by the boundedness of the density and (2.2). Applying the same argument to the other half of the integral in the left-hand side of (3.19), we conclude that

$$(3.21) \quad \int_{-\infty}^{\infty} (P(W < x) - P(W < 0)) \nu_X(dx) = o(\sigma^{-1}) \quad \text{as } \sigma \rightarrow 0.$$

Similarly,

$$(3.22) \quad \int_{-\infty}^{\infty} (P(W < x) - P(W < 0)) \nu_Y(dx) = o(\sigma^{-1}) \quad \text{as } \sigma \rightarrow 0.$$

Recalling that $f_W(0) = 1/\sigma\sqrt{2\pi}$, we conclude immediately from (3.19), (3.21) and (3.22) that $c_X \geq c_Y$.

This completes the proof of part (i) of the theorem. The proof of part (ii) is identical. \square

When the Lévy measures of the infinitely divisible random vectors \mathbf{X} and \mathbf{Y} satisfy (2.2), Theorems 2.1 and 3.2 imply that (2.5), (2.6) and (2.8) are the necessary and sufficient conditions for the right Slepian inequality (2.7) and (2.9), and (2.10) and (2.11) are the necessary and sufficient conditions for the left Slepian inequality (2.12).

Assuming only (1.1) instead of assuming (2.7) for all $t > 0$ is not, in general, sufficient for the conclusions of Theorem 3.2, even under the assumption (2.2). We show this through the following two examples.

EXAMPLE 3.1 [(1.1) implies neither (2.5) nor (2.6)]. We modify an example of Samorodnitsky and Taqqu [14] as follows. Let $d = 1$ and let Y be a mean 1 Poisson random variable. Let X be a (nonnegative) infinitely divisible random variable such that $X =_d X_1 + X_2$, where X_1 and X_2 are independent infinitely divisible random variables with Lévy measures

$$(3.23) \quad \begin{aligned} \nu_{X_1}(dx) &= n\delta_{1/2}(dx), \\ \nu_{X_2}(dx) &= c\mathbf{1}(x \geq 1)\exp(-x(\log x)^{1/2})dx, \end{aligned}$$

where n is a positive integer to be specified later, and $c > 0$ is chosen in such a way that $\nu_{X_2}([1, \infty)) = 1/2$. (The two random variables have zero shifts c_{X_1} and c_{X_2} .)

Observe that as $\lambda \rightarrow \infty$ (through integer values),

$$P(Y \geq \lambda) \sim P(Y = \lambda) = (e\lambda)^{-1} = o\left(\exp(-\lambda(\log \lambda)^{1/2})\right).$$

On the other hand, $\int_1^\infty \nu_{X_2}(dx) < \infty$ and hence X_2 is compound Poisson, that is, $X_2 = Z_1 + \dots + Z_N$, where N is a Poisson random variable and the Z_i 's are i.i.d. random variables independent of N . Because the distribution of Z_i is $\nu_{X_2}(R)^{-1}\nu_{X_2}$, there is a constant C such that, as $\lambda \rightarrow \infty$,

$$P(X_2 \geq \lambda) > C(Z_1 \geq \lambda) \geq C^2 \exp(-\lambda(\log \lambda)^{1/2}).$$

Therefore,

$$(3.24) \quad P(Y > \lambda) = o(P(X_2 > \lambda))$$

as $\lambda \rightarrow \infty$. Hence there is a $\lambda_0 > 0$ such that for every $\lambda > \lambda_0$,

$$P(X > \lambda) \geq P(X_2 > \lambda) \geq P(Y > \lambda).$$

On the other hand, one can choose n so large that

$$P(X > \lambda) \geq P(X_1 > \lambda) \geq P(Y > 0) \geq P(Y > \lambda)$$

for every $0 < \lambda < \lambda_0$, implying (1.1). However,

$$\nu_X\left(\left(\frac{3}{4}, \infty\right)\right) = \nu_{X_2}\left(\left(\frac{3}{4}, \infty\right)\right) = 1/2 < 1 = \nu_Y\left(\left(\frac{3}{4}, \infty\right)\right)$$

and so (2.5) fails.

By taking $\tilde{X} = -Y$ and $\tilde{Y} = -X$, we have an example of a situation where (1.1) holds in the absence of (2.6).

EXAMPLE 3.2 [(1.1) does not imply (2.8)]. We take once again $d = 1$. Let E be a mean 1 exponential random variable. Let $X = E$ and $Y = -E + \log 2$. Then the Lévy measures of X and Y are $\nu_X(dx) = x^{-1}e^{-x}1(x > 0)dx$ and $\nu_Y(dx) = |x|^{-1}e^{x}1(x < 0)dx$ ([6], Section XVII.3(d)). They satisfy (2.2) and, moreover, $c_X = 0 < \log 2 = c_Y$, which means that (2.8) fails. However, it is elementary to verify that (1.1) holds in this case.

REMARKS.

1. Example 3.2 notwithstanding, (1.1) *does* imply (2.8) in the case (2.2) under the additional assumption that the Lévy measures ν_X and ν_Y are *concentrated on the same quadrant of R^d* . To see this, suppose, for example, that ν_X and ν_Y are concentrated on R_+^d . Then $X_1 = U_1 + (c_X)_1$ and $Y_1 = U_2 + (c_Y)_1$, where $U_1 \geq 0$ and $U_2 \geq 0$. Because (2.2) holds, we have $P(U_2 > \delta) < 1$ because U_2 is a limit in distribution of compound Poisson variables. If $(c_X)_1 < (c_Y)_1$ (say), then $P(Y_1 \geq (c_Y)_1) = 1$, but $P(X_1 \geq (c_Y)_1) < 1$, contradicting (1.1).
2. In view of Theorem 3.2, Examples 3.1 and 3.2 illustrate situations where the right (say) Slepian inequality (2.7) holds for some $t > 0$ and fails for other $t > 0$. In the strictly α -stable case with $0 < \alpha \leq 2$ (including centered Gaussian case $\alpha = 2$), because of the relation $\mathbf{X}^{*t} =_d t^{1/\alpha}\mathbf{X}$, all the distinction between (1.1) and (2.7) disappears. In particular, strictly α -stable random vectors \mathbf{X} and \mathbf{Y} with $0 < \alpha < 1$ satisfy the right Slepian inequality (1.1) if and only if (2.5), (2.6) and (2.8) hold, and similarly with the left Slepian inequality (1.2).

4. Symmetric α -stable case: Ornstein-Uhlenbeck processes. In this section we specialize the results of Section 2 to the *symmetric α -stable (S α S)* case $0 < \alpha < 2$. The example of Ornstein-Uhlenbeck S α S processes presented in the end of the section is, to the best of our knowledge, the first nontrivial example where Slepian inequalities can be checked in the S α S case with $0 < \alpha < 2$. In the S α S case the Lévy measure ν of a random vector \mathbf{X} is given in the form

$$(4.1) \quad \nu(A) = \int_{S^d} \int_0^\infty \mathbf{1}(r\mathbf{s} \in A) r^{-(1+\alpha)} dr \Gamma(d\mathbf{s}),$$

where Γ is a finite symmetric measure on Borel subsets of the unit sphere $S^d = \{\mathbf{x} \in R^d: \|\mathbf{x}\| = 1\}$ in *some* norm $\|\cdot\|$ on R^d . The measure Γ is commonly called *the spectral measure of \mathbf{X}* . We will use in this section the maximum norm $\|\mathbf{x}\| = \max_{i=1, \dots, d} |x_i|$.

Let, therefore, \mathbf{X} and \mathbf{Y} be two $S\alpha S$ random vectors, $0 < \alpha < 2$, with spectral measures $\Gamma_{\mathbf{X}}$ and $\Gamma_{\mathbf{Y}}$ accordingly. Choose the following sequences $\{A_n, n \geq 1\}$ and $\{B_n, n \geq 1\}$ of sets deflating to the origin:

$$(4.2) \quad A_n = \prod_{i=1}^d (-\delta_{n,i}, \delta_{n,i}) \quad \text{and} \quad B_n = \prod_{i=1}^d (-\theta_{n,i}, \theta_{n,i}),$$

where for a fixed $i \in \{1, \dots, d\}$, $\{\delta_{n,i}\}_{n=1}^\infty$ and $\{\theta_{n,i}\}_{n=1}^\infty$ are two sequences of positive numbers decreasing to zero (refer to the remark following Definition 3.1). Because these sets are symmetric around the origin, conditions (3.6) and (3.9) hold automatically (see Remark 4 following Theorem 3.1). A straightforward computation shows the following proposition.

PROPOSITION 4.1. *In the $S\alpha S$ case, conditions (3.4) and (3.5) take, respectively, the following forms. For every $\lambda \in R^d - R_-^d$,*

$$(4.3) \quad \int_{S^d} \mathbf{1}(\mathcal{P}_\lambda \subseteq \mathcal{P}_s) \left[\left(\max_{i \in \mathcal{P}_\lambda} \frac{\lambda_i}{s_i} \vee \min_{i \in \{1, \dots, d\}} \frac{\delta_{n,i}}{|s_i|} \right)^{-\alpha} - \left(\min_{i \notin \mathcal{P}_s} \frac{\lambda_i}{s_i} \right)^{-\alpha} \right] \Gamma_{\mathbf{X}}(ds) \\ \geq \int_{S^d} \mathbf{1}(\mathcal{P}_\lambda \subseteq \mathcal{P}_s) \left[\left(\max_{i \in \mathcal{P}_\lambda} \frac{\lambda_i}{s_i} \vee \min_{i \in \{1, \dots, d\}} \frac{\theta_{n,i}}{|s_i|} \right)^{-\alpha} - \left(\min_{i \notin \mathcal{P}_s} \frac{\lambda_i}{s_i} \right)^{-\alpha} \right] \Gamma_{\mathbf{Y}}(ds)$$

and for every $\lambda \in R_-^d$,

$$(4.4) \quad \int_{S^d} \left(\min_{i \notin \mathcal{P}_s} \frac{\lambda_i}{s_i} \vee \min_{i \in \{1, \dots, d\}} \frac{\delta_{n,i}}{|s_i|} \right)^{-\alpha} \Gamma_{\mathbf{X}}(ds) \\ \leq \int_{S^d} \left(\min_{i \notin \mathcal{P}_s} \frac{\lambda_i}{s_i} \vee \min_{i \in \{1, \dots, d\}} \frac{\theta_{n,i}}{|s_i|} \right)^{-\alpha} \Gamma_{\mathbf{Y}}(ds).$$

Here $\mathcal{P}_{\mathbf{X}} = \{i: x_i \geq 0\}^1$.

Stationary $S\alpha S$ moving averages are stationary $S\alpha S$ stochastic processes that are important in applications (e.g., [4]; see also [16] for a recent study). They can be represented as

$$(4.5) \quad X(t) = \int_{-\infty}^\infty f(t+x)M(dx), \quad -\infty < t < \infty,$$

where $f \in L^\alpha(-\infty, \infty)$ and M is an independently scattered $S\alpha S$ random measure on $(-\infty, \infty)$ with Lebesgue control measure. The d -dimensional random vector $(X(t_1), \dots, X(t_d))$, $t_1, \dots, t_d \in R$, is $S\alpha S$ with spectral measure given by

$$(4.6) \quad \Gamma(A) = \left(\frac{1}{2}\right)m_1(T^{-1}(A)) + \left(\frac{1}{2}\right)m_1(T^{-1}(-A)),$$

where m_1 is a finite measure on Borel subsets of R defined by

$$m_1(dx) = \|f(\cdot + x)\|^\alpha dx,$$

$\|f(\cdot + x)\| = \max_{i=1, \dots, d} |f(t_i + x)|$ and $T: R \rightarrow S^d$ is given by

$$T(x) = \begin{cases} (f(t_1 + x)/\|f(\cdot + x)\|, \dots, f(t_d + x)/\|f(\cdot + x)\|) & \text{if } \|f(\cdot + x)\| \neq 0, \\ (1, 0, \dots, 0), & \text{if } \|f(\cdot + x)\| = 0. \end{cases}$$

See, for example, Samorodnitsky and Taqqu [13]. In particular, if $f(x) \geq 0$ for all $x \in R$, then the spectral measure given by (4.6) is concentrated on $R_-^d \cup R_+^d$. Substituting (4.6) into (4.3) and (4.4), we get the following corollary.

COROLLARY 4.1. *Let $\{X(t), t \in R\}$ and $\{Y(t), t \in R\}$ be two S α S moving averages as in (4.5), defined by two nonnegative functions f and g in $L^\alpha(-\infty, \infty)$. For given $t_1, \dots, t_d \in R$ consider the two S α S random vectors $\mathbf{X} = (X(t_1), \dots, X(t_d))$ and $\mathbf{Y} = (Y(t_1), \dots, Y(t_d))$. Then the Slepian inequalities (1.1) and (1.2) hold for these vectors, if for every $\lambda > 0$ and $n = 1, 2, \dots$,*

$$(4.7) \quad \int_{-\infty}^{\infty} \left(\max_{i \in \{1, \dots, d\}} \frac{\lambda_i}{f(t_i + x)} \vee \min_{i \in \{1, \dots, d\}} \frac{\delta_{n,i}}{f(t_i + x)} \right)^{-\alpha} dx \geq \int_{-\infty}^{\infty} \left(\max_{i \in \{1, \dots, d\}} \frac{\lambda_i}{g(t_i + x)} \vee \min_{i \in \{1, \dots, d\}} \frac{\theta_{n,i}}{g(t_i + x)} \right)^{-\alpha} dx$$

and

$$(4.8) \quad \int_{-\infty}^{\infty} \left(\min_{i \in \{1, \dots, d\}} \frac{\lambda_i}{f(t_i + x)} \vee \min_{i \in \{1, \dots, d\}} \frac{\delta_{n,i}}{f(t_i + x)} \right)^{-\alpha} dx \leq \int_{-\infty}^{\infty} \left(\min_{i \in \{1, \dots, d\}} \frac{\lambda_i}{g(t_i + x)} \vee \min_{i \in \{1, \dots, d\}} \frac{\theta_{n,i}}{g(t_i + x)} \right)^{-\alpha} dx.$$

EXAMPLE 4.1. *S α S Ornstein–Uhlenbeck processes.* The conditions of the corollary are relatively easy to verify for S α S Ornstein–Uhlenbeck processes. These are stationary S α S moving averages of the type (4.5), with

$$(4.9) \quad f(x) = \gamma^{1/\alpha} e^{-\gamma x} \mathbf{1}(x > 0) \quad \text{and} \quad g(x) = \mu^{1/\alpha} e^{-\mu x} \mathbf{1}(x > 0),$$

with $\gamma, \mu > 0$. (The purpose of the normalization $\gamma^{1/\alpha}$ and $\mu^{1/\alpha}$ is to give the two processes the same scale.) The Ornstein–Uhlenbeck processes are also Markov (see [2]).

We will see that if $0 < \gamma < \mu$, then the two processes satisfy Slepian inequalities in the sense that for every $t_1, \dots, t_d \in R$ and all real numbers $\lambda_1, \dots, \lambda_d$,

$$(4.10) \quad P(X(t_1) > \lambda_1, \dots, X(t_d) > \lambda_d) \geq P(Y(t_1) > \lambda_1, \dots, Y(t_d) > \lambda_d)$$

(and by symmetry, the left Slepian inequality as well). This result is known in the Gaussian case because for Gaussian Ornstein–Uhlenbeck processes,

$$E(X(t)X(s)) = \exp(-\gamma|t - s|) \geq \exp(-\mu|t - s|) = E(Y(t)Y(s))$$

for all $t, s \in R$, and so (1.5) are trivially verified. Intuitively, a large μ causes $Y(t)$ to be affected mainly by the increments of the random measure M “near t .” Thus, independence of the increments of M leads us to suspect that the components of the process Y “cluster together” less than the components of X do.

For a rigorous verification, assume that $-\infty = t_0 < t_1 < \dots < t_d < t_{d+1} = \infty$ and choose

$$\begin{aligned} \delta_{n,i} &= \delta_n \exp(-\gamma(t_i - t_1)), \\ \theta_{n,i} &= \delta_n \exp(-\mu(t_i - t_1)), \end{aligned}$$

$\delta_n \downarrow 0$ and $i = 1, \dots, d$. In the subsequent computations we will drop the subscript n .

In our case, the left-hand side of (4.7) is

$$\begin{aligned} & \int_{-t_1}^{\infty} \left(\max_{i \in \{1, \dots, d\}} \frac{\lambda_i}{\gamma^{1/\alpha} \exp(-\gamma(t_i + x))} \vee \min_{i \in \{1, \dots, d\}} \frac{\delta \exp(-\gamma(t_i - t_1))}{\gamma^{1/\alpha} \exp(-\gamma(t_i + x))} \right)^{-\alpha} dx \\ &= \int_{-t_1}^{\infty} \gamma \exp(-\gamma\alpha x) \left(\max_{i \in \{1, \dots, d\}} \lambda_i \exp(\gamma t_i) \vee \delta \exp(\gamma t_1) \right)^{-\alpha} dx \\ &= \alpha^{-1} \left(\max_{i \in \{1, \dots, d\}} \lambda_i \exp(\gamma(t_i - t_1)) \vee \delta \right)^{-\alpha}. \end{aligned}$$

Because the last expression is clearly nonincreasing in γ , (4.7) follows.

We now check (4.8). Because $t_0 = -\infty$, the left-hand side of the latter equals

$$\begin{aligned} & \sum_{j=0}^{d-1} \int_{-t_{j+1}}^{-t_j} \left(\min_{i \in \{j+1, \dots, d\}} \frac{\lambda_i}{\gamma^{1/\alpha} \exp(-\gamma(t_i + x))} \vee \min_{i \in \{j+1, \dots, d\}} \frac{\delta \exp(-\gamma(t_i - t_1))}{\gamma^{1/\alpha} \exp(-\gamma(t_i + x))} \right)^{-\alpha} dx \\ (4.11) \quad &= \sum_{j=0}^{d-1} \int_{-t_{j+1}}^{-t_j} \gamma \exp(-\gamma\alpha x) \left(\min_{i \in \{j+1, \dots, d\}} \lambda_i \exp(\gamma t_i) \vee \delta \exp(\gamma t_1) \right)^{-\alpha} dx \\ &= \alpha^{-1} \sum_{j=0}^{d-1} (\exp(\gamma\alpha t_{j+1}) - \exp(\gamma\alpha t_j)) \\ & \quad \times \left(\min_{i \in \{j+1, \dots, d\}} \lambda_i \exp(\gamma t_i) \vee \delta \exp(\gamma t_1) \right)^{-\alpha}. \end{aligned}$$

Now (4.8) follows from (4.11) and the following lemma.

LEMMA 4.1. *Let $-\infty < t_1 < \dots < t_d < \infty$, λ_i , $i = 1, \dots, d$, positive numbers. Then for every $\delta > 0$ and $u, v \leq t_1$ the function*

$$(4.12) \quad k(\gamma) = \sum_{j=0}^{d-1} (\exp(\gamma \alpha t_{j+1}) - \exp(\gamma \alpha t_j)) \times \left(\min_{i \in \{j+1, \dots, d\}} \lambda_i \exp(\gamma t_i) \vee \delta \exp(\gamma u) \right)^{-\alpha}.$$

with $t_0 = v$ is nondecreasing in $\gamma > 0$.

PROOF. The proof is by induction in d . For $d = 1$ we have

$$\begin{aligned} k(\gamma) &= (\exp(\gamma \alpha t_1) - \exp(\gamma \alpha v)) (\lambda_1 \exp(\gamma t_1) \vee \delta \exp(\gamma u))^{-\alpha} \\ &= (1 - \exp(-\gamma \alpha (t_1 - v))) (\lambda_1 \vee \delta \exp(-\gamma (t_1 - u)))^{-\alpha} \end{aligned}$$

and this is nondecreasing in γ because $t_1 \geq u \vee v$.

Suppose now that the statement of the lemma is true for a $d \geq 1$ and let us prove it for $d + 1$. We have now

$$k(\gamma) = \sum_{j=0}^d (\exp(\gamma \alpha t_{j+1}) - \exp(\gamma \alpha t_j)) \left(\min_{i \in \{j+1, \dots, d+1\}} \lambda_i \exp(\gamma t_i) \vee \delta \exp(\gamma u) \right)^{-\alpha}.$$

Consider two cases.

Case 1. $\lambda_1 e^{\gamma t_1} \leq \min_{i \in \{2, \dots, d+1\}} \lambda_i e^{\gamma t_i}$. Then

$$\begin{aligned} k(\gamma) &= (\exp(\gamma \alpha t_1) - \exp(\gamma \alpha v)) (\lambda_1 \exp(\gamma t_1) \vee \delta \exp(\gamma u))^{-\alpha} \\ &\quad + \sum_{j=1}^d (\exp(\gamma \alpha t_{j+1}) - \exp(\gamma \alpha t_j)) \left(\min_{i \in \{j+1, \dots, d+1\}} \lambda_i \exp(\gamma t_i) \vee \delta \exp(\gamma u) \right)^{-\alpha}. \end{aligned}$$

The fact that the first term in the preceding sum is nondecreasing has been proved when we considered the case $d = 1$, whereas the second term is nondecreasing by the assumption of the induction with $v = t_1$.

Case 2. $\lambda_1 e^{\gamma t_1} > \min_{i \in \{2, \dots, d+1\}} \lambda_i e^{\gamma t_i}$. Then combining together the first two terms in the sum, we obtain

$$\begin{aligned} k(\gamma) &= (\exp(\gamma \alpha t_2) - \exp(\gamma \alpha v)) \left(\min_{i \in \{2, \dots, d+1\}} \lambda_i \exp(\gamma t_i) \vee \delta \exp(\gamma u) \right)^{-\alpha} \\ &\quad + \sum_{j=2}^d (\exp(\gamma \alpha t_{j+1}) - \exp(\gamma \alpha t_j)) \left(\min_{i \in \{j+1, \dots, d+1\}} \lambda_i \exp(\gamma t_i) \vee \delta \exp(\gamma u) \right)^{-\alpha} \end{aligned}$$

and this is nondecreasing once again by the assumption of induction (we preserve the same u and v , delete t_1 and let t_2 play the role of t_1).

This completes the proof of the lemma. \square

5. The Brown–Rinott family. In 1988, Brown and Rinott [3] obtained Slepian inequalities for a particular family of infinitely divisible random vectors. The authors of the present paper have wanted to understand for some time how to place their result in a general theory of Slepian inequalities for infinitely divisible random vectors. We show in this section where the results of Brown and Rinott fit in the general theory, and we also provide some extensions.

Brown and Rinott consider a particular subclass of the infinitely divisible distributions, defined by $2^d - 1$ numbers, labeled t_A , where $A \neq \emptyset$ runs through all subsets of the set $\{1, 2, \dots, d\}$.

Let Q be an infinitely divisible probability law on R , not necessarily symmetric, and let $\{Q^t, t \geq 0\}$ be the corresponding convolution semigroup. The Brown–Rinott family \mathcal{BR}_Q of infinitely divisible random vectors in R^d is constructed as follows. Let $\mathcal{A} = \{A: A \subseteq \{1, \dots, d\}\}$. Choose a vector of nonnegative numbers $\mathbf{t} = \{t_A, A \in \mathcal{A}, A \neq \emptyset\}$ and let $\{Z_A, A \in \mathcal{A}, A \neq \emptyset\}$ be independent (real-valued) random variables, with $Z_A \sim Q^{t(A)}$, $A \in \mathcal{A}$ and $A \neq \emptyset$. Then define an infinitely divisible random vector $\mathbf{X} = (X_1, \dots, X_d)$ by

$$(5.1) \quad X_i = \sum_{A: i \in A} Z_A, \quad i = 1, \dots, d.$$

The family \mathcal{BR}_Q is obtained by allowing \mathbf{t} to vary. It is indeed a family of infinitely divisible random vectors because any linear combination of components of members of the family can be expressed as a linear combination of independent infinitely divisible random variables. It is easy, moreover, to identify the parameters of the random vector \mathbf{X} in \mathcal{BR}_Q for fixed \mathbf{t} . Namely, suppose that the infinitely divisible law Q has [in the representation (3.1)] one-dimensional Lévy measure μ and shift a . Then the d -dimensional Lévy measure ν of \mathbf{X} is given by

$$(5.2) \quad \nu = \sum_{\substack{A \in \mathcal{A} \\ A \neq \emptyset}} t_A \mu^{(A)},$$

where for an $A \in \mathcal{A}$, $A \neq \emptyset$, $\mu^{(A)}$ is the measure μ placed on the line $l_A = \{\mathbf{x} \in R^d: x_i = 0 \forall i \notin A, x_{i_1} = x_{i_2} \forall i_1, i_2 \in A\}$. That is, for every nonnegative measurable function $h: R^d \rightarrow R$,

$$\int_{R^d - \{0\}} h(\mathbf{x}) \nu(d\mathbf{x}) = \sum_{\substack{A \in \mathcal{A} \\ A \neq \emptyset}} t_A \int_{-\infty}^{\infty} h(\dots, x, \dots, 0, \dots, x, \dots, 0, \dots) \mu(dx)$$

when the arguments of h in the integral under the sum are equal to x for all i 's in A and to 0 for all i 's not in A . Further, by (5.1), the shift vector \mathbf{b} of \mathbf{X} is given by

$$(5.3) \quad b_i = a \sum_{A: i \in A} t_A$$

[if the norm $\|\cdot\|$ in (3.1) is chosen to be the maximum norm as well].

Brown and Rinott give as an example the multivariate Poisson distribution (X_1, \dots, X_d) , which is defined by (5.1) with Z_A Poisson with mean t_A . They also consider the $M/G/\infty$ queue (Poisson arrivals, general service time, infinite number of servers). If $X(\tau)$ denotes the number of customers in the queue at time τ , then $(X(\tau_1), \dots, X(\tau_d))$, $0 < \tau_1 < \dots < \tau_d$, is multivariate Poisson with $t_A \neq 0$ if A consists of consecutive numbers and $t_A = 0$ otherwise. (They give an explicit expression for t_A that involves the rate of the Poisson arrival process and the distribution of the service times.)

Let \mathbf{X} and \mathbf{Y} be two infinitely divisible random vectors in \mathcal{BR}_Q defined by vectors \mathbf{t} and \mathbf{t}^* accordingly. We want to use our results to derive Slepian inequalities for \mathbf{X} and \mathbf{Y} based on a proper comparison of the vectors \mathbf{t} and \mathbf{t}^* .

We start by obtaining an explicit form of (3.4) and (3.5) in this case. Choose $A_n = B_n = \{\mathbf{x} \in R^d: \|\mathbf{x}\| \leq \delta\}$ with $\delta = \delta_n > 0$ and $\|\cdot\|$ our usual maximum norm. It turns out to be somewhat more convenient to work with the equivalent condition (3.10) here. For every random vector $\mathbf{W} \in R^d$, we have by (5.2),

$$\begin{aligned}
 & \int_{A_n^c} (P(\mathbf{W} < \mathbf{x}) - P(\mathbf{W} < \mathbf{0})) \nu_{\mathbf{x}}(d\mathbf{x}) \\
 &= \sum_{\substack{A \in \mathcal{A} \\ A \neq \emptyset}} t_A \int_{\|\mathbf{x}\| > \delta} \left[P\left(\left(\bigcap_{i \in A} \{W_i < x\}\right) \cap \left(\bigcap_{i \notin A} \{W_i < 0\}\right)\right) \right. \\
 & \qquad \qquad \qquad \left. - P\left(\bigcap_{i=1}^d \{W_i < 0\}\right) \right] \mu(d\mathbf{x}) \\
 &= \sum_{\substack{A \in \mathcal{A} \\ A \neq \emptyset}} t_A \int_{\delta}^{\infty} P\left[\left(\left(\bigcap_{i \in A} \{W_i < x\}\right) \cap \left(\bigcap_{i \notin A} \{W_i < 0\}\right)\right) \right. \\
 (5.4) \qquad \qquad \qquad & \qquad \qquad \left. - \left(\bigcap_{i=1}^d \{W_i < 0\}\right) \right] \mu(d\mathbf{x}) \\
 & \quad - \sum_{\substack{A \in \mathcal{A} \\ A \neq \emptyset}} t_A \int_{-\infty}^{-\delta} P\left[\left(\bigcap_{i=1}^d \{W_i < 0\}\right) - \left(\left(\bigcap_{i \in A} \{W_i < x\}\right) \right. \right. \\
 & \qquad \qquad \qquad \left. \left. \cap \left(\bigcap_{i \notin A} \{W_i < 0\}\right)\right)\right] \mu(d\mathbf{x}) \\
 &= \sum_{\substack{B \in \mathcal{A} \\ B \neq \emptyset}} \int_{\delta}^{\infty} P\left(\left(\bigcap_{i \in B} \{0 \leq W_i < x\}\right) \right. \\
 & \qquad \qquad \qquad \left. \cap \left(\bigcap_{i \notin B} \{W_i < 0\}\right)\right) \mu(d\mathbf{x}) \sum_{A: A \supseteq B} t_A
 \end{aligned}$$

$$\begin{aligned}
 & - \sum_{\substack{B \in \mathcal{A} \\ B \neq \emptyset}} \int_{-\infty}^{-\delta} P \left(\left(\bigcap_{i \in B} \{x \leq W_i < 0\} \right) \right. \\
 & \qquad \left. \cap \left(\bigcap_{i \notin B} \{W_i < x\} \right) \right) \mu(dx) \sum_{A: A \cap B \neq \emptyset} t_A.
 \end{aligned}$$

We immediately conclude that if for all $B \in \mathcal{A}$ and $B \neq \emptyset$, the two conditions

$$(5.5) \quad \sum_{A: A \supseteq B} t_A \geq \sum_{A: A \supseteq B} t_A^*$$

and

$$(5.6) \quad \sum_{A: A \cap B \neq \emptyset} t_A \leq \sum_{A: A \cap B \neq \emptyset} t_A^*$$

hold, then (3.10) holds. Moreover, use (5.5) and (5.6) with B running over singletons to conclude, using (5.2) and (5.3), that (3.6) holds (with \geq replaced by an equality). In the same way, one can easily check that for any random vector $\mathbf{W} \in R^d$, we have

$$\begin{aligned}
 & \int_{A_n^c} (P(\mathbf{W} > \mathbf{x}) - P(\mathbf{W} > \mathbf{0})) \nu_{\mathbf{x}}(d\mathbf{x}) \\
 & = \sum_{\substack{B \in \mathcal{A} \\ B \neq \emptyset}} \int_{-\infty}^{-\delta} P \left(\left(\bigcap_{i \in B} \{x < W_i \leq 0\} \right) \cap \left(\bigcap_{i \notin B} \{W_i > 0\} \right) \right) \mu(dx) \sum_{A: A \supseteq B} t_A \\
 (5.7) \quad & - \sum_{\substack{B \in \mathcal{A} \\ B \neq \emptyset}} \int_{\delta}^{\infty} P \left(\left(\bigcap_{i \in B} \{0 < W_i \leq x\} \right) \right. \\
 & \qquad \left. \cap \left(\bigcap_{i \notin B} \{W_i > x\} \right) \right) \mu(dx) \sum_{A: A \cap B \neq \emptyset} t_A.
 \end{aligned}$$

We conclude that (5.5) and (5.6) imply (3.11) and (3.9) as well. Applying Theorem 3.1, we recover Theorem 1.1 of Brown and Rinott:

PROPOSITION 5.1. *Relations (5.5) and (5.6) imply the right and left Slepian inequalities (2.7) and (2.12) for all $t > 0$.*

Furthermore, suppose that the distribution Q is supported by $[0, \infty)$. This is well known to be equivalent to (i) μ satisfies (2.2) and is supported by $(0, \infty)$ and (ii) $\alpha \geq 0$ (Feller [6], Section XV11.3(f)). By using (5.4) and (5.3) (with B once again running over the singletons), we recover the direct part of Theorem 1.2 of Brown and Rinott:

PROPOSITION 5.2. *If the distribution Q is supported by $[0, \infty)$, then (5.5) \Rightarrow (2.7) for all $t > 0$ and (5.6) \Rightarrow (2.12) for all $t > 0$.*

Turning to converse statements and using our Theorem 3.2, we obtain the following refinement of the results of Brown and Rinott:

PROPOSITION 5.3. *Suppose that the right Slepian inequalities (2.7) hold for all $t > 0$. Then $\mu((0, \infty)) > 0 \Rightarrow$ condition (5.5) and $\mu((-\infty, 0)) > 0 \Rightarrow$ condition (5.6).*

Suppose that the left Slepian inequalities (2.12) hold for all $t > 0$. Then $\mu((0, \infty)) > 0 \Rightarrow$ condition (5.6) and $\mu((-\infty, 0)) > 0 \Rightarrow$ condition (5.5).

PROOF. Setting $\mathbf{W} = \lambda$ in the identity (5.4), we get, for any $\lambda \in R^d - R_-^d$,

$$(5.8) \quad \nu_{\mathbf{X}}\{\mathbf{x} \in R^d: \mathbf{x} > \lambda\} = \sum_{A: A \supseteq \mathcal{P}_\lambda} t_A \mu\left\{x > 0: x > \max_{i \in \mathcal{P}_\lambda} \lambda_i\right\}$$

and for any $\lambda < \mathbf{0}$,

$$(5.9) \quad \begin{aligned} & \nu_{\mathbf{X}}\{\mathbf{x} \in R^d: \mathbf{x} \not> \lambda\} \\ &= \sum_{\substack{B \in \mathcal{A} \\ B \neq \emptyset}} \sum_{A: A \cap B \neq \emptyset} t_A \mu\{x < 0: x \leq \lambda_i \forall i \in B, x > \lambda_i \forall i \notin B\}. \end{aligned}$$

We can now apply Theorem 3.2. Because of (5.8), the right Slepian inequalities (2.7) for all $t > 0$ implies (5.5) provided $\mu((0, \infty)) > 0$. It is almost as easy to see that because of (5.9), they also imply (5.6) provided $\mu((-\infty, 0)) > 0$: for a fixed $B \in \mathcal{A}$, $B \neq \emptyset$, use (5.9) with $\lambda_i \downarrow -\infty \forall i \notin B$ and $\lambda_i \uparrow 0 \forall i \in B$.

The other statements follow in a similar way if we start with (5.7) and set $\mathbf{W} = \lambda$. \square

Brown and Rinott [3] also discuss the extent to which (1.1) alone implies (5.5), and (1.2) alone implies (5.6), when Q is supported by $[0, \infty)$. Although we do not have a complete answer to this problem, we are able to shed some additional light. The following proposition generalizes Proposition 1.3 of Brown and Rinott and Theorem 1 of Ellis [5]: we remove the compound Poisson assumption (while retaining, in the first part, the assumption of existence of exponential moments of the Lévy measure).

PROPOSITION 5.4. *Suppose that Q is supported by $[0, \infty)$, and that $\mu((0, \infty)) > 0$.*

(i) *Assume that for all $\theta > 0$,*

$$(5.10) \quad \int_1^\infty e^{\theta x} \mu(dx) < \infty.$$

Then the right Slepian inequality (1.1) implies (5.5).

(ii) Assume that the shift $a = 0$ and that the Lévy measure μ has slowly varying tails at 0; that is, for any $r > 0$,

$$(5.11) \quad \lim_{x \rightarrow 0} \frac{\mu((rx, \infty))}{\mu((x, \infty))} = 1.$$

Then the left Slepian inequality (1.2) implies (5.6).

PROOF. Because μ satisfies (2.2), so do $\nu_{\mathbf{X}}$ and $\nu_{\mathbf{Y}}$. We will assume, therefore, that all relevant characteristic functions are given in the form (2.1). In particular, we assume that a is the shift corresponding to the representation (2.1). Then by analogy with (5.3) we conclude that the shift vector $\mathbf{c}_{\mathbf{X}}$ is given by

$$(5.12) \quad (\mathbf{c}_{\mathbf{X}})_i = a \sum_{A: i \in A} t_A,$$

and similarly with $\mathbf{c}_{\mathbf{Y}}$. Now, choose $\theta \geq \mathbf{0}$. For part (i), observe that (5.10) implies

$$(5.13) \quad \begin{aligned} E \exp(\theta, \mathbf{X}) &= \exp\left(\int_{R^d - \{0\}} (\exp((\theta, \mathbf{x})) - 1) \nu_{\mathbf{X}}(d\mathbf{x}) + (\theta, \mathbf{c}_{\mathbf{X}})\right) \\ &= \exp\left\{\sum_{\substack{A \in \mathcal{A} \\ A \neq \emptyset}} t_A \left[\int_0^\infty \left(\exp\left(x \sum_{i \in A} \theta_i\right) - 1\right) \mu(dx) + a \sum_{i \in A} \theta_i\right]\right\} \\ &< \infty. \end{aligned}$$

Because (1.1) implies

$$\begin{aligned} E \exp(\theta, \mathbf{X}) &= \prod_{i=1}^d \theta_i \int_{R^d} \exp(\theta, \mathbf{x}) P(\mathbf{X} > \mathbf{x}) d\mathbf{x} \\ &\geq \prod_{i=1}^d \theta_i \int_{R^d} \exp(\theta, \mathbf{x}) P(\mathbf{Y} > \mathbf{x}) d\mathbf{x} = E \exp(\theta, \mathbf{Y}), \end{aligned}$$

we conclude by (5.13) that

$$(5.14) \quad \begin{aligned} &\sum_{\substack{A \in \mathcal{A} \\ A \neq \emptyset}} t_A \left[\int_0^\infty \left(\exp\left(x \sum_{i \in A} \theta_i\right) - 1\right) \mu(dx) + a \sum_{i \in A} \theta_i\right] \\ &\geq \sum_{\substack{A \in \mathcal{A} \\ A \neq \emptyset}} t_A^* \left[\int_0^\infty \left(\exp\left(x \sum_{i \in A} \theta_i\right) - 1\right) \mu(dx) + a \sum_{i \in A} \theta_i\right]. \end{aligned}$$

Take now any $B \in \mathcal{A}$, $B \neq \emptyset$, and choose

$$\theta_i = \begin{cases} \theta, & \text{if } i \in B, \\ 0, & \text{if } i \notin B, \end{cases}$$

$\theta > 0$. Then (5.14) reduces to

$$\begin{aligned}
 (5.15) \quad & \left(\sum_{A \supseteq B} t_A \right) \left[\int_0^\infty (\exp(x\theta|B|) - 1) \mu(dx) + a\theta|B| \right] \\
 & + \sum_{\substack{A \not\supseteq B \\ A \neq \emptyset}} t_A \left[\int_0^\infty (\exp(x\theta|A \cap B|) - 1) \mu(dx) + a\theta|A \cap B| \right] \\
 & \geq \left(\sum_{A \supseteq B} t_A^* \right) \left[\int_0^\infty (\exp(x\theta|B|) - 1) \mu(dx) + a\theta|B| \right] \\
 & \quad + \sum_{\substack{A \not\supseteq B \\ A \neq \emptyset}} t_A^* \left[\int_0^\infty (\exp(x\theta|A \cap B|) - 1) \mu(dx) + a\theta|A \cap B| \right],
 \end{aligned}$$

where $|A|$ stands for the cardinality of A . Setting

$$c_n(\theta) = \int_0^\infty (\exp(x\theta n) - 1) \mu(dx) + a\theta n, \quad \theta > 0, n = 0, 1, 2, \dots,$$

we can rewrite (5.15) as

$$c_{|B|}(\theta) \sum_{A \supseteq B} t_A + \sum_{\substack{A \not\supseteq B \\ A \neq \emptyset}} c_{|A \cap B|}(\theta) t_A \geq c_{|B|}(\theta) \sum_{A \supseteq B} t_A^* + \sum_{\substack{A \not\supseteq B \\ A \neq \emptyset}} c_{|A \cap B|}(\theta) t_A^*.$$

Now (5.5) follows from the easily verifiable fact

$$\lim_{\theta \rightarrow \infty} \frac{c_{n+1}(\theta)}{c_n(\theta)} = \infty,$$

by letting $\theta \uparrow \infty$.

For part (ii), we have $Ee^{-(\theta, \mathbf{X})} \geq Ee^{-(\theta, \mathbf{Y})}$ and, hence, an argument identical to that leading to (5.15) shows that (1.2) implies, in our case, that for every $\theta > 0$,

$$\begin{aligned}
 (5.16) \quad & \sum_{A \cap B \neq \emptyset} t_A \left[\int_0^\infty (1 - \exp(-x\theta|A \cap B|)) \mu(dx) \right] \\
 & \leq \sum_{A \cap B \neq \emptyset} t_A^* \left[\int_0^\infty (1 - \exp(-x\theta|A \cap B|)) \mu(dx) \right].
 \end{aligned}$$

Denote $b(\theta) = \int_0^\infty (1 - e^{-\theta x}) \mu(dx)$, $\theta > 0$. Using (5.11) one can easily check that for every $c > 0$,

$$\lim_{\theta \rightarrow \infty} \frac{b(c\theta)}{b(\theta)} = \lim_{\theta \rightarrow \infty} \frac{\int_0^\infty e^{-t} \mu\left(\frac{t}{c\theta}, \infty\right) dt}{\int_0^\infty e^{-t} \mu\left(\frac{t}{\theta}, \infty\right) dt} = 1.$$

Now (5.6) follows from (5.16) upon letting $\theta \uparrow \infty$. \square

REMARKS.

1. If Q is supported on $[0, \infty)$ and is *compound Poisson*, then it is the distribution of the random variable $V = \sum_{i=1}^N U_i$, where the $\{U_i\}$ are i.i.d. nonnegative with distribution $\mu/\mu(R_+)$ (μ is the Lévy measure of V), and N is a Poisson random variable independent of the U_i 's. Then (5.10) reduces to $Ee^{\theta U_1} < \infty$, which is the assumption of Proposition 1.3 of Brown and Rinott [3] and Theorem 1 of Ellis [5].
2. For the \mathcal{BR}_Q family,

$$\begin{aligned} (1.1) &\Rightarrow (5.5) \quad [\text{Proposition 5.4(i)}] \\ &\Rightarrow (2.7) \quad \text{for all } t > 0 \text{ (Theorem 3.2)} \\ &\Rightarrow (2.5) \quad \text{and (2.6).} \end{aligned}$$

In general, however, (1.1) does not imply (2.5) or (2.6) *even in the presence of all exponential moments*, as Example 3.1 demonstrates.

In their Theorem 1.2(ii), Brown and Rinott [3] state that for the \mathcal{BR}_Q family, conditions (1.2) and (5.6) are equivalent when Q is supported by $[0, \infty)$ without any additional assumptions. We do not find their proof convincing, and the following example seems to provide a counterexample to that statement.

EXAMPLE 5.1. Let $d = 2$ and $\mu(dx) = x^{-1}e^{-x} dx$, $x > 0$, with shift $a \geq 0$ to be chosen later. (μ is the Lévy measure of a unit mass exponential random variable.) Let

$$(5.17) \quad \begin{aligned} t_1 = t_2 = 3, & \quad t_{12} = 0, \\ t_1^* = t_2^* = 0, & \quad t_{12}^* = 5. \end{aligned}$$

Observe that (5.6) fails for $B = \{1, 2\}$. Let $\Gamma(5)$, $\Gamma_1(3)$ and $\Gamma_2(3)$ be independent random variables with gamma (5), gamma (3) and gamma (3) distributions accordingly (all with the scale parameter equal to 1). Then we can represent the vectors \mathbf{X} and \mathbf{Y} (in law) as

$$X_1 = \Gamma_1(3) + 3a, \quad X_2 = \Gamma_2(3) + 3a, \quad Y_1 = Y_2 = \Gamma(5) + 5a.$$

We claim that one can choose $a > 0$ so large that for every $\lambda \in R^2$, (1.2) holds.

Because $Y_1 = Y_2$ a.s., the "worst case" for (1.2) is the case $\lambda_1 = \lambda_2 = \lambda$. In that case, (1.2) reduces to $P(X_1 < \lambda, X_2 < \lambda) \geq P(Y_1 < \lambda)$ or, equivalently, to

$$(5.18) \quad \left(P(\Gamma_1(3) + 3a < \lambda) \right)^2 \geq P(\Gamma(5) + 5a < \lambda).$$

Observe that (5.18) holds trivially for every $\lambda \leq 5a$. We need, therefore, only to consider the case $\lambda > 5a$. Letting $x = \lambda - 5a > 0$, we see that we only need to exhibit an $a > 0$ for which

$$(5.19) \quad \left(P(\Gamma_1(3) < x + 2a) \right)^2 \geq P(\Gamma(5) < x)$$

for every $x > 0$. Choose an $x_0 > 0$ such that for every $x \geq x_0$,

$$(5.20) \quad P(\Gamma(5) > x) \geq \frac{1}{48}e^{-x}x^4 \quad \text{and} \quad P(\Gamma_1(3) > x) \leq e^{-x}x^2.$$

Let $x_1 = x_0 \vee \sqrt{96}$. Choose now a so large that

$$\left(P(\Gamma_1(3) \leq 2a)\right)^2 \geq P(\Gamma(5) \leq x_1).$$

Then (5.19) holds trivially for all $0 < x \leq x_1$, while its truth for $x > x_1$ is a simple consequence of (5.20).

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SCHOOL OF OPERATIONS RESEARCH
AND INDUSTRIAL ENGINEERING
CORNELL UNIVERSITY
ITHACA, NEW YORK 14853

BOSTON UNIVERSITY
111 CUMMINGTON STREET
BOSTON, MASSACHUSETTS 02215-2411