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NECESSARY AND SUFFICIENT CONDITIONS FOR THE STRASSEN LAW OF THE ITERATED LOGARITHM IN NONUNIFORM TOPOLOGIES

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We give a necessary and sufficient condition for the Strassen functional law of the iterated logarithm for the Wiener process to hold when the topology is defined by a general class of norms on $\mathbb{C}(0,1)$.

1. Introduction and results. Let $\{W(t), t \geq 0\}$ be a standard Wiener process, and consider the set $\mathbb{Y} = \{Y_T(s), T \geq 0\}$ of functions of $s \in [0,1]$, where

$$Y_T(s) = (2T \log_2 T)^{-1/2} W(Ts), \tag{1.1}$$

$\log_2 T = \log_+ \log_+ T$ and $\log_+ T = \log(\max(T, e))$. Let $(\mathbb{C}_0(0,1), \mathbb{U})$ denote the space $\mathbb{C}_0(0,1)$ of all continuous functions h on $[0,1]$ with $h(0) = 0$, endowed with the uniform topology $\mathbb{U}$, generated by the norm $|h| = \sup_{0 \leq s \leq 1} |h(s)|$. The functional law of the iterated logarithm due to Strassen (1964) shows that $\mathbb{Y}$ is almost surely relatively compact in $(\mathbb{C}_0(0,1), \mathbb{U})$ and that the limit set of $\mathbb{Y}$ in $(\mathbb{C}_0(0,1), \mathbb{U})$, defined as the set of all limit points of the sequences $\{Y_{T_n}, n \geq 1\}$ with $T_n \to \infty$, is almost surely equal to the Strassen set

$$\mathbb{K} = \left\{ h \in \mathbb{C}_0(0,1) \colon h(s) = \int_0^s \dot{h}(s)\, ds \text{ with } \int_0^1 \dot{h}(s)^2\, ds \leq 1 \right\}. \tag{1.2}$$

Given the Strassen law, a simple argument shows that with probability 1, for any continuous functional Ψ on $(\mathbb{C}_0(0,1), \mathbb{U})$, one has

$$\limsup_{T \to \infty} \Psi(Y_T) = \sup_{h \in \mathbb{K}} \Psi(h). \tag{1.3}$$

In particular, the choice of $\Psi(h) = |h(1)|$ in (1.3) yields the usual law of the iterated logarithm for the Wiener process due to Lévy (1937).

In view of extending the validity of (1.3) to a larger class of functionals than that of the continuous functions on $(\mathbb{C}_0(0,1), \mathbb{U})$, a natural question is to characterize the topologies τ on $\mathbb{C}_0(0,1)$ for which this statement holds with $\mathbb{U}$ replaced by τ. This problem has received some attention lately. Particularly, Baldi, Ben Arous and Kerkyacharian (1992) have shown that the Strassen law holds for the topology generated by the Hölder norm

$$\|h\|_\alpha^{(H)} = \sup_{0 \leq s < t \leq 1} |h(s) - h(t)|/|t - s|^\alpha \tag{1.4}$$

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if and only if $\alpha < 1/2$. Their arguments are based on Schilder-type large deviation principles [Schilder (1966)] under the Hölder topology. Beibel and Lerche (1994) have established similar large deviation principles when the topology is defined by a weighted sup-norm of the form

$$\|h\|_q^{(W)} = \sup_{0<s\leq 1} |h(s)|/q(s), \tag{1.5}$$

where q is a monotone increasing function on $[0, 1]$, positive on $(0, 1]$ and satisfying

$$\limsup_{s\downarrow 0} |W(s)|/q(s) < \infty \quad \text{a.s.} \tag{1.6}$$

A nontopological approach to the Strassen law was followed by Ben Arous and Ledoux (1993). Recently, some sufficient conditions for the Strassen law were provided by Deheuvels and Lifshits (1993). The purpose of this paper is to give a simple necessary and sufficient condition for that property. In order to state our main theorem, we will need the following notation and assumptions.

In the sequel and unless otherwise specified, $\|\cdot\|$ will denote a norm on $\mathbb{C}_0(0,1)$, with values in $[0,\infty]$. We assume, namely, that for any $f_1, f_2 \in \mathbb{C}_0(0,1)$ and $\lambda_1, \lambda_2 \in \mathbb{R}$,

$$0 \leq \|\lambda_1 f_1 + \lambda_2 f_2\| \leq \|\lambda_1 f_1\| + \|\lambda_2 f_2\| = |\lambda_1|\,\|f_1\| + |\lambda_2|\,\|f_2\| \leq \infty, \tag{1.7}$$

with the conventions that $(0 \times \infty) = 0$ and $c + \infty = \infty$ for $-\infty < c \leq \infty$. Throughout, $\mathbb{B}$ will denote the σ-field of Borel subsets of $(\mathbb{C}_0(0,1), \mathbb{U})$ and $\|\cdot\|$ will be supposed to be a $\mathbb{B}$-measurable mapping of $\mathbb{C}_0(0,1)$ onto $[0,\infty]$. We will denote by $(\mathbb{C}_0(0,1), \tau)$ the topological space obtained by endowing $\mathbb{C}_0(0,1)$ with the topology τ induced by $\|\cdot\|$.

We will say that a topology τ *obeys the Strassen law* if the following statement (SL) holds with probability 1.

(SL) The *set of values of each subsequence* $\{Y_{T_n}, n \geq 1\}$, with $T_n \to \infty$, *is relatively compact in* $(\mathbb{C}_0(0,1), \tau)$. *The set of all limit points of sequences of this type is equal to* $\mathbb{K}$.

REMARK 1.1. The statement (SL) is slightly weaker than that proved by Strassen (1964) for the uniform topology. Namely, it does not imply that the sets of the form $\{Y_T, 0 \leq T \leq U\}$ are relatively compact in $(\mathbb{C}_0(0,1), \tau)$ for $U < \infty$. On the other hand, (SL) is still sufficient for the main implication of the Strassen law, since it implies that (1.3) holds for each τ-continuous functional Ψ.

REMARK 1.2. If τ is generated by a measurable norm $\|\cdot\|$ as above, and whenever $\mathbb{K}$ is τ-compact, (SL) is equivalent to

$$\lim_{T\to\infty} \left\{ \inf_{h\in\mathbb{K}} \|Y_T - h\| \right\} = 0 \quad \text{a.s.} \tag{1.8}$$

and

(1.9) $$\liminf_{T\to\infty} \|Y_T - h\| = 0 \quad \text{a.s. for each } h \in \mathbb{K}.$$

Denote by I the identity mapping of [0, 1] onto itself. Our main result is as follows.

THEOREM 1.1. *Let the norm* $\|\cdot\|$ *be lower semicontinuous on* $(\mathbb{C}_0(0,1),\mathbb{U})$. *Then, the Strassen law* (SL) *holds for the topology* τ *generated by* $\|\cdot\|$ *if and only if the condition*

(C) $$P\Big(\sup_{0\le\theta\le\varepsilon} \|W((1-\theta)I)\| < \infty\Big) > 0 \quad \textit{for some } \varepsilon > 0$$

is satisfied.

REMARK 1.3. The above-mentioned result of Baldi, Ben Arous and Kerkyacharian (1992) is an easy corollary of Theorem 1.1, since the Hölder norm $\|\cdot\|_\alpha^{(H)}$ is readily verified to be lower semicontinuous on $(\mathbb{C}_0(0,1),\mathbb{U})$ and satisfying (C) iff $\alpha < 1/2$. Likewise, under the assumptions of Beibel and Lerche (1994), the weighted sup-norm $\|\cdot\|_q^{(W)}$ is lower semicontinuous on $(\mathbb{C}_0(0,1),\mathbb{U})$ and satisfies (C) iff (1.6) holds. Beibel and Lerche (1994) do not prove that (SL) is valid under (1.6) in the weighted sup-norm topology, but Theorem 1.1 implies that such is the case.

REMARK 1.4. The importance of the assumption that $\|\cdot\|$ is lower semicontinuous on $(\mathbb{C}_0(0,1),\mathbb{U})$ originates from the following observation. Since the map $\theta \to W((1-\theta)I)$ is continuous with respect to $\mathbb{U}$, then if $\|\cdot\|$ is lower semicontinuous, the maps $\theta \to \|W((1-\theta)I)\|$ and $\varepsilon \to \sup_{0\le\theta\le\varepsilon} \|W((1-\theta)I)\|$ are also lower semicontinuous with respect to $\mathbb{U}$. Hence, the supremum in (C) can be replaced by the supremum over any dense countable subset of $[0,\varepsilon]$, which in turn proves *measurability* of $\sup_{0\le\theta\le\varepsilon} \|W((1-\theta)I)\|$. We are not aware of any interesting example (with respect to the Wiener process) of a norm which is not lower semicontinuous. However, our arguments enable us to prove the following result when this condition does not hold. Under the assumption that $\sup_{0\le\theta\le\varepsilon} \|W((1-\theta)I)\|$ is measurable with respect to the σ-field $\mathbb{B}$ completed with respect to the Wiener measure [see, e.g., Dellacherie (1972)], (C) is necessary for (SL) and sufficient for (1.8) and (1.9). However, we are unable to prove in this general case that $\mathbb{K}$ is compact in $(\mathbb{C}_0(0,1),\tau)$. This means that, under this measurability assumption, (C) implies a weak version of (SL), with relative compactness changed into total boundedness. In this case, (1.3) turns out to hold for all functionals Ψ that are uniformly continuous with respect to $\|\cdot\|$ on $\mathbb{K}$ (see Theorem 3.1 and Corollary 3.1 in the sequel).

REMARK 1.5. The label "C" in the necessary and sufficient condition of Theorem 1.1 refers to consistency of the norm $\|\cdot\|$ with respect to the Wiener process (see Section 2 below).

The remainder of our paper is organized as follows. In Section 2, we discuss some additional consistency conditions, which turn out to be intimately connected with the functional law of the iterated logarithm. We prove Theorem 1.1 in Section 3. In Section 4, we give examples of norms commonly used in statistics to illustrate the meaning and implications of our results.

We refer to Goodman and Kuelbs (1991) and the references therein for related results on the law of the iterated logarithm.

2. Consistency conditions. A $\mathbb{B}$-measurable norm $\|\cdot\|$, generating the topology τ on $\mathbb{C}_0(0,1)$, will be said to be consistent if the first two of the following conditions are satisfied, where c_3, c_4 and c_5 are some appropriate finite constants.

(C.1) $P(\|W\| < \infty) = 1$.
(C.2) $\mathbb{K}$ is a compact subset of $(\mathbb{C}_0(0,1), \tau)$.
(C.3) $P(\lim_{\varepsilon\downarrow 0}\{\sup_{0<\theta<\varepsilon} \|W((1-\theta)I) - W\|\} \le c_3) = 1$.
(C.4) $P(\lim_{\varepsilon\downarrow 0}\{\sup_{0<\theta_1,\theta_2<\varepsilon} \|W(\theta_1 + (1-\theta_2)I) - W(\theta_1) - W\|\} \le c_4) = 1$.
(C.5) $P(\limsup_{\varepsilon\downarrow 0} \|W(\min(I,\varepsilon))\| \le c_5) = 1$.

Variants of (C.1)–(C.5) have been used by Deheuvels and Lifshits (1993) to obtain sufficient conditions for the validity of Strassen's law and of some of its generalizations. In the forthcoming Section 3, we will discuss the relationships of these conditions with the Strassen law (SL) and limit ourselves to (C.1), (C.2), (C.3) and (C.5). The aim of the remainder of the present section is to establish the following theorem.

THEOREM 2.1. *Let the norm* $\|\cdot\|$ *be lower semicontinuous on* $(\mathbb{C}_0(0,1), \mathbb{U})$ *and satisfy* (C). *Then, the conditions* (C.1), (C.2), (C.3) *and* (C.5) *are valid.*

The proof of Theorem 2.1 will be captured in the following sequence of facts and lemmas. We first note that (C) obviously implies (C.1), so that we will concentrate on proving that it also implies (C.2), (C.3) and (C.5). We will make use of the following framework [see, e.g., Borell (1976)]. Let Z denote a random vector taking values in a Hausdorff locally convex topological space $\mathbf{X}$. Denote by $\mathbb{B}_\mathbf{X}$ the σ-field of Borel subsets of $\mathbf{X}$, and assume that the probability law $P_Z(B) = P(Z \in B)$, $B \in \mathbb{B}_\mathbf{X}$, is a centered Gaussian Radon measure. Then, there exists a *kernel* $\mathbb{H}$, which is a linear subspace of $\mathbf{X}$, endowed with a Hilbert norm $|\cdot|_\mathbb{H}$, such that the following property holds. For any $h \in \mathbb{H}$, there exists a linear measurable form $\widetilde{h}$ on $\mathbf{X}$ satisfying the equalities

$$P_Z(B+h) = \int_B \exp\left(\widetilde{h}(x) - \tfrac{1}{2}|h|_\mathbb{H}^2\right) P_Z(dx) \quad \text{for each } B \in \mathbb{B}_\mathbf{X} \tag{2.1}$$

and

$$\int_\mathbf{X} \widetilde{h}^2(x) P_Z(dx) = |h|_\mathbb{H}^2. \tag{2.2}$$

In the sequel, we will specialize in the case where $\mathbf{X} = (\mathbb{C}_0(0,1), \mathbb{U})$, $Z = W$ is the standard Wiener process and $P_Z = P_W$ is the Wiener measure [see, e.g., Itô and McKean (1965), Kuo (1975) or Hida (1980)], in which case $\mathbb{H}$ consists of all absolutely continuous functions h of the form $h(u) = \int_0^u \phi(t)\,dt$ with $\int_0^1 \phi^2(t)\,dt < \infty$. The corresponding Hilbert norm is defined on $\mathbb{H}$ by the expression $|\cdot|_{\mathbb{H}} = \{\int_0^1 \phi^2(t)\,dt\}^{1/2}$ and the Strassen set $\mathbb{K}$ is the unit ball of $\mathbb{H}$. It is noteworthy that in this case, (2.1) reduces to the celebrated Cameron–Martin formula [Cameron and Martin (1945); see, e.g., Kuo (1975)]. The next lemma [Borell (1977)] states a useful simple consequence of (2.1).

LEMMA 2.1. *Let $h \in \mathbb{H}$ and let $A \in \mathbb{B}_{\mathbf{X}}$ denote a symmetric subset of $\mathbb{X}$. Then*

$$P_Z(A+h) \geq \exp\left(-\tfrac{1}{2}|h|_{\mathbb{H}}^2\right) P_Z(A). \tag{2.3}$$

PROOF. By setting $B = A$ in (2.1), the Jensen inequality gives, by symmetry of A and linearity of $\tilde{h}$,

$$\int_A \exp\big(\tilde{h}(x)\big) P_Z(dx) \geq P_Z(A) \exp\left\{ \int_A \tilde{h}(x) P_Z(dx)/P_Z(A) \right\} = P_Z(A). \qquad \square$$

Lemma 2.2 yields the key argument in the proof that (C) implies (C.2).

LEMMA 2.2. *Let $(\mathbf{C}, |\cdot|_0)$ denote a normed space, let $\|\cdot\|_0$ denote a lower semicontinuous norm on $(\mathbf{C}, |\cdot|_0)$ and let τ_0 denote the topology defined by $\|\cdot\|_0$ on $\mathbf{C}$. Then, whenever a compact subset of $(\mathbf{C}, |\cdot|_0)$ is totally bounded with respect to $\|\cdot\|_0$, it is also compact with respect to τ_0.*

PROOF. See Proposition 4.1 of Deheuvels and Lifshits (1993). □

LEMMA 2.3. *Let the norm $\|\cdot\|$ be lower semicontinuous on $(\mathbb{C}_0(0,1), \mathbb{U})$ and satisfy (C). Then the condition (C.2) holds.*

PROOF. By Lemma 2.2, all we need is to show that, under the assumptions of the lemma, $\mathbb{K}$ is totally bounded with respect to the norm $\|\cdot\|$. For any $h \in \mathbb{C}_0(0,1)$ and $M > 0$, let $\mathbb{D}_{M,h} = \{g \in \mathbb{C}_0(0,1): \|h - g\| < M\}$ denote the ball of radius M centered on h. Since (C) implies (C.1), we may use the latter property to establish the existence of M so large that

$$P_W(\mathbb{D}_{M,\mathbf{0}}) = P(W \in \mathbb{D}_{M,\mathbf{0}}) = P\big(\|W\| < M\big) > 0, \tag{2.4}$$

where $\mathbf{0}$ denotes the null function. Let $\varepsilon > 0$ and $h_1, \dots, h_n \in \mathbb{K}$ be such that $\|h_i - h_j\| > \varepsilon$ for $1 \leq i < j \leq n$. Since the balls $\mathbb{D}_{M,2Mh_i/\varepsilon}$ are then disjoint, $\sum_{i=1}^n P_W(\mathbb{D}_{M,2Mh_i/\varepsilon}) \leq 1$, whence

$$n \leq \left\{ \min_{1 \leq i \leq n} P_W\big(\mathbb{D}_{M,2Mh_i/\varepsilon}\big) \right\}^{-1} \leq \left\{ \inf_{h \in \mathbb{K}} P_W\big(\mathbb{D}_{M,2Mh/\varepsilon}\big) \right\}^{-1}. \tag{2.5}$$

On the other hand, (2.3) and the symmetry of the ball $\mathbb{D}_{M,\mathbf{0}}$ imply that, for any $h \in \mathbb{K}$,

$$P_W(\mathbb{D}_{M,2Mh/\varepsilon}) \geq \exp\left(-(2M/\varepsilon)^2 \tfrac{1}{2}|h|_{\mathbb{H}}^2\right) P_W(\mathbb{D}_{M,\mathbf{0}}). \tag{2.6}$$

By combining (2.4), (2.5) and (2.6) with the fact that $|h|_{\mathbb{H}} \leq 1$ for $h \in \mathbb{K}$, it follows that

$$n \leq \exp\left(-(2M/\varepsilon)^2 \tfrac{1}{2}|h|_{\mathbb{H}}^2\right) / P_W(\mathbb{D}_{M,\mathbf{0}}). \tag{2.7}$$

By (2.7), the total boundedness of $\mathbb{K}$ with respect to the norm $\|\cdot\|$ is immediate. □

We now turn to the more delicate part of the proof of Theorem 2.1 related to the conditions (C.3) and (C.5).

LEMMA 2.4. *Let the norm* $\|\cdot\|$ *be lower semicontinuous on* $(\mathbb{C}_0(0,1), \mathbb{U})$ *and satisfy* (C). *Then the condition* (C.3) *holds.*

PROOF. The triangle inequality implies that

$$\begin{aligned} C_3 &:= \lim_{\varepsilon \downarrow 0} \left\{ \sup_{0 \leq \theta \leq \varepsilon} \|W((1-\theta)I) - W\| \right\} \\ &\leq \lim_{\varepsilon \downarrow 0} \left\{ \sup_{0 \leq \theta \leq \varepsilon} \|W((1-\theta)I)\| \right\} + \|W\|. \end{aligned} \tag{2.8}$$

Next, we make use of the fact that, provided $\|\cdot\|$ is lower semicontinuous, for any fixed $\theta \in [0,1]$ the map $h \to \|h((1-\theta)I)\|$ is lower semicontinuous on $(\mathbb{C}_0(0,1), \mathbb{U})$. This, in turn, implies that, for any $\varepsilon \in [0,1]$, the map $h \to \sup_{0 \leq \theta \leq \varepsilon} \|h((1-\theta)I)\|$ is also lower semicontinuous. It follows that the set

$$E_\varepsilon := \left\{ h \in \mathbb{C}_0(0,1) \colon \sup_{0 \leq \theta \leq \varepsilon} \|h((1-\theta)I)\| < \infty \right\} \tag{2.9}$$

is a $\mathbb{B}$-measurable linear subspace of $(\mathbb{C}_0(0,1), \mathbb{U})$. The zero-one law [see Cameron and Graves (1951), Kallianpur (1970) and Jain (1971)] implies therefore that $P_W(E_\varepsilon) = 0$ or 1 for each $\varepsilon \in [0,1]$. In view of (2.9), it follows that the assumption (C) is equivalent to

$$P_W(E_\varepsilon) = P\left(\sup_{0 \leq \theta \leq \varepsilon} \|h((1-\theta)I)\| < \infty \right) = 1 \quad \text{for some } \varepsilon > 0. \tag{2.10}$$

By combining (2.8) with (2.10) and the fact that (C) implies (C.1), we see that (C) implies almost sure finiteness of C_3. All we need therefore is to set $c_3 = C_3$ in (C.3) and to prove that C_3 is nonrandom. Toward this goal, we will make use of the almost surely uniformly convergent on [0, 1] orthogonal Karhunen–Loève expansion [see, e.g., Example 1.4.4 in Ash and Gardner (1975), page 42]

$$\begin{aligned} W(t) &= \sum_{n=1}^{m} + \sum_{n=m+1}^{\infty} \left\{\left(n-\tfrac{1}{2}\right)\pi\right\}^{-1} \xi_n e_n(t) \\ &=: W^{\le m}(t) + W^{>m}(t) \quad \text{for } 0 \le t \le 1, \end{aligned} \tag{2.11}$$

with $e_n(t) = 2^{1/2}\sin((n-\frac{1}{2})\pi t)$, $n = 1,2,\dots,$ and where $\{\xi_n,\, n \ge 1\}$ is a sequence of independent $N(0,1)$ random variables. Noting that, almost surely for $n = 1,2,\dots,$

$$\xi_n = \left(n-\tfrac{1}{2}\right)\pi \int_0^1 W(t)e_n(t)\,dt, \tag{2.12}$$

we will show that the limit c_3 is independent of $\xi_1,\dots,\xi_m$ for each $m \ge 1$. Since the σ-algebra of events $\bigcap_{m\ge 1}\sigma\{\xi_{m+1},\xi_{m+2},\dots\}$ is degenerate, this, in turn, will suffice to prove the nonrandomness of C_3. Toward this goal, we make use of the triangle inequality to obtain from (2.11) that

$$\begin{aligned} &\left| \sup_{0\le\theta\le\varepsilon} \left\|W((1-\theta)I) - W\right\| - \sup_{0\le\theta\le\varepsilon} \left\|W^{>m}((1-\theta)I) - W^{>m}\right\| \right| \\ &\quad \le \sup_{0\le\theta\le\varepsilon} \left\|W^{\le m}((1-\theta)I) - W^{\le m}\right\|. \end{aligned} \tag{2.13}$$

In view of (2.13) and of the fact that $W^{>m}$ and $W^{\le m}$ are independent, the proof boils down to showing that for each $m \ge 1$,

$$\lim_{\theta\downarrow 0} \left\|W^{\le m}((1-\theta)I) - W^{\le m}\right\| = 0 \quad \text{a.s.} \tag{2.14}$$

Let $m \ge 1$ be fixed. We see from (2.11) that

$$\begin{aligned} &\left\|W^{\le m}((1-\theta)I) - W^{\le m}\right\| \\ &\quad \le \left\{ \sum_{n=1}^{m} \left\{\left(n-\tfrac{1}{2}\right)\pi\right\}^{-1} |\xi_n| \right\} \max_{1\le n\le m} \left\|e_n((1-\theta)I) - e_n\right\|. \end{aligned} \tag{2.15}$$

Moreover, since $e_n((1-\theta)I) - e_n \in \mathbb{H}$ for $n = 1,2,\dots,$

$$\begin{aligned} &\max_{1\le n\le m} \left\|e_n((1-\theta)I) - e_n\right\| \\ &\quad \le \left\{ \max_{1\le n\le m} \left|e_n((1-\theta)I) - e_n\right|_{\mathbb{H}} \right\} \sup_{h\in\mathbb{H},\, h\ne 0} \{\|h\|/|h|_{\mathbb{H}}\} \\ &\quad = \left\{ \max_{1\le n\le m} \left|e_n((1-\theta)I) - (1-\theta)e_n\right|_{\mathbb{H}} + \theta \max_{1\le n\le m} |e_n|_{\mathbb{H}} \right\} \sup_{h\in\mathbb{K}} \|h\|, \end{aligned} \tag{2.16}$$

where we have used the triangle inequality. Lemma 2.3 implies that, under our assumptions, $\mathbb{K}$ is compact with respect to the topology defined by $\|\cdot\|$, so that

$\sup_{h \in \mathbb{K}} \|h\| < \infty$. On the other hand, it follows from the definitions of e_n and $|\cdot|_{\mathbb{H}}$ (for Wiener measure) that, uniformly over $n = 1, \dots, m$,

$$\begin{aligned}
&\left|e_n((1-\theta)I) - (1-\theta)e_n\right|_{\mathbb{H}}^2 \\
&\quad \le 2\left(n - \tfrac{1}{2}\right)^2 \pi^2 (1-\theta)^2 \int_0^1 \Big|\cos\Big(\left(n - \tfrac{1}{2}\right)\pi(1-\theta)t\Big) \\
&\qquad\qquad - \cos\Big(\left(n - \tfrac{1}{2}\right)\pi t\Big)\Big|^2 dt \\
&\quad \le 8\left(m - \tfrac{1}{2}\right)^2 \pi^2 \int_0^1 \sin^2\Big(\left(n - \tfrac{1}{2}\right)\pi\theta/2\Big) dt \to 0 \quad \text{as } \theta \to 0.
\end{aligned} \tag{2.17}$$

Thus, by combining (2.16) with (2.17), we see that for any fixed $m \ge 1$, the right-hand side of (2.16) tends to 0 as $\theta \to 0$. The proof of Lemma 2.4 is now complete. □

LEMMA 2.5. *Let the norm* $\|\cdot\|$ *be lower semicontinuous on* $(\mathbb{C}_0(0,1), \mathbb{U})$ *and satisfy* (C). *Then the condition* (C.5) *holds.*

PROOF. Let $W_\gamma = W(\min(I, \gamma))$ for $\gamma \ge 0$. The map $\gamma \to W_\gamma$ of $[0,1]$ onto the linear space $\mathbb{C}_0(0,1)$ endowed with $\|\cdot\|$ defines a nonhomogeneous $\mathbb{C}_0(0,1)$-valued process with independent increments. Therefore, we may apply to any finite (or countable) subset Π of $[0,1]$ the Lévy inequality

$$P\Big(\sup_{\gamma \in \Pi} \|W_\gamma\| > r\Big) \le 2P\big(\|W_1\| > r\big) = 2P\big(\|W\| > r\big) \quad \text{for } r \ge 0. \tag{2.18}$$

Since the map $\gamma \in [0,1] \to W_\gamma$ is continuous with respect to the uniform topology, the map $\gamma \in [0,1] \to \|W_\gamma\|$ is lower semicontinuous, whence, by (2.18),

$$P\Big(\sup_{\gamma \in [0,1]} \|W_\gamma\| > r\Big) \le 2P\big(\|W\| > r\big) \quad \text{for } r \ge 0. \tag{2.19}$$

Next, the condition (C) implies (C.1), which in turn implies the existence of an $r_5 > 0$ so large that $2P(\|W\| > r_5) < 1$. Let

$$C_5 := \limsup_{\gamma \downarrow 0} \|W_\gamma\|. \tag{2.20}$$

It follows from (2.19), (2.20) and the inequality $2P(\|W\| > r_5) < 1$ that $P(C_5 \le r_5) > 0$, which in turn implies that $P(C_5 < \infty) > 0$. On the other hand, C_5 is measurable with respect to the σ-field $\bigcap_{\varepsilon > 0} \sigma\{W(s), 0 \le s \le \varepsilon\}$, which is degenerate by Blumenthal's zero-one law [Blumenthal (1957)]. Hence, C_5 is a finite constant and (C.5) holds with $c_5 = C_5$. □

PROOF OF THEOREM 2.1. In view of the obvious implication that (C) implies (C.1), the statements of Lemmas 2.3, 2.4 and 2.5 prove the theorem. □

3. Proof of Theorem 1.1. and related results.

3.1. *Necessity of* (C). Denote by $\mathbb{B}^*$ the σ-field $\mathbb{B}$ completed with respect to the Wiener measure [see, e.g., Dellacherie (1972)]. We will say that the norm $\|\cdot\|$ is sup-measurable if the following condition is satisfied:

$$\text{(M)}\qquad \text{The map } h \in \mathbb{C}_0(0,1) \to \sup_{0\le\theta\le\varepsilon} \|h((1-\theta)I\| \text{ is } \mathbb{B}^*\text{-measurable for each } \varepsilon \in [0,1].$$

We will say likewise that the norm $\|\cdot\|$ is lower semicontinuous if it is lower semicontinuous with respect to the uniform topology $\mathbb{U}$. Clearly, this assumption implies (M). Recalling that $E_\varepsilon = \{h \in \mathbb{C}_0(0,1)\colon \sup_{0\le\theta\le\varepsilon} \|h((1-\theta)I)\| < \infty\}$, the arguments following (2.9) imply that, under (M), the assumption (C) is equivalent to $P(W \in E_\varepsilon) = 1$ for some $\varepsilon \in (0,1]$. Moreover $P(W \in E_\varepsilon) = 0$ or 1 for each $\varepsilon \in [0,1]$. Let further

$$E_0 := \bigcup_{\varepsilon>0} E_\varepsilon = \left\{h \in \mathbb{C}_0(0,1)\colon \limsup_{\theta\downarrow 0} \left\|h\big((1-\theta)I\big)\right\| < \infty\right\}. \tag{3.1}$$

In view of (3.1), we have $E_{\varepsilon_1} \subseteq E_{\varepsilon_2} \subseteq E_0$ for $0 < \varepsilon_2 \le \varepsilon_1 \le 1$. Therefore, if (C) does not hold (resp. holds), then $P(W \in E_\varepsilon) = 0$ for all $\varepsilon \in (0,1]$ (resp. $P(W \in E_\varepsilon) = 1$ for some $\varepsilon \in (0,1]$). This in turn is equivalent to $P(W \in E_0) = 0$ [resp. $P(W \in E_0) = 1$]. Observe from (1.1) and (3.1) that if $P(W \in E_0) = 0$, then, for each $n = 1, 2, \dots,$

$$\limsup_{\theta\downarrow 0} \left\|(2\log_2 n)^{1/2} Y_{n(1-\theta)}\right\| = \infty \quad \text{a.s.} \tag{3.2}$$

This readily implies with probability 1 the existence of a sequence $\theta_n \in (0,1]$ with $\theta_n \uparrow 1$ such that $\|(2\log_2 n)^{1/2} Y_{n(1-\theta_n)}\| \ge 2\log_2 n$. It follows that there exists almost surely a random sequence $0 < T_1 < \cdots < T_n \to \infty$ such that

$$\lim_{n\to\infty} \|Y_{T_n}\| = \infty. \tag{3.3}$$

Since (3.3) is in contradiction to the fact that $\{Y_{T_n},\, n \ge 1\}$ would be, according to (SL), relatively compact in $(\mathbb{C}_0(0,1), \tau)$, the topology τ defined by $\|\cdot\|$ obeys the Strassen law only if (C) holds.

By all this, we have proved the necessity of (C) in Theorem 1.1. This result is stated in the stronger setting of (M) as follows.

LEMMA 3.1. *Let the norm* $\|\cdot\|$ *satisfy* (M). *Then the sequence of values of* $\|Y_{T_n}\|$ *is bounded with probability* 1 *for each positive sequence* $\{T_n,\, n \ge 1\}$ *with* $T_n \to \infty$ *only if* (C) *holds. In particular*, (SL) *holds only if* (C) *holds.*

3.2. *Sufficiency of* (C). A rough outline of our proof is as follows. In view of Theorem 2.1, all we need is to establish that the norm obeys Strassen's law when the conditions (C.1), (C.2), (C.3) and (C.5) are satisfied. This will require the following technical arguments.

We will make an instrumental use of the isoperimetric inequality [Borell (1975); Sudakov and Tsyrelson (1978)]. In the general setting of (2.1) and (2.2), with a space $(\mathbf{X}, \mathbb{B}_{\mathbf{X}})$, an $\mathbf{X}$-valued random vector Z and kernel $\mathbb{H}$ with unit ball $\mathbf{K}$ with respect to the norm $|\cdot|_{\mathbb{H}}$, for any $r \geq 0$, $A \in \mathbb{B}_{\mathbf{X}}$ and $B \in \mathbb{B}_{\mathbf{X}}$ with $B \cap (A + r\mathbf{K}) = \emptyset$, we have

$$P_Z(B) \leq 1 - \Phi\Big\{\Phi^{-1}\big(P_Z(A)\big) + r\Big\}, \tag{3.4}$$

where Φ (resp. Φ^{-1}) denotes the distribution function (resp. the quantile function) of the standard normal $N(0, 1)$ law. For a proof and discussion of (3.4), we refer to Ledoux and Talagrand [(1991), page 17]. The following lemma states some useful consequences of (3.4).

LEMMA 3.2. *Let $\|\cdot\|$ be a measurable seminorm defined on a Hausdorff locally convex topological space $\mathbf{X}$, and let Z denote an $\mathbf{X}$-valued centered Gaussian random vector with Radon distribution P_Z such that $P(\|Z\| < \infty) = 1$. Let m be a median of the distribution of $\|Z\|$, and let $c > m$, $\sigma = \sup_{h \in \mathbb{K}} \|h\|$, $\beta = \beta(c, Z) := c/\Phi^{-1}(P(\|Z\| < c))$.*

Then, for any $R \geq m$, the following inequalities hold:

$$\begin{gathered} P\big(\|Z\| > R\big) \leq 1 - \Phi\big((R-m)/\sigma\big), \\ \sigma \leq \beta, \\ P\big(\|Z\| > R\big) \leq 1 - \Phi\big((R-m)/\beta\big). \end{gathered} \tag{3.5}$$

Moreover, for $R > R_0(Z, \|\cdot\|)$ sufficiently large, we have

$$P\big(\|Z\| > R\big) \leq \exp\Big(-R^2/(3\sigma^2)\Big) \quad \text{and} \quad P\big(\|Z\| > R\big) \leq \exp\Big(-R^2/(3\beta^2)\Big). \tag{3.6}$$

PROOF. By the triangle inequality, we see that, for any $x = y + rh$ with $\|y\| \leq m, h \in \mathbf{K}$, the inequality $\|x\| \leq m + r\|h\| \leq m + r\sigma$ holds. The isoperimetric inequality (3.4), applied to $A = \{y\colon \|y\| \leq m\}$ and $B = \{x\colon \|x\| > m + r\sigma\}$ yields $P_Z\{x\colon \|x\| > m + r\sigma\} \leq 1 - \Phi(r)$, which gives the first well-known inequality [Talagrand (1984)] in (3.5) after a change of variables.

For the second inequality in (3.5), we let $h \in \mathbf{K}$ be nonnull and expand Z in a vector of the form $Z = Z_h + \xi h$, where Z_h is a centered Gaussian vector and ξ is a centered $N(0, |h|_{\mathbb{H}}^{-2})$ Gaussian random variable independent of Z_h. In view of the symmetry of the distribution of Z_h, we have, for any $r \geq 0$,

$$P\big(\|Z\| \geq r\big) \geq \tfrac{1}{2}P\big(\|\xi h\| \geq r\big) = \tfrac{1}{2}P\big(|\xi| \geq r/\|h\|\big) = 1 - \Phi\big(r|h|_{\mathbb{H}}/\|h\|\big).$$

By taking the supremum of the latter expression over $h \in \mathbf{K}$, $h \neq 0$, we obtain

$$P\big(\|Z\| \geq r\big) \geq \sup_{h \in \mathbb{H}, h \neq 0} \Big\{1 - \Phi\big(r|h|_{\mathbb{H}}/\|h\|\big)\Big\} = 1 - \Phi\big(r/\sigma\big);$$

hence, $\sigma \le r/\Phi^{-1}(P(\|Z\| < r))$. By setting $r = c$ in this last expression, we get $\sigma \le \beta$. The remainder of the proof is straightforward and therefore will be omitted. □

We now turn to the proof of the sufficiency of (C) via the following three steps.

Step 1. *Geometric sequence argument.* Let $\lambda > 1$ and set $T_n = \lambda^n$ for $n = 1, 2, \dots$. The following lemma holds.

LEMMA 3.3. *Let the norm* $\|\cdot\|$ *satisfy* (C.1). *Then for any fixed* $\lambda > 1$, *we have*

$$\lim_{n\to\infty}\left\{\inf_{h\in\mathbb{K}}\|Y_{T_n} - h\|\right\} = 0 \quad a.s. \tag{3.7}$$

PROOF. By the Borel–Cantelli lemma, it is enough to prove that for any $\varepsilon > 0$, we have

$$\sum_n P\Big(\inf_{h\in\mathbb{K}}\|Y_{T_n} - h\| > \varepsilon\Big) < \infty. \tag{3.8}$$

Toward this aim, we set $L_T = (2\log_2 T)^{1/2}$, $\mathbb{D}_{M,h} = \{g \in \mathbb{C}_0(0,1): \|g - h\| < M\}$ and observe that (C.1) implies the existence of a large $M > 0$ such that

$$P_W(\mathbb{D}_{M,\mathbf{0}}) = P\big(\|W\| < M\big) \ge \Phi(1). \tag{3.9}$$

Since for any $T > 0$, $L_T Y_T$ and W are identically distributed, we have ultimately, as $T \to \infty$,

$$\begin{aligned} P\Big(\inf_{h\in\mathbb{K}}\|Y_T - h\| > \varepsilon\Big) &= P\Big(W \notin \mathbb{D}_{\varepsilon L_T,\mathbf{0}} + L_T\mathbb{K}\Big)\\ &\le P\Big(W \notin \mathbb{D}_{M,\mathbf{0}} + L_T\mathbb{K}\Big). \end{aligned} \tag{3.10}$$

We now apply the inequality (3.4) with $Z = W$, $\mathbf{X} = \mathbb{C}_0(0,1)$, $\mathbf{K} = \mathbb{K}$, $r = L_T$, $A = \mathbb{D}_{M,\mathbf{0}}$, $\mathbb{B}_{\mathbf{X}} = \mathbb{B}$ and $B = \mathbb{C}_0(0,1) - \{\mathbb{D}_{M,\mathbf{0}} + L_T\mathbb{K}\}$. We note that $\mathbb{K}$, being compact in $(\mathbb{C}_0(0,1), \mathbb{U})$, belongs to $\mathbb{B}$, the same being true for $\mathbb{D}_{M,\mathbf{0}}$ and B by the measurability assumption on $\|\cdot\|$. This, in combination with (3.10), yields

$$P\Big(\inf_{h\in\mathbb{K}}\|Y_T - h\| > \varepsilon\Big) \le 1 - \Phi(1 + L_T). \tag{3.11}$$

By setting $T = T_n$ in (3.11), one can easily infer (3.8). □

REMARK 3.1. One readily extends (3.8) via (3.11) to all sequences $T_n \to \infty$ such that the series $\sum_n (\log T_n)^{-1}(\log_2 T_n)^{-1/2}\exp(-\sqrt{2\log_2 T_n}) < \infty$. This shows that, under the sole assumption (C.1), Y_{T_n} clusters into $\mathbb{K}$ with respect to the topology τ for all sequences satisfying this condition.

Step 2. *Continuity argument.* Let $\lambda > 1$ and let $T_n = \lambda^n$ for $n = 1, 2, \dots$. We have the following lemma.

LEMMA 3.4. *Let the norm* $\|\cdot\|$ *satisfy* (M), (C.1) *and* (C.3). *Then for any fixed* $\varepsilon > 0$, *there exists a* $\lambda_0 > 1$ *such that for all* $1 < \lambda \leq \lambda_0$,

$$\limsup_{n\to\infty}\left\{\sup_{T\in[T_n,T_{n+1}]}\|Y_T - Y_{T_n}\|\right\} < \varepsilon \quad a.s. \tag{3.12}$$

PROOF. Set for convenience $U = T_{n+1}$ and recall that $L_T = (2\log_2 T)^{1/2}$. The triangle inequality entails that, for all large n,

$$\begin{aligned}
&\sup_{U/\lambda\leq T\leq U}\|Y_T - Y_U\| \\
&\quad\leq \sup_{U/\lambda\leq T\leq U}\left\{\|Y_T - T^{-1/2}W(UI)/L_T\| + \|T^{-1/2}W(UI)/L_T - Y_U\|\right\} \\
&\quad\leq \sup_{U/\lambda\leq T\leq U} U^{-1/2}\|W(TI) - W(UI)\|/L_T \\
&\qquad + \sup_{U/\lambda\leq T\leq U}\left|(L_U/L_T)(U/T)^{1/2} - 1\right|U^{-1/2}\|W(UI)\|/L_U \\
&\quad\leq 2^{1/2}U^{-1/2}\sup_{U/\lambda\leq T\leq U}\|W(TI) - W(UI)\|/L_U \\
&\qquad + (\lambda - 1)U^{-1/2}\|W(UI)\|/L_U \\
&\quad\leq 2^{1/2}L_U^{-1}\{\eta' + \eta''\},
\end{aligned} \tag{3.13}$$

where η' is identical in distribution to $\sup_{0\leq\theta\leq 1-1/\lambda}\|W((1-\theta)I) - W\|$ and η'' is identical in distribution to $(\lambda - 1)\|W\|$. It follows readily from (3.13) that

$$\begin{aligned}
&P\left(\sup_{T\in[T_n,T_{n+1}]}\|Y_T - Y_{T_n}\| \geq \varepsilon\right) \\
&\quad\leq P\left(\sup_{0\leq\theta\leq 1-1/\lambda}\|W((1-\theta)I) - W\| \geq \frac{\varepsilon}{2}(\log_2 U)^{1/2}\right) \\
&\qquad + P\left(\|W\| \geq \frac{\varepsilon}{2}(\log_2 U)^{1/2}/(\lambda - 1)\right) \\
&\quad =: D_n' + D_n''.
\end{aligned} \tag{3.14}$$

To evaluate D_n', observe by (M) that the seminorm

$$N_\lambda(h) = \sup_{0\leq\theta\leq 1-1/\lambda}\|h((1-\theta)I) - h\|$$

is measurable. Moreover, the assumption (C.3) ensures that $P(N_\lambda(W) < \infty) = 1$ for all $\lambda > 1$ sufficiently close to 1. In addition, if c_3 is as in (C.3), we see that this assumption also implies that

$$\beta(\lambda) := (c_3 + 1)\Big/\Phi^{-1}\Big(P\big(N_\lambda(W) < c_3 + 1\big)\Big) \to 0 \quad \text{as } \lambda \downarrow 1.$$

By choosing $\lambda > 1$ sufficiently close to 1 and then by applying (3.6) to N_λ with $Z = W$ and $R = (\varepsilon/2)(\log_2 U)^{1/2}$, we obtain that for all large n,

$$D'_n \leq \exp\left(-\frac{\varepsilon^2}{12\beta^2(\lambda)}\log_2 U\right) \leq n^{-\delta'(\lambda,\varepsilon)}, \tag{3.15}$$

where $\delta'(\lambda,\varepsilon) = \varepsilon^2/(16\beta^2(\lambda))$.

Consider now D''_n. By (C.1), $P(\|W\| < \infty) = 1$, so that we may apply (3.6) with $Z = W$, and $R = \varepsilon(\log_2 U)^{1/2}/(\lambda - 1)$ and $\sigma = \sup_{h\in\mathbb{K}}\|h\|$ to obtain that, as $n \to \infty$,

$$D''_n \leq \exp\left(-\frac{\varepsilon^2}{12\sigma^2(\lambda-1)^2}\log_2 U\right) \leq n^{-\delta''(\lambda,\varepsilon)}, \tag{3.16}$$

where $\delta''(\lambda,\varepsilon) = \varepsilon^2/(16\sigma^2(\lambda-1)^2)$.

By choosing further $\lambda > 1$ so close to 1 that $\min\{\delta'(\lambda,\varepsilon), \delta''(\lambda,\varepsilon)\} > 1$, we obtain (3.12) by combining (3.14), (3.15), (3.16) and the Borel–Cantelli lemma. □

REMARK 3.2. As follows from Lemmas 3.3 and 3.4, the relation (1.8), which is the first half of Strassen's law, is implied by (M), (C.1) and (C.3), the latter conditions being satisfied via Theorem 2.1 if $\|\cdot\|$ is a lower semicontinuous norm fulfilling the assumption (C).

Step 3. Occupancy argument. Let $\lambda > 1$ and let $T_n = \lambda^n$ for $n = 1, 2, \ldots$. We have the following lemma.

LEMMA 3.5. *Let the norm $\|\cdot\|$ satisfy* (C.1) *and* (C.5). *Then for any fixed $h \in \mathbb{K}$ with $|h|_{\mathbb{H}} < 1$ and $\varepsilon > 0$, there exists a $\lambda_1 > 1$ such that for all $\lambda \geq \lambda_1$,*

$$\liminf_{n\to\infty}\left\{\|Y_{T_n} - h\|\right\} < \varepsilon \quad a.s. \tag{3.17}$$

PROOF. Let $\lambda > 1$, set $T_n = \lambda^n$ for $n = 1, 2, \ldots$ and $L_{T_n} = (2\log_2 T_n)^{1/2}$. Consider the decomposition

$$\begin{aligned} Y_{T_n} &= L_{T_n}^{-1}T_n^{-1/2}\big(W(T_n I) - W(T_{n-1})\big)1_{\{I > T_{n-1}/T_n\}} \\ &\quad + L_{T_n}^{-1}T_n^{-1/2}W\Big(T_n \min\big(I, T_{n-1}/T_n\big)\Big) \\ &:= \xi'_n + \xi''_n. \end{aligned} \tag{3.18}$$

Observe first that ξ''_n follows the same distribution as $L_{T_n}^{-1}W(\min(I,\lambda^{-1}))$. Moreover, by (C.5), there exists a $\lambda_1 > 1$ so large that, for all $\lambda \geq \lambda_1$, $P(\|W\min(I, \lambda^{-1}))\| < \infty) = 1$ and $\beta = \beta(\lambda) := (c_5+1)/\Phi^{-1}(P(\|W(\min(I,\lambda^{-1}))\| < c_5+1)) < \varepsilon/5$. This enables us to apply (3.6) with $Z = W(\min(I,\lambda^{-1}))$ and $R = \varepsilon L_{T_n}/2$ to obtain that for all n sufficiently large,

$$P\big(\|\xi'_n - Y_{T_n}\| \geq \varepsilon/2\big) = P\big(\|\xi''_n\| \geq \varepsilon/2\big) \leq \exp\big(-\varepsilon^2 L_{T_n}^2/12\beta^2\big) \leq n^{-2}. \tag{3.19}$$

An application of the triangle inequality to (3.18) yields $\|\xi_n' - h\| \le \|Y_{T_n} - h\| + \|\xi_n''\|$, which implies, in turn,

$$P(\|\xi_n' - h\| < \varepsilon/2) \le P(\|Y_{T_n} - h\| < \varepsilon/4) + P(\|\xi_n''\| < \varepsilon/4). \tag{3.20}$$

Since $L_{T_n} Y_{T_n}$ and W are identically distributed, we have

$$P(\|Y_{T_n} - h\| < \varepsilon/4) = P(\|W - L_{T_n} h\| < L_{T_n}\varepsilon/4) = P_W(A_n + h_n), \tag{3.21}$$

where $A_n := \mathbb{D}_{L_{T_n}\varepsilon/4,\,0}$ and $h_n := L_{T_n} h$. Observe that $\frac{1}{2}L_{T_n}^2 = (1 + o(1))\log n$, and, by (C.1), that $P_W(A_n) \to 1$ as $n \to \infty$. It follows therefore from (3.21) and (2.3), taken with $Z = W, A = A_n$ and $h = h_n$, that for all large n,

$$P(\|Y_{T_n} - h\| < \varepsilon/4) \ge \tfrac{1}{2}\exp\left(-\tfrac{1}{2}|h|_{\mathbb{H}}^2 L_{T_n}^2\right) \ge \tfrac{1}{2}\exp(-|h|_{\mathbb{H}} \log n). \tag{3.22}$$

By combining the estimates (3.19), (3.20), (3.21) and (3.22) with the assumption that $|h|_{\mathbb{H}} < 1$, we readily obtain that

$$\sum_n P(\|\xi_n' - h\| < \varepsilon/2) = \infty \quad \text{and} \quad \sum_n P(\|\xi_n' - Y_{T_n}\| \ge \varepsilon/2) < \infty. \tag{3.23}$$

Since the ξ_n' are independent, based on nonoverlapping increments of the Wiener process, a repeated application of the Borel–Cantelli lemma in (3.23) implies (3.17). □

We may now combine Steps 1, 2 and 3 to obtain the following variant of the sufficient part of Theorem 1.1.

THEOREM 3.1. *Assume that the norm* $\|\cdot\|$ *satisfies* (M), (C.1), (C.3) *and* (C.5). *Then*

$$\lim_{T\to\infty} \left\{ \inf_{h\in\mathbb{K}} \|Y_T - h\| \right\} = 0 \quad a.s. \tag{3.24}$$

and, for any specified $h \in \mathbb{K}$,

$$\liminf_{T\to\infty} \|Y_T - h\| = 0 \quad a.s. \tag{3.25}$$

In addition, if $\mathbb{K}$ *is totally bounded with respect to* $\|\cdot\|$, *then*

$$\sup_{h\in\mathbb{K}} \left\{ \liminf_{T\to\infty} \|Y_T - h\| \right\} = 0 \quad a.s. \tag{3.26}$$

PROOF. The assertions (3.24) and (3.25) are direct consequences of Lemmas 3.3, 3.4 and 3.5. Note that an application of Lemma 3.1 in conjunction with (C.1) suffices to show that $\sigma = \sup_{h\in\mathbb{K}} \|h\| < \infty$. Given (3.24), (3.25) and total boundedness of $\mathbb{K}$, (3.26) is immediate. □

The following corollary of Theorem 3.1 shows the usefulness of the weak form of the Strassen law obtained by combining (3.24) and (3.25) [or equivalently (1.8) and (1.9)].

COROLLARY 3.1. *Assume that the norm* $\|\cdot\|$ *satisfies* (M), (C.1), (C.3) *and* (C.5). *Assume further that* $\mathbb{K}$ *is totally bounded with respect to* $\|\cdot\|$. *Then for any specified mapping* Ψ *of* $\mathbb{C}_0(0,1)$ *onto* $\mathbb{R}$, *uniformly continuous with respect to* $\|\cdot\|$ *on* $\mathbb{K}$, *we have*

$$\limsup_{T\to\infty} \Psi(Y_T) = \sup_{h\in\mathbb{K}} \Psi(h) \quad a.s. \tag{3.27}$$

PROOF. Assume first that $L_1 = \limsup_{T\to\infty} \Psi(Y_T) < \infty$ and $L_2 = \sup_{h\in\mathbb{K}} \Psi(h) < \infty$, and let t_n be a sequence such that $t_n \to \infty$ with $\Psi(Y_{t_n}) \to L$. By combining the total boundedness assumption with (3.24), it follows that for each fixed $\eta > 0$, we may find an $h_\eta \in \mathbb{K}$ and extract a subsequence t'_n from t_n such that ultimately $\|Y_{t'_n} - h_n\| < \eta$. The uniform continuity assumption shows that, given any $\varepsilon > 0$, we may choose $\varepsilon > 0$ so small that $\|g - h_\eta\| < \eta$ implies $|\Psi(g) - \Psi(h_\eta)| < \varepsilon$. Therefore, we have $L_1 < \varepsilon + L_2$ almost surely. Conversely, if h is such that $\Psi(h) > L_2 - \varepsilon/2$, we may use (3.25) to obtain likewise that $L_1 > L_2 - \varepsilon$ almost surely. Since $\varepsilon > 0$ can be chosen arbitrarily small, the conclusion is immediate. □

PROOF OF THEOREM 1.1. The proof follows directly from a joint application of Theorem 2.1, Lemma 3.1, Theorem 3.1 and the facts that (C.2) implies total boundedness of $\mathbb{K}$ with respect to the norm $\|\cdot\|$ and that the lower semicontinuity of $\|\cdot\|$ implies (M). □

4. Applications and examples.

4.1. *Weighted sup-norms.* Let q be a positive measurable function defined on $(0,1)$ and bounded away from 0 on each interval of the form $[\varepsilon, 1]$ for $\varepsilon > 0$. Assume further that

$$\limsup_{s\downarrow 0} |W(s)|/q(s) < \infty \quad \text{a.s.} \tag{4.1}$$

For instance, (4.1) is satisfied when

$$\limsup_{s\downarrow 0} \left\{ q^{-1}(s)\Big(s \log_2 (1/s)\Big)^{1/2} \right\} < \infty. \tag{4.2}$$

It is readily verified that under (4.1) the weighted sup-norm $\|h\| = \sup_{s\in(0,1]} |h(s)|/q(s)$ satisfies (C). Recalling the results of Beibel and Lerche (1994), we see that we need not assume [as they do in (1.6)] that q is monotone to ensure the validity of (SL).

4.2. *Integral norms.* Let q is positive measurable function defined on $(0,1)$ and assume that

$$\int_0^1 |W(s)|/q(s)\,ds < \infty \quad \text{a.s.,} \tag{4.3}$$

which is easily checked to hold when

$$\int_0^1 q^{-1}(s)s^{1/2}\,ds < \infty. \tag{4.4}$$

Then the integral norm $\|h\| = \int_0^1 (|h(s)|/q(s))\,ds$ satisfies (C).

4.3. *Hölder-type norms.* Let q be a positive locally bounded function defined on $(0,1)$ and assume that

$$\sup_{0\le s<t\le 1} |W(s) - W(t)|/q(s-t) < \infty \quad \text{a.s.,} \tag{4.5}$$

which is easily checked to hold when

$$\limsup_{s\downarrow 0} \left\{ q^{-1}(s)\Big(s\log\big(1/s\big)\Big)^{1/2} \right\} < \infty. \tag{4.6}$$

Then the Hölder-type norm $\|h\| = \sup_{0\le s<t\le 1} |h(t) - h(s)|/q(t-s)$ satisfies (C) (see Remark 1.4).

4.4. *Norms based on the Karhunen–Loève expansion of W.* We return to the Karhunen–Loève expansion given in (2.11). Introduce the norm

$$\|h\| = \sup_{n\ge 1} \left| \left(n - \tfrac{1}{2}\right)\pi \int_0^1 h(t)e_n(t)\,dt \right| \Big/ (\log_+ n)^{1/2}. \tag{4.7}$$

It is easily checked through the Borel–Cantelli lemma that this norm satisfies (C.1) since, via (2.12),

$$\|W\| = \sup_{n\ge 1} \left\{ |\xi_n|/(\log_+ n)^{1/2} \right\} < \infty \quad \text{a.s.} \tag{4.8}$$

On the other hand, it can be seen likewise that

$$\limsup_{\theta\downarrow 0} \big\|W\big((1-\theta)I\big) - W\big\| \ge 2 \quad \text{a.s.} \tag{4.9}$$

The assertions (4.8) and (4.9) show that $\theta \to W((1-\theta)I)$ is a rare example of a bounded but discontinuous $(\mathbb{C}_0(0,1), \|\cdot\|)$-valued process. We have the following proposition.

PROPOSITION 4.1. *The norm* $\|\cdot\|$ *defined in* (4.7) *satisfies* (C).

PROOF. Fix $n \geq 1$ and introduce the centered Gaussian process defined by

$$\zeta_n(\theta) = \left(n - \tfrac{1}{2}\right)\pi \int_0^1 W(\theta t) e_n \, dt \quad \text{for } \theta \in \left[\tfrac{1}{2}, 1\right]. \tag{4.10}$$

In view of the definition of (C) and of (4.7), all we need is to check that

$$\sup_{n \geq 1} \left\{ \sup_{\theta \in [1/2, 1]} |\zeta_n(\theta)|/(\log_+ n)^{1/2} \right\} < \infty. \tag{4.11}$$

Toward this goal, we first represent $\zeta_n(\theta)$ as a stochastic integral by letting

$$\begin{aligned} \zeta_n(\theta) &= 2^{1/2}\left(n - \tfrac{1}{2}\right)\pi \int_0^1 W(\theta t) \sin\left(\left(n - \tfrac{1}{2}\right)\pi t\right) dt \\ &= 2^{1/2}\left(n - \tfrac{1}{2}\right)\pi \int_0^\theta W(u) \sin\left(\left(n - \tfrac{1}{2}\right)\pi u/\theta\right) du/\theta \\ &= 2^{1/2} \int_0^\theta \cos\left(\left(n - \tfrac{1}{2}\right)\pi u/\theta\right) dW(u), \end{aligned} \tag{4.12}$$

so that

$$E\left(\zeta_n^2(\theta)\right) = 2 \int_0^\theta \cos^2\left(\left(n - \tfrac{1}{2}\right)\pi u/\theta\right) du = \theta \leq 1. \tag{4.13}$$

Next, we consider the natural metric $\rho_n(s,t) = \{E((\zeta_n(s) - \zeta_n(t))^2)\}^{1/2}$ generated by the process $\zeta_n(\cdot)$. We obtain from (4.12) that for $1/2 \leq \theta \leq \theta + \delta \leq 1$,

$$\begin{aligned} \rho_n^2(\theta, \theta + \delta) &= 2 \int_0^\theta \left(\cos\left(\left(n - \frac{1}{2}\right)\frac{\pi u}{\theta}\right) - \cos\left(\left(n - \frac{1}{2}\right)\frac{\pi u}{\theta + \delta}\right)\right)^2 du \\ &\quad + 2 \int_\theta^{\theta + \delta} \cos^2\left(\left(n - \frac{1}{2}\right)\frac{\pi u}{\theta + \delta}\right) \\ &\leq 2\left(n - \frac{1}{2}\right)^2 \pi^2 \theta^2 \left(\frac{1}{\theta} - \frac{1}{\theta + \delta}\right)^2 + 2\delta \sin^2\left(\left(n - \frac{1}{2}\right)\frac{\pi\delta}{\theta + \delta}\right) \\ &\leq \frac{2n^2\pi^2\delta^2}{\theta^2} + 2\delta\left\{\frac{n\pi\delta}{\theta}\right\}^2 = 2n^2\pi^2\frac{\delta^2 + \delta^3}{\theta^2}, \end{aligned} \tag{4.14}$$

whence $\rho_n(\theta, \theta + \delta) \leq 4n\pi\delta$. It follows that for any $\varepsilon \in (0, 2^{1/2})$, the 2δ grid with $\delta = \varepsilon/(4n\pi)$ provides an ε-net of the metric space $([1/2, 1], \rho_n)$. Let $N_n(\varepsilon)$ be the minimal number of balls of radius ε covering this space. We have $N_n(\varepsilon) \leq (4\delta)^{-1} \leq n\pi/\varepsilon$. This yields an estimate for the Dudley integral D_n [see Dudley (1973) and Adler (1990)]:

$$\begin{aligned} D_n &:= \int_0^{2^{1/2}} \left(\log N_n(\varepsilon)\right)^{1/2} d\varepsilon \\ &\leq (2 \log n\pi)^{1/2} + \int_0^1 |\log \varepsilon|^{1/2} \, d\varepsilon \leq 2^{1/2}(\log n)^{1/2} + 3. \end{aligned} \tag{4.15}$$

It is well known [see, e.g., Dmitrowskii (1990)] that for each bounded centered Gaussian process ζ, the median m of the random variable $\sup|\zeta|$ can be evaluated from the corresponding Dudley integral D via the inequality $m \le 4(2^{1/2}D)$. When applied to $\zeta_n(\cdot)$, this estimate yields the following upper bound for the median m_n of $\sup_{\theta \in [1/2,1]} |\zeta_n(\theta)|$:

$$m_n \le 4\big(2^{1/2}D_n\big) \le 16(\log n)^{1/2} + 18. \tag{4.16}$$

We finally use the basic upper bound (3.5) with $\mathbf{X} = \mathbb{C}(1/2, 1)$ ($\mathbb{C}(a,b)$ denoting the set of continuous functions on $[a,b]$), $Z = \zeta_n(\cdot)$ and the norm $\|h\| = \sup_{\theta \in (1/2,1]} |h(\theta)|$. This, in combination with (4.13) and (4.16), shows that, for any $A > 1$,

$$P\bigg(\sup_{\theta \in [1/2,1]} |\zeta_n(\theta)| > (16 + A)(\log n)^{1/2} + 18\bigg) \le 1 - \Phi\big(A(\log n)^{1/2}\big).$$

Thus, for $A > 1$,

$$\sum_n P\bigg(\sup_{\theta \in [1/2,1]} |\zeta_n(\theta)|/(\log n)^{1/2} > 34 + A\bigg) \le \sum_n 1 - \Phi\big(A(\log n)^{1/2}\big) < \infty.$$

An application of the Borel–Cantelli lemma completes the proof of (4.11). □

REMARK 4.1. A similar construction shows the delicate difference between the conditions (C) and (C.1). Let $k_n = 2^n$ for $n \ge 1$ and set

$$\|h\| = \sup_{n \ge 1} \left| \left(k_n - \tfrac{1}{2}\right)\pi \int_0^1 h(t) e_{k_n}(t)\, dt \right| \Big/ (\log_+ n)^{1/2}. \tag{4.17}$$

It is immediate from the fact that the random variables in (2.12) are independent that the seminorm defined via (4.17) satisfies (C.1). However, a careful evaluation shows that it does not satisfy the condition (C).

5. Conclusion. It is obvious that our arguments can be used, in the spirit of Goodman and Kuelbs (1991) to obtain results similar to those given above for general Gaussian processes, or likewise to treat the related problem of partial sums. This will be considered elsewhere.

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