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A UNIVERSAL CHUNG-TYPE LAW OF THE ITERATED LOGARITHM[1]

BY UWE EINMAHL AND DAVID M. MASON

Indiana University and University of Delaware

Let $X_1, X_2, \ldots,$ be a sequence of independent and identically distributed random variables. We find sequences of norming and centering constants α_n and β_n such that a universal Chung-type law of the iterated logarithm holds, namely, $\liminf_{n\to\infty} \max_{1\le k\le n} |S_k - k\beta_n|/\alpha_n < \infty$ almost surely, where S_k denotes the sum of the first k of $X_1, X_2, \ldots, k \ge 1$. If the underlying distribution function is in the Feller class, we show that this lim inf is strictly positive with probability 1.

1. Introduction. Let $X, X_1, X_2, \ldots$ be independent identically distributed (i.i.d.) nondegenerate random variables and set $S_0 := 0, S_n := X_1 + \cdots + X_n$, $n \ge 1$. Write Lt for $\log(t \vee e)$ and put $LLt := L(Lt)$, $t \ge 0$. It follows from the so-called other law of the iterated logarithm (LIL) due to Chung (1948) and Jain and Pruitt (1975) that under assumption

$$EX^2 < \infty \quad \text{and} \quad EX = 0, \tag{1.1}$$

one has almost surely (a.s.)

$$\liminf_{n\to\infty} \max_{1\le k\le n} |S_k|/\sqrt{n/LLn} = \left(\pi/\sqrt{8}\right)\left(EX^2\right)^{1/2}. \tag{1.2}$$

It is known that condition (1.1) is necessary for (1.2) to hold. [Refer to Csáki (1978).] More general results, however, are attainable if one uses different norming sequences. Jain and Pruitt (1973) obtained analogues of (1.2) for certain random variables in the domain of attraction of a stable law of index $\alpha \in (0, 2]$. Among other things they showed that if one has for a suitable sequence $a_n \uparrow \infty$,

$$S_n/a_n \to Z \quad \text{in distribution}, \tag{1.3}$$

where Z is a stable r.v. of index $\alpha \in (0, 2]$, which is not supported on a half-line, then there exists a positive constant c such that

$$\liminf_{n\to\infty} \max_{1\le k\le n} |S_k|/a_{[n/LLn]} = c < \infty \quad \text{a.s.}, \tag{1.4}$$

where as usual $[x]$ denotes the integer part of x.

Using large deviation techniques, Jain (1982) was finally able to identify the constant c in (1.4), thereby showing that the classical Chung LIL is a special

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case of a general result on the lim inf behavior of suitably normalized sums of i.i.d. random variables in the domain of attraction of a stable law as above. This of course excludes random variables in the domain of attraction of completely asymmetric stable laws if $\alpha < 1$, and one might ask what type of lim inf behavior one has in such situations. Jain and Pruitt (1973) pointed out that if both X and the limiting variable Z are nonnegative (or nonpositive) with probability 1, then the statement on the maximum reduces to a one-sided LIL result for the sums S_n (see Corollary 5). For some related results refer to Fristedt and Pruitt (1971), Klass (1976, 1977), Mason (1994), Pruitt (1981, 1990), Wichura (1974) and Zhang (1986). This still leaves open the question of what is the proper analogue of (1.4) when a general r.v. is attracted to a completely asymmetric stable law if $\alpha < 1$.

Another problem that has not yet been solved is to find out what is the lim inf behavior if relation (1.3) is only satisfied after a suitable centering, that is, if one has for some sequences $\{b_n\}$ and $a_n \uparrow \infty$,

$$(S_n - b_n)/a_n \to Z \quad \text{in distribution,} \tag{1.5}$$

which is more general than (1.3) if $\alpha = 1$. Jain and Pruitt (1973) obtained in that situation only a partial answer.

More recently Weiner (1985) showed that it is possible to extend result (1.4) to certain random variables in the Feller class. Recall that a r.v. X is in the Feller class if there exist sequences $\{b_n\}$ and $a_n \uparrow \infty$, such that

$$(S_n - b_n)/a_n \quad \text{is tight with nondegenerate subsequential limits.} \tag{1.6}$$

He proved that if one has (1.6) with $b_n = 0$ and if all the limit laws are in a certain sense uniformly "nonasymmetric," then (1.4) holds true.

The purpose of the present paper is to show that there exists a *universal* Chung-type LIL that is applicable for *any* random variable in the Feller class. To prove such a general result, it will be necessary to introduce appropriate centering constants in statement (1.4). To be more specific, we are interested in a result of the type

$$0 < \liminf_{n\to\infty} \max_{1\le k\le n} |S_k - \beta_{n,k}|/\Psi_n < \infty \quad \text{a.s.,} \tag{1.7}$$

where $\{\beta_{n,k}, 1 \le k \le n, n \ge 1\}$ is an array and $\Psi_n \uparrow \infty$.

Here one might think that it would be most natural to work with $\beta_{n,k} = 0$, but given the work of Jain and Pruitt (1973), this is clearly impossible. One of our most difficult tasks is to find the right centering constants. Somewhat surprisingly it will turn out that a centering exists such that the lim inf in (1.7) is always finite, no matter whether X is in the Feller class or not, thereby making the upper bound part of (1.7) a *universal* result.

We will employ the so-called quantile transformation method, which has been successfully used by Csörgő, Haeusler and Mason (1988a, b) to study weak convergence results for trimmed sums, and also by Mason (1994) for his work on one-sided LIL results. If F is the right-continuous distribution function of

X, define the *quantile function* Q by setting $Q(u) := \inf\{x\colon F(x) \geq u\}, 0 < u < 1$, $Q(0) := Q(0+)$. Then two functions—the *truncated mean function* $\mu(s, 1-t)$, $0 < s, t < \frac{1}{2}$, and the *truncated variance function* $\sigma^2(s, 1-t), 0 < s, t < \frac{1}{2}$—will play a crucial role. They are defined as

$$\mu(s, 1-t) := \int_s^{1-t} Q(u)\,du,$$

$$\sigma^2(s, 1-t) := \int_s^{1-t}\int_s^{1-t} (u \wedge v - uv)\,dQ(u)\,dQ(v), \qquad 0 < s, t < \tfrac{1}{2}.$$

Obviously, the σ-function increases as s and t decrease so that if we define for $\gamma_1, \gamma_2 > 0, \sigma_n(\gamma_1, \gamma_2) := \sigma(\gamma_1 LLn/n, 1 - \gamma_2 LLn/n)$, $n \geq 1$, we obtain nondecreasing sequences (in n).

Finally set for $\gamma_1, \gamma_2 > 0$, $\mu_n(\gamma_1, \gamma_2) := \mu(\gamma_1 LLn/n, 1 - \gamma_2 LLn/n), n \geq 1$. Then our first result is as follows.

THEOREM 1. *Let X be a nondegenerate random variable. Given $\gamma_1, \gamma_2 > 0$ with $\gamma_1 + \gamma_2 < 1$, there exists a positive constant C_1 such that*

$$\liminf_{n \to \infty} \max_{1 \leq k \leq n} |S_k - k\mu_n(\gamma_1, \gamma_2)| / \sqrt{n/LLn}\,\sigma_n(\gamma_1, \gamma_2) \leq C_1 < \infty \quad a.s. \tag{1.8}$$

Here of course it might happen that the lim inf is equal to zero. Our next result, which also gives more information about the centering, shows that this is not the case if X is in the Feller class. As already indicated in statement (1.7), we will consider general arrays $\{\beta_{n,k}, 1 \leq k \leq n,\ n \geq 1\}$. For technical reasons we need the mild regularity condition

$$\max_{n_j \leq n \leq n_{j+1}} \max_{1 \leq k \leq n_j} |\beta_{n,k} - \beta_{n_j,k}| = o\Big(\sqrt{n_j/LLn_j}\,\sigma_{n_j}\Big) \quad \text{as } j \to \infty, \tag{1.9}$$

where $n_j := [\exp(j/(LLj)^2)], j \geq 1$, and $\sigma_n := \sigma_n(1,1), n \geq 1$. This condition of course is trivially true if $\beta_{n,k} = 0$. It will also turn out that it is satisfied if $\beta_{n,k} = k\mu_n(\gamma_1, \gamma_2), \gamma_1, \gamma_2 > 0$ (see Lemma 6). We then have the following theorem.

THEOREM 2. *Let X be a nondegenerate random variable in the Feller class. There exists a positive constant C_2 such that we have for any array $\{\beta_{n,k}, 1 \leq k \leq n, n \geq 1\}$ satisfying* (1.9),

$$\liminf_{n \to \infty} \max_{1 \leq k \leq n} |S_k - \beta_{n,k}| / \sqrt{n/LLn}\,\sigma_n \geq C_2 \quad a.s. \tag{1.10}$$

Using the fact that the two sequences $\sigma_n(\gamma_1, \gamma_2)$ and σ_n are of the same order if X is in the Feller class (see Lemma 4), it is evident that (1.7) holds true in this case if we set $\beta_{n,k} = k\mu_n(\gamma_1, \gamma_2)$, where $0 < \gamma_1, \gamma_2$, $\gamma_1 + \gamma_2 < 1$ and $\Psi_n = (n/LLn)^{1/2}\sigma_n$.

COROLLARY 1. *Let X be a nondegenerate random variable in the Feller class and let γ_1, γ_2 be positive constants satisfying $\gamma_1 + \gamma_2 < 1$. Then we have for a suitable positive constant C_3,*

$$\liminf_{n\to\infty} \max_{1\le k\le n} |S_k - k\mu_n(\gamma_1,\gamma_2)|/\sqrt{n/LLn}\,\sigma_n = C_3 \quad a.s. \tag{1.11}$$

Recalling Theorem 2, it is now clear that the preceding centering in terms of the truncated mean function is *optimal* in the sense that it enables us to use the smallest possible norming sequence. If X is in the domain of attraction of a stable law of index $\alpha \in (0, 2]$, which is not supported on a half-line, then of course the result (1.4) of Jain and Pruitt (1973) applies too. Comparing (1.4) to Corollary 1, we first note that the norming sequences in (1.4) and (1.10) have the same order. To see this, use the convergence of types theorem and recall that whenever X is in the Feller class, then (1.6) is satisfied with $b_n = n\mu(1/n, 1-1/n)$ and $a_n = n^{1/2}\sigma(1/n, 1-1/n)$. [Refer to Corollary 10 in Csörgő, Haeusler and Mason (1988a).] The centering, however, may be quite different. What one can always show is that under condition (1.3),

$$n|\mu_n(\gamma_1,\gamma_2)| = O\big((nLLn)^{1/2}\sigma_n\big). \tag{1.12}$$

At first sight, this does not appear too useful because it clearly indicates that the order of our centering might be much larger than the order of the lim inf of the maximum of the partial sums. As a matter of fact, one can do slightly better. We will show that if the stable r.v. Z in (1.3) is not completely asymmetric, then one can improve (1.12) after an appropriate choice of the constants γ_1, γ_2 to

$$n|\mu_n(\gamma_1,\gamma_2)| = o\big((nLLn)^{1/2}\sigma_n\big), \tag{1.13}$$

and we can apply the subsequent Theorem 3 with $\beta_n = 0$.

THEOREM 3. *Let X be a nondegenerate random variable in the Feller class. Given $\gamma_1, \gamma_2 > 0$ with $\gamma_1 + \gamma_2 < 1$, there exist positive constants C_4, C_5 such that if $\{\beta_n\}$ is a sequence satisfying*

$$\limsup_{n\to\infty} |\beta_n - \mu_n(\gamma_1,\gamma_2)|/\sqrt{LLn/n}\,\sigma_n \le C_4 < \infty, \tag{1.14}$$

then we have

$$\liminf_{n\to\infty} \max_{1\le k\le n} |S_k - k\beta_n|/\sqrt{n/LLn}\,\sigma_n \le C_5 \quad a.s. \tag{1.15}$$

One might now wonder whether it is possible to obtain a similar improvement of Theorem 1 in general. It will turn out that this is impossible. To see this we first formulate a corollary to Theorem 3 that might be of independent interest. It shows that the lim inf in the one-sided LIL of Mason (1994) cannot be equal to 0 if X is in the Feller class.

COROLLARY 2. *Let $X \geq 0$ be a nondegenerate random variable in the Feller class. There exists a positive constant C_6 such that*

$$-\sqrt{2} \leq \liminf_{n \to \infty} (S_n - n\mu_n)/\sqrt{nLLn}\, \tilde{\sigma}_n \leq -C_6 \quad a.s. \tag{1.16}$$

where $\tilde{\sigma}_n := \sigma(0, LLn/n)$, $\mu_n := \mu(0, LLn/n)$.

If Theorem 3 were true for general random variables, we would also obtain statement (1.16) for nonnegative random variables whose distribution functions have slowly varying upper tails. However, this would be in contradiction to Lemma 4 in Mason (1994), which says that in this case the lim inf in (1.16) is equal to 0 almost surely. Moreover, we will show in Section 5 that it is not even possible to prove Theorem 3 for random variables outside the Feller class in general if one only assumes

$$|\beta_n - \mu_n(\gamma_1, \gamma_2)|/\sqrt{LLn/n}\, \sigma_n(\gamma_1, \gamma_2) \to 0 \quad \text{as } n \to \infty. \tag{1.17}$$

One might also ask whether one can choose a constant in (1.14) that depends only on γ_1 and γ_2 such that (1.15) holds for all random variables in the Feller class. If this were true, our proof of Corollary 2 would lead to a universal constant C_6 in (1.16), which is impossible in view of Lemma 4 in Mason (1994).

After having indicated how the foregoing theorems relate to the result (1.4) of Jain and Pruitt (1973), we now would like to turn to random variables in the domain of attraction of completely asymmetric stable random variables in the case $\alpha < 1$. If X is nonnegative, one can use the aforementioned one-sided LIL results to obtain a Chung-type LIL with zero centering and the norming sequence $\{(nLLn)^{1/2}\sigma_n\}$, which misses the optimal sequence by a factor LLn. The same phenomenon will occur if one uses the zero centering in (1.7) for a general random variable attracted to a positive stable random variable, though the resulting norming sequence ψ_n might be somewhat closer to the optimal one.

Finally, if X is in the domain of attraction of an asymmetric stable law of index $\alpha = 1$, then Jain and Pruitt (1973) have shown that under an additional condition one can prove a lim inf result with zero centering, where the resulting norming sequence has the same order as the sequence $\{n|\mu(1/n, 1 - 1/n)|\}$. One can show, however, that if their condition is satisfied, then this norming sequence is at least of order $\{n^{1/2}\sigma(1/n, 1 - 1/n)\}$, which misses the optimal norming sequence by a power of LLn. As a matter of fact, one can easily show that in the cases covered by the corollary in Jain and Pruitt (1973), the above norming sequence even leads to an additional factor of order $O((Ln)^c)$, where c is a positive constant. We thus see that in many cases the zero centering requires much larger norming sequences than in Corollary 1.

The remainder of the paper is organized as follows. We will first prove Theorem 1 in Section 2. Our proof is based on the Skorokhod embedding in conjunction with an appropriate truncation argument. Here it will turn out to be extremely helpful to represent the random variables $\{X_n\}$ as quantile transformations of uniformly distributed random variables. We then prove Theorem

2 and Corollary 1 in Section 3. We will show that after a symmetrization argument one can obtain Theorem 2 by a modification of the proof given by Jain and Pruitt (1973). The proof of Theorem 3 will be carried out in Section 4. We will employ the same truncation argument as in Section 2, but then we will use a skillful conditioning argument due to Jain and Pruitt (1973) that enables us to eventually reduce the proof to studying small deviation probabilities for certain arrays of rowwise independent random variables. Because we will be dealing with truncated random variables at that point, we can show via the Berry–Esseen inequality that the normal approximation will be good enough for our purposes. Finally we will demonstrate in Section 5 how one can infer the results for random variables in the domain of attraction to a stable law from the foregoing theorems. We will also construct a random variable where the tail probabilities are slowly varying and condition (1.17) is satisfied for a suitable sequence $\{\beta_n\}$, but (1.15) fails.

2. Proof of Theorem 1.

It is enough to prove Theorem 1 for random variables with infinite variance. (The finite variance case will be included in the proof of Theorem 3.)

We first need the following auxiliary result.

LEMMA 1. *If $EX^2 = \infty$, we have*:

(a) $\sigma^2(s, 1-t) \sim sQ^2(s) + tQ^2(1-t) + \tau^2(s, 1-t)$ *as* $s, t \downarrow 0$, *where* $\tau^2(s, 1-t) := \int_s^{1-t} Q^2(u)\,du, 0 < s, t < \frac{1}{2}$.

(b) $\mu(s, 1-t)/\tau(s, 1-t) \to 0$ *as* $s, t \downarrow 0$.

Both statements are contained in the proof of Lemma 2.1 of Csörgő, Haeusler and Mason (1988b).

To simplify our notation, we set $\psi_n := \sqrt{n/LLn}\,\sigma_n(\gamma_1, \gamma_2)$ and $M_n := \max_{1 \le k \le n} |S_k - k\mu_n(\gamma_1, \gamma_2)|$. Without loss of generality, we can and actually do assume that $X_j = Q(U_j)$, $j \ge 1$, where $\{U_j\}$ is a sequence of i.i.d. uniform $(0,1)$ random variables. The next lemma will be crucial for the proof of Theorem 1.

LEMMA 2. *Given $\gamma_1, \gamma_2 > 0$ with $\gamma_1 + \gamma_2 < 1$, there exists a constant $C_7 > 0$ (depending on γ_1 and γ_2 only) such that for large enough n*,

$$P\{M_n \le C_7\psi_n\} \ge (Ln)^{-1+\delta/2},$$

where $\delta := 1 - \gamma_1 - \gamma_2$.

PROOF. (i) Define the event

$$E_n := \{\gamma_1 LLn/n \le U_j \le 1 - \gamma_2 LLn/n,\ 1 \le j \le n\}.$$

Then it is obvious that

$$P\{M_n \le C_7\psi_n\} \ge P(\{M_n \le C_7\psi_n\} \mid E_n)P(E_n). \tag{2.1}$$

Moreover, we have,

$$(2.2)\quad P\big(\{M_n \le C_7\psi_n\} \mid E_n\big) = P\left\{ \max_{1\le k\le n} \left| \sum_{j=1}^{k} \{Q(V_{n,j}) - \mu_n(\gamma_1,\gamma_2)\} \right| \le C_7\psi_n \right\},$$

where $V_{n,j}, 1 \le j \le n$, are independent uniform $(\gamma_1 LLn/n, 1-\gamma_2 LLn/n)$ r.v.'s.

Next observe that on account of Lemma 1(b),

$$\begin{aligned} EQ(V_{n,j}) &= \mu_n(\gamma_1,\gamma_2)/\big(1-(\gamma_1+\gamma_2)LLn/n\big) \\ &= \mu_n(\gamma_1,\gamma_2) + o\big(\sigma_n(\gamma_1,\gamma_2)LLn/n\big), \qquad 1 \le j \le n. \end{aligned}$$

Setting $Z_{n,j} = Q(V_{n,j}) - EQ(V_{n,j}), 1 \le j \le n$, and $T_{n,k} := \sum_{j=1}^{k} Z_{n,j}, 1 \le k \le n$, we can infer that for large enough n,

$$(2.3)\quad \begin{aligned} &P\left\{ \max_{1\le k\le n} \left| \sum_{j=1}^{k} \{Q(V_{n,j}) - \mu_n(\gamma_1,\gamma_2)\} \right| \le C_7\psi_n \right\} \\ &\quad \ge P\left\{ \max_{1\le k\le n} |T_{n,k}| \le \frac{C_7}{2}\psi_n \right\}. \end{aligned}$$

Because

$$\begin{aligned} P(E_n) &= \big(1-(1-\delta)LLn/n\big)^n \\ &= \exp\Big(-(1-\delta)LLn + O\big((LLn)^2/n\big)\Big) \end{aligned}$$

which is, for large n, greater than or equal to

$$\tfrac{1}{2}(Ln)^{-1+\delta},$$

we can infer from (2.1)–(2.3) that

$$(2.4)\quad P\{M_n \le C_7\psi_n\} \ge \frac{1}{2} P\left\{ \max_{1\le k\le n} |T_{n,k}| \le \frac{C_7}{2}\psi_n \right\} (Ln)^{-1+\delta},$$

provided n is large enough.

(ii) We now use the Skorokhod embedding. Without loss of generality, we can assume that there exists a Brownian motion W and a triangular array of rowwise independent stopping times $\{\nu_{n,j}: 1 \le j \le n, n \ge 1\}$ such that for any $n \ge 1$,

$$(2.5)\quad (T_{n,1},\dots,T_{n,n}) =_d \big(W(\nu_{n,1}),\dots,W(\nu_{n,1}+\cdots+\nu_{n,n})\big),$$

$$(2.6)\quad E\nu_{n,j} = EZ_{n,j}^2, \qquad 1 \le j \le n,$$

and for any integer $r \ge 2$,

$$(2.7)\quad E|\nu_{n,j}|^r \le 2\big(8/\pi^2\big)^{r-1} r!\, E(Z_{n,j})^{2r}, \qquad 1 \le j \le n.$$

[Refer to Theorem A.1 in Hall and Heyde (1980).]

Setting $\tau_n^2(\gamma_1,\gamma_2) := \tau^2(\gamma_1 LLn/n, 1-\gamma_2 LLn/n), \gamma_1,\gamma_2 > 0, n \geq 1$, it is easy to see that

$$EZ_{n,j}^2 \leq \tau_n^2(\gamma_1,\gamma_2)/\big(1-(\gamma_1+\gamma_2)LLn/n\big) \sim \tau_n^2(\gamma_1,\gamma_2), \qquad 1 \leq j \leq n. \tag{2.8}$$

Moreover, we have for any constant $\alpha > 0$ and large enough n,

$$P\Big\{\max_{1\leq k\leq n}|T_{n,k}| \leq \tfrac{1}{2}C_7\psi_n\Big\} \geq P_n(1) - P_n(2), \tag{2.9}$$

where

$$P_n(1) := P\Big\{\max_{0\leq t\leq (2+\alpha)n\sigma_n^2(\gamma_1,\gamma_2)}|W(t)| \leq \tfrac{1}{2}C_7\psi_n\Big\}$$

and

$$P_n(2) := P\Big\{\sum_{j=1}^n(\nu_{n,j}-E\nu_{n,j}) \geq \alpha n\sigma_n^2(\gamma_1,\gamma_2)\Big\}.$$

Now

$$P_n(1) = P\Big\{\max_{0\leq t\leq 1}|W(t)| \leq C_7(4\alpha+8)^{-1/2}(LLn)^{-1/2}\Big\},$$

and using relation (2.1) of Jain and Pruitt (1975), we get

$$P_n(1) \geq \frac{8}{3\pi}(Ln)^{-\pi^2(2+\alpha)/2C_7^2}. \tag{2.10}$$

Next we need an upper bound for $P_n(2)$. Toward that end, we first note that by Lemma 1(a), for large n,

$$\begin{aligned}|Z_{n,j}| &\leq 2\big|Q(\gamma_1 LLn/n)\big| \vee 2\big|Q(1-\gamma_2 LLn/n)\big|\\ &\leq 3\gamma_0^{-1/2}\sqrt{n/LLn}\,\sigma_n(\gamma_1,\gamma_2), \qquad 1\leq j\leq n,\end{aligned}$$

where $\gamma_0 := \gamma_1 \wedge \gamma_2$.

Using the c_r-inequality, (2.7) and Lemma 1, we get, for large n,

$$\begin{aligned}E|\nu_{n,j}-E\nu_{n,j}|^r &\leq 4\big(16/\pi^2\big)^{r-1}r!EZ_{n,j}^{2r}\\ &\leq 5\Big\{\big(144/\pi^2\gamma_0\big)\psi_n^2\Big\}^{r-1}r!\sigma_n^2(\gamma_1,\gamma_2), \qquad 1\leq j\leq n.\end{aligned}$$

In particular,

$$\mathrm{Var}(\nu_{n,j}) \leq \big(1440/\pi^2\gamma_0\big)\psi_n^2\sigma_n^2(\gamma_1,\gamma_2) =: \lambda_n^2, \qquad 1\leq j\leq n, \tag{2.11}$$

and

$$E|\nu_{n,j}-E\nu_{n,j}|^r \leq \tfrac{1}{2}\lambda_n^2 r! c_n^{r-2}, \qquad 1\leq j\leq n, \tag{2.12}$$

where $c_n := (144/\pi^2\gamma_0)\psi_n^2$.

Using a version of the Bernstein inequality given as Exercise 14 on page 111 of Chow and Teicher (1988), we find that for large n,

$$P_n(2) \le \exp\Big(-\alpha^2\psi_n^4(LLn)^2/\big(2n\lambda_n^2 + 2\alpha c_n\psi_n^2 LLn\big)\Big),$$

which is

$$= (Ln)^{-\pi^2\gamma_0\alpha^2/(288(10+\alpha))}.$$

Choosing $\alpha = \alpha(\gamma_1, \gamma_2)$ large enough so that

$$\pi^2\gamma_0\alpha^2/\big(288(10+\alpha)\big) \ge 1,$$

we get for large n,

$$P_n(2) \le (Ln)^{-1}, \tag{2.13}$$

Setting $C_7 := 2\pi\sqrt{2+\alpha}\,\delta^{-1/2}$, we finally can infer from (2.4), (2.9), (2.10) and (2.13) that

$$\begin{aligned} P\Big\{\max_{1\le k\le n}|T_{n,k}| \le \frac{1}{2}C_7\psi_n\Big\} &\ge \frac{8}{3\pi}(Ln)^{-\delta/8} - (Ln)^{-1} \\ &\ge 2(Ln)^{-\delta/2}, \end{aligned}$$

which in combination with (2.4) implies the assertion of Lemma 2. □

Set $m_l := [\exp(l^\rho)]$, $l \ge 1$, where $\rho > 1$.

LEMMA 3. *Given* $\gamma_1, \gamma_2 > 0$ *with* $\gamma_1 + \gamma_2 \le 1$, *we have*,

$$\max_{1\le k\le m_{l-1}} \big|S_k - k\mu_{m_l}(\gamma_1,\gamma_2)\big|/\psi_{m_l} \to 0 \quad a.s. \tag{2.14}$$

PROOF. Let $X'_{l,j} := Q(U_j)1\{\gamma_1 LLm_l/m_l \le U_j \le 1 - \gamma_2 LLm_l/m_l\}, 1 \le j \le m_{l-1}$, and set $S'_{l,k} := \sum_{j=1}^k X'_{l,j}$, $1 \le k \le m_{l-1}$. Then it is obvious that

$$\begin{aligned} P\{S_k \ne S'_{l,k} \text{ for some } 1 \le k \le m_{l-1}\} &\le P\Bigg(\bigcup_{j=1}^{m_{l-1}}\{X_j \ne X'_{l,j}\}\Bigg) \\ &\le m_{l-1}LLm_l/m_l. \end{aligned}$$

Because by the choice of the subsequence $\{m_l\}$,

$$\sum_l m_{l-1}LLm_l/m_l < \infty, \tag{2.15}$$

we obtain via the Borel–Cantelli lemma

$$\max_{1\le k\le n}|S_k - S'_{l,k}| = O(1) \quad \text{a.s.} \tag{2.16}$$

Next observe that $EX'_{l,j} = \mu_{m_l}(\gamma_1, \gamma_2)$, $1 \le j \le m_{l-1}$. Thus a straightforward application of Kolmogorov's inequality yields for any $\varepsilon > 0$

$$P\Big\{\max_{1 \le k \le m_{l-1}} |S'_{l,k} - k\mu_{m_l}(\gamma_1, \gamma_2)| \ge \varepsilon\psi_{m_l}\Big\}$$
$$\le \varepsilon^{-2} m_{l-1} LLm_l \operatorname{Var}(X'_{l,1}) / \big(m_l \sigma^2_{m_l}(\gamma_1, \gamma_2)\big).$$

Recalling Lemma 1(a), it is clear that

$$\operatorname{Var}(X'_{l,1}) / \sigma^2_{m_l}(\gamma_1, \gamma_2) = O(1).$$

Using once more (2.15), we get for any $\varepsilon > 0$,

$$\sum_l P\Big\{\max_{1 \le k \le m_{l-1}} |S'_{l,k} - k\mu_{m_l}(\gamma_1, \gamma_2)| \ge \varepsilon\psi_{m_l}\Big\} < \infty. \tag{2.17}$$

This of course implies

$$\max_{1 \le k \le m_{l-1}} \big|S'_{l,k} - k\mu_{m_l}(\gamma_1, \gamma_2)\big| / \psi_{m_l} \to 0 \quad \text{a.s.} \tag{2.18}$$

Combining (2.16) and (2.18), we obtain the assertion of Lemma 3. □

We are now ready to conclude the proof of Theorem 1. To further simplify our notation, we set for $l \ge 2$,

$$M_1^{(l)} := M_{m_{l-1}},$$
$$M_2^{(l)} := \max_{m_{l-1} \le k \le m_l} \big|(S_k - S_{m_{l-1}}) - (k - m_{l-1})\mu_{m_l}(\gamma_1, \gamma_2)\big|.$$

Obviously,

$$M_{m_l}/\psi_{m_l} \le M_1^{(l)}/\psi_{m_l} + M_2^{(l)}/\psi_{m_l}.$$

In view of Lemma 3, it will be enough to show that after an appropriate choice of $\rho > 1$,

$$\liminf_{l \to \infty} M_2^{(l)}/\psi_{m_l} \le C_7 \quad \text{a.s.} \tag{2.19}$$

Noticing that the r.v.'s $M_2^{(l)}, l \ge 2$, are independent, and using the Borel–Cantelli lemma, we only have to check that

$$\sum_l P\{M_2^{(l)} \le C_7\psi_{m_l}\} = \infty. \tag{2.20}$$

Because $P\{M_2^{(l)} \le t\} \ge P\{M_{m_l} \le t\}$, $t \ge 0$, we can infer from Lemma 2 that for large enough l,

$$P\{M_2^{(l)} \le C_7\psi_{m_l}\} \ge (Lm_l)^{-1+\delta/2}, \tag{2.21}$$

which implies statement (2.20) provided we choose $\rho = (1 - \delta/2)^{-1}$. This completes the proof of Theorem 1. □

3. Proof of Theorem 2 and Corollary 1. The following lemma is an immediate consequence of relation (1.42b) in Csörgő, Haeusler and Mason (1988a).

LEMMA 4. *Let X be in the Feller class and let $0 < \lambda_1, \lambda_2 \le 1$. Then we have*

$$\limsup_{t \downarrow 0} \sigma^2(\lambda_1 t, 1 - \lambda_2 t)/\sigma^2(t, 1-t) \le K < \infty,$$

where K is a positive constant depending on λ_1, λ_2 and the distribution of X.

The next lemma will be crucial for the proof of Theorem 2. It can be inferred from a result of Hall (1983) on concentration functions [see also Griffin, Jain and Pruitt (1984), relation (1.3)], but we prefer to give a somewhat more direct argument.

LEMMA 5. *Let X be in the Feller class. Then we have*

$$\lim_{c \downarrow 0} \limsup_{m \to \infty} \sup_{\beta} P\Big\{ |S_m - \beta| \le c\sqrt{m}\,\sigma(1/m) \Big\} = 0,$$

where we set $\sigma^2(s) := \sigma^2(s, 1-s), 0 < s < \frac{1}{2}$.

PROOF. We use a symmetrization argument. Let $\{Y_j\}$ be a sequence of independent copies of X, which is also independent of the X_j's. Set $X_j^* := X_j - Y_j$, $j \ge 1$, and $S_m^* := \sum_{j=1}^m X_j^*$, $m \ge 1$. Then it is obvious that

$$P\Big\{ |S_m - \beta| \le c\sqrt{m}\,\sigma(1/m) \Big\}^2 \le P\Big\{ |S_m^*| \le 2c\sqrt{m}\,\sigma(1/m) \Big\} \tag{3.1}$$

and it is enough to prove

$$\lim_{c \downarrow 0} \limsup_{m \to \infty} P\Big\{ |S_m^*| \le c\sqrt{m}\,\sigma(1/m) \Big\} = 0. \tag{3.2}$$

To that end, we first note that by Corollary 10 in Csörgő, Haeusler and Mason (1988a),

$$\Big\{ \Big(S_m - m\mu(1/m)\Big) \Big/ \sqrt{m}\,\sigma(1/m) \Big\} \tag{3.3}$$

is tight with nondegenerate subsequential limits,

which of course implies

(3.4) $\Big\{ S_m^* / \sqrt{m}\sigma(1/m) \Big\}$ is tight with nondegenerate subsequential limits.

Denoting the class of all subsequential limit distributions by $\mathcal{L}^*$, we can infer that for any $c > 0$,

$$\limsup_{m \to \infty} P\Big\{ |S_m^*| \le c\sqrt{m}\,\sigma(1/m) \Big\} \le \sup_{Q \in \mathcal{L}^*} Q\big([-c, c]\big). \tag{3.5}$$

By the theorem of Pruitt (1983) we know that all p-measures Q in $\mathcal{L}^*$ have C^∞-densities, and we can use the same argument as in Lemma 1 of Weiner (1985) to infer that

$$\lim_{c \downarrow 0} \sup_{Q \in \mathcal{L}^*} Q\big([-c,c]\big) = 0. \tag{3.6}$$

Combining (3.5) and (3.6), we obtain (3.2) and consequently the assertion of Lemma 5. □

We are now ready to prove Theorem 2. To simplify our notation, we set

$$M_n := \max_{1 \le k \le n} |S_k - \beta_{n,k}|, \qquad n \ge 1.$$

Because of Lemma 5, we can find a positive constant C_8 such that for large enough m,

$$P_m := \sup_\beta P\Big\{|S_m - \beta| \le 2C_8\sqrt{m}\,\sigma\big(1/m\big)\Big\} \le e^{-3}. \tag{3.7}$$

Setting $m_n := [2n/LLn]$ and arguing as in Jain and Pruitt (1973), we find that for large enough n,

$$\begin{aligned} &P\Big\{M_n \le C_8\sqrt{n/LLn}\,\sigma_n\Big\} \\ &\quad \le P\Big\{M_n \le C_8\sqrt{m_n}\,\sigma\big(1/m_n\big)\Big\} \\ &\quad \le \prod_{j=1}^{[LLn/2]} P\Big\{\big|S_{jm_n} - S_{(j-1)m_n} - \big(\beta_{n,jm_n} - \beta_{n,(j-1)m_n}\big)\big| \le 2C_8\sqrt{m_n}\,\sigma\big(1/m_n\big)\Big\} \\ &\quad \le P_{m_n}^{[LLn/2]} \le (Ln)^{-3/2}. \end{aligned}$$

Letting $n_j := [\exp(j/(Lj)^2)], j \ge 1$, we readily obtain

$$\sum_{j=1}^{\infty} P\Big\{M_{n_j} \le C_8\sqrt{n_j/LLn_j}\,\sigma\big(1/n_j\big)\Big\} < \infty. \tag{3.8}$$

Using Borel–Cantelli and (1.9) we find that we have almost surely for any $\varepsilon > 0$, $n_j \le n \le n_{j+1}$ and $j \ge j_0(\varepsilon, \omega)$,

$$\begin{aligned} M_n(\omega) &\ge M_{n_j}(\omega) - \max_{n_j \le n \le n_{j+1}} \max_{1 \le k \le n_j} |\beta_{n,k} - \beta_{n_j,k}| \\ &\ge (C_8 - \varepsilon)\sqrt{n_j/LLn_j}\,\sigma_{n_j} \\ &\ge \frac{C_8 - \varepsilon}{K_0(1+\varepsilon)}\sqrt{n/LLn}\sigma_n, \end{aligned}$$

where $K_0 := \limsup_{n \to \infty} \sigma_{n_{j+1}}/\sigma_{n_j}$, which is finite on account of Lemma 4.

Because ε can be made arbitrarily small, we obtain the assertion of Theorem 2 with $C_2 := C_8/K_0$.

We now turn to the proof of Corollary 1. In light of Theorems 1 and 2, we only have to check that condition (1.9) is satisfied. The assertion then follows from the Hewitt–Savage 0-1 law.

LEMMA 6. *Let X be in the Feller class. Then we have for any* $\gamma_1, \gamma_2 > 0$,

$$n_j \max_{n_j \le n \le n_{j+1}} |\mu_n(\gamma_1, \gamma_2) - \mu_{n_j}(\gamma_1, \gamma_2)| = o\Big(\sqrt{n_j/LLn_j}\,\sigma_{n_j}\Big) \quad \text{as } j \to \infty.$$

PROOF. We first prove Lemma 6 in the infinite variance case. Recalling Lemma 1, we obtain for large j,

$$\begin{aligned}
&\max_{n_j \le n \le n_{j+1}} |\mu_n(\gamma_1, \gamma_2) - \mu_{n_j}(\gamma_1, \gamma_2)| \\
&\quad \le \int_{\gamma_1 LLn_{j+1}/n_{j+1}}^{\gamma_1 LLn_j/n_j} |Q(u)|\,du + \int_{\gamma_2 LLn_{j+1}/n_{j+1}}^{\gamma_2 LLn_j/n_j} |Q(1-u)|\,du \\
&\quad \le 2\sigma_{n_{j+1}}(\gamma_1, \gamma_2)\left\{ \int_{\gamma_1 LLn_{j+1}/n_{j+1}}^{\gamma_1 LLn_j/n_j} u^{-1/2}\,du + \int_{\gamma_2 LLn_{j+1}/n_{j+1}}^{\gamma_2 LLn_j/n_j} u^{-1/2}\,du \right\} \\
&\quad \le 4\big[(\gamma_1)^{1/2} + (\gamma_2)^{1/2}\big]\sigma_{n_{j+1}}(\gamma_1, \gamma_2)\big(n_j^{-1/2} - n_{j+1}^{-1/2}\big)(LLn_{j+1})^{1/2},
\end{aligned}$$

which is of order

$$o\big(\sigma_{n_{j+1}}(\gamma_1, \gamma_2) n_j^{-1/2} (LLn_j)^{-1/2}\big) \quad \text{as } j \to \infty.$$

Because by Lemma 4, $\sigma_{n_{j+1}}(\gamma_1, \gamma_2)/\sigma_{n_j} = O(1)$ for any $\gamma_1, \gamma_2 > 0$, we obtain the assertion of Lemma 6 for the case $EX^2 = \infty$.

If $EX^2 < \infty$, it is easy to see that

$$\limsup_{s, t \downarrow 0} \big(\sqrt{s}|Q(s)| + \sqrt{t}|Q(1-t)|\big) = 0,$$

and we can prove Lemma 6 in this case by an obvious modification of the above argument. □

4. Proof of Theorem 3 and Corollary 2. We need a further lemma that shows that the σ-function and the τ-function are of the same order if X is in the Feller class. Recall that $\tau^2(s, 1-t)$ is defined as $\int_s^{1-t} Q^2(u)\,du$, $0 < s, t < \frac{1}{2}$.

LEMMA 7. *Let X be in the Feller class. Then we have for all* $\lambda_1, \lambda_2 > 0$,

$$\limsup_{t \downarrow 0} \sigma^2(\lambda_1 t, 1 - \lambda_2 t)/\tau^2(\lambda_1 t, 1 - \lambda_2 t) \le \widetilde{K},$$

where $\widetilde{K}$ *is a positive constant depending on* λ_1, λ_2 *and the distribution of* X.

PROOF. We only have to prove Lemma 7 for r.v.'s with $EX^2 = \infty$. Recalling Lemma 4, we see that it is enough to show

$$\limsup_{t\downarrow 0} \sigma^2(\lambda_3 t, 1-\lambda_3 t)/\tau^2(\lambda_1 t, 1-\lambda_2 t) < \infty, \tag{4.1}$$

where $\lambda_3 = \lambda_1 \vee \lambda_2$.

By the monotonicity of the τ-function, we have

$$\tau^2(\lambda_1 t, 1-\lambda_2 t) \geq \tau^2(\lambda_3 t, 1-\lambda_3 t), \tag{4.2}$$

whence (4.1) follows from

$$\limsup_{t\downarrow 0} \sigma^2(t, 1-t)/\tau^2(t, 1-t) < \infty, \tag{4.3}$$

which is always true if X is the Feller class. To see this, use relation (1.42c) in Csörgő, Haeusler and Mason (1988a) in conjunction with our Lemma 1(a). □

We are now ready to show how the proof of Theorem 1 has to be modified so as to obtain the improved statement for r.v.'s in the Feller class. The next lemma is the counterpart to Lemma 2. To simplify our notation, we set

$$\tau_n^2(\gamma_1,\gamma_2) := \tau^2\big(\gamma_1 LLn/n,\ 1-\gamma_2 LLn/n\big), \quad \gamma_1,\gamma_2 > 0,\ n \geq 1.$$

LEMMA 8. *Given $\gamma_1, \gamma_2 > 0$ with $\gamma_1 + \gamma_2 = 1-\delta$, where $\delta > 0$, there exist positive constants C_9, C_{10} such that if $\{\beta_n\}$ is a sequence of real numbers satisfying, for large n,*

$$|\beta_n - \mu_n(\gamma_1,\gamma_2)| \leq C_9 \sqrt{LLn/n}\, \tau_n(\gamma_1,\gamma_2), \tag{4.4}$$

we have for large enough n,

$$P\Big\{\max_{1\leq k\leq n} |S_k - k\beta_n| \leq C_{10}\sqrt{n/LLn}\,\sigma_n(\gamma_1,\gamma_2)\Big\} \geq (Ln)^{-1+\delta/2}. \tag{4.5}$$

PROOF. If $EX^2 = \infty$, arguing as in part (i) of the proof of Lemma 2, we can obtain for all large enough n,

$$\begin{aligned} &P\Big\{\max_{1\leq k\leq n} |S_k - k\beta_n| \leq C_{10}\psi_n\Big\} \\ &\quad \geq P\Big\{\max_{1\leq k\leq n} |T_{n,k} - k\Delta_n| \leq \frac{C_{10}}{2}\psi_n\Big\}(Ln)^{-1+\delta}, \end{aligned} \tag{4.6}$$

where $T_{n,k} := \sum_{j=1}^{k}\{Q(V_{n,j}) - EQ(V_{n,j})\}$, $1 \leq k \leq n$, $\Delta_n := \beta_n - \mu_n(\gamma_1,\gamma_2)$ and $\psi_n = \sqrt{n/LLn}\,\sigma_n(\gamma_1,\gamma_2)$.

Moreover, noticing that if $EX^2 < \infty$,

$$\begin{aligned} EQ(V_{n,j}) &= \mu_n(\gamma_1,\gamma_2) + O\big(\mu_n(\gamma_1,\gamma_2)LLn/n\big) \\ &= \mu_n(\gamma_1,\gamma_2) + O\big(\sigma_n(\gamma_1,\gamma_2)LLn/n\big), \end{aligned}$$

it is easy to see that one can also prove (4.6) in the finite variance case.

Next set $m_n := [\alpha n/LLn]$, where $\alpha > 0$ will be specified later and consider the events

$$F_{n,j} := \{|T_{n,(j+1)m_n} - (j+1)m_n\Delta_n| \le \tfrac{1}{4}C_{10}\psi_n,\ M_n(j) \le \tfrac{1}{4}C_{10}\psi_n\},$$

where $M_n(j) := \max_{jm_n \le k \le (j+1)m_n} |T_{n,k} - T_{n,jm_n} - (k - jm_n)\Delta_n|$, $j \ge 0$. Let $l_n := \min\{j \ge 0\colon (j+1)m_n \ge n\}$ and note that

$$l_n \sim LLn/\alpha \quad \text{as } n \to \infty. \tag{4.7}$$

From the definition of the events $F_{n,j}$ it is easy to see that

$$P\left(\bigcap_{j=0}^{l_n} F_{n,j}\right) \le P\Big\{\max_{1\le k\le n} |T_{n,k} - k\Delta_n| \le \tfrac{1}{2}C_{10}\psi_n\Big\}. \tag{4.8}$$

Furthermore, we have for $1 \le l \le l_n$,

$$P\left(\bigcap_{j=0}^{l} F_{n,j}\right) = E\left[\prod_{j=0}^{l-1} I_{F_{n,j}} P(F_{n,l}\|\mathcal{G}_{lm_n}^{(n)})\right], \tag{4.9}$$

where $\mathcal{G}_k^{(n)}$ is the σ-field generated by the r.v.'s $T_{n,1},\dots,T_{n,k}$, $1 \le k \le n$.

Because $M_n(l)$ is independent of $\mathcal{G}_{lm_n}^{(n)}$, it is obvious that

$$\begin{aligned} P(F_{n,l}\|\mathcal{G}_{lm_n}^{(n)}) \ge{}& P\left(\left\{|T_{n,(l+1)m_n} - (l+1)m_n\Delta_n| \le \frac{C_{10}}{4}\psi_n\right\}\|\mathcal{G}_{lm_n}^{(n)}\right)\\ &- P\left\{M_n(l) \ge \frac{1}{4}C_{10}\psi_n\right\}. \end{aligned} \tag{4.10}$$

Letting $K_1 := \limsup_{n\to\infty} \tau_n(\gamma_1,\gamma_2)/\sigma_n(\gamma_1,\gamma_2)$, which is bounded above by 1 if $EX^2 = \infty$ [use Lemma 1(a)] and equal to $(EX^2/\mathrm{Var}(X))^{1/2}$ if $EX^2 < \infty$, we find that

$$\begin{aligned} &P\{M_n(l) \ge \tfrac{1}{4}C_{10}\psi_n\}\\ &\quad \le P\Big\{\max_{1\le k\le m_n} |T_{n,k}| \ge \big(\tfrac{1}{4}C_{10} - 2\alpha K_1 C_9\big)\sqrt{n/LLn}\,\sigma_n(\gamma_1,\gamma_2)\Big\}, \end{aligned}$$

which by Kolmogorov's inequality is less than or equal to

$$16\alpha(C_{10} - 8\alpha K_1 C_9)^{-2}\,\mathrm{Var}\big(Q(V_{n,1})\big)/\sigma_n^2(\gamma_1,\gamma_2).$$

As in (2.8) we have

$$\limsup_{n\to\infty} \mathrm{Var}\big(Q(V_{n,1})\big)/\tau_n^2(\gamma_1,\gamma_2) \le 1$$

so that the last term is less than or equal to

$$17K_1^2\alpha(C_{10} - 8\alpha K_1 C_9)^{-2}$$

provided we have chosen n large enough.

Setting $C_{10} := 8\alpha K_1 C_9 + 12(\alpha K_1^2)^{1/2}$, we can infer that for large n,

$$P\{M_n(l) \geq \tfrac{1}{4} C_{10}\psi_n\} \leq \tfrac{1}{8}, \qquad 0 \leq l \leq l_n. \tag{4.11}$$

Next observe that

$$\begin{aligned}
&P\Big(\{|T_{n,(l+1)m_n} - (l+1)m_n\Delta_n| \leq \tfrac{1}{4}c_{10}/4\psi_n\} \,\|\, \mathcal{G}_{lm_n}^{(n)}\Big) I_{F_{n,l-1}} \\
&\quad \geq P\{-\tfrac{1}{4}C_{10}\psi_n \leq T_{n,m_n} - m_n\Delta_n \leq 0\} I_{\{0 \leq T_{n,lm_n} - lm_n\Delta_n \leq (C_{10}/4)\psi_n\}} I_{F_{n,l-1}} \\
&\qquad + P\{0 \leq T_{n,m_n} - m_n\Delta_n \leq \tfrac{1}{4}C_{10}\psi_n\} I_{\{0 > T_{n,lm_n} - lm_n\Delta_n \geq -(C_{10}/4)\psi_n\}} I_{F_{n,l-1}},
\end{aligned}$$

which in turn, by (4.11), is greater than or equal to

$$q_n I_{F_{n,l-1}},$$

where

$$q_n := \big(P\{T_{n,m_n} - m_n\Delta_n \geq 0\} \wedge P\{T_{n,m_n} - m_n\Delta_n \leq 0\}\big) - \tfrac{1}{8}.$$

If $EX^2 = \infty$, we have $\operatorname{Var} Q(V_{n,1}) \sim \tau_n^2(\gamma_1, \gamma_2)$ as $n \to \infty$, so that an application of the Berry–Esseen inequality in conjunction with (4.4) yields, for large n,

$$\begin{aligned}
q_n &\geq 1 - \Phi\big(2\sqrt{m_n}\,|\Delta_n|/\tau_n(\gamma_1,\gamma_2)\big) - \frac{E|Q(V_{n,1}) - EQ(V_{n,1})|^3}{\sqrt{m_n}\,\tau_n^3(\gamma_1,\gamma_2)} - \frac{1}{8} \\
&\geq 1 - \Phi\big(2C_9\alpha^{1/2}\big) - \frac{3\Big(|Q(\gamma_1 LLn/n)| \vee |Q(1-\gamma_2 LLn/n)|\Big)}{\sqrt{m_n}\,\tau_n(\gamma_1,\gamma_2)} - \frac{1}{8} \\
&\geq 1 - \Phi\big(2C_9\alpha^{1/2}\big) - 4K_2(\alpha\gamma_0)^{-1/2} - \frac{1}{8},
\end{aligned}$$

where $\gamma_0 := \gamma_1 \wedge \gamma_2$ and $K_2 := \limsup_{n\to\infty} \sigma_n(\gamma_1,\gamma_2)/\tau_n(\gamma_1,\gamma_2)$, which is finite because of Lemma 7. Choosing $C_9 \leq (1/32)\alpha^{-1/2}$ and $\alpha \geq \gamma_0^{-1} 64 K_2^2$, we have, in this case,

$$q_n \geq 1 - \Phi\left(\frac{1}{16}\right) - \frac{3}{8} \geq \frac{1}{16}.$$

If $EX^2 < \infty$, we have $\sigma_n^2(\gamma_1,\gamma_2) \uparrow \operatorname{Var}(X), \gamma_1, \gamma_2 > 0$. Moreover, it is easy to see that in this case the array $\{Z_{n,j} = Q(V_{n,j}) - EQ(V_{n,j}); 1 \leq j \leq m_n, n \geq 1\}$ satisfies Lyapounov's condition for the CLT. It follows that as $n \to \infty$,

$$\sup_x \big|P\{\pm T_{n,m_n} \leq x\sqrt{m_n}\,\sigma_n(\gamma_1,\gamma_2)\} - \Phi(x)\big| \to 0, \tag{4.12}$$

which in turn implies

$$\liminf_{n\to\infty} P\{\pm T_{n,m_n} \geq m_n\Delta_n\} \geq 1 - \Phi\big(\alpha^{1/2}K_1C_9\big). \tag{4.13}$$

Setting $C_9 = (1/32 \wedge (4K_1)^{-1})\alpha^{-1/2}$, we see that we have in both cases, for large n,

(4.14)
$$q_n \geq \frac{1}{16}.$$

Recalling (4.10) and (4.11), we get for $1 \leq l \leq l_n$ and large n,

(4.15)
$$P(F_{n,l} \| \mathcal{G}_{lm_n}^{(n)}) I_{F_{n,l-1}} \geq \frac{1}{16} I_{F_{n,l-1}},$$

which of course implies in combination with (4.8) and (4.9),

(4.16)
$$P\Big\{ \max_{1 \leq k \leq n} |T_{n,k} - k\Delta_n| \leq \tfrac{1}{2} C_{10} \psi_n \Big\} \geq 16^{-l_n} P(F_{n,0}).$$

Noticing that $F_{n,0} = \{M_n(0) \leq \frac{1}{4} C_{10} \psi_n\}$, we can infer from (4.11) that the last term is greater than or equal to

$$\frac{7}{8} 16^{-l_n},$$

which in turn by (4.7) is, for large n, greater than or equal to

$$\exp(-5(\log 2) LLn/\alpha) \geq (Ln)^{-\delta/2},$$

provided $\alpha \geq 10(\log 2)\delta^{-1}$.

Setting $\alpha = 10(\log 2)\delta^{-1} \vee \gamma_0^{-1} 64^2 K_2^2$ and defining C_9 and C_{10} as before, we obtain (4.5). □

The next lemma corresponds to Lemma 3.

LEMMA 9. *Let $\{\beta_n\}$ be as in Lemma 8 and let $\gamma_1, \gamma_2 > 0$ be real numbers satisfying $\gamma_1 + \gamma_2 \leq 1$. Set $m_l := [\exp(l^\rho)], l \geq 1$, where $\rho > 1$. Then we have, as $l \to \infty$,*

(4.17)
$$\max_{1 \leq k \leq m_{l-1}} |S_k - k\beta_{m_l}| / \sqrt{m_l / LLm_l}\, \sigma_{m_l}(\gamma_1, \gamma_2) \to 0 \quad a.s.$$

PROOF. Noticing that the proof of Lemma 3 also applies to r.v.'s with a finite variance, we only need to show that

(4.18)
$$m_{l-1} |\beta_{m_l} - \mu_{m_l}(\gamma_1, \gamma_2)| / \sqrt{m_l / LLm_l}\, \sigma_{m_l}(\gamma_1, \gamma_2) \to 0.$$

To prove (4.18), simply note that $m_{l-1} = o(m_l / LLm_l)$ and use (4.4) in conjunction with the fact that as $l \to \infty$,

$$\tau_{m_l}(\gamma_1, \gamma_2) / \sigma_{m_l}(\gamma_1, \gamma_2) = O(1).$$ □

Using the two foregoing lemmas and arguing as in the proof of Theorem 1, we obtain, for any sequence $\{\beta_n\}$ as before,

(4.19)
$$\liminf_{n \to \infty} \max_{1 \leq k \leq n} |S_k - k\beta_n| / \sqrt{n/LLn}\, \sigma_n(\gamma_1, \gamma_2) \leq C_{10} \quad \text{a.s.},$$

which in combination with Lemma 4 implies (1.15).

Finally, observe that by Lemma 7,

$$\sigma_n^2/\tau_n^2(\gamma_1,\gamma_2) = O(1). \tag{4.20}$$

It is now obvious that condition (1.14) with an appropriate choice for the constant C_4 implies (4.4) and consequently (1.15), and the proof of Theorem 3 is complete.

To verify Corollary 2, we apply Theorem 3 with $\beta_n = \mu_n(\gamma_1,\gamma_2) - C_6\sqrt{LLn/n}\,\sigma_n$, where $\gamma_1 = \gamma_2 = \frac{1}{3}$ (say) and $C_6 = C_4(\frac{1}{3},\frac{1}{3},X)$. It follows that

$$\begin{aligned}&\liminf_{n\to\infty}(S_n - n\beta_n)/\sqrt{nLLn}\,\sigma_n\\&\qquad\le \liminf_{n\to\infty}\max_{1\le k\le n}|S_k - k\beta_n|/\sqrt{nLLn}\,\sigma_n = 0 \quad \text{a.s.,}\end{aligned}$$

but we also have

$$\begin{aligned}&\liminf_{n\to\infty}(S_n - n\beta_n)/\sqrt{nLLn}\sigma_n\\&\qquad= \liminf_{n\to\infty}\big(S_n - n\mu_n(\gamma_1,\gamma_2)\big)/\sqrt{nLLn}\sigma_n + C_6.\end{aligned}$$

Because X is nonnegative, it is obvious that

$$\mu_n(\gamma_1,\gamma_2) \le \mu_n$$

and we can conclude that

$$\liminf_{n\to\infty}(S_n - n\mu_n)/\sqrt{nLLn}\,\sigma_n \le -C_6 \quad \text{a.s.} \tag{4.21}$$

Finally noting that $\widetilde{\sigma}_n/\sigma_n \to 1$ because X is nonnegative, we obtain the upper bound in Corollary 2. The lower bound follows from the Theorem in Mason (1994). □

5. Examples and discussion. We first consider random variables X whose distribution functions F are in the domain of attraction of a stable law of index $0 < \alpha < 2$, written $F \in D(\alpha)$. It is known [cf. Corollary 3 of Csörgő, Haeusler and Mason (1988a)] that $F \in D(\alpha)$, $0 < \alpha < 2$, if and only if there exists a positive function h defined on $(0,1)$, slowly varying at 0, and constants $\delta_1 \ge 0, \delta_2 \ge 0$ with at least one being greater than 0 such that as $s \downarrow 0$,

$$Q(s+) = \big(-\delta_1 + o(1)\big)s^{-1/\alpha}h(s) \tag{5.1.i}$$

and

$$Q(1-s) = \big(\delta_2 + o(1)\big)s^{-1/\alpha}h(s). \tag{5.1.ii}$$

Using elementary properties of regularly varying functions, it can be readily shown that for all $0 < \alpha < 2$ as $s,t \downarrow 0$,

$$\tau^2(s,1-t) \sim \left(\frac{2}{\alpha} - 1\right)^{-1}\big\{\delta_1^2 s^{-2/\alpha+1}h^2(s) + \delta_2^2 t^{-2/\alpha+1}h^2(t)\big\}, \tag{5.2}$$

$$(5.3)\qquad \sigma^2(s,1-t) \sim 2(2-\alpha)^{-1}\{\delta_1^2 s^{-2/\alpha+1}h^2(s) + \delta_2^2 t^{-2/\alpha+1}h^2(t)\},$$

$$(5.4)\qquad \mu(s,1-t) \sim \left(\frac{1}{\alpha}-1\right)^{-1}\{-\delta_1 s^{-1/\alpha+1}h(s) + \delta_2 t^{-1/\alpha+1}h(t)\},$$

for $0 < \alpha < 1$, and

$$(5.5)\quad \mu(0,1) - \mu(s,1-t) \sim \left(1-\frac{1}{\alpha}\right)^{-1}\{-\delta_1 s^{-1/\alpha+1}h(s) + \delta_2 t^{-1/\alpha+1}h(t)\},$$

for $1 < \alpha < 2$.

Set for $\nu_1, \nu_2 > 0$ and $n \geq 1$,

$$(5.6)\qquad \Delta_n(\nu_1,\nu_2) = \begin{cases} \mu_n(\nu_1,\nu_2), & \text{if } 0 < \alpha < 1,\\ \mu(0,1) - \mu_n(\nu_1,\nu_2), & \text{if } 1 < \alpha < 2. \end{cases}$$

Using (5.3), (5.4) and (5.5) we see that for all $0 < \alpha < 2, \alpha \neq 1$ and $\nu_1, \nu_2 > 0$,

$$(5.7)\qquad \begin{aligned} &\left(\frac{n}{LLn}\right)^{1/2} \frac{\Delta_n(\nu_1,\nu_2)}{\sigma_n(\nu_1,\nu_2)} \\ &= \frac{(2-\alpha)^{1/2}}{2^{1/2}|1-1/\alpha|} \frac{\{-\delta_1 \nu_1^{-1/\alpha+1} + \delta_2 \nu_2^{-1/\alpha+1}\}}{(\delta_1^2 \nu_1^{-2/\alpha+1} + \delta_2^2 \nu_2^{-2/\alpha+1})^{1/2}} + o(1) \quad \text{as } n \to \infty. \end{aligned}$$

We are now prepared to derive versions of the results of Jain and Pruitt (1973).

COROLLARY 3. *Assume that $F \in D(\alpha), 0 < \alpha < 2$, $\alpha \neq 1$, with both δ_1 and $\delta_2 > 0$ in* (5.1.i) *and* (5.1.ii) *or $F \in D(2)$. Then there exists a positive constant $c > 0$ such that*

$$(5.8)\qquad \liminf_{n\to\infty} \max_{1\leq m\leq n} |S_m - m\mu_\alpha| / \left(\sqrt{n/LLn}\,\sigma_n\right) = c \quad a.s.,$$

where $\mu_\alpha = 0$ if $0 < \alpha < 1$ and $\mu_\alpha = \mu(0,1) = EX_1$ if $1 < \alpha < 2$.

PROOF. First assume $\alpha \neq 2$. Choose $\gamma_1 > 0$ and $\gamma_2 > 0$ such that $\gamma_1 + \gamma_2 < 1$ and

$$(5.9)\qquad -\delta_1 \gamma_1^{-1/\alpha+1} + \delta_2 \gamma_2^{-1/\alpha+1} = 0.$$

Setting $\beta_n = \mu_\alpha$ we see by (5.7), (5.3) and (5.9) that as $n \to \infty$,

$$(5.10)\qquad \beta_n - \mu_n(\gamma_1,\gamma_2) = o\left(\sqrt{LLn/n}\,\sigma_n\right).$$

Thus by (1.15), for some positive constant C_5,

$$(5.11)\qquad \liminf_{n\to\infty} \max_{1\leq k\leq n} |S_k - k\mu_\alpha| / \left(\sqrt{n/LLn}\,\sigma_n\right) \leq C_5 \quad \text{a.s.}$$

and, further, because (1.9) is trivially satisfied with $\beta_{n,k} = k\mu_\alpha$, by Theorem 2,

$$\liminf_{n\to\infty} \max_{1\le k\le n} |S_k - k\mu_\alpha| / \left(\sqrt{n/LLn}\,\sigma_n\right) \ge C_2 \quad \text{a.s.}$$

for some positive C_2, which in combination with (5.11) and the Hewitt–Savage 0-1 law implies (5.8).

When $F \in D(2)$, then by Corollary 1 of Csörgő, Haeusler and Mason (1988a), $\sigma(s) =: \sigma(s, 1-s)$ is slowly varying at 0 and

$$s^{1/2}\left\{|Q(s)| + |Q(1-s)|\right\}/\sigma(s) \to 0 \quad \text{as } s \downarrow 0,$$

from which it is easy to infer that (5.10) holds for any $\gamma_1, \gamma_2 > 0$ and (5.8) follows as before. □

Suppose now that $F \in D(1)$. Further assume that

$$a_n^{-1} S_n \to_d Y \quad \text{as } n \to \infty, \tag{5.12}$$

for a suitable norming sequence $a_n \to \infty$, where Y is a stable random variable of index 1. We can assume without loss of generality that $a_n = \sqrt{n}\,\sigma(1/n, 1-1/n)$. Moreover, we also have

$$a_n^{-1}\left\{S_n - n\mu\left(1/n, 1-1/n\right)\right\} \to_d Y' \quad \text{as } n \to \infty, \tag{5.13}$$

where Y' is also stable of index 1. [See Corollary 3 of Csörgő, Haeusler and Mason (1988a).] From (5.12) and (5.13) in conjunction with the convergence of types theorem, we can conclude that for some $b \in \mathbb{R}$,

$$n\mu\left(1/n, 1-1/n\right) \Big/ \left(\sqrt{n}\,\sigma\left(1/n, 1-1/n\right)\right) \to b \quad \text{as } n \to \infty,$$

from which it is straightforward to infer that

$$s^{-1/2}\mu(s, 1-s)/\sigma(s, 1-s) \to b \quad \text{as } s \downarrow 0. \tag{5.14}$$

Using (5.1.i) and (5.1.ii) it is easy to show that for all $\nu_1, \nu_2 > 0$,

$$\mu(\nu_1 s, 1-\nu_2 s) - \mu(s, 1-s) \sim \{\delta_1 \log \nu_1 - \delta_2 \log \nu_2\} h(s) \quad \text{as } s \downarrow 0. \tag{5.15}$$

Also by setting $\alpha = 1$ in (5.3) we have

$$\sigma(s, 1-s) \sim \left(2\left(\delta_1^2 + \delta_2^2\right)\right)^{1/2} s^{-1/2} h(s) \quad \text{as } s \downarrow 0. \tag{5.16}$$

Thus from (5.15) and (5.16) we get

$$\begin{aligned} &s^{-1/2}\left\{\mu(\nu_1 s, 1-\nu_2 s) - \mu(s, 1-s)\right\}/\sigma(s, 1-s) \\ &\quad \sim (\delta_1 \log \nu_1 - \delta_2 \log \nu_2) \Big/ \left(2\left(\delta_1^2 + \delta_2^2\right)\right)^{1/2} \quad \text{as } s \downarrow 0. \end{aligned} \tag{5.17}$$

COROLLARY 4. *Assume that $F \in D(1)$ and* (5.12) *is satisfied. Further assume that* (5.1.i) *and* (5.1.ii) *hold with both δ_1 and $\delta_2 > 0$. Then there exists a positive constant c such that*

$$\liminf_{n\to\infty} \max_{1\le m\le n} |S_m| / \left(\sqrt{n/LLn}\,\sigma_n\right) = c \quad a.s. \tag{5.18}$$

PROOF. Choose $\gamma_1 > 0$ and $\gamma_2 > 0$ such that $\gamma_1 + \gamma_2 < 1$, satisfying the equation

$$\delta_1 \log \gamma_1 - \delta_2 \log \gamma_2 = -b\left(2\left(\delta_1^2 + \delta_2^2\right)\right)^{1/2}. \tag{5.19}$$

Next from (5.14) and (5.17) we have

$$\begin{aligned}\sqrt{\frac{n}{LLn}}\,\frac{\mu_n(\gamma_1,\gamma_2)}{\sigma_n} &= \sqrt{\frac{n}{LLn}}\,\frac{\{\mu_n(\gamma_1,\gamma_2) - \mu_n(1,1)\}}{\sigma_n} + \sqrt{\frac{n}{LLn}}\,\frac{\mu_n(1,1)}{\sigma_n} \\ &= -b + b + o(1) = o(1) \quad \text{as } n \to \infty.\end{aligned} \tag{5.20}$$

Setting $\beta_n = 0$ we see that (1.14) holds, so that the proof of Corollary 4 now proceeds exactly like that of Corollary 3. □

COROLLARY 5. *Assume that $F \in D(\alpha)$ with $0 < \alpha < 1$ and that* (5.1.i) *and* (5.1.ii) *hold with either $\delta_1 = 0$ and $\delta_2 > 0$ or $\delta_1 > 0$ and $\delta_2 = 0$. Then there exists a positive constant c such that*

$$\liminf_{n\to\infty} \max_{1\le k\le n} |S_k| / \left(\sqrt{nLLn}\,\sigma_n\right) \le c \quad a.s. \tag{5.21}$$

Moreover, if $F(0-) = 0$, if $\delta_2 > 0$, and $F(0) = 1$, if $\delta_2 = 0$, then the positive constant c can be chosen so that equality holds almost surely in (5.21).

PROOF. We will assume that $\delta_2 > 0$. The proof for the case $\delta_1 > 0$ is the same. Notice that for any choice of $\gamma_1, \gamma_2 > 0$ such that $\gamma_1 + \gamma_2 < 1$,

$$\begin{aligned}&\max_{1\le k\le n} |S_k| / \left(\sqrt{nLLn}\,\sigma_n\right) \\ &\quad\le \max_{1\le k\le n} |S_k - k\mu_n(\gamma_1,\gamma_2)| / \left(\sqrt{nLLn}\,\sigma_n\right) \\ &\qquad+ \sqrt{n}\,|\mu_n(\gamma_1,\gamma_2)| / \left(\sqrt{LLn}\,\sigma_n\right).\end{aligned} \tag{5.22}$$

It follows from (5.3) and (5.4) that

$$\lim_{n\to\infty} \sqrt{n}\,|\mu_n(\gamma_1,\gamma_2)| / \left(\sqrt{LLn}\,\sigma_n\right) =: K > 0, \tag{5.23}$$

and from Corollary 1 we get

$$\liminf_{n\to\infty} \max_{1\le k\le n} |S_k - k\mu_n(\gamma_1,\gamma_2)| / \left(\sqrt{nLLn}\,\sigma_n\right) = 0 \quad \text{a.s.} \tag{5.24}$$

Now (5.21) obviously follows from (5.22), (5.23) and (5.24).

If we assume further that $F(0-) = 0$, then

$$\max_{1\le k\le n} |S_k|/\left(\sqrt{nLLn}\,\sigma_n\right) = S_n/\left(\sqrt{nLLn}\,\sigma_n\right), \tag{5.25}$$

so that in this case equality in (5.22) for a suitable positive constant is obtained from Theorem 3.1 of Wichura (1974). □

Corollaries 3 and 4 correspond to Theorem 1 of Jain and Pruitt (1973), assuming that the limiting stable law is not completely asymmetric. Corollary 5 includes their Theorem 2.

We finally show that Theorem 3 is no longer true for a random variable whose distribution function has a slowly varying upper tail, written $F \in D(0)$, even if one assumes (1.17). Let $X \ge 0$ be a random variable such that

$$1 - F(x) = \exp\left(-Lx/\sqrt{LLLx}\right), \qquad x \ge 1.$$

Then an easy calculation shows that

$$Q(1-t) \sim \exp\left(L(1/t)\sqrt{LLL(1/t)}\right) \quad \text{as } t \downarrow 0.$$

Because $Q(1-t)$ is rapidly varying at 0, arguing as in the proof of Lemma 4 of Mason (1994), we get

$$\sigma^2(s, 1-t) \sim tQ^2(1-t) \quad \text{as } s, t \downarrow 0,$$

from which we can infer that for any $\gamma_1 \ge 0$ and $\gamma_2 > 0$,

$$\sigma_n^2(\gamma_1,\gamma_2) \sim \gamma_2\left(\frac{LLn}{n}\right)\exp\left(2L\left(\frac{n}{\gamma_2 LLn}\right)\sqrt{LLL\left(\frac{n}{\gamma_2 LLn}\right)}\right).$$

Setting $\sigma_n^2(\gamma) = \sigma_n^2(0,\gamma)$, we conclude after some calculation that if $\gamma > 1$ and $\gamma_1, \gamma_2 > 0$ are such that $\gamma_1 + \gamma_2 < 1$,

$$\sigma_n^2(\gamma_1,\gamma_2)/\sigma_n^2(\gamma) \le \exp(\varepsilon_n LLLn)$$

with $\varepsilon_n \to 0$.

Therefore,

$$\limsup_{n\to\infty} \sigma_n^2(\gamma_1,\gamma_2)/\left(\sigma_n^2(\gamma)LLn\right) = 0. \tag{5.26}$$

Because we have in the slowly varying case

$$n\mu_n(\gamma_1,\gamma_2) = o\left(\sqrt{nLLn}\,\sigma_n(\gamma_1,\gamma_2)\right),$$

condition (1.17) holds with $\beta_n = 0, n \ge 1$. If Theorem 3 were to hold for this example, statement (1.15) would imply

$$\liminf_{n\to\infty} S_n/\left(\sqrt{n/LLn}\,\sigma_n(\gamma_1,\gamma_2)\right) \le C_5 < \infty \quad \text{a.s.},$$

which by (5.26) gives

$$\liminf_{n\to\infty} S_n/\big(\sqrt{n}\,\sigma_n(\gamma)\big) = 0 \quad \text{a.s.}$$

This would contradict (1.6) in Mason (1994), which implies that the preceding lim inf is equal to infinity almost surely.

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DEPARTMENT OF MATHEMATICS
INDIANA UNIVERSITY
BLOOMINGTON, INDIANA 47405

DEPARTMENT OF MATHEMATICAL SCIENCES
UNIVERSITY OF DELAWARE
NEWARK, DELAWARE 19716