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ON THE FRACTAL NATURE OF EMPIRICAL INCREMENTS

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We prove that the set of points where exceptional oscillations of empirical and related processes occur infinitely often is a random fractal, and evaluate its Hausdorff dimension.

1. Introduction and motivation. Let $\alpha_n(t) = n^{1/2}(F_n(t) - t)$ for $0 \le t \le 1$ denote the *uniform empirical process*, where $F_n(t) = n^{-1}\#\{U_i \le t: 1 \le i \le n\}$ is the *uniform empirical distribution function* based on the first n observations from a sequence $U_1, U_2, \dots$ of independent uniform $(0,1)$ random variables, and $\#E$ denotes the number of elements in the set E.

We will be mainly concerned with the study of local oscillations of α_n. These are conveniently described by introducing a sequence $\{h_n,\ n \ge 1\}$ of positive constants, which we will assume to satisfy assumptions among (H.1)–(H.4) listed below. Let $\log_j$ denote the jth iterated logarithm.

(H.1) $h_n \downarrow 0$, $nh_n \uparrow \infty$ and $0 < h_n < 1$.

(H.2) $nh_n/\log n \to \infty$.

(H.3) $\log(1/h_n)/\log_2 n \to \infty$.

(H.4) $nh_n/\log_2 n \to \infty$.

The following Theorems A and B give a partial description, suitable for our needs, of the limiting behavior of the extremal h_n-increments of α_n.

THEOREM A. *Under* (H.1) *and* (H.4), *for any prescribed* $t_0 \in [0,1)$, *we have*

$$(1.1)\qquad \limsup_{n\to\infty} \pm (2h_n \log_2 n)^{-1/2}(\alpha_n(t_0 + h_n) - \alpha_n(t_0)) = 1 \quad a.s.$$

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THEOREM B. *Under* (H.1), (H.2) *and* (H.3), *we have*

$$(1.2)\quad \lim_{n\to\infty}\left\{\sup_{0\le t\le 1-h_n} \pm\,(2h_n\log(1/h_n))^{-1/2}(\alpha_n(t+h_n)-\alpha_n(t))\right\} = 1 \quad a.s.$$

Theorem A was proved for $t_0 = 0$ by Kiefer (1972), and an easy argument shows that it holds equivalently for any prescribed $t_0 \in [0,1)$. Theorem B coincides with (1.3) in Deheuvels and Mason (1992) and extends previous results due to Stute (1982) and Mason, Shorack and Wellner (1983).

Keeping (1.1) and (1.2) in mind, introduce the sequence of random sets defined for $n \ge 1$ and $\Lambda \ge 0$ by

$$(1.3)\quad E_n^{\pm}(\Lambda) = \left\{t\in[0,1-h_n]: \pm(2h_n\log(1/h_n))^{-1/2}(\alpha_n(t+h_n)-\alpha_n(t)) \ge \Lambda\right\}.$$

Observe that (H.2) $\Rightarrow$ (H.4). Therefore, Theorem A implies that, under (H.1)–(H.2)–(H.3), for any $\Lambda > 0$ and prescribed $t_0 \in [0,1)$,

$$(1.4)\quad P\big(t_0 \in E_n^{\pm}(\Lambda)\text{ i.o.}\big) = 0,$$

whereas Theorem B implies under the same assumptions that for any $\Lambda \in (0,1)$,

$$(1.5)\quad P\big(E_n^{\pm}(\Lambda) = \varnothing\text{ i.o.}\big) = 0.$$

In other words, (1.4) and (1.5) show that, under (H.1), (H.2) and (H.3), with probability 1 for any $\Lambda \in (0,1)$ and prescribed $t_0 \in [0,1)$, the sets $E_n^{\pm}(\Lambda)$, $n = 1,2,\dots$, are ultimately *nonvoid and not containing* t_0. The main purpose of the present paper is to study this strange phenomenon by giving a precise description of the limiting behavior of such random sets of *exceptional points*, where large oscillations of α_n occur infinitely often.

To motivate further our investigations, we shall survey the very much related problem of local oscillations of the Wiener process. First, we recall from the Komlós, Major and Tusnády (1975, 1976) strong approximations that we may define $\{U_n,\ n\ge 1\}$ on a probability space on which sits a sequence $\{W_n(t),\ t\ge 0\}$, $n = 1,2,\dots$, of standard Wiener processes such that

$$(1.6)\quad \sup_{0\le t\le 1}|\alpha_n(t) - W_n(t) + tW_n(1)| = O(n^{-1/2}\log n) \quad \text{a.s. as } n\to\infty.$$

Making use of the easily proven fact (which follows from an application of the Borel–Cantelli lemma) that $|W_n(1)| = O(\sqrt{\log n})$ a.s. as $n \to \infty$, it is readily verified from (1.6) that both (1.1) and (1.2) hold under (H.1)–(H.2)–(H.3), with the formal replacement of α_n by W_n. This gives heuristic support to the idea that one should obtain useful guidance toward the solution of our problem by considering the much simpler case where α_n is replaced by a single Wiener process W. This question has been investigated in detail by Orey and Taylor (1974) [see also Kôno (1977)], whose results we now briefly discuss. The following Theorem C [Lévy (1948); see, e.g., (1.1)–(1.2) in Orey and Taylor (1974)] gives the analogues of Theorems A and B in this case.

THEOREM C. *For any prescribed* $t_0 \in [0,1)$, *we have*

$$(1.7)\quad \limsup_{h\downarrow 0} \pm (2h\log_2(1/h))^{-1/2}(W(t_0+h)-W(t_0)) = 1 \quad a.s.,$$

whereas

$$(1.8)\quad \lim_{h\downarrow 0}\left\{\sup_{0\le t\le 1-h} \pm (2h\log(1/h))^{-1/2}(W(t+h)-W(t))\right\} = 1 \quad a.s.$$

In spite of the fact that (1.7) does not fully coincide with (1.1) when, in the latter statement, h and W are formally replaced by h_n and W_n (or α_n), respectively, the similarities of (1.7)–(1.8) and (1.1)–(1.2) are very striking. It is therefore natural to introduce the Brownian motion analogue of $E_n^{\pm}(\Lambda)$ by setting, for $\Lambda \ge 0$,

$$(1.9)\quad B^{\pm}(\Lambda) = \left\{t\in[0,1): \limsup_{h\downarrow 0} \pm (2h\log(1/h))^{-1/2}(W(t+h)-W(t)) \ge \Lambda\right\}.$$

Orey and Taylor (1974) showed that $B^{\pm}(\Lambda)$ is a *random fractal*. Their main result, stated in Theorem D below, provides the Hausdorff dimension of this set. Recall [see, e.g., Falconer (1985), Chapter 2 in Falconer (1990) and Taylor (1986)] that the Hausdorff (or Hausdorff–Besicovitch) dimension dim B of a subset B of $[0,1]$ may be defined by setting

$$(1.10)\quad \dim B = \inf\{c>0: s^c\text{-mes } B = 0\},$$

where the s^c-measure of B is, in turn, defined for each $c>0$ by

$$(1.11)\quad s^c\text{-mes } B = \lim_{n\downarrow 0}\left(\inf\left(\sum_i |I_i|^c: B\subseteq \cup_i I_i, |I_i|\le h\right)\right).$$

Here, the I_i constitute an h-cover of B (i.e., a collection of intervals with lengths not exceeding h, whose union includes B), we set $|I|$ for the Lebesgue measure of I and the infimum in (1.11) is taken over all h-covers of B.

THEOREM D. *For any* $\Lambda \in [0,1]$ *we have almost surely*

$$(1.12)\quad \dim B^{\pm}(\Lambda) = 1-\Lambda^2.$$

A statement like that of Theorem D is clearly meaningless with the replacement of $B^{\pm}(\Lambda)$ by $E_n^{\pm}(\Lambda)$ since that latter sets depend upon n. We are therefore led to define, for any $\Lambda \ge 0$,

$$(1.13)\quad E^{\pm}(\Lambda) = \left\{t\in[0,1): \limsup_{n\to\infty} \pm (2h_n\log(1/h_n))^{-1/2}(\alpha_n(t+h_n)-\alpha_n(t)) \ge \Lambda\right\}.$$

One of the purposes of this paper is to establish the following theorem, in the spirit of Theorem D, where we prove that $E^{\pm}(\Lambda)$ is a *random fractal*, and evaluate its Hausdorff dimension.

THEOREM 1.1. *Under* (H.1), (H.2) *and* (H.3), *for any* $\Lambda \in [0,1]$, *we have almost surely*

$$\dim E^{\pm}(\Lambda) = 1 - \Lambda^2. \tag{1.14}$$

Moreover, for any $\Lambda \in [0,1)$, $E^{\pm}(\Lambda)$ *is almost surely everywhere dense in* $[0,1]$.

Theorem 1.1 seems paradoxical in the sense that it implies, among all points of $[0,1]$, the existence of some exceptional locations in the vicinity of which occur infinitely often large fluctuations of α_n. One might have expected some kind of uniformity principle to hold among the points of $[0,1]$, which would be in contradiction with the existence of such exceptional sites. This however, is not the case, since Theorem 1.1 implies that, with probability 1, $E^{\pm}(\Lambda) \neq \varnothing$ for each $\Lambda \in [0,1)$. We note that a similar phenomenon was discovered by Hawkes (1981) for uniform spacings.

Up to now, we have only considered the extremal fluctuations of α_n corresponding to large values of $\pm(\alpha_n(t+h_n) - \alpha_n(t))$. One could, however, have obtained very similar results by replacing this expression by other types of statistics based upon $\{\alpha_n(u)\colon u \in [t, t+h_n]\}$. For instance [see e.g., Deheuvels and Mason (1992), Mason, Shorack and Wellner (1983) and Stute (1982)], versions of Theorems A and B are known to be true when $\pm(\alpha_n(t+h_n) - \alpha_n(t))$ is replaced by $\sup_{0 \le u, v \le h_n} |\alpha_n(t+u) - \alpha_n(t+v)|$. In the following Section 2, we will state an extended form of Theorem 1.1, where a general functional version of large fluctuations of α_n is used. That will enable us to describe in the same way all exceptional oscillations of α_n within a large class. We will deduce all of our results from the very general setup of Theorem 3.1 in Section 3, where the proofs are detailed.

2. Statement of main results for the uniform empirical process.

2.1. *Preliminary facts and notation.* In what follows, $B(0,1)$ will denote the set of all bounded functions on $[0,1]$ and AC(0, 1) will denote the set of all absolutely continuous functions $f \in B(0,1)$ of the form $f(s) = \int_0^s \dot{f}(u)\,du$, with $\int_0^1 \dot{f}(u)^2\,du < \infty$, for $s \in [0,1]$. $\mathbb{U}$ will denote the topology defined on $B(0,1)$ by the sup-norm $\|f\| = \sup_{0 \le s \le 1}|f(s)|$. A set E, endowed with a topology T will be denoted by (E, T).

The following notation will be convenient. Set, for any $f \in B(0,1)$ and $\varepsilon \ge 0$,

$$\begin{aligned} N_\varepsilon(f) &= \{g \in B(0,1)\colon \|f - g\| < \varepsilon\}, \\ \overline{N}_\varepsilon(f) &= \{g \in B(0,1)\colon \|f - g\| \le \varepsilon\} \end{aligned} \tag{2.1}$$

and for any (nonvoid) $A \subseteq B(0,1)$ and $\varepsilon \geq 0$,

$$A^{\varepsilon} = \bigcup_{f \in A} N_{\varepsilon}(f). \tag{2.2}$$

For any $\theta \geq 0$, set

$$\mathbb{S}_{\theta} = \left\{ f \in AC(0,1): \int_0^1 \dot{f}(u)^2 \, du \leq \theta^2 \right\}. \tag{2.3}$$

Note that $\mathbb{S}_{\theta} = \theta \mathbb{S}_1$, where $\mathbb{S}_1$ is the so-called *Strassen set*, introduced by Strassen (1964) in the statement of his famous law of the iterated logarithm. An application of the Arzela–Ascoli theorem shows that $\mathbb{S}_{\theta}$ is a compact subset of $(B(0,1), \mathbb{U})$ for each $\theta \geq 0$.

Let $\mathbf{I}$ denote the identity mapping of $[0,1]$ onto $[0,1]$. We will make use of the following Theorems E and F.

THEOREM E. *Under* (H.1) *and* (H.4), *for any prescribed* $t_0 \in [0,1)$, *the sequence of functions* $\{(2h_n \log_2 n)^{-1/2}(\alpha_n(t_0 + h_n\mathbf{I}) - \alpha_n(t_0)),\ n \geq 1\}$ *is almost surely relatively compact in* $(B(0,1), \mathbb{U})$ *with limit set equal to* $\mathbb{S}_1$.

THEOREM F. *Let* $J_n \subseteq [0, 1 - h_n]$ $n = 1, 2, \dots,$ *be a sequence of intervals such that* $(\log(|J_n|/h_n))/\log(1/h_n) \to 1$ *as* $n \to \infty$, *and let* $K_n(J_n) = \{2h_n \log(1/h_n))^{-1/2}(\alpha_n(t + h_n\mathbf{I}) - \alpha_n(t))$: $t \in J_n\}$. *Then, under* (H.1), (H.2) *and* (H.3), *for any* $\varepsilon > 0$, *there exists almost surely an* $n(\varepsilon) < \infty$ *such that, for all* $n \geq n(\varepsilon)$,

$$\mathbb{S}_1 \subseteq K_n^{\varepsilon}(J_n) \subseteq \mathbb{S}_1^{2\varepsilon}. \tag{2.4}$$

Theorem E was proved by Mason (1988) for $t_0 = 0$, and an easy argument shows that it holds equivalently for any prescribed $t_0 \in [0,1)$ [see, e.g., Deheuvels and Mason (1994a)]. Theorem F is due to Deheuvels and Mason (1992). The following corollary is a direct consequence of Theorems E and F and is stated without proof [see, e.g., Deheuvels and Mason (1992)].

COROLLARY A. *Let* Θ: $B(0,1) \to \mathbb{R}$ *be continuous with respect to* $\mathbb{U}$. *Then: Whenever* (H.1) *and* (H.4) *hold, we have for each prescribed* $t_0 \in [0,1)$,

$$\begin{aligned} &\limsup_{n \to \infty} \Theta\Big((2h_n \log_2 n)^{-1/2}\big(\alpha_n(t_0 + h_n\mathbf{I}) - \alpha_n(t_0)\big)\Big) \\ &\quad = \sup_{f \in \mathbb{S}_1} \Theta(f) \quad a.s. \end{aligned} \tag{2.5}$$

Whenever (H.1)–(H.3) *hold and* $\{J_n,\ n \geq 1\}$ *is as in Theorem F, we have*

$$\begin{aligned} &\lim_{n \to \infty} \left\{ \sup_{t \in J_n} \Theta\Big((2h_n \log(1/h_n))^{-1/2}\big(\alpha_n(t + h_n\mathbf{I}) - \alpha_n(t)\big)\Big) \right\} \\ &\quad = \sup_{f \in \mathbb{S}_1} \Theta(f) \quad a.s. \end{aligned} \tag{2.6}$$

Observe that Corollary A implies Theorems A and B, by choosing $\Theta(f) = \pm f(1)$ in (2.5) and (2.6) and $J_n = [0, 1 - h_n]$ in (2.6).

The following well-known facts about s^c-measures and Hausdorff dimension, as defined in (1.10) and (1.11), will be useful. In the first place, for any $c > 0$, the s^c-measure is a *metric outer measure* [see, e.g., Falconer (1990), page 25]. This means that $s^c(\varnothing) = 0$, $0 \le s^c(A) \le s^c(B) \le \infty$ for each pair of Borel sets $A \subseteq B \subseteq \mathbb{R}$, and, for any sequence $\{A_n,\ n \ge 1\}$ of Borel subsets of $\mathbb{R}$,

$$s^c\text{-mes}\left(\bigcup_{n=1}^{\infty} A_n\right) \le \sum_{n=1}^{\infty} s^c\text{-mes}(A_n), \tag{2.7}$$

with equality when the $\{A_n,\ n \ge 1\}$ are disjoint. Moreover [see, e.g., Falconer (1990), page 29],

$$s^c\text{-mes } A = \begin{cases} \infty, & \text{if } c < \dim A, \\ 0, & \text{if } c > \dim A. \end{cases} \tag{2.8}$$

These properties, when combined with the definition (1.10) of the Hausdorff dimension, imply that, for any sequence $\{A_n,\ n \ge 1\}$ of Borel subsets of $\mathbb{R}$,

$$\dim\left(\bigcup_{n \ge 1} A_n\right) = \sup_{n \ge 1}\{\dim A_n\}. \tag{2.9}$$

2.2. *Main results for the uniform empirical process.* For an arbitrary $f \in B(0,1)$, we set

$$L(f) = \Big\{t \in [0,1)\colon \liminf_{n \to \infty} \|(2h_n \log(1/h_n))^{-1/2} \times (\alpha_n(t + h_n\mathbf{I}) - \alpha_n(t)) - f\| = 0\Big\}. \tag{2.10}$$

An application of Theorem F shows that, with probability 1, $L(f) = \varnothing$ for all $f \notin \mathbb{S}_1$, so that we need only consider $L(f)$ for $f \in \mathbb{S}_1$. Set, accordingly, for an arbitrary $\Lambda \in [0,1]$,

$$L_\Lambda = \bigcup\left\{L(f)\colon f \in \mathbb{S}_1,\ \int_0^1 \dot{f}(u)^2\,du \ge \Lambda^2\right\}. \tag{2.11}$$

The main result of this section is stated in the following theorem.

THEOREM 2.1. *Under* (H.1), (H.2) *and* (H.3), *for any* $f \in \mathbb{S}_1$ *such that* $\int_0^1 \dot{f}(u)^2\,du \in (0,1)$ *and* $\Lambda \in [0,1)$, *the sets* $L(f)$ *and* L_Λ *are almost surely everywhere dense in* $[0,1]$ *and satisfy*

$$\dim L(f) = 1 - \int_0^1 \dot{f}(u)^2\,du \quad \text{and} \quad \dim L_\Lambda = 1 - \Lambda^2. \tag{2.12}$$

REMARK 2.1. Theorem 1.1 is a consequence of Theorem 2.1, as follows from the arguments below. First, observe, by (1.10) and (1.11), that

$$A \subseteq B \subseteq [0,1] \quad \Rightarrow \quad 0 \le \dim A \le \dim B \le 1. \tag{2.13}$$

For any $\Lambda \geq 0$, the function defined by $f_\Lambda^{\pm}(t) = \pm\Lambda t$, for $0 \leq t \leq 1$, satisfies $\int_0^1 \dot{f}_\Lambda^{\pm}(t)^2\,dt = \Lambda^2$. Recalling (1.13) and (2.10), we see then that

$$L(f_\Lambda^{\pm}) \subseteq E^{\pm}(\Lambda). \tag{2.14}$$

Conversely, let $t \in E^{\pm}(\Lambda)$. By (1.13) and Theorem F, there exists a $\lambda \geq \Lambda$ and a sequence $1 \leq n_1 < n_2 < \cdots$, along which $\pm(2h_n \log(1/h_n))^{-1/2}(\alpha_n(t + h_n) - \alpha_n(t)) \to \lambda$. Another application of Theorem F shows that there exists an event Ξ of probability 1, independent of t, λ and of $\{n_j, j \geq 1\}$, such that the following property holds. For each $\omega \in \Xi$, there exists an $f \in \mathbb{S}_1$ (depending upon ω) and an unbounded subsequence $\{n'_j, j \geq 1\}$ of $\{n_j, j \geq 1\}$ (depending upon ω), along which $\| \pm (2h_n \log(1/h_n))^{-1/2}(\alpha_n(t + h_n\mathbf{I}) - \alpha_n(t)) - f\| \to 0$. This, in turn, entails $f(1) = \pm\lambda$, and hence, by the Schwarz inequality,

$$\Lambda \leq \lambda = |f(1)| = \left|\int_0^1 \dot{f}(u)\,du\right| \leq \left|\int_0^1 \dot{f}(u)^2\,du\right|^{1/2}. \tag{2.15}$$

It follows from (2.11) and (2.15) that $t \in L(f) \subseteq L_\Lambda$. Since this last property holds for each $\omega \in \Xi$ and for all $t \in E^{\pm}(\Lambda)$, we have with probability 1,

$$E^{\pm}(\Lambda) \subseteq L_\Lambda. \tag{2.16}$$

Given the conclusion of Theorem 2.1, we obtain readily from (2.12), (2.13), (2.14) and (2.16) that, for any $\Lambda \in [0, 1)$, the set $E^{\pm}(\Lambda)$ is almost surely everywhere dense in $[0, 1]$ and satisfies $\dim E^{\pm}(\Lambda) = 1 - \Lambda^2$, which establishes Theorem 1.1

REMARK 2.2. It will become obvious from the arguments used in the sequel for the proof of Theorem 2.1 that the statement of this theorem remains valid when, in the definition (2.10) of $L(f)$, $[0, 1)$ is replaced by any fixed interval $(c, d) \subseteq [0, 1)$ with $c < d$.

REMARK 2.3. It follows from Theorem F that, with probability 1, for any $t \in [0, 1]$ there exists an $f \in \mathbb{S}_1$ such that $t \in L(f)$. This, in turn, implies that $L_0 = [0, 1]$ a.s.

3. A more general setup and proofs.

3.1. *Preliminaries*. We will frame our proofs in a more general setup than that of the uniform empirical process. Denote by $(B(0, 1), \mathbb{U})$ the set $B(0, 1)$ of all bounded functions on $[0, 1]$ endowed with the uniform topology.

Let $\{\Gamma_n(t), t \in [0, 1]\}$ be a sequence of processes taking values in $B(0, 1)$. For any $0 < h < 1$, define the corresponding *increment process*

$$\xi_n(h, t; \mathbf{I}) = \begin{cases} \Gamma_n(t + h\mathbf{I}) - \Gamma_n(t), & \text{for } t \in [0, 1 - h], \\ 0, & \text{otherwise,} \end{cases} \tag{3.1}$$

and *oscillation modulus*

$$\omega_n(h) = \sup_{0 \leq s \leq h} \sup_{0 \leq t \leq 1-s} |\Gamma_n(t + s) - \Gamma_n(t)|. \tag{3.2}$$

Throughout this section and unless otherwise specified, we define $L(f)$ and L_Λ as in (2.10) and (2.11), respectively, with the formal replacement of α_n by Γ_n. Namely, we set

$$(3.3)\qquad L(f) = \Big\{t \in [0,1): \liminf_{n\to\infty} \|(2h_n \log(1/h_n))^{-1/2} \times (\Gamma_n(t + h_n\mathbf{I}) - \Gamma_n(t)) - f\| = 0\Big\}$$

and

$$(3.4)\qquad L_\Lambda = \bigcup \Big\{L(f): f \in \mathbb{S}_1, \int_0^1 \dot{f}(u)^2\,du \ge \Lambda^2\Big\}.$$

For any $\gamma > 0$, set $\nu_k = \lfloor (1+\gamma)^k \rfloor$, $k = 0, 1, \dots,$ with $\lfloor u \rfloor$ denoting the integer part of u and for any sequence $\{h_n, n \ge 1\}$ satisfying (H.1), (H.2) and (H.3), set

$$(3.5)\qquad b_n = (2h_n \log(1/h_n))^{1/2}.$$

Introduce the following assumptions. There exists a function $\psi(\gamma)$ of $\gamma > 0$, where $\psi(\gamma) \to 0$ and $\gamma \downarrow 0$, such that, for all sequences $\{h_n, n \ge 1\}$ satisfying (H.1), (H.2) and (H.3) and $\gamma > 0$, we have:

(I) $\limsup_{j\to\infty} b_n^{-1}\omega_n(\gamma h_n) = \psi(\gamma)$ a.s.

(II) For any $\varepsilon > 0$ and $\Lambda > 0$, we have for each $t \in [0,1)$ and all j sufficiently large,

$$(3.6)\qquad P\Big((n/\nu_j)^{1/2} b_{\nu_j}^{-1}\xi_n(h_{\nu_j}, t; \mathbf{I}) \notin \mathbb{S}_\Lambda^\varepsilon \text{ for some } \nu_{j-1} < n \le \nu_j\Big) \le C_1 P\Big(b_{\nu_j}^{-1}\xi_{\nu_j}(h_{\nu_j}, t; \mathbf{I}) \notin \mathbb{S}_\Lambda^{\varepsilon/2}\Big),$$

for some universal constant C_1.

Let $\{\Pi(t), t \ge 0\}$ denote a process with stationary and independent increments satisfying:

(i) $E(\Pi(t)) = 0$ and $\mathrm{Var}(\Pi(t)) = t$ for all $t \ge 0$,

and for which the moment generating function

(ii) $\phi(s) = E(\exp(s\Pi(1)))$

is finite in a neighborhood of 0. Set for $0 < h < 1$, $0 \le s \le 1$, $t \ge 0$ and integers $n \ge 1$,

$$(3.7)\qquad L_n(h, t; s) = n^{-1/2}\{\Pi(n(t + sh)) - \Pi(nt)\}.$$

(III) Assume that, for some $\Pi(\cdot)$ as above, for all n sufficiently large and any $m \ge 1$, for any $t_1, \dots, t_m \in [0, \frac{1}{2}]$ (respectively, $t_1, \dots, t_m \in [\frac{1}{2}, 1]$) and Borel sets $B_1, \dots, B_m \in (B(0,1), \mathbb{U})$,

$$(3.8)\qquad P\big(b_n^{-1}\xi_n(h_n, t_i; \mathbf{I}) \notin B_i, i = 1, \dots, m\big) \le C_2 P\big(b_n^{-1}L_n(h_n, t_i; \mathbf{I}) \notin B_i, i = 1, \dots, m\big),$$

for some universal constant C_2.

We record for later use the following auxiliary results which follow from Assumption (I), just as in Lemmas 3.5 and 3.6 of Deheuvels and Mason (1992). There exist functions $\psi_i(\gamma)$, $i = 1, 2$, of $\gamma > 0$, where each $\psi_i(\gamma) \to 0$ as $\gamma \downarrow 0$, such that, for all sequences $\{h_n,\ n \geq 1\}$ satisfying (H.1), (H.2) and (H.3) and $\gamma > 0$, we have

$$\text{(A.1)}\qquad \limsup_{j\to\infty}\left\{\max_{\nu_{j-1}<n\leq\nu_j}\ \sup_{0\leq t\leq 1}\left|(n/\nu_j)^{1/2}b_{\nu_j}^{-1} - b_n^{-1}\right|\left\|\xi_n(h_{\nu_j}, t; \mathbf{I})\right\|\right\} = \psi_1(\gamma)\quad \text{a.s.}$$

$$\text{(A.2)}\qquad \limsup_{j\to\infty}\left\{\max_{\nu_{j-1}<n\leq\nu_j}\ \sup_{0\leq t\leq 1} b_n^{-1}\left\|\xi_n(h_{\nu_j}, t; \mathbf{I}) - \xi_n(h_n, t; \mathbf{I})\right\|\right\} = \psi_2(\gamma)\quad \text{a.s.}$$

We will establish the following result, which we will show contains Theorem 2.1.

THEOREM 3.1. *Under the assumptions* (H.1)–(H.2)–(H.3) *and* (I)–(II)–(III), *for any* $f \in \mathbb{S}_1$ *such that* $\int_0^1 \dot{f}(u)^2\,du \in (0,1)$ *and* $\Lambda \in [0,1]$, *the sets* $L(f)$ *and* L_Λ *are almost surely dense in* $[0,1]$ *and satisfy*

$$\text{(3.9)}\qquad \dim L(f) = 1 - \int_0^1 \dot{f}(u)^2\,du \quad \textit{and} \quad \dim L_\Lambda = 1 - \Lambda^2.$$

To see how Theorem 2.1 follows from Theorem 3.1, set $\Gamma_n = \alpha_n$ for $n \geq 1$. That Assumption (I) holds is readily inferred from Theorem 02 of Stute (1982) or Theorem 1 of Mason, Shorack and Wellner (1983), which implies that for all $\gamma > 0$,

$$\text{(3.10)}\qquad \lim_{n\to\infty} b_n^{-1}\omega_n(\gamma h_n) = \gamma^{1/2}\quad \text{a.s.}$$

The proof that Assumption (II) holds is easily achieved by a slight variation of the proof of Lemma 3.4 of Deheuvels and Mason (1992). Finally, from Lemma 3.1 of Deheuvels and Mason (1992), we conclude that Assumption (III) is fulfilled with $\{\Pi(t) + t,\ t \geq 0\}$ being a standard Poisson process with rate 1.

Assumptions (I), (II) and (III) also hold when

$$\text{(3.11)}\qquad \Gamma_n = n^{-1/2}\sum_{m=1}^{n}\Pi_m \quad \text{for } n \geq 1,$$

where $\Pi_1, \Pi_2, \ldots,$ is an i.i.d. sequence of processes with stationary and independent increments satisfying (i) and (ii) above. In particular, they hold when $\Pi_1, \Pi_2, \ldots,$ are i.i.d. standard Wiener processes. The details of showing this are worked out in Deheuvels and Mason (1994b).

3.2. *Upper bounds.* In this subsection, we will establish the "upper bound" half of Theorem 3.1. Namely, we will show that, under its assumptions, we have for all $0 \le \Lambda \le 1$,

$$\dim L_\Lambda \le 1 - \Lambda^2. \tag{3.12}$$

We will assume throughout that (H.1)–(H.2)–(H.3) hold together with assumptions (I)–(II)–(III) and make use of the following notation. Recalling (3.1) and (3.2), for each $n \ge 1$, $\Lambda \ge 0$ and $\varepsilon > 0$, set

$$U_n(\varepsilon, \Lambda) = \{t \in [0,1] : b_n^{-1}\xi_n(h_n, t; \mathbf{I}) \notin \mathbb{S}_\Lambda^\varepsilon\} \tag{3.13}$$

and write

$$L(\varepsilon, \Lambda) = \{t \in [0,1] : b_n^{-1}\xi_n(h_n, t; \mathbf{I}) \notin \mathbb{S}_\Lambda^\varepsilon \text{ i.o.}\}. \tag{3.14}$$

Let further, for each $f \in B(0,1)$,

$$L(f) = \left\{t \in [0,1] : \liminf_{n\to\infty} \|b_n^{-1}\xi_n(h_n, t; \mathbf{I}) - f\| = 0\right\} \tag{3.15}$$

and, for each $\Lambda \ge 0$,

$$L_{\Lambda+} = \bigcup \left\{L(f) : f \in \mathbb{S}_1, \int_0^1 \dot{f}(u)^2\,du > \Lambda^2\right\}. \tag{3.16}$$

Note for further use that, for any $0 \le \Lambda_1 < \Lambda < \Lambda_2$, $L_{\Lambda_2+} \subseteq L_\Lambda \subseteq L_{\Lambda_1+}$, so that, by (2.13),

$$\dim L_{\Lambda_2+} \le \dim L_\Lambda \le \dim L_{\Lambda_1+}. \tag{3.17}$$

LEMMA 3.1. *For every integer $m_0 \ge 1$, we have*

$$L_{\Lambda+} \subseteq \bigcup_{m=m_0}^{\infty} L(1/m, \Lambda). \tag{3.18}$$

PROOF. In view of (2.10) and (2.11), it suffices to show that, whenever $f \in \mathbb{S}_1$ is such that $\int_0^1 \dot{f}(t)^2\,dt := \lambda^2 > \Lambda^2$, we have $L(f) \subseteq \bigcup_{m=m_0}^{\infty} L(1/m, \Lambda)$. Since, in this case, $f \notin \mathbb{S}_\Lambda$, there is an $\varepsilon(f) > 0$ such that $f \notin \mathbb{S}_\Lambda^\varepsilon$ for all $\varepsilon \in [0, \varepsilon(f)]$. Therefore, by (3.14) and (3.15), for any $\varepsilon \in [0, \varepsilon(f)]$, $L(f) \subseteq L(\varepsilon, \Lambda)$. Since, by (3.14), for every $0 < \varepsilon_1 \le \varepsilon_2 < \infty$, $L(\varepsilon_2, \Lambda) \subseteq L(\varepsilon_1, \Lambda)$, the conclusion is straightforward. □

Recalling definition (1.10), we will show that, for every $\Lambda \in (0,1)$, $\eta \in (0, 1 - \Lambda^2)$ and $\varepsilon > 0$,

$$s^{1-\Lambda^2+\eta}\text{-mes}(L(\varepsilon, \Lambda)) = 0 \quad \text{a.s.}, \tag{3.19}$$

which by (2.7) and (3.18) implies

$$s^{1-\Lambda^2+\eta}\text{-mes}(L_{\Lambda+}) = 0 \quad \text{a.s.} \tag{3.20}$$

Since $\eta \in (0, 1-\Lambda^2)$ can be made as small as desired in (3.20), this says that

$$\dim L_{\Lambda+} \le 1 - \Lambda^2 \quad \text{a.s.} \tag{3.21}$$

Now, since (3.21) shows that $\dim L_{\Lambda_1+} \le 1 - \Lambda_1^2$ a.s. for each $0 \le \Lambda_1 < \Lambda$, we will obtain via (3.17) and (3.12) holds for all $0 < \Lambda < 1$.

Note that inequality (3.12) is trivial for $\Lambda = 0$ by (2.13). To show that it also holds for $\Lambda = 1$, observe that $L_1 \subseteq L_{1-\varepsilon}$ for each $\varepsilon \in (0,1)$, which, by (2.13) and (3.12) holding for all $0 < \Lambda < 1$ implies that $0 \le \dim L_1 \le \dim L_{1-\varepsilon} = 1 - (1-\varepsilon)^2$. This, in turn, implies that $\dim L_1 = 0$. Summing up the cases $\Lambda \in (0,1)$, $\Lambda = 0$ and $\Lambda = 1$, we see that (3.12) holds for all $\Lambda \in [0,1]$.

In order to establish (3.19), we must first fix some notation and gather some facts together.

Denote by $\lfloor u \rfloor \le u < \lfloor u \rfloor + 1$ the integer part of u, and by $\mathbb{1}\{E\}$ the indicator function of the event E. For $\gamma > 0$ and $\theta > 0$, set

$$\nu_k = \lfloor (1+\gamma)^k \rfloor, \qquad k = 1, 2, \dots, \tag{3.22}$$

and

$$\begin{aligned} & t_j(i,\theta) = i\theta h_{\nu_j}, \\ & 0 \le i \le m_j := \#\{i : i \ge 0,\ [t_j(i,\theta), t_j(i+1,\theta)] \subseteq [0,1]\} \\ & \qquad = \lfloor (\theta h_{\nu_j})^{-1} \rfloor - 1, \qquad j \ge 1. \end{aligned} \tag{3.23}$$

Furthermore, let for $0 < \varepsilon < \Lambda/2$, $\theta > 0$, $\gamma > 0$ and $0 \le i \le m_j$, $j \ge 1$,

$$\begin{aligned} \mathbb{1}_{i,j}(\varepsilon,\Lambda) = \mathbb{1}\Big\{ & (n/\nu_j)^{1/2} b_{\nu_j}^{-1} \xi_n(h_{\nu_j}, t_j(i,\theta); \mathbf{I}) \notin \mathbb{S}_\Lambda^{\varepsilon/2} \\ & \qquad \text{for some } \nu_{j-1} < n \le \nu_j \Big\} \end{aligned} \tag{3.24}$$

and

$$I_{i,j}(\varepsilon) = \begin{cases} [t_j(i,\theta) - \theta h_{\nu_j}, t_j(i,\theta) + \theta h_{\nu_j}], & \text{if } \mathbb{1}_{i,j}(\varepsilon,\Lambda) = 1, \\ \varnothing, & \text{otherwise.} \end{cases} \tag{3.25}$$

Note for further use that the distribution of $\mathbb{1}_{i,j}(\varepsilon,\Lambda)$ does not depend upon $i \in \{0,\dots,m_m\}$.

Recall (3.13), (3.24) and (3.25). From (A.1), (A.2) and Assumption (I), we easily infer that, for all $\gamma > 0$ and $\theta > 0$ small enough, there exists almost surely an $N = N(\varepsilon,\Lambda;\gamma,\theta) < \infty$ such that, for all $j \ge N$,

$$U_n(\varepsilon,\Lambda) \subseteq \bigcup_{i=1}^{m_j} I_{i,j}(\varepsilon). \tag{3.26}$$

Notice that, by definition (1.11) and (3.26), for any $c > 0$,

$$s^c\text{-mes}(U_n(\varepsilon,\Lambda)) \le \sum_{j=0}^{m_j} (2\theta h_{\nu_j})^c \mathbb{1}_{i,j}(\varepsilon,\Lambda). \tag{3.27}$$

Let $E_j = E\mathbb{1}_{i,j}(\varepsilon, \Lambda)$. We shall show that, for any $\Lambda > 0$,

$$\sum_{j=1}^{\infty} (m_j + 1)\big(2\theta h_{\nu_j}\big)^{1-\Lambda^2} E_j < \infty, \tag{3.28}$$

which since, by (3.13) and (3.14),

$$L(\varepsilon, \Lambda) = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} U_n(\varepsilon, \Lambda), \tag{3.29}$$

the finiteness of the series in (3.28) will imply that, for any $\Lambda \in (0, 1]$,

$$s^{1-\Lambda^2}\text{-mes}(L(\varepsilon, \Lambda)) < \infty \quad \text{a.s.}, \tag{3.30}$$

as long as $\theta > 0$ and $\gamma > 0$ are such that (3.26) holds. In view of (2.8), this, in turn suffices for (3.19).

In order to prove (3.28), we shall require a number of lemmas.

LEMMA 3.2. *For every $\varepsilon > 0$, $\Lambda \geq 0$ and $\gamma > 0$, there exists a $j(\varepsilon, \Lambda, \gamma) < \infty$ such that, for all $j \geq j(\varepsilon, \Lambda, \gamma)$ and $0 \leq i \leq m_j$,*

$$E_j = P\big(\mathbb{1}_{i,j}(\varepsilon, \Lambda) = 1\big) \leq C_1 P\Big(b_{\nu_j}^{-1}\xi_{\nu_j}\big(h_{\nu_j}, t_j(i,\theta); \mathbf{I}\big) \notin \mathbb{S}_{\Lambda}^{\varepsilon/4}\Big). \tag{3.31}$$

PROOF. It follows directly from Assumption (II) [see, e.g., (3.6)], taken with the formal replacement of ε by $\varepsilon/2$. □

From Lemma 3.2 we see that the remainder of the proof of (3.28) boils down to obtaining an appropriate upper bound for the right-hand side of (3.31). The next lemmas are directed toward this aim.

Let $\{\Pi(t),\ t \geq 0\}$ be the stationary independent increment process of Assumption (III). In view of (i) and (ii) of Assumption (II), we can make use of the strong approximation results of Komlós, Major and Tusnády (1975, 1976) to construct on the same probability space a version of the process $\{\Pi(t),\ t \geq 0\}$ and a standard Wiener process $\{W(t),\ t \geq 0\}$ such that, for universal constants $C_3 > 0$, $C_4 > 0$ and $C_5 > 0$,

$$P\Big(\sup_{0 \leq x \leq T} |\Pi(x) - W(x)| \geq C_3 \log T + z\Big) \leq C_4 \exp(-C_5 z), \tag{3.32}$$

for all $T \geq 1$ and $-\infty < z < \infty$.

The following large-deviation result will also be useful. Let $\{W(t),\ t \geq 0\}$ be a standard Wiener process, and, for any $\lambda > 0$, set $W_{(\lambda)}(t) = 2^{-1/2}\lambda^{-1}W(\lambda s)$ for $0 \leq s \leq 1$. For each $f \in B(0,1)$, set

$$J(f) = \begin{cases} \int_0^1 \dot{f}(u)^2\, du, & \text{if } f \in AC(0,1) \text{ with Lebesgue derivative } \dot{f}, \\ \infty, & \text{otherwise}, \end{cases} \tag{3.33}$$

and for any $B \subseteq B(0,1)$, define

$$J(B) = \inf_{f \in B} J(f) \quad \text{if } B \neq \varnothing \quad \text{and} \quad J(\varnothing) = \infty. \tag{3.34}$$

LEMMA 3.3. *The function J is a lower semicontinuous mapping of $(B(0,1),\mathbb{U})$ onto $[0,\infty]$. Moreover, for each closed subset F of $(B(0,1),\mathbb{U})$, we have*

$$\limsup_{\lambda\to\infty} \lambda^{-1}\log P\big(W_{(\lambda)}\in F\big) \le -J(F), \tag{3.35}$$

and for each open subset G of $(B(0,1),\mathbb{U})$, we have

$$\liminf_{\lambda\to\infty} \lambda^{-1}\log P\big(W_{(\lambda)}\in G\big) \ge -J(G). \tag{3.36}$$

PROOF. This result is due to Schilder (1966). We refer to Deuschel and Stroock [(1989), page 12] and to Varadhan [(1966), pages 262–263] for details concerning this fact. □

LEMMA 3.4. *For any $\eta>0$ and $\Lambda\in(0,1]$, there exists a $\kappa>0$ such that for all large n,*

$$P\big(b_n^{-1}L_n(h_n,t;\mathbf{I})\notin \mathbb{S}_\Lambda^\eta\big)\le 2h_n^{\Lambda^2+\kappa}\quad \textit{for } 0\le t\le 1-h_n. \tag{3.37}$$

PROOF. First, observe that the complement $F=B(0,1)-\mathbb{S}_\Lambda^{\eta/2}$ of $\mathbb{S}_\Lambda^{\eta/2}$ in $B(0,1)$ is closed with respect to $\mathbb{U}$ and such that $\rho := J(F)>\Lambda^2$. Next, making use of Assumption (III), (3.32) and the triangle inequality, we obtain for $nh_n\ge 1$ and $0\le t\le 1-h_n$ the chain of inequalities [see (3.5)–(3.7)]

$$\begin{aligned}
&P\big(b_n^{-1}L_n(h_n,t;\mathbf{I})\notin \mathbb{S}_\Lambda^\eta\big)\\
&\quad= P\big(n^{-1/2}b_n^{-1}\Pi(nh_n\mathbf{I})\notin \mathbb{S}_\Lambda^\eta\big)\\
&\quad\le P\left(\left(2nh_n\log\left(\frac{1}{h_n}\right)\right)^{-1/2}W(nh_n\mathbf{I})\notin \mathbb{S}_\Lambda^{\eta/2}\right)\\
&\qquad+P\left(\sup_{0\le x\le nh_n}|\Pi(x)-W(x)|\ge\frac{\eta}{2}n^{1/2}b_n\right)\\
&\quad\le P\big(W_{(\log(1/h_n))}\in F\big)\\
&\qquad+C_4\exp\left(-C_5\left\{\frac{\eta}{2}\left(2nh_n\log\left(\frac{1}{h_n}\right)\right)^{1/2}-C_3\log(nh_n)\right\}\right)\\
&\quad=: P_{n,1}(\eta)+P_{n,2}(\eta).
\end{aligned}\tag{3.38}$$

Let κ be an arbitrary constant such that $\kappa\in(0,\rho-\Lambda^2)$. By (3.35), we have for all large n,

$$P_{n,1}(\eta)\le \exp\big(-(\Lambda^2+\kappa)\log(1/h_n)\big)=h_n^{\Lambda^2+\kappa}, \tag{3.39}$$

where we have used from (H.1) that $h_n\to 0$ [and hence, that $\log(1/h_n)\to\infty$] and $nh_n\to\infty$. Further, it follows from (H.1) that $\log(nh_n)=o((nh_n\log(1/h_n))^{1/2})$, and from (H.2) that $\log(1/h_n)=o((nh_n\log(1/h_n))^{1/2})$ as $n\to-\infty$.

Thus, for any $\tau > \Lambda^2 + \kappa$, we have for all large n,

$$P_{n,2}(\eta) \le \exp(-\tau \exp(1/h_n)) = h_n^{\tau} < h_n^{\Lambda^2+\kappa}. \tag{3.40}$$

The conclusion (3.37) follows directly from (3.38), (3.39) and (3.40). □

We are now ready to prove (3.28). Recalling (3.31), an application of Assumption (III) and (3.8) taken with $n = \nu_j$ and $B = \mathbb{S}_\Lambda^{\varepsilon/4}$ shows that $E_j \le C_1 C_2 P(b_{\nu_j}^{-1} L_{\nu_j}(h_{\nu_j}, t_j(i)\mathbf{I}) \notin \mathbb{S}_\Lambda^{\varepsilon/4})$ for all j sufficiently large. By (3.37), taken with $n = \nu_j$ and $\eta = \varepsilon/4$, this yields in turn the inequality $E_j \le 2C_1C_2 h_{\nu_j}^{\Lambda^2+\kappa}$ for some $\kappa > 0$. Recalling from (3.23) that $(m_j + 1) \le (\theta h_{\nu_j})^{-1}$ for all large j, we thus obtain that, for all $\gamma > 0$ and k large,

$$\sum_{j=k}^{\infty} (m_j + 1)(2\theta h_{\nu_j})^{1-\Lambda^2} E_j \le 2^{2-\Lambda^2} C_1 C_2 \theta^{-\Lambda^2} \sum_{j=k}^{\infty} h_{\nu_j}^{\kappa}. \tag{3.41}$$

Making use of (H.3), we see that the inequality $h_n \le (\log n)^{-2/\kappa}$ holds ultimately as $n \to \infty$. Thus, $h_{\nu_j}^{\kappa} \le (\log \nu_j)^{-2} = O(j^{-2})$ as $j \to \infty$, which implies that the right-hand side of (3.41) is finite, and establishes (3.28).

REMARK 3.1. We have just proved the following statement, which is slightly stronger than the inequality $\dim L_\Lambda \le 1 - \Lambda^2$. Namely, we have, for each $\Lambda \in (0, 1]$,

$$\lim_{\varepsilon \downarrow 0} \dim L(\varepsilon, \Lambda) \le 1 - \Lambda^2. \tag{3.42}$$

3.3. *Lower bounds.* We inherit the notation and assumptions of Sections 3.1 and 3.2, and turn to showing that, for any function $f \in \mathbb{S}_1$ with

$$0 < \lambda^2 = \int_0^1 \dot{f}(u)^2 \, du < 1, \tag{3.43}$$

we have

$$\dim L(f) \ge 1 - \lambda^2. \tag{3.44}$$

First, we will discuss some consequences of this property. Let $\Lambda \in (0, 1)$. By choosing $\lambda = \Lambda \in (0, 1)$ and taking $f(t) = \Lambda t$ for $t \in [0, 1]$, the obvious inclusion $L(f) \subseteq L_\Lambda$ implies by (2.13) and (3.44) that $\dim L_\Lambda \ge 1 - \Lambda^2$. In view of (3.12), this shows that

$$\dim L_\Lambda = 1 - \Lambda^2 \quad \text{for each } \Lambda \in (0, 1). \tag{3.45}$$

We next observe that (2.13) and (3.12) jointly imply that (3.45) holds for $\Lambda = 1$. For the case $\Lambda = 0$, we combine (2.9) with (3.45) to obtain that

$$\dim L_0 = \dim\Big(\bigcup_{m \ge 1} L_{1/m}\Big) = \sup_{m \ge 1} \dim L_{1/m} = \sup_{m \ge 1} (1 - m^{-2}) = 1.$$

Therefore, to prove that (3.45) holds for each $\Lambda \in [0, 1]$, it suffices to show that (3.44) holds for all $\lambda \in (0, 1)$.

Given that (3.45) holds, and since, by (2.11), $L(f) \subseteq L_\lambda$ whenever $\lambda^2 = \int_0^1 \dot{f}(u)\,du \in [0,1]$, we see from (2.13) that, whenever $0 \le \lambda^2 < 1$,

$$0 \le \dim L(f) \le \dim L_\lambda = 1 - \lambda^2. \tag{3.46}$$

By combining (3.44) (for $0 < \lambda^2 < 1$) and (3.46) (for $0 \le \lambda^2 \le 1$), we readily obtain that, for $0 < \lambda^2 \le 1$, we have

$$\dim L(f) = 1 - \lambda^2 = 1 - \int_0^1 \dot{f}(u)^2\,du, \tag{3.47}$$

which completes the proof of the "lower bound" half of Theorem 3.1, given that (3.44) holds.

The following lemma will turn out to be an instrumental tool in proving (3.44).

LEMMA 3.5. *Let $\mathbb{K} \subseteq [0,1]$ be such that $\mathbb{K} = \bigcap_{m=1}^\infty E_m$, where $E_1 \supseteq \cdots \supseteq E_m \supseteq \cdots$ for $m = 1, 2, \dots$, and $E_m = \bigcup_{k=1}^{M_m} I_{m,k}$, with $\{I_{m,k},\ 1 \le k \le M_m\}$ being, for each $m \ge 1$, a collection of disjoint closed subintervals of $[0, \frac{1}{2}]$ such that $\max_{1 \le k \le M_m} |I_{m,k}| \to 0$ and $M_m \to \infty$ as $m \to \infty$. Then, if there exist two constants $\Delta > 0$ and $d > 0$ such that, for every interval $I \subseteq [0, \frac{1}{2}]$ with $|I| \le \Delta$ there is a constant $m(I)$ such that for all $m \ge m(I)$,*

$$M_m(I) := \#\{I_{m,k} \subseteq I : 1 \le k \le M_m\} \le d|I|^c M_m, \tag{3.48}$$

we have s^c-mes$(\mathbb{K}) > 0$.

PROOF. This is a version of Lemma 2.2 of Orey and Taylor (1974). The corresponding proof being simple, we give details for the sake of completeness. Consider an h-cover $\bigcup_{i=1}^N I_i \supseteq \mathbb{K}$ of $\mathbb{K} = \bigcap_{m=1}^\infty \{\bigcup_{k=1}^{M_m} I_{m,k}\}$, the I_i being open intervals with $|I_i| \le h$ for $i = 1, \dots, N$. By (3.48), we have for each I_i and $m \ge m_0 := \max_{1 \le i \le N} m(I_i)$, $\#\{I_{m,k} \subseteq I_i:\ 1 \le k \le M_m\} \le d|I_i|^c M_m$, so that $\sum_{i=1}^N \#\{I_{m,k} \subset I_i:\ 1 \le k \le M_m\} \le d \sum_{i=1}^N |I_i|^c M_m$. Since there exists an m_1 such that $\bigcup_{k=1}^{M_m} I_{m,k} \subseteq \bigcup_{i=1}^N I_i$, for all $m \ge m_1$, the $I_{m,k}$ which are not included in any $I_i = (c_i, d_i)$ for $i = 1, \dots, N$ must contain at least one point among $c_1, \dots, c_N, d_1, \dots, d_N$. The $I_{m,k}$, $k = 1, \dots, M_m$, being disjoint, this entails that $\sum_{i=1}^N \#\{I_{m,k} \subseteq I_i:\ 1 \le k \le M_m\} \ge M_m - 2N$. Therefore, for all $m \ge \max\{m_0, m_1\}$ such that $M_m \ge 4N$, we have $\sum_{i=1}^N |I_i|^c \ge d^{-1}(1 - 2NM_m^{-1}) \ge \frac{1}{2} d^{-1} > 0$. □

We will apply Lemma 3.5 with $\mathbb{K}$ chosen as a suitable subset of $L(f)$ and $c = 1 - \lambda^2 - \eta$, for a small $\eta > 0$. This will enable us to show that

$$s^{1-\lambda^2-\eta}\text{-mes}(L(f)) \ge s^{1-\lambda^2-\eta}\text{-mes}(\mathbb{K}) > 0. \tag{3.49}$$

By (2.8) and making use of the fact that $\eta > 0$ in (3.49) may be chosen arbitrarily small, this will suffice for (3.44). The remainder of this section is devoted to the construction of $\mathbb{K}$ and was inspired by the arguments in Section 4 of Orey and Taylor (1974).

Throughout the sequel, we will assume that

$$\eta \in \left(0, \max\{\tfrac{1}{2}\lambda^2, 1 - \lambda^2\}\right) \quad \text{and } \gamma > 0 \text{ are fixed.}$$

We require some additional notation. For any $V \subseteq \mathbb{R}$ and $\varepsilon > 0$, let

$$\mathbb{N}(\varepsilon, V) = \bigcup_{x \in V} (x - \varepsilon, x + \varepsilon) \tag{3.50}$$

be an open ε-neighborhood of V in $\mathbb{R}$. For fixed $\theta > 0$ and $j \geq 1$, we let ν_j be as in (3.22) and $t_j(i) = t_j(i,1)$, for $1 \leq i \leq 2m_j$, be an in (3.23). For any $f \in B(0,1)$, $\varepsilon > 0$ and $j \geq 1$, we set

$$U_{j,f}(\varepsilon) = \left\{ t \in \left[0, \tfrac{1}{2}\right] : \|b_{\nu_j}^{-1}\xi_{\nu_j}\left(h_{\nu_j}, t; \mathbf{I}\right) - f\| < \varepsilon \right\} \tag{3.51}$$

and

$$W_{j,f}(\varepsilon) = \left\{ t_j(i) : 1 \leq i \leq m_j, \|b_{\nu_j}^{-1}\xi_{\nu_j}\left(h_{\nu_j}, t_j(i); \mathbf{I}\right) - f\| < \varepsilon \right\}. \tag{3.52}$$

Let ψ be as in Assumption (I).

LEMMA 3.6. *For any $\varepsilon \in (0,1)$ and $\theta = \theta(\varepsilon) < 1$ satisfying $\psi(\theta) \leq \frac{1}{4}\varepsilon$, there exists almost surely a $j_0(\varepsilon, \theta)$ such that, for all $j \geq j_0(\varepsilon, \theta)$,*

$$\mathbb{N}\left(\theta h_{\nu_j}, W_{j,f}(\varepsilon)\right) \subseteq U_{j,f}(2\varepsilon). \tag{3.53}$$

PROOF. Let $t_j(i) \in W_{j,f}(\varepsilon)$ with $1 \leq i \leq m_j$ and let t be such that $|t - t_j(i)| \leq \theta h_{\nu_j}$. By (3.23), the fact that $\theta < 1$ obviously implies that $t \in [0,1]$. Recalling the notation (3.52), the triangle inequality entails

$$\begin{aligned} \|b_{\nu_j}^{-1}\xi_{\nu_j}\left(h_{\nu_j}, t; \mathbf{I}\right) - f\| &\leq b_{\nu_j}^{-1}\|\xi_{\nu_j}\left(h_{\nu_j}, t; \mathbf{I}\right) - \xi_{\nu_j}\left(h_{\nu_j}, t_j(i); \mathbf{I}\right)\| \\ &\quad + \|b_{\nu_j}^{-1}\xi_{\nu_j}\left(h_{\nu_j}, t_j(i); \mathbf{I}\right) - f\| \\ &\leq 2b_{\nu_j}^{-1}\omega_{\nu_j}\left(\theta h_{\nu_j}\right) + \varepsilon. \end{aligned} \tag{3.54}$$

By combining (3.54) and Assumption (I), we obtain that the inequality

$$\|b_{\nu_j}^{-1}\xi_{\nu_j}\left(h_{\nu_j}, t; \mathbf{I}\right) - f\| \leq 4\psi(\theta) + \varepsilon \tag{3.55}$$

holds almost surely for all j sufficiently large. Our choice of θ in this last inequality readily implies (3.53). □

For each $\varepsilon > 0$ we set from now on $\theta = \theta(\varepsilon)$ as in Lemma 3.6 $j_0(\varepsilon) = j_0(\varepsilon, \theta(\varepsilon))$ and

$$t_j(i) = t_j(i,1) = \#ih_{\nu_j} \quad \text{for } j \geq 1 \text{ and } 1 \leq i \leq m_j := \lfloor h_{\nu_j}^{-1} \rfloor - 1. \tag{3.56}$$

Further for $j \geq 1$ let

$$N_{j,f}(\varepsilon) = \#W_{j,f}(\varepsilon) = \#\left\{ i : 1 \leq i \leq m_j, \|b_{\nu_j}^{-1}\xi_{\nu_j}\left(h_{\nu_j}, t_j(i); \mathbf{I}\right) - f\| \leq \varepsilon \right\}, \tag{3.57}$$

and for any set $I \subseteq [0,1]$,

$$\begin{aligned} N_{j,f}(\varepsilon; I) &= \#\left\{ W_{j,f}(\varepsilon) \cap I \right\} \\ &= \#\Big\{ i : 1 \leq i \leq m_j, t_j(i) \in I, \\ &\qquad \|b_{\nu_j}^{-1}\xi_{\nu_j}\left(h_{\nu_j}, t_j(i); \mathbf{I}\right) - f\| < \varepsilon \Big\}. \end{aligned} \tag{3.58}$$

By the Assumption (III) and (3.8), we see that, for j sufficiently large and all $r_{j,1}, r_{j,2}$,

$$P\big(N_{j,f}(\varepsilon;I) \le r_{j,2}\big) \le C_2 P\big(N'_{j,f}(\varepsilon;I) \le r_{j,2}\big), \tag{3.59(i)}$$

$$P\big(N_{j,f}(\varepsilon;I) > r_{j,1}\big) \le C_1 P\big(N'_{j,f}(\varepsilon;I) > r_{j,1}\big), \tag{3.59(ii)}$$

where

$$N'_{j,f}(\varepsilon, I) = \sum_{i:\, 1 \le i \le m_j,\, t_j(i) \in I} X_i, \tag{3.60}$$

and, for 0, 1, ..., m_j,

$$X_i = \mathbb{1}\Big\{\|b_{\nu_j}^{-1} L_{\nu_j}\big(h_{\nu_j}, t_j(i); \mathbf{I}\big) - f\| < \varepsilon\Big\}. \tag{3.61}$$

Observe that $X_1, X_2, \ldots$ are independent and identically distributed Bernoulli random variables with probability of success

$$p_j(\varepsilon) = P(X_0 = 1) = P\Big(b_{\nu_j}^{-1} L_{\nu_j}\big(h_{\nu_j}, 0; \mathbf{I}\big) \in N_\varepsilon(f)\Big), \tag{3.62}$$

with $N_\varepsilon(f)$ being as in (2.1). The following lemma evaluates $p_j(\varepsilon)$, when f satisfies (3.43).

LEMMA 3.7. *For any $\delta \in (0, \lambda^2)$, there exist $0 < \delta'' < \delta' < \delta$, an $\varepsilon_0 = \varepsilon_0(\delta) \in (0, \frac{1}{2})$ and a $j_1(\varepsilon, \delta) \ge 1$ such that for each $\varepsilon \in (0, \varepsilon_0]$ and $j \ge j_1(\varepsilon, \delta)$, we have*

$$h_{\nu_j}^{\lambda^2} \le h_{\nu_j}^{\lambda^2 - \delta''} \le p_j(\varepsilon) \le h_{\nu_j}^{\lambda^2 - \delta'} \le h_{\nu_j}^{\lambda^2 - \delta}. \tag{3.63}$$

PROOF. Let $\Pi(\cdot)$ be as in Assumption (III) and $W(\cdot)$ be as in (3.32). Observe that $p_j(\varepsilon) = P'_{\nu_j}$, where

$$P'_n = P\Big(\big(2hn_n \log(1/h_n)\big)^{-1/2} \Pi(nh_n \mathbf{I}) \in N_\varepsilon(f)\Big). \tag{3.64}$$

Next, we make use of the obvious inequalities

$$\begin{aligned} P'_n \ge\ & P\Big(\big(2nh_n \log(1/h_n)\big)^{-1/2} W(nh_n \mathbf{I}) \in N_{\varepsilon/2}(f)\Big) \\ & - P\bigg(\sup_{0 \le x \le nh_n} |\Pi(x) - W(x)| \ge \frac{\varepsilon}{2} n^{1/2} b_n\bigg) =: P'_{n,1} - P'_{n,2}. \end{aligned} \tag{3.65}$$

By Lemma 3.3 applied with $G = N_{\varepsilon/2}(f)$, which is open with respect to $\mathbb{U}$, we see that, for any specified $\rho \in (J(N_{\varepsilon/2}(f)), 1)$, we have for all large n,

$$P'_{n,1} = P\big(W_{(\log(1/h_n))} \in G\big) \ge \exp(-\rho \log(1/h_n)) = h_n^\rho. \tag{3.66}$$

On the other hand, a comparison of (3.38) with (3.65) shows that $P'_{n,2} \le P_{n,2}(\varepsilon)$. By choosing $\eta = \varepsilon$ and $\tau > 1 > \rho$ in (3.40), it follows that for all large n,

$$P'_{n,2} \le \tfrac{1}{2} h_n^\rho. \tag{3.67}$$

Since $h_n \to 0$, for any fixed $\rho' \in (\rho, 1)$, we have ultimately $\frac{1}{2}h_n^{\rho} \geq h_n^{\rho'}$. This, in combination with (3.65), (3.66), (3.67) and the fact that $\rho \in (J(N_{\varepsilon/2}(f)), 1)$ is arbitrary, suffices to show that the inequality

$$P_n' \geq h_n^{\rho'} \tag{3.68}$$

holds ultimately in n for each $\rho' \in (J(N_{\varepsilon/2}(f)), 1)$.

Observe (see, e.g., Lemma 3.3) that J, as defined in (3.33), is lower semicontinuous with respect to $\mathbb{U}$, so that $\liminf_{\varepsilon \downarrow 0} J(N_{\varepsilon/2}(f)) \geq J(f) = \lambda^2$. Since obviously $J(N_{\varepsilon/2}(f)) < J(f)$, it follows that $\lim_{\varepsilon \downarrow 0} J(N_{\varepsilon/2}(f)) = \lambda^2$. Thus, for any fixed $\delta > 0$, there exists $\varepsilon' > 0$ so small that, for all $\varepsilon \in (0, \varepsilon']$,

$$\lambda^2 - \delta < J\big(N_{\varepsilon/2}(f)\big) < \lambda^2. \tag{3.69}$$

By (3.68) and (3.69) we obtain the first inequality in (3.63) by first choosing an $\varepsilon \in (0, \varepsilon']$ and then by selecting a $\delta' \in (0, \lambda^2 - J(N_{\varepsilon/2}(f)))$. It follows from (3.69) that $0 < \delta' < \delta$.

For the second inequality in (3.63), we observe that $p_j(\varepsilon) \leq P_{\nu_j}''$, where

$$P_n'' = P\Big(\big(2nh_n \log(1/h_n)\big)^{-1/2}\Pi(nh_n\mathbf{I}) \in \overline{N}_\varepsilon(f)\Big) \tag{3.70}$$

and $\overline{N}_\varepsilon(f) \supseteq \mathrm{N}_\varepsilon(f)$ is as in (2.1). By a similar argument as that used for (3.65), we obtain that

$$\begin{aligned} P_n'' &\leq P\Big(\big(2nh_n \log(1/h_n)\big)^{-1/2}W(hn_n\mathbf{I}) \in \overline{N}_{\varepsilon/2}(f)\Big) \\ &\quad + P\bigg(\sup_{0 \leq x \leq nh_n} |\Pi(x) - W(x)| \geq \frac{\varepsilon}{2}n^{1/2}b_n\bigg) =: P_{n,1}'' + P_{n,2}'. \end{aligned} \tag{3.71}$$

We now apply Lemma 3.3 with $F = \overline{N}_{\varepsilon/2}(f)$, which is closed with respect to $\mathbb{U}$, to obtain that, for any specified $\rho \in (0, J(\overline{N}_{\varepsilon/2}(f)))$, we have for all large n,

$$P_{n,1}' = P\big(W_{(\log(1/h_n))} \in F\big) \leq \exp\big(-\rho \log(1/h_n)\big) = h_n^{\rho}. \tag{3.72}$$

By (3.67), (3.71) and (3.72), we see that for all n sufficiently large, $P_n'' \leq \frac{3}{2}h_n^{\rho}$, which is ultimately less than $h_n^{\rho''}$ if $\rho'' \in (0, \rho)$. Since $\rho \in (0, J(\overline{N}_{\varepsilon/2}(f)))$ is arbitrary, the inequality

$$P_n'' \leq h_n^{\rho''} \tag{3.73}$$

holds ultimately in n for each $\rho'' \in (0, J(\overline{N}_{\varepsilon/2}(f)))$. Since $N_{\varepsilon/2}(f) \subseteq \overline{N}_{\varepsilon/2}(f) \subseteq N_\varepsilon(f)$, we have $J(\overline{N}_{\varepsilon/2}(f)) < \lambda^2$ and $\lim_{\varepsilon \downarrow 0} J(\overline{N}_{\varepsilon/2}(f)) = \lambda^2$. Thus, for any fixed $\delta > 0$, there exists an $\varepsilon'' \in (0, \varepsilon']$ so small that, for all $\varepsilon \in (0, \varepsilon'']$,

$$\lambda^2 - \delta < J\big(\overline{N}_{\varepsilon/2}(f)\big) < \lambda^2. \tag{3.74}$$

By (3.73) and (3.74) we obtain the second inequality in (3.63) by first choosing an $\varepsilon \in (0, \varepsilon'']$ and the by selecting a $\delta'' \in (\lambda^2 - J(\overline{N}_{\varepsilon/2}(f), \delta))$. Since the choice of $\varepsilon \in (0, \varepsilon']$ which was required for the first inequality in (3.63) was arbitrary, and $\varepsilon'' \leq \varepsilon'$, we see that we obtain both halves of (3.63) by setting $\varepsilon_0(\delta) = \varepsilon''$. □

Let

$$\mathsf{h}(u) = \begin{cases} u \log u - u + 1, & \text{for } u > 0, \\ 1, & \text{for } u = 0. \end{cases}$$

LEMMA 3.8. *Let S_N follow a binomial distribution with parameters N and p. Then, for all $r \in [1, 1/p]$,*

$$P(S_N \geq Nrp) \leq \exp(-Np\mathsf{h}(r)), \tag{3.75}$$

and for all $r \in [0, 1]$,

$$P(S_N \leq Nrp) \leq \exp(-Np\mathsf{h}(r)), \tag{3.76}$$

PROOF. The inequalities are trivial when either $p = 0$ or $p = 1$. When $0 < p < 1$ the Markov inequality implies that for $1 \leq r \leq 1/p$, $P(S_N \geq Nrp) \leq \exp(-NH(r, p))$, and for $0 \leq r \leq 1$, $P(S_N \leq Nrp) \leq \exp(-NH(r, p))$, where in both cases

$$H(r,p) = \begin{cases} (rp-1)\log\left(\dfrac{1-p}{1-rp}\right) + rp\log r, & \text{for } 0 < r < 1/p, \\ -\log(1-p), & \text{for } r = 0, \\ -\log p, & \text{for } r = 1/p. \end{cases}$$

Making use of the inequality $\log s \leq s - 1$ for $s > 0$, we obtain that, for $0 < r < 1/p$ and $0 < p < 1$,

$$H(r,p) \geq (rp-1)\left(\frac{1-p}{1-rp} - 1\right) + rp\log r = p\mathsf{h}(r),$$

and likewise, $H(0, p) = -\log(1-p) \geq p = p\mathsf{h}(0)$ and $H(1/p, p) = -\log p \geq -\log p - 1 + p = p\mathsf{h}(1/p)$, from which (3.75) and (3.76) are immediate. □

Let $m_j(I) = \#\{i\colon\ 0 \leq i \leq m_j,\ [t_j(i), t_j(i+1)] \subseteq I\}$. Recalling (3.23) and (3.56), we see that $m_j([0,1]) = m_j$ and, whenever $E \subseteq [0, \frac{1}{2}]$ is an union of disjoint intervals of lengths greater than $L \geq 3h_{\nu_j}$,

$$\frac{|E|}{3h_{\nu_j}} \leq \left(1 - \frac{2h_{\nu_j}}{L}\right)\frac{|E|}{h_{\nu_j}} \leq m_j(E) \leq \frac{|E|}{h_{\nu_j}}. \tag{3.77}$$

LEMMA 3.9. *Let $\varepsilon \in (0, \varepsilon_0(\frac{1}{2}\lambda^2))$, where $\varepsilon_0(\cdot)$ is as in Lemma 3.7, and let $\delta \in (0, 1-\lambda^2)$ be fixed. For any $\sigma \in (0,1)$, there exists almost surely a $j_2(\varepsilon, \sigma, \delta) \geq 1$ such that, for all $j \geq j_2(\varepsilon, \sigma, \delta)$, we have*

$$|N_{j,f}(\varepsilon; E) - m_j(E)p_j(\varepsilon)| < \sigma m_j(E)p_j(\varepsilon), \tag{3.78}$$

for all $E \subseteq [0,1]$ which are unions of disjoint intervals with lengths $L \geq h_{\nu_j}^{1-\lambda^2-\delta}$.

PROOF. Since $N_{j,f}(\varepsilon; \cdot)$ and $m_j(\cdot)$ are additive set functions, it suffices to prove (3.78) when $E = I$ is an interval with $|I| \geq h_{\nu_j}^{1-\lambda^2-\delta}$. Fix $0 < \delta' < \delta$ and

$0 < \sigma' < \sigma$. We next show that we need only prove that the conclusion of the lemma holds when $E = J$ is an interval of the form $[t_j(i), t_j(i + \kappa(j)))$ for some $0 \le i \le \frac{3}{2}m_j$, where $\kappa(j) := \lfloor h_{\nu_j}^{-\lambda^2-\delta'} \rfloor$ (note here that $|J| = \kappa(j)h_{\nu_j} \in [h_{\nu_j}^{1-\lambda^2-\delta'} - h_{\nu_j}, h_{\nu_j}^{1-\lambda^2-\delta'}]$). To see this, assume accordingly that (3.78) is satisfied with the formal replacements of E and σ by J and σ', respectively, and for each interval J as above. Observe that $h_{\nu_j}^{1-\lambda^2-\delta'} = o(h_{\nu_j}^{1-\lambda^2-\delta})$, $h_{\nu_j} = o(h_{\nu_j}^{1-\lambda^2-\delta'})$, $K = K(I,j) := \lfloor |I|/h_{\nu_j}^{1-\lambda^2-\delta'} \rfloor \ge \lfloor h_{\nu_j}^{\delta'-\delta} \rfloor := K(j) \to \infty$ as $j \to \infty$. Choose j' so large that, for all $j \ge j'$, $K(j) \ge 3$, $3h_{\nu_j} < h_{\nu_j}^{1-\lambda^2-\delta'}$ and

$$(3.79)\qquad \begin{aligned} &1 - \sigma \le (1 - \sigma')\left\{\frac{1 - 2h_{\nu_j}^{\lambda^2+\delta'}}{1 + 4/(K(j) - 2)}\right\} \quad \text{and} \\ &(1 + \sigma')\left\{\frac{1 + 4/(K(j) - 2)}{1 - 2h_{\nu_j}^{\lambda^2+\delta}}\right\} \le (1 + \sigma). \end{aligned}$$

For $j \ge j'$ and for each interval $I \subseteq [0, \frac{1}{2}]$, with $|I| \ge h_{\nu_j}^{1-\lambda^2-\delta}$, there exists $(K + 2)$ disjoint intervals $J_1, \dots, J_{K+2}$ of the form $[t_j(i), t_j(i + \kappa(j)))$ such that $\bigcup_{k=1}^{K-2} J_k \subseteq I \subseteq \bigcup_{k=1}^{K+2} J_k$ and $|J_k| = |J_1| \sim h_{\nu_j}^{1-\lambda^2-\delta'}$ for $k = 1, \dots, K + 2$ (here we use the notation $a_j \sim b_j$ when $a_j/b_j \to 1$ as $j \to \infty$). Moreover, we have the inequalities $(K - 2)|J_1| \le |I| \le (K + 2)|J_1|$. Also, our assumptions imply that we may apply the right side of (3.77) to either $E = I$ or $E = J_k$. Therefore, it follows from (3.79) that almost surely for all j large enough,

$$(3.80)\qquad \begin{aligned} N_{j,f}(\varepsilon; I) &\le \sum_{k=1}^{K+2} N_{j,f}(\varepsilon; J_k) \le (1 + \sigma')\left\{\sum_{k=1}^{K+2} m_j(J_k)\right\} p_j(\varepsilon) \\ &\le (1 + \sigma')\left\{h_{\nu_j}^{-1}(K + 2)|J_1|\right\} p_j(\varepsilon) \\ &\le (1 + \sigma')\left(\frac{K + 2}{K - 2}\right)\left\{h_{\nu_j}^{-1}|I|\right\} p_j(\varepsilon) \\ &\le (1 + \sigma')\left\{\frac{1 + 4/(K - 2)}{1 - 2h_{\nu_j}^{\lambda^2+\delta}}\right\} m_j(I) p_j(\varepsilon) \\ &\le (1 + \sigma) m_j(I) p_j(\varepsilon). \end{aligned}$$

A similar argument shows likewise that almost surely for all j large enough,

$$(3.81)\qquad \begin{aligned} N_{j,f}(\varepsilon; I) &\ge \sum_{k=1}^{K} N_{j,f}(\varepsilon; J_k) \\ &\ge (1 - \sigma')\left\{h_{\nu_j}^{-1}\left(1 - 2h_{\nu_j}^{\lambda^2+\delta'}\right)(K - 2)|J_1|\right\} p_j(\varepsilon) \\ &\ge (1 - \sigma')\left\{\frac{1 - 2h_{\nu_j}^{\lambda^2+\delta'}}{1 + 4/(K - 2)}\right\} m_j(I) p_j(\varepsilon) \\ &\ge (1 - \sigma) m_j(I) p_j(\varepsilon). \end{aligned}$$

Thus, by combining (3.80) and (3.81), we obtain that (3.78) is satisfied for $E = I$.

In the remainder of our proof, we set of convenience $\sigma = \sigma'$, $\delta = \delta'$ and restrict our attention to the intervals I of the form $I = [t_j(i), t_j(i + \kappa(j)))$ for some $0 \le i \le \frac{3}{2}m_j$. Noting that the total number of such intervals is bounded above by $h_{\nu_j}^{-1}$, we let $Q_j = P(N_{j,r}(\varepsilon; I) > r_{j,1})$, where $r_{j,1} = (1 + \sigma)m_j(I)p_j(\varepsilon)$. By (3.59), the inequality $Q_j \le C_2 Q_j' := C_2 P(N_{j,f}'(\varepsilon; I) > r_{j,1})$ holds ultimately as $j \to \infty$. Recalling the notations (3.60), (3.61) and (3.62), we apply (3.75) to $S_N = N_{j,f}'(\varepsilon; I)$, $N = m_J(I)$, $r = r_{j,1}$ and $p = p_j(\varepsilon)$, to obtain in turn that

$$Q_j \le C_2 \exp\big(-m_j(I)p_j(\varepsilon)\mathsf{h}(1 + \sigma)\big). \tag{3.82}$$

Observe from (3.23) that $m_j(I) \ge h_{\nu_j}^{-1}|I| - 2 \ge \frac{1}{2}h_{\nu_j}^{-\lambda^2-\delta}$ for all large j. On the other hand, by (3.63), the assumption that $\varepsilon \in (0, \varepsilon_0(\frac{1}{2}\lambda^2))$ entails that $p_j(\varepsilon) \ge h_{\nu_j}^{\lambda^2}$ for all large j. By combining these two inequalities with (3.82), we obtain that, for all large j,

$$h_{\nu_j}^{-1}Q_j < C_2 \exp\Big(-\mathsf{h}(1 + \sigma)h_{\nu_j}^{-\delta/2}\Big). \tag{3.83}$$

Making use of (H.3), we see that the inequality $h_n^{-1} \ge (\log n)^{4/\delta}$ holds ultimately as $n \to \infty$. This, in turn, implies that $n_{\nu_j}^{-\delta/2} \ge \frac{1}{2}(j\log(1 + \gamma))^2$ for all large j, whence $\Sigma_j h_{\nu_j}^{-1}Q_j < \infty$. The Borel–Cantelli lemma completes the proof of the "upper half" of (3.78). The proof of the "lower half" is very similar and therefore is omitted. □

We shall now show that for any $f \in \mathbb{S}_1$ satisfying (3.43) with probability 1 there exists a sequence of sets $E_1 \supseteq E_2 \supseteq \dots$ fulfilling the assumptions of Lemma 3.5 and such that $\mathbb{K} = \bigcap_{m=1}^{\infty} E_m \subseteq L(f)$. The following arguments are directed toward the construction of such a sequence of sets. In a first step, we establish the existence of this sequence via an induction argument. In a second step, we will show that $\{E_m, m \ge 1\}$ satisfies (3.48). In a third concluding step, we will apply Lemma 3.5 to prove that (3.44) holds.

STEP 1. *Existence of* E_m. Choose an $f \in \mathbb{S}_1$ satisfying (3.43). We first fix an arbitrary

$$\eta \in \big(0, \min(\tfrac{1}{2}\lambda^2, \tfrac{1}{2}(1 - \lambda^2))\big)$$

and choose two auxiliary sequences of nonnegative constants $\{\sigma_m, m \ge 1\}$ and $\{\delta_m, m \ge 0\}$ such that

$$0 < \sigma_m < \frac{1}{2} \quad \text{for } m \ge 1, \tag{3.84)(i}$$

$$\prod_{m=1}^{\infty}\left(\frac{1 + \sigma_m}{1 - \sigma_m}\right) \le 2, \tag{3.84)(ii}$$

$$\delta_0 = 0,\ \delta_m > 0 \quad \text{for } m \ge 1 \quad \text{and}$$

$$\sum_{m=1}^{\infty} \delta_m \le \frac{1}{3}\eta < \min\left(\frac{1}{6}\lambda^2, \frac{1}{6}(1 - \lambda^2)\right). \tag{3.84)(iii}$$

Note for further use that the assumption that $0 < \eta < 1$ [see, e.g., (3.43)] and (3.84)(iii) entail that

$$(3.85)\qquad \begin{aligned} \Delta_m := \sum_{k=1}^{m} \delta_k < \min\{\tfrac{1}{6}\lambda^2, \tfrac{1}{6}(1-\lambda^2)\} \quad \text{and} \\ 0 < \delta_m < \Delta_m < 1 \quad \text{for } m \geq 1. \end{aligned}$$

Let $\varepsilon_0(\delta)$ be defined as in Lemma 3.7 for $\delta \in (0, \lambda^2)$, $\theta(\varepsilon)$ be as in Lemma 3.6 for $\varepsilon \in (0,1)$ and observe that we may assume $\varepsilon_0(\cdot)$ to be nondecreasing. We select two decreasing sequences $\{\varepsilon_m,\ m \geq 1\}$ and $\{\theta_m,\ m \geq 1\}$ of positive constants such that the following conditions hold for $m \geq 1$:

$$(3.86)(\text{i})\qquad \varepsilon_m \downarrow 0 \quad \text{as } m \uparrow \infty,$$

$$(3.86)(\text{ii})\qquad 0 < \varepsilon_m < \min\left\{\frac{1}{2}, \varepsilon_0(\delta_m)\right\},$$

$$(3.86)(\text{iii})\qquad \theta_m \leq \min\left\{\theta(\varepsilon_m), \frac{1}{16}\varepsilon_m^2\right\},$$

$$(3.86)(\text{iv})\qquad 2\theta_m^{1-\lambda^2-\eta} < \min\{\sigma_m, \sigma_{m+1}\},$$

$$(3.86)(\text{v})\qquad \left(\frac{\theta_m}{1-\theta_m}\right)^{2(1-\lambda^2)/3} \leq \frac{1}{2}\sigma_{m+1},$$

$$(3.86)(\text{vi})\qquad \left(\frac{1}{1-\theta_m}\right)^{1-\lambda^2} \leq \frac{1+(5/6)\sigma_{m+1}}{1+(1/2)\sigma_{m+1}}.$$

Next, for each $m \geq 1$, we choose $j_2(\varepsilon_m, \sigma_m, \delta_m) \geq j_1(\varepsilon_m, \delta_m) \geq j_0(\varepsilon_m) = j_0(\varepsilon_m, \theta(\varepsilon_m))$ in Lemmas 3.6, 3.7 and 3.9, and observe that all the terms of the random sequence $\{j_2(\varepsilon_m, \sigma_m, \delta_m),\ m \geq 1\}$ are finite with probability 1. Our construction of the sets $\{E_m,\ m \geq 1\}$ relies on an induction argument which, given $\{M_{m-1}, j_{m-1}, E_{m-1}, E_{m-1}^*\}$, evaluates $\{M_m, j_m, E_m, E_m^*\}$. Here, M_m and j_m denote positive integers, whereas E_m and E_m^* are nonvoid subsets of $[0,1]$. It is helpful to first describe the choice of j_m given $\{M_{m-1}, j_{m-1}\}$ for $m \geq 1$. Set

$$(3.87)\qquad \begin{aligned} j_0 = 1, \quad & L_0 = \tfrac{1}{2} \quad \text{and} \quad M_0 = 1, \\ \text{and for } m \geq 1, \quad & L_m = \theta_m h_{\nu_{j_m}} \quad \text{and} \quad L_{m-1}^* = L_{m-1} - L_m. \end{aligned}$$

We choose j_m to be any positive integer such that the following set of conditions hold:

$$(3.88)(\text{i})\qquad j_m > \max\{j_{m-1}, j_2(\varepsilon_m, \sigma_m, \delta_m)\},$$

$$(3.88)(\text{ii})\qquad 3L_m < 3h_{\nu_{j_m}} \leq h_{\nu_{j_m}}^{1-\lambda^2} \leq h_{\nu_{j_m}}^{1-\lambda^2-\delta_m} \leq L_{m-1}^* \leq L_{m-1},$$

$$(3.88)(\text{iii})\qquad \frac{L_{m-1}}{L_{m-1}^*} \leq 1 + \frac{1}{2}\sigma_m,$$

$$(3.88)(\text{iv}) \qquad 1 - \sigma_m \le 1 - \frac{2h_{\nu_{j_m}}}{L^*_{m-1}} < 1,$$

$$(3.88)(\text{v}) \qquad \frac{2h^{\delta_m}_{\nu_{j_m}}}{h^{\lambda^2+\delta_{m-1}}_{\nu_{j_{m-1}}}} \le \theta_m \quad \text{for } m \ge 2,$$

$$(3.88)(\text{vi}) \qquad 6h^{\delta_m}_{\nu_{j_m}} \le \theta_m \quad \text{for } m \ge 1,$$

$$(3.88)(\text{vii}) \qquad 6h^{\delta_m}_{\nu_{j_m}} \le \theta_m M_{m-1} L^*_{m-1} \le \frac{2}{3} h^{-\delta_m}_{\nu_{j_m}}.$$

The fact that $h_{\nu_j} \to 0$ as $j \to \infty$ readily implies that (3.88) holds by selecting j_m sufficiently large, so the existence of j_m (and hence, of L_m and L^*_{m-1}) is guaranteed with probability 1. Moreover, (3.88)(ii) implies that $L^*_{m-1} \ge 3L_m > 0$. To evaluate M_m, E_m and E^*_m, given j_m and $\{M_{m-1}, j_{m-1}, E_{m-1}, E^*_{m-1}\}$, we set

$$(3.89) \qquad E_m = \bigcup_{\substack{i:\, 1 \le i \le m_{j_m} \\ t_{j_m}(i) \in W_{j_m,f}(\varepsilon_m) \cap E^*_{m-1}}} \left[t_{j_m}(i), t_{j_m}(i) + L_m \right]$$

for $m \ge 1$, with $E_0 = [0, L_0] = [0, \frac{1}{2}]$,

$$(3.90) \qquad E^*_m = \bigcup_{\substack{i:\, 1 \le i \le m_{j_m} \\ t_{j_m}(i) \in W_{j_m,f}(\varepsilon_m) \cap E^*_{m-1}}} \left[t_{j_m}(i), t_{j_m}(i) + L^*_m \right]$$

for $m \ge 1$, with $E^*_0 = [0, L^*_0]$,

and, in view of (3.52), (3.58), (3.89) and (3.90), let

$$(3.91) \qquad \begin{aligned} M_m &= N_{j_m,f}(\varepsilon_m, E^*_{m-1}) \\ &= \#\{i: 1 \le i \le m_{j_m}, t_{j_m}(i) \in W_{j_m,f}(\varepsilon_m) \cap E^*_{m-1}\} \end{aligned}$$

for $m \ge 1$, $M_0 = 1$.

It is obvious from (3.89), (3.90), (3.91) and the definitions of M_0, E_0 and E^*_0 that, for each $m \ge 0$, E_m (resp. E^*_m) is the union of M_m closed intervals of length L_m (resp. L^*_m), which will be denoted by $I_{m,k}$ (resp. $I^*_{m,k}$) for $k = 1, \dots, M_m$. Moreover, these intervals are disjoint, since, by (3.88)(ii), $t_{j_m}(i) + L_m < t_{j_m}(i+1)$ and $t_{j_m}(i) + L^*_m < t_{j_m}(i+1)$. We have for $m \ge 0$,

$$(3.92) \qquad E_m = \bigcup_{k=1}^{M_m} I_{m,k} \quad \text{and} \quad E^*_m = \bigcup_{k=1}^{M_m} I^*_{m,k}.$$

Also by construction note that for $m \ge 1$,

$$(3.93) \qquad E_m \subseteq E_{m-1} \quad \text{and} \quad E^*_m \subseteq E^*_{m-1}.$$

To prove that the induction assumption carries over from stage $m-1$ to stage m, we need only check that $M_m \ge 1$, given that $M_{m-1} \ge 1$. This property will be proved by first establishing the following lemma.

LEMMA 3.10. *We have*

$$M_0 L_0^* \geq \tfrac{1}{3} \tag{3.94}$$

and, whenever $M_{m-1} \geq 1$ *for some* $m \geq 1$,

$$M_m L_m^* \geq h_{\nu_{j_m}}^{\lambda^2+\delta_m} \quad \text{and} \quad M_m L_m^* < h_{\nu_{j_m}}^{\lambda^2-2\delta_m}. \tag{3.95}$$

PROOF. Statement (3.94) follows from (3.88)(ii) which implies that

$$M_0 L_0^* = L_0^* = L_0 - L_1 \geq L_0 - \tfrac{1}{3}L_0^* \geq \left(1 - \tfrac{1}{3}\right)L_0 = \tfrac{1}{3}.$$

The first part of (3.95), is proved by induction on $m \geq 1$, assuming that $\{M_k, j_k, E_k, E_k^*\}$ have been defined for $k = 0, \ldots, m-1$ and that $M_{m-1} \geq 1$ (recall that $M_0 = 1$). If such is the case, the notations (3.90) and (3.91), with the formal change of m into $m-1$, say that E_{m-1}^* is the union of M_{m-1} disjoint intervals of lengths equal to L_{m-1}^*. By (3.84)(iii) and (3.86)(i) (which jointly imply that $\varepsilon_m < \varepsilon_0(\delta_m) < \varepsilon_0(\frac{1}{3}\eta) \leq \varepsilon_0(\frac{1}{2}\lambda^2)$), (3.85) (which implies that $\delta_m < 1 - \lambda^2$) and (3.88)(ii), we may apply Lemma 3.9 with $\varepsilon = \varepsilon_m$, $\delta = \delta_m$, $\sigma = \sigma_m$, $E = E_{m-1}^*$ and $L = L_{m-1}^*$, to obtain, via (3.78) and (3.91), that

$$1 - \sigma_m \leq \frac{M_m}{m_{j_m}(E_{m-1}^*)\, p_{j_m}(\varepsilon_m)} \leq 1 + \sigma_m. \tag{3.96}$$

Recall by (3.88)(ii) that $L_{m-1}^* \geq 3h_{\nu_{j_m}}$, and by (3.88)(iv) that $(1 - 2h_{\nu_{j_m}}/L_{m-1}^*) \geq 1 - \sigma_m$. This allows us to apply inequality (3.77) with $E = E_{m-1}^*$, $|E| = M_{m-1}L_{m-1}^*$ and $L = L_{m-1}^*$, to get

$$(1 - \sigma_m) h_{\nu_{j_m}}^{-1} \leq \frac{m_{j_m}(E_{m-1}^*)}{M_{m-1}L_{m-1}^*} < h_{\nu_{j_m}}^{-1}, \tag{3.97}$$

Since (3.84)(iii), (3.85), (3.86)(i) and (ii) and (3.88)(i) jointly imply that (3.63) of Lemma 3.7 holds with $\delta = \delta_m$, $\varepsilon = \varepsilon_m$ and $j = j_m$, this, in combination with (3.96) and (3.97), shows that

$$\begin{aligned}
(1 - \sigma_m)^2 h_{\nu_{j_m}}^{-1+\lambda^2} &\leq (1 - \sigma_m)^2 h_{\nu_{j_m}}^{-1} p_{j_m}(\varepsilon_m) \\
&\leq \frac{M_m}{M_{m-1}L_{m-1}^*} \leq (1 + \sigma_m) h_{\nu_{j_m}}^{-1} p_{j_m}(\varepsilon_m) \\
&\leq (1 + \sigma_m) h_{\nu_{j_m}}^{-1+\lambda^2-\delta_m}.
\end{aligned} \tag{3.98}$$

Now (3.88)(iii) and the definitions (3.87) of L_m and L_m^* imply that

$$L_m/(1 + \sigma_{m+1}) = \theta_m h_{\nu_{j_m}}/(1 + \sigma_{m+1}) \leq L_m^* \leq L_m = \theta_m h_{\nu_{j_m}},$$

and (3.84)(i) implies that

$$\tfrac{1}{2} \leq 1 - \sigma_m, \qquad 1 + \sigma_m \leq \tfrac{3}{2} \quad \text{and} \quad 1 + \sigma_{m+1} \leq \tfrac{3}{2}.$$

These inequalities when combined with (3.98) imply that

$$\begin{aligned}
&\frac{1}{6}\{\theta_m M_{m-1}L^*_{m-1}\}h^{\lambda^2}_{\nu_{j_m}} \\
&\le \left\{(1-\sigma_m)^2 M_{m-1}L^*_{m-1}h^{-1+\lambda^2}_{\nu_{j_m}}\right\}\frac{\theta_m h_{\nu_{j_m}}}{1+\sigma_{m+1}} \\
&\le M_m L^*_m \le \left\{(1+\sigma_m)M_{m-1}L^*_{m-1}h^{-1+\lambda^2-\delta_m}_{\nu_{j_m}}\right\}\theta_m h_{\nu_{j_m}} \\
&\le \frac{3}{2}\{\theta_m M_{m-1}L^*_{m-1}\}h^{\lambda^2-\delta_m}_{\nu_{j_m}}.
\end{aligned} \tag{3.99}$$

Making use of (3.88)(vii), we obtain readily from (3.99) the first part of (3.95). Moreover, we also get from (3.88)(vii) and (3.99), for each $m \ge 1$, the upper bound $M_m L^*_m < h^{\lambda^2-2\delta_m}_{\nu_{j_m}}$, which is the second part of (3.95). This completes the proof of the lemma. □

Now, armed with Lemma 3.10, we can complete the induction argument. For this, we note that (3.85) implies $1-\lambda^2-\delta_m \ge \frac{2}{3}(1-\lambda^2) > 0$, whereas (3.86)(i) and (ii) show that $\theta_m \le (1/16)\varepsilon_m^2 \le 1/64$. Therefore, $L^*_m \le L_m = \theta_m h_{\nu_{j_m}} \le (1/64)h_{\nu_{j_m}}$, so that, by (3.94) and (3.95), we have for $m \ge 1$,

$$M_m \ge 64 h^{-2(1-\lambda^2)/3}_{\nu_{j_m}}, \tag{3.100}$$

which converges to infinity as $m \to \infty$. Moreover, (3.100) and $h_{\nu_{j_m}} < 1$ show that $M_m \ge 64$ for $m \ge 1$, which, in view of $M_0 = 1$, obviously implies that $M_m \ge 1$ for all $m \ge 1$.

This finishes the proof of our induction argument, and establishes the existence of $\{M_m, j_m, E_m, E^*_m\}$ for each $m \ge 0$. This completes Step 1.

STEP 2. *Properties of* E_m. We now turn to the more difficult proof that, for appropriate choices of c, d and Δ, (3.48) holds for all intervals $I \subseteq [0, \frac{1}{2}]$ such that $|I| \le \Delta$. This will be achieved by considering separately a series of special cases depending upon the length and endpoints of I.

Recalling (3.48), (3.58), (3.89) and (3.92), we see that, for any interval $I \subseteq [0, \frac{1}{2}]$ and for each $m \ge 1$,

$$M_m(I) = \#\{I_{m,k} \subseteq I: 1 \le k \le M_m\} \le N_{j_m,f}(\varepsilon_m, I). \tag{3.101}$$

Case 1. We assume that $m \ge 1$ and that $I \subseteq [0, \frac{1}{2}]$ is an interval such that $I \subseteq I_{m-1,k_0}$ for some $k_0 \in \{1,\dots,M_{m-1}\}$. We will consider the following three possibilities, denoted by 1a, 1b and 1c.

Possibility 1a. For $m \ge 1$, let $I \subseteq [0, \frac{1}{2}]$ be an interval such that $I \subseteq I_{m-1,k_0}$ for some $k_0 \in \{1,\dots,M_{m-1}\}$ and $|I| \ge h^{1-\lambda^2-\delta_m}_{\nu_{j_m}}$ [this is possible by (3.88)(i) and (ii)]. By applying Lemma 3.9 with $\varepsilon = \varepsilon_m$, $\delta = \delta_m$, $\sigma = \sigma_m$, $j = j_m$, $E = I$ and $L = |I|$, and then the right side of (3.77) with $E = I$, we obtain via (3.101) that

$$(3.102)\quad \begin{aligned} M_m(I) &\le N_{j_m,f}(\varepsilon_m, I) \\ &\le (1+\sigma_m) m_{j_m}(I) p_{j_m}(\varepsilon_m) \le (1+\sigma_m) h_{\nu_{j_m}}^{-1} |I| p_{j_m}(\varepsilon_m). \end{aligned}$$

Consider first the case where $m \ge 2$. Since $I \subseteq I_{m-1,k_0}$, we have

$$|I| \le |I_{m-1,k_0}| = L_{m-1} = \theta_{m-1} h_{\nu_{j_{m-1}}} \le h_{\nu_{j_{m-1}}} \quad \text{for } m \ge 2.$$

Thus, by (3.94), we have

$$M_{m-1} L^*_{m-1} \ge h_{j_{m-1}}^{\lambda^2 + \delta_{m-1}} \ge |I|^{\lambda^2 + \delta_{m-1}} \quad \text{for } m \ge 2.$$

This, in combination with the left side of (3.98) and (3.102), entails that, for $m \ge 2$,

$$(3.103)\quad \begin{aligned} \frac{M_m(I)}{M_m} &= \frac{M_{m-1}L^*_{m-1}}{M_m} \times M_m(I) \times \frac{1}{M_{m-1}L^*_{m-1}} \\ &\le \frac{1}{(1-\sigma_m)^2 h_{\nu_{j_m}}^{-1} p_{j_m}(\varepsilon_m)} \\ &\quad \times (1+\sigma_m) h_{\nu_{j_m}}^{-1} |I| p_{j_m}(\varepsilon_m) \times \frac{1}{M_{m-1}L^*_{m-1}} \\ &= \frac{1+\sigma_m}{(1+\sigma_m)^2} |I| \times \frac{1}{M_{m-1}L^*_{m-1}} \\ &\le \frac{1+\sigma_m}{(1-\sigma_m)^2} \times \frac{|I|}{h_{\nu_{j_{m-1}}}^{\lambda^2+\delta_{m-1}}} \\ &\le \frac{1+\sigma_m}{(1-\sigma_m)^2} \times \frac{|I|}{|I|^{\lambda^2+\delta_{m-1}}} = \frac{1+\sigma_m}{(1-\sigma_m)^2} |I|^{1-\lambda^2-\delta_{m-1}}. \end{aligned}$$

For $m = 1$, (3.94) yields $M_0 L^*_0 \ge \frac{1}{3}$, so that in this case the first inequality in (3.103) yields

$$(3.104)\quad \begin{aligned} \frac{M_m(I)}{M_m} &\le \frac{1+\sigma_m}{(1-\sigma_m)^2} |I| \times \frac{1}{M_{m-1}L^*_{m-1}} \\ &\le 3\frac{1+\sigma_m}{(1-\sigma_m)^2} |I| \le 3\frac{1+\sigma_m}{(1-\sigma_m)^2} |I|^{1-\lambda^2}. \end{aligned}$$

Possibility 1b. For $m \ge 1$, let $I \subseteq [0, \frac{1}{2}]$ be an interval with $I \subseteq I_{m-1,k_0}$ for some $k_0 \in \{1, \dots, M_{m-1}\}$ and $\frac{1}{2}\theta_m h_{\nu_{j_m}} \le |I| \le h_{\nu_{j_m}}^{1-\lambda^2-\delta_m}$. In this case, there exists an interval I' with $I \subseteq I' \subset [0, \frac{1}{2}]$ and $|I'| = h_{\nu_{j_m}}^{1-\lambda^2-\delta_m}$. Making use of the assumption that $h_{\nu_{j_m}} \le 2\theta_m^{-1}|I|$ and of (3.84)(ii), which implies that $0 < 1 - \lambda^2 - 2\delta_m < 1$, we obtain that

$$h_{\nu_{j_m}}^{1-\lambda^2-2\delta_m} \le \left(2\theta_m^{-1}|I|\right)^{1-\lambda^2-\delta_m} \le 2\theta_m^{-1}|I|^{1-\lambda^2-2\delta_m}. \tag{3.105}$$

For $m \ge 2$, we apply (3.103) with the formal change of I into I', which yields

$$\begin{aligned}\frac{M_m(I)}{M_m} \le \frac{M_m(I')}{M_m} &\le \frac{1+\sigma_m}{(1-\sigma_m)^2}\frac{|I'|}{h_{\nu_{j_{m-1}}}^{\lambda^2+\delta_{m-1}}}\\ &= \frac{1+\sigma_m}{(1-\sigma_m)^2}h_{\nu_{j_m}}^{1-\lambda^2-2\delta_m}\left\{\frac{h_{\nu_{j_m}}^{\delta_m}}{h_{\nu_{j_{m-1}}}^{\lambda^2+\delta_{m-1}}}\right\}\\ &\le \frac{1+\sigma_m}{(1-\sigma_m)^2}|I|^{1-\lambda^2-2\delta_m}\left\{\frac{2\theta_m^{-1}h_{\nu_{j_m}}^{\delta_m}}{h_{\nu_{j_{m-1}}}^{\lambda^2+\delta_{m-1}}}\right\}\\ &\le \frac{1+\sigma_m}{(1-\sigma_m)^2}|I|^{1-\lambda^2-2\delta_m} \quad \text{for } m \ge 2,\end{aligned} \tag{3.106}$$

where we have made use of (3.105) and (3.88)(v) for the last two steps.

Recall from (3.85) that $0 < \delta_m < 1$. It follows from the assumption $h_{\nu_{j_m}} \le 2\theta_m^{-1}|I|$ that

$$h_{\nu_{j_m}}^{\delta_m} \le \left(2\theta_m^{-1}|I|\right)^{\delta_m} \le 2\theta_m^{-1}|I|^{\delta_m}.$$

For $m = 1$, this last inequality, in combination with (3.104) implies that

$$\begin{aligned}\frac{M_m(I)}{M_m} \le \frac{M_m(I')}{M_m} &\le 3\frac{1+\sigma_m}{(1-\sigma_m)^2}|I'|\\ &= \frac{1+\sigma_m}{(1-\sigma_m)^2}h_{\nu_{j_m}}^{1-\lambda^2-2\delta_m} \times 3h_{\nu_{j_m}}^{\delta_m}\\ &\le \frac{1+\sigma_m}{(1-\sigma_m)^2}|I|^{1-\lambda^2-2\delta_m}\left\{6\theta_m^{-1}h_{\nu_{j_m}}^{\delta_m}\right\}\\ &\le \frac{1+\sigma_m}{(1-\sigma_m)^2}|I|^{1-\lambda^2-2\delta_m} \quad \text{for } m = 1,\end{aligned} \tag{3.107}$$

where we have made use of (3.88)(vi).

Possibility 1c. For $m \ge 1$, let $I \subseteq [0, \frac{1}{2}]$ be an interval with $I \subseteq I_{m-1,k_0}$ for some $k_0 \in \{1, \dots, M_{m-1}\}$ and $|I| < \frac{1}{2}\theta_m h_{\nu_{j_m}}$. In this case, we obtain that $M_m(I) = 0$ since $L_m = \theta_m h_{\nu_{j_m}}$.

By combining the inequalities (3.103), (3.104) of Possibility 1a, (3.106), (3.107) of Possibility 1b and $M_m(I) = 0$ in Possibility 1c, we see that, whenever $m \ge 1$ and $I \subseteq [0, \frac{1}{2}]$ is an interval such that $I \subseteq I_{m-1,k_0}$ for some

$K_0 \in \{1, \ldots, M_{m-1}\}$, we have

$$\frac{M_m(I)}{M_m} \le \frac{1+\sigma_m}{(1-\sigma_m)^2}|I|^{1-\lambda^2-2\delta_m} \quad \text{for } m \ge 2; \tag{3.108)(i}$$

$$\frac{M_m(I)}{M_m} \le 3\frac{1+\sigma_m}{(1-\sigma_m)^2}|I|^{1-\lambda^2-2\delta_m} \quad \text{for } m = 1. \tag{3.108)(ii}$$

We now turn to the cases when I is not necessarily a subset of some I_{m-1,k_0}. It will be convenient for $m = 1, 2, \ldots$ to denote by $H(m)$ the statement that, for each interval $I \subseteq [0, \frac{1}{2}]$,

$$\frac{M_m(I)}{M_m} \le 3\left\{\prod_{i=1}^{m}\left(\frac{1+\sigma_i}{1-\sigma_i}\right)^2\right\}|I|^{1-\lambda^2-2\Delta_m}. \tag{3.109}$$

Note for further use that (3.108) implies (3.109), which therefore holds for all $m \ge 1$ under the assumptions of Case 1, that is, if $I \subseteq [0, \frac{1}{2}]$ is a subset of some I_{m-1,k_0} for some $k_0 \in \{1, \ldots, M_{m-1}\}$. In the particular case $m = 1$, we have $M_{m-1} = M_0 = 1$ and $I_{m-1,1} = I_{0,1} = [0, \frac{1}{2}]$, so that (3.109) then holds for each interval $I \subseteq [0, \frac{1}{2}]$ and $H(1)$ is satisfied. Thus, to show that $H(m)$ is satisfied for each $m \ge 1$, it suffices to prove that $H(m-1)$ implies $H(m)$ for each $m \ge 2$.

Assume accordingly that $H(m-1)$ holds for some $m \ge 2$, that is, that, for each interval $I \subseteq [0, \frac{1}{2}]$, we have

$$\frac{M_{m-1}(I)}{M_{m-1}} \le 3\left\{\prod_{i=1}^{m-1}\left(\frac{1+\sigma_i}{1-\sigma_i}\right)^2\right\}|I|^{1-\lambda^2-2\Delta_{m-1}}. \tag{3.110}$$

Toward the aim of showing that $H(m)$ is satisfied, given that it is the case for $H(m-1)$, it will be convenient to first establish the following lemma.

LEMMA 3.11. *Let $m \ge 2$. Assume that $I \subseteq [0,1]$ is an interval with $|I| \ne 0$ and endpoints belonging to the set $\{t_{j_{m-1}}(i), 1 \le i \le m_{j_{m-1}}\}$ and that $H(m-1)$ holds. Then*

$$\frac{M_m(I)}{M_m} \le 3\frac{1+(1/2)\sigma_m}{1+\sigma_m}\left\{\prod_{i=1}^{m}\left(\frac{1+\sigma_i}{1-\sigma_i}\right)^2\right\}|I|^{1-\lambda^2-2\Delta_{m-1}}. \tag{3.111}$$

PROOF. Recall from (3.48) that for $m \ge 2$, $M_m(I) = \#\{I_{m,k} \subseteq I\colon 1 \le k \le M_m\}$ and $M_{m-1}(I) = \#\{I_{m-1,k} \subseteq I\colon 1 \le k \le M_{m-1}\}$. Recalling the notation (3.89) and (3.92), with the change of m to $m-1$, if I has endpoints belonging to $\{t_{j_{m-1}}(i), 1 \le i \le m_{j_{m-1}}\}$, then $I \cap E_{m-1}$ is the disjoint union of the $M_{m-1}(I)$ intervals $I_{m-1,k}$ such that $I_{m-1,k} \subseteq I$ and $1 \le k \le M_{m-1}$. It follows that

$$\begin{aligned}\frac{M_m(I)}{M_m} &\sum_{\substack{k:\,1\le k\le M_{m-1}\\ I_{m-1,k}\subseteq I}} \frac{M_m(I_{m-1,k})}{M_m}\\ &\le M_{m-1}(I) \times \max_{1\le k\le M_{m-1}}\left(\frac{M_m(I_{m-1,k})}{M_m}\right).\end{aligned} \tag{3.112}$$

By (3.88)(ii), for any $k \in \{1, \dots, M_{m-1}\}$, $|I_{m-1,k}| = L_{m-1} \geq h_{\nu_{j_m}}^{1-\lambda^2-\delta_m}$. It follows that $I_{m-1,k}$ satisfies the assumptions of Possibility 1a of Case 1. By applying the first inequality in (3.103) with the formal replacements of I and $|I|$ by $I_{m-1,k}$ and L_{m-1}, respectively, in combination with (3.88)(iii), we therefore obtain the inequalities

$$(3.113)\qquad \begin{aligned} \frac{M_m(I_{m-1,k})}{M_m} &\leq \frac{1+\sigma_m}{(1-\sigma_m)^2} \frac{L_{m-1}}{M_{m-1}L^*_{m-1}} \\ &\leq \frac{1+\sigma_m}{(1-\sigma_m)^2} \times \left(1 + \frac{1}{2}\sigma_m\right) \times \frac{1}{M_{m-1}}. \end{aligned}$$

By combining (3.110), (3.112) and (3.113), we see that, under the assumptions of the lemma,

$$\begin{aligned} \frac{M_m(I)}{M_m} &\leq \frac{M_{m-1}(I)}{M_{m-1}} \times \left(\frac{1+\sigma_m}{1-\sigma_m}\right)^2 \times \frac{1+(1/2)\sigma_m}{1+\sigma_m} \\ &\leq 3\frac{1+(1/2)\sigma_m}{1+\sigma_m}\left\{\prod_{i=1}^{m}\left(\frac{1+\sigma_i}{1-\sigma_i}\right)^2\right\}|I|^{1-\lambda^2-2\Delta_{m-1}}, \end{aligned}$$

which is (3.111). □

Recalling the notations (3.89) and (3.92), we see that for $m \geq 2$, an arbitrary interval $I \subseteq [0, \frac{1}{2}]$ must be of one of the following types:

Type (i): $I \cap I_{m-1,k} = \varnothing$ for each $k = 1, \dots, M_{m-1}$, in which case we will set $I' = \varnothing$.

Type (ii): $I \cap I_{m-1,k_0} \neq \varnothing$ for some $k_0 \in \{1, \dots, M_{m-1}\}$, in which case we will let I' denote the smallest possible closed interval containing all $I_{m-1,k}$ with $k \in \{1, \dots, M_{m-1}\}$ such that $I \cap I_{m-1,k} \neq \varnothing$.

Observe that we have only the following two possibilities for type (ii):

Type (iia): $I' = I_{m-1,k_0}$, that is, I' contains at most one interval $I_{m-1,k}$ with $k \in \{1, \dots, M_{m-1}\}$.

Type (iib): I' contains at least two intervals $I_{m-1,k}$ with $k \in \{1, \dots, M_{m-1}\}$, in which case I' may be decomposed into the disjoint union of two intervals, $I' = I_1 \cup I_2$, where $I_1 \subseteq I_{m-1,k_1}$ for some $k_1 \in \{1, \dots, M_{m-1}\}$ and I_2 has endpoints belonging to $\{t_{\nu_{j_{m-1}}}(i), 1 \leq i \leq m_{j_{m-1}}\}$.

Let $I'' = I \cap I'$ and note for further use that in all cases, we have

$$(3.114)\qquad M_m(I) = M_m(I'') \leq M(I') \quad \text{and} \quad |I''| \leq |I|.$$

We will show that $H(m-1) \Rightarrow H(m)$ for $m \geq 2$ by considering successively the cases where I belongs to either Type (i), Type (iia) or Type (iib).

Case 2. We assume that $m \geq 2$ and that I is of Type (i), that is, that

$I \cap I_{m-1,k} = \varnothing$ for each $k = 1, \dots, M_{m-1}$. In this case, $I' = \varnothing$, whence (3.114) yields

$$\frac{M_m(I)}{M_m} \le \frac{M_m(I')}{M_m} = 0 \tag{3.115}$$

and $H(m)$ holds for each $m \le 2$.

Case 3. We assume that $m \ge 2$ and that I is of Type (iia), that is, that $I' = I_{m-1,k_0}$ for some $k_0 \in \{1, \dots, M_{m-1}\}$. In this case, the interval $I'' = I \cap I' \subseteq I_{m-1,k_0}$ satisfies the assumptions of Case 1. It therefore follows readily from (3.108)(i), when applied with the formal replacement of I by I'', and (3.114), that

$$\frac{M_m(I)}{M_m} = \frac{M_m(I'')}{M_m} \le \frac{1+\sigma_m}{(1-\sigma_m)^2}|I''|^{1-\lambda^2-2\delta_m} \le \frac{1+\sigma_m}{(1-\sigma_m)^2}|I|^{1-\lambda^2-2\delta_m},$$

so that $H(m)$ holds also in this case for each $m \ge 2$.

Case 4. We finally assume that $m \ge 2$, $H(m-1)$ holds and I is of Type (iib). Let $I' = I_1 \cup I_2$ be as above and note for further use that these assumptions imply that $I'' = I \cap I'$ satisfies

$$|I_2| - L_{m-1} \le |I''| \le |I| \tag{3.116)(i}$$

$$|I''| \le |I'| = |I_1| + |I_2|, \tag{3.116)(ii}$$

$$|I_1| = L_{m-1} = \theta_{m-1} h_{\nu_{j_{m-1}}} \quad \text{and}$$
$$|I_2| - L_{m-1} \ge h_{\nu_{j_{m-1}}} - L_{m-1} = (1-\theta_{m-1}) h_{\nu_{j_{m-1}}}. \tag{3.116)(iii}$$

By first applying (3.108)(i) with the formal replacement of I by I_1 and then (3.116)(i) and (iii), we obtain the inequalities

$$\begin{aligned}
\frac{M_m(I_1)}{M_m} &\le \frac{1+\sigma_m}{(1-\sigma_m)^2}|I_1|^{1-\lambda^2-2\delta_m} \\
&= \frac{1+\sigma_m}{(1-\sigma_m)^2}|I|^{1-\lambda^2-2\delta_m}\left(\frac{|I_1|}{|I|}\right)^{1-\lambda^2-2\delta_m} \\
&\le \frac{1+\sigma_m}{(1-\sigma_m)^2}|I|^{1-\lambda^2-2\delta_m}\left(\frac{|I_1|}{|I_2| - L_{m-1}}\right)^{1-\lambda^2-2\delta_m} \\
&\le \frac{1+\sigma_m}{(1-\sigma_m)^2}|I|^{1-\lambda^2-2\delta_m}\left(\frac{\theta_{m-1}}{1-\theta_{m-1}}\right)^{1-\lambda^2-2\delta_m} \\
&\le \frac{1+\sigma_m}{(1-\sigma_m)^2}|I|^{1-\lambda^2-2\delta_m}\left(\frac{\theta_{m-1}}{1-\theta_{m-1}}\right)^{2(1-\lambda^2)/3} \\
&\le \frac{(1/2)\sigma_m}{1+\sigma_m}\left(\frac{1+\sigma_m}{1-\sigma_m}\right)^2 |I|^{1-\lambda^2-2\delta_m}
\end{aligned} \tag{3.117}$$

$$\leq 3\frac{(1/6)\sigma_m}{1+\sigma_m}\left\{\prod_{i=1}^{m}\left(\frac{1+\sigma_i}{1-\sigma_i}\right)^2\right\}|I|^{1-\lambda^2-2\Delta_m},$$

where we have used (3.86)(v), in combination with the observation that by (3.84)(iii),

$$0 < \tfrac{2}{3}(1-\lambda^2) < 1-\lambda^2-2\Delta_m < 1-\lambda^2-2\delta_m < 1.$$

Likewise, by applying (3.111) with the formal replacement of I by I_2 and then (3.116)(i) and (iii), we obtain the inequalities

$$\begin{aligned}
\frac{M_m(I_2)}{M_m} &\leq 3\frac{1+(1/2)\sigma_m}{1+\sigma_m}\left\{\prod_{i=1}^{m}\left(\frac{1+\sigma_i}{1-\sigma_i}\right)^2\right\}|I_2|^{1-\lambda^2-2\Delta_{m-1}} \\
&\leq 3\frac{1+(1/2)\sigma_m}{1+\sigma_m}\left\{\prod_{i=1}^{m}\left(\frac{1+\sigma_i}{1-\sigma_i}\right)^2\right\} \\
&\quad\times\left(\frac{|I_2|}{|I|}\right)^{1-\lambda^2-2\Delta_m}|I|^{1-\lambda^2-2\Delta_m} \\
&\leq 3\frac{1+(1/2)\sigma_m}{1+\sigma_m}\left\{\prod_{i=1}^{m}\left(\frac{1+\sigma_i}{1-\sigma_i}\right)^2\right\} \\
&\quad\times\left(\frac{|I_2|}{|I_2|-L_m}\right)^{1-\lambda^2-2\Delta_m}|I|^{1-\lambda^2-2\Delta_m} \\
&\leq 3\frac{1+(1/2)\sigma_m}{1+\sigma_m}\left\{\prod_{i=1}^{m}\left(\frac{1+\sigma_i}{1-\sigma_i}\right)^2\right\} \\
&\quad\times\left(\frac{1}{1-\theta_{m-1}}\right)^{1-\lambda^2}|I|^{1-\lambda^2-2\Delta_m} \\
&\leq 3\frac{1+(5/6)\sigma_m}{1+\sigma_m}\left\{\prod_{i=1}^{m}\left(\frac{1+\sigma_i}{1-\sigma_i}\right)^2\right\}|I|^{1-\lambda^2-2\Delta_m},
\end{aligned} \tag{3.118}$$

where we have used (3.86)(vi).

By combining (3.116)(ii) with (3.117) and (3.118), we readily obtain that

$$\begin{aligned}
\frac{M_m(I)}{M_m} &= \frac{M_m(I'')}{M_m} \leq \frac{M_m(I')}{M_m} = \frac{M_m(I_1)}{M_m} + \frac{M_m(I_2)}{M_m} \\
&\leq 3\left\{\prod_{i=1}^{m}\left(\frac{1+\sigma_i}{1-\sigma_i}\right)^2\right\}|I|^{1-\lambda^2-2\Delta_m},
\end{aligned} \tag{3.119}$$

so that $H(m)$ holds, as sought.

By the above remark that $H(1)$ holds, it follows from (3.108) in Case 1 (3.115) in Case 2, (3.117) in Case 3 and (3.119) in Case 4 that $H(m)$ [see, e.g., (3.109)] holds for each $m \geq 1$. By (3.84)(ii) and (3.85), this implies in turn that, for every interval $I \subseteq [0, \frac{1}{2}]$ and $m \geq 1$, we have

$$M_m(I) \leq 6|I|^{1-\lambda^2-\eta} M_m, \tag{3.120}$$

which completes Step 2.

STEP 3. *The Hausdorff dimension of $L(f)$*. By (3.92), (3.93) and (3.120), we see that the sequence of random sets $\{E_m, m \geq 1\}$ satisfies with probability 1 the assumptions of Lemma 3.5. By applying this lemma with $c = 1 - \lambda^2 - \eta$, $d = 6$, $\Delta = \frac{1}{2}$ and $m(I) = 1$, we obtain that $s^{1-\lambda^2-\eta}\text{-mes}(\mathbb{K}) > 0$ a.s., where $\mathbb{K} := \bigcap_{m=1}^{\infty} E_m$.

In view of Lemma 3.6 and of the inequality $L_m = \theta_m h_{\nu_{j_m}} \leq \theta(\varepsilon_m) h_{\nu_{j_m}}$, the fact [following from (3.88)(i)] that $j_m \geq j_2(\varepsilon_m, \sigma_m, \delta_m) \geq j_0(\varepsilon_m, \theta(\varepsilon_m))$, implies via (3.53) that for each $m \geq 1$, $\mathbb{N}(L_m, W_{j_m, f}(\varepsilon_m)) \subseteq U_{j_m, f}(2\varepsilon_m)$. Since, by definitions (3.50) and (3.89), $E_m \subseteq \mathbb{N}(L_m, W_{j_m, f}(\varepsilon_m))$, it follows that, for each $m \geq 1$, $E_m \subseteq U_{j_m, f}(2\varepsilon_m)$. In view of (3.51), this means that, for each $m \geq 1$ and $t \in E_m$, $\|b_{\nu_{j_m}}^{-1} \xi_{\nu_{j_m}}(h_{\nu_{j_m}}, t; \mathbf{I}) - f\| \leq 2\varepsilon_m$. Since $\mathbb{K} = \bigcap_{m=1}^{\infty} E_m$, this last inequality also holds for each $t \in \mathbb{K}$ and all $m \geq 1$. The fact [see, e.g., (3.86)(i)] that $\varepsilon_m \to 0$ implies in turn that $t \in L(f)$, and hence, that $\mathbb{K} \subseteq L(f)$. Thus (3.49) holds with probability 1. In view of the arguments following (3.49), this suffices for (3.44) and completes Step 3.

REMARK 3.2. Note that to prove (3.44), we have only made use of the Assumptions (I) and (III). In particular, if we assume that $\Gamma_n = W$ is a single Wiener process, the Assumption (I) holds by Lévy (1948), whereas Assumption (III) is trivial. In this case, our arguments show readily that, if $\{W(t), t \geq 0\}$ is a Wiener process, for any $f \in \mathbb{S}_1$ with $\lambda^2 = \int_0^1 \dot{f}(u)^2\,du \in (0,1)$, we have

$$\dim B(f) \geq 1 - \lambda^2, \tag{3.121}$$

where

$$B(f) = \Big\{ t \in [0,1) : \liminf_{h \downarrow 0} \|(2h\log(1/h))^{-1/2} \times (W(t + h\mathbf{I}) - W(t)) - f\| = 0 \Big\}. \tag{3.122}$$

In Deheuvels and Mason (1994b) it is shown that in fact we have $\dim B(f) = 1 - \lambda^2$.

REMARK 3.3. It is obvious from the arguments used in the preceding proof of (3.44) that, for any $f \in \mathbb{S}_1$ with $\lambda^2 = \int_0^1 \dot{f}(u)^2\,du \in (0,1)$ and each nonvoid open subinterval I of $[0,1]$, we have $\dim\{L(f) \cap I\} = 1 - \eta^2$ a.s. This, in turn, implies that $L(f) \cap I \neq \varnothing$ a.s., whence an easy argument shows that $L(f)$ is almost surely everywhere dense in $[0,1]$. Since, by (3.4), L_Λ is the union of the $L(f)$'s such that $\lambda^2 \geq \Lambda$, it follows that L_Λ is almost surely everywhere dense in $[0,1]$ for each $\Lambda \in [0,1)$. This completes the proofs of Theorems 2.1 and 3.1, and, in view of Remark 2.1, of Theorem 1.1.

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